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Vaidya Spacetime as an Evaporating Black Hole

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The energy-momentum tensor of a scalar field for an evaporating black hole modeled by the two-dimensional Vaidya metric is examined. It is shown that the energy flux from a naked singularity, formed when a black hole disappears, can be divergent even when the mass M disappears with the condition $dM/dv \to 0$ (v the usual advanced time).

It is widely believed that a black hole which is formed by collapse of matter radiates the thermal radiation whose temperature is proportional to the surface gravity.[1] In its original derivation, it is assumed that the spacetime is static or stationary to calculate this Hawking radiation. This assumption will be valid only when the radiated energy is negligibly small compared with the mass energy of the black hole. When the radiation becomes sufficiently large, it will be modified via the Einstein equation. However, we have no formula to describe such a situation. If we want to consider the back-reaction effectively, we may consider a dynamical geometry as a background spacetime as the first step.

Hiscock[2] considered the energy-momentum tensor of a massless scalar field in the two-dimensional Vaidya spacetime without angular part. This procedure corresponds to the geometrical optic approximation in four-dimensional spacetime[3] and we have a general formula for calculating the energy-momentum tensor in two-dimensional spacetime.[4] He examined the situation that the black hole evaporates out and disappears at the finite time without creation of infinite radiation, and concluded that the finite radiation is created if the black hole decreases its mass $M(v)$ to zero with the condition $\dot{M}(=dM/dv)=0$. We cannot, however, conclude in such a way. In this letter we shall examine the energy-momentum tensor of a scalar field in an explicit model spacetime of which we can put the metric into double null form, and show that there is a counter-example against Hiscock's condition.

The two-dimensional imploding Vaidya metric is given by

$$ds^2=\left(1-\frac{2M(v)}{r}\right)dv^2-2dvdr. \tag{1}$$

The model spacetime is as follows:

$$M(v)=\begin{cases} 0, & v<-v_0(\text{region I}) \\ m(v), & -v_0<v<0(\text{region II}) \\ 0, & v>0(\text{region III}), \end{cases} \tag{2}$$

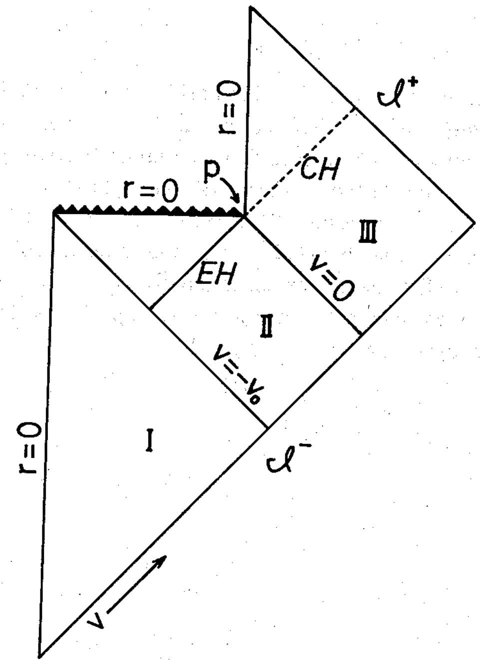

Fig. 1. Penrose diagram of the model for evaporating black hole. Regions I and III are flat spacetimes, and region II is an imploding Vaidya spacetime. The singularity is represented by a jagged line and the point **p** is the naked singularity. Event horizon and Cauchy horizon are represented by ***EH*** and ***CH*** respectively.

where

$$m(v)=\frac{\beta x^2}{2(1-2\beta x)} \tag{3}$$

with

$$x=\frac{1}{2\beta}(1-\sqrt{1+2\beta v}). \tag{4}$$

From the regularity of the metric we must restrict β and v_0 so that the condition $2\beta v_0<1$ is satisfied. Note the mass satisfies the condition $m(0)=\dot{m}(0)=0$. The discontinuity of the mass at the time $v=-v_0$ must be considered as the collapse of δ-functional shell. The spacetime structure is shown in Fig. 1. We can transform the metric in each region into the double null form as

$$ds^2=\begin{cases} dudv, & \text{(region I)} \\ C(\tau,v)d\tau dv, & \text{(region II)} \\ dUdv. & \text{(region III)} \end{cases} \tag{5}$$

In region II, C and τ are given by

$$C(v,\tau)=-\frac{2\pi z}{\beta x^2+z}(\sqrt{z}J_1(2\sqrt{\beta z})+x\sqrt{\beta}J_0(2\sqrt{\beta z}))^2 \tag{6}$$

with

$$z=r-\beta x^2 \tag{7}$$

and

$$\tau=\frac{\sqrt{z}N_1(2\sqrt{\beta z})+x\sqrt{\beta}N_0(2\sqrt{\beta z})}{\sqrt{z}J_1(2\sqrt{\beta z})+x\sqrt{\beta}J_0(2\sqrt{\beta z})}, \tag{8}$$

where J_ν and N_ν are the Bessel functions of the first and second kind of order ν. For the case $z<0$, we must analytically continue the metric coefficient C and the coordinate τ. In such a region, the regular retarded time is given by $