Vagueness – a Rough Set View

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Abstract. Vagueness for a long time has been studied by philosophers, logicians and linguists. Recently researchers interested in AI contributed essentially to this area.
In this paper we present a new approach to vagueness, called rough set theory. The starting of the theory theory is the assumption that fundamental mechanisms of human reasoning are based on the ability to classify object of interest, i.e. group objects into similarity classes, which form granules (basic concepts) of knowledge about the universe of discourse (e.g. color, height, weight etc.). Every union of basic concepts is called a precise (crisp) concept, otherwise the concept is called imprecise (rough). Thus rough concepts (sets) cannot be expressed in terms of elementary concepts (set). Therefore with each imprecise concept a pair of precise concepts, called its lower and upper approximation, is associated. Approximations are basic operations of rough set theory.
The paper contains basics of rough set theory, shows some of its applications, and the relationship to fuzzy sets, the theory of evidence, discriminant analysis and boolean reasoning methods are pointed out.

1 Introduction

Vagueness has been studied for many years by researchers interested in mathematics, philosophical logic and philosophy of language (see e.g. [1, 2, 6, 8, 12, 13, 30, 36, 47, 59, 69, 70, 72]). Recently, researchers interested in AI contributed essentially to this area of research. The most important contributions seemingly are fuzzy set theory (see [92]) and the theory of evidence (see [74]).

This paper presents another approach to vagueness based on rough set theory (see [60]).

Rough set theory bears on the assumption that we have initially some information (knowledge) about elements of the universe we are interested in. Evidently to some elements of the universe the same information can be associated and consequently the elements can be *similar* or *indiscernible*, in view of the available information. Similarity is assumed to be a reflexive and symmetric relation, whereas the indiscernibility relation - also transitive. Thus similarity is a tolerance relation and indiscernibility is an equivalence relation.

The concepts of similarity and indiscernibility attracted attention of philosophers and logicians for many years (see e.g. [88, 91]), nevertheless these concepts are still far of being understood fully.

2 The Boundary-line Approach to Vagueness

The idea of vagueness is usually connected with the so called "boundary-line" approach first formulated by Frege (see [21]), who writes:

"The concept must have a sharp boundary. To the concept without a sharp boundary there would correspond an area that had not a sharp boundary-line all around ".

Thus according to Frege "the concept without a sharp boundary", i.e. vague concept, must have boundary-line examples which cannot be classified, on the basis of available information, neither to the concept nor to its complement. For example the concept of an *odd (even) number* is precise, because every number is either odd or even - whereas the concept of a *beautiful women* is vague, because for some women we cannot decide whether they are beautiful or not (there are boundary-line cases).

In the rough set approach vagueness is due to the lack of information about some elements of the universe. If with some elements the same information is associated, in view of this information these elements are indiscernible. For example if some patients suffering from a certain disease display the same symptoms, they are indiscernible with respect to these symptoms. It turns out that the indiscernibility leads to the boundary-line cases, i.e. some elements cannot be classified neither to the concept nor to its complement, in view of the available information and thus form the boundary-line cases.

Now let us present these ideas more formally.

Suppose we are given a finite not empty set U called the *universe*, and let I be a binary relation on U. By $I(x)$ we mean the set of all $y \in U$ such that yIx. If I is reflexive and symmetric, i.e.

$$xIx, \text{ for every } x \in U,$$

$$xIy, \text{ implies } yIx \text{ for every } x, y \in U,$$

then I is a tolerance relation. If I is also transitive, i.e. xIy and yIz implies xIz, then I is an equivalence relation. In this case $I(x) = [x]_I$, i.e. $I(x)$ is an equivalence class of the relation I containing element x. If I is a tolerance relation and xIy, then x, y are called *similar* with respects to I (*I-similar*), whereas if I is an equivalence relation and xIy, then x, y are referred to as *indiscernible* with respect to I (*I-indiscernible*). For the sake of simplicity we will assume in this paper that I is an equivalence relation.

Let us define now two following operations on sets

$$I_*(X) = \{x \in U : I(x) \subseteq X\},$$

$$I^*(X) = \{x \in U : I(x) \cap X \neq \emptyset\},$$

assigning to every subset X of the universe U two sets $I_*(X)$ and $I^*(X)$ called the *I-lower* and the *I-upper approximation* of X respectively. The set

$$BN_I(X) = I^*(X) - I_*(X)$$

will be referred to as the *I-boundary region* of X.

If the boundary region of X is the empty set, i.e. $BN_I(X) = \emptyset$, then X will be called *crisp* (*exact*) with respect to I; in the opposite case, i.e. if $BN_I(X) \neq \emptyset$, X will be referred to as *rough* (*inexact*) with respect to I.

Thus rough sets seems to be a natural mathematical model of vague concepts.

One can easily show the following properties of approximations:

1) $I_*(X) \subseteq X \subseteq I^*(X)$,
2) $I_*(\emptyset) = I^*(\emptyset) = \emptyset, I_*(U) = I^*(U) = U$,
3) $I^*(X \cup Y) = I^*(X) \cup I^*(Y)$,
4) $I_*(X \cap Y) = I_*(X) \cap I_*(Y)$,
5) $X \subseteq Y$ implies $I_*(X) \subseteq I_*(Y)$ and $I^*(X) \subseteq I^*(Y)$,
6) $I_*(X \cup Y) \supseteq I_*(X) \cup I_*(Y)$,
7) $I^*(X \cap Y) \subseteq I^*(X) \cap I^*(Y)$,
8) $I_*(-X) = -I^*(X)$,
9) $I^*(-X) = -I_*(X)$,
10) $I_*(I_*(X)) = I^*(I_*(X)) = I_*(X)$,
11) $I^*(I^*(X)) = I_*(I^*(X)) = I^*(X)$.

It is easily seen that the lower and the upper approximation of a set are interior and closure operations in a topology generated by the indiscernibility relation. Thus vagueness is related to some topological properties of inexact concepts.

Vagueness can be also characterized numerically by defining the following coefficient, called the *accuracy* of *approximation*

$$\alpha_I(X) = \frac{|I_*(X)|}{|I^*(X)|},$$

where $|X|$ denotes the cardinality of X.

Obviously $0 \leq \alpha_I(X) \leq 1$. If $\alpha_I(X) = 1$, X is *crisp* with respect to I (the concept X is *precise* with respect to I), and otherwise, if $\alpha_I(X) < 1$, X is *rough* with respect to I (the concept X is vague with respect to I).

3 Topological Classification of Vagueness

It turns out that the above considerations give rise to the following four basic classes of rough sets, i.e. four classes of vagueness:

a) $I_*(X) \neq \emptyset$ and $I^*(X) \neq U$, iff X is *roughly I-observable,*
b) $I_*(X) = \emptyset$ and $I^*(X) \neq U$, iff X is *internally I-unobservable,*
c) $I_*(X) \neq \emptyset$ and $I_*(X) = U$, iff X is *externally I-unobservable,*
d) $I_*(X) = \emptyset$ and $I^*(X) = U$, iff X is *totally I-unobservable.*

The intuitive meaning of this classification is the following.

If X is roughly I-observable we are able to decide for some elements of U whether they belong to X or $-X$.

If X is internally I-unobservable we are able to decide whether some elements of U belong to $-X$, but we are unable to decide for any element of U whether it belongs to X or not.

If X is externally I-unobservable we are able to decide for some elements of U whether they belong to X, but we are unable to decide for any element of U whether it belongs to $-X$ or not.

If X is totally I-unobservable, we are unable to decide for any element of U whether it belongs to X or $-X$.

That means, that X is roughly observable if there are some elements in the universe which can be positively classified, to X or $-X$.

External I-unobservability of a set refers to a situation when positive classification is possible for some elements, but it is impossible to determine that an element does not belong to X.

4 An Example

In this section we will illustrate the above ideas intuitively, by means of an indiscernibility relation generated by data.

Data are often presented as a table, columns of which are labeled by *attributes*, rows by *objects* of interest and entries of the table are *attribute values.* For example, in a table containing information about patients suffering from a certain disease objects are *patients* (strictly specking their ID's), attributes can be, for example, *blood pressure, body temperature* etc., whereas the entry corresponding to object *Smiths* and the attribute *blood preasure* can be *normal.* Such tables are known as *information systems.*

Table 1. Example of an information system

Patient	Headache	Muscle-pain	Temperature	Flu
p1	no	yes	high	yes
p2	yes	no	high	yes
p3	yes	yes	very high	yes
p4	no	yes	normal	no
p5	yes	no	high	no
p6	no	yes	very high	yes

Each row of the table can be seen as information about specific patient. For example patient p2 is characterized in the table by the following attribute-value set

(Headache, yes), (Muscle-pain, no), (Temperature, high), (Flu, yes),

which form information about the patient.

Obviously each subset of attributes defines an indiscernibility (equivalence) relation on the set of patients. Patients are indiscernible by a set of attributes if they have the same values of the attributes.

For example, patients p2, p3 and p5 are indiscernible with respect to the attribute Headache, patients p3 and p6 are indiscernible with respect to the attributes Muscle-pain and Flu, and patients p2 and p5 are indiscernible with respect to the attributes Headache, Muscle-pain and Temperature.

Patient p2 has flu, whereas patient p5 does not, and they are indiscernible with respect to the attributes Headache, Muscle-pain and Temperature, hence flu cannot be characterized in terms of the attributes Headache, Muscle-pain and Temperature. Thuse p2 and p5 are the boundary-line cases, which cannot be properly classified in view of the available knowledge. The remaining patients p1, p3 and p6 display symptoms which enable us to classify them with certainty as having flu, patients p2 and p5 cannot be excluded as having flu and patient p4 for sure does not have flu, in view of the displayed symptoms. Thus the lower approximation of the set of patients having flu is the set {p1, p3, p6} and the upper approximation of this set is the set {p1, p2, p3, p5, p6}, whereas the boundary-line cases are patients p2 and p5. Similarly p4 does not have flu and p2, p5 cannot be excludes as having flu, thus the lower approximation of this concept is the set {p4}, whereas – the upper approximation – is the set {p2, p4, p5} and the boundary region of the concept "not flu" is the set {p2, p5}, the same as in the previous case. Hence the accuracy of approximation of "flu", $\alpha(X_{flu}) = 3/5$ and $\alpha(X_{notflu}) = 1/3$.

5 Vagueness and Uncertainty

A vague concept has a boundary-line cases, i.e. elements which cannot be with *certainty* classified as elements of the concept i.e., we are uncertain whether the boundary-line cases belong to the concept or not. Hence *uncertainty* is related to the *membership* of elements to a set. Therefore in order to discuss the problem of uncertainty from the rough set perspective we have to define a *rough membership function*, and investigate its properties.

The rough membership function can be defined employing the relation I in the following way (see [63]):

$$\mu_X^I(x) = \frac{|X \cap I(x)|}{|I(x)|}.$$

Obviously $0 \leq \mu_X^I(x) \leq 1$.

The rough membership has a probabilistic flavour and can be interpreted as a conditional probability which expresses a degree to which an element belongs to a set. For example, patient p1 can be classified as having flu on the basis of his body temperature with probability 3/5.

The rough membership function can be used to define the approximations and the boundary region of a set, as shown below:

$$I_*(X) = \{x \in U : \mu_X^I(x) = 1\},$$

$$I^*(X) = \{x \in U : \mu_X^I(x) > 0\},$$
$$BN_I(X) = \{x \in U : 0 < \mu_X^I(x) < 1\}.$$

Thus there exists a strict connection between vagueness and uncertainty. As we mentioned above vagueness is related to sets (concepts), whereas uncertainty is related to elements of sets, and the rough set approach shows clear connection between the two concepts.

It can be shown (see [63]) that the rough membership function has the following properties:

a) $\mu_X^I(x) = 1$ iff $x \in I_*(X)$,
b) $\mu_X^I(x) = 0$ iff $x \in U - I^*(X)$,
c) $0 < \mu_X^I(x) < 1$ iff $x \in BN_I(X)$,
d) If $I = \{(x,x) : x \in U\}$, then $\mu_X^I(x)$ is the characteristic function of X,
e) If xIy, then $\mu_X^I(x) = \mu_X^I(y)$,
f) $\mu_{U-X}^I(x) = 1 - \mu_X^I(x)$ for any $x \in U$,
g) $\mu_{X\cup Y}(x) \geq \max(\mu_X^I(x), \mu_Y^I(x))$ for any $x \in U$,
h) $\mu_{X\cap Y}^I(x) \leq \min(\mu_X^I(x), \mu_Y^I(x))$ for any $x \in U$,
i) If $\mathbf{X}$ is a family of pair wise disjoint sets of U, then $\mu_{\cup\mathbf{X}}^I(x) = \sum_{X\in\mathbf{X}} \mu_X^I(x)$ for any $x \in U$.

The above properties show clearly the difference between fuzzy and rough memberships. In particular properties g) and h) show that the rough membership can be regarded as a generalization of of fuzzy membership.

6 Applications

Rough set theory has found many interesting applications. The rough set approach seems to be of fundamental importance to AI and cognitive sciences, especially in the areas of machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, inductive reasoning and pattern recognition. It seems of particular importance to decision support systems.

The main advantage of rough set theory is that it does not need any preliminary or additional information about data – like probability in statistics, or basic probability assignment in Dempster-Shafer theory and grade of membership or the value of possibility in fuzzy set theory.

Rough set theory has been successfully applied in many real-life problems e.g., in medicine, pharmacology, engineering, banking, financial and market analysis and others. Some exemplary applications are listed below.

Medicine turned out to be a very interesting domain of application of rough sets (see e.g., [28, 82, 83, 84, 85, 87]). In pharmacology the analysis of relationships between the chemical structure and the antimicrobial activity of drugs (see [43, 44, 45, 46]) has been successfully investigated. Banking applications include evaluation of a bankruptcy risk (see [80, 81]) and market research (see [24, 95]). Very interesting results have been also obtained in speaker independent speech recognition (see [10, 14, 15, 16, 17]) and acoustics (see [40, 41]). The rough set approach seems also important for various engineering applications, like diagnosis of machines using vibroacoustics symptoms (noise, vibrations) (see [55, 56, 57]), material sciences (see [33]) and process control (see e.g., [53, 54, 67, 86, 96]). Application in linguistics (see e.g., [26, 27, 39, 51]) and environment (see [29]), databses (see e.g., [3, 4, 5, 73]) are other important domains, where rough set proved to be a valuable tool.

More about applications of rough set theory can be found in the references (see e.g., [48, 49, 78, 93]). Besides, many other fields of application, e.g., time series analysis, image processing and character recognition, are being extensively explored.

7 Conclusion

Rough set theory seems to be well suited as a mathematical model of vagueness and uncertainty. Vagueness is a property of sets (concepts) and is strictly related to the existence to the boundary region of a set, whereas uncertainty is a property of elements of sets and is related to the rough membership function. In the rough set approach both concepts are closely related and are due to the indiscernibility caused by insufficient information about the world.

Rough set theory overlaps to a certain degree many other mathematical theories. Particularly interesting is the relationship with fuzzy set theory and Dempster-Shafer theory of evidence. The concepts of rough set and fuzzy set are different since they refer to various aspects of imprecision (see [63]) whereas the connection with theory of evidence is more substantial (see [76]). Besides, rough set theory is related to discriminant analysis (see [42]), Boolean reasoning methods (see [77]) and others. The relationship between rough set theory and decision analysis is presented in (see [64, 79]). More details concerning these relationships can be found in the references. Nevertheless rough set theory can be viewed in its own rights as an independent discipline with considerable achievements to its credit.

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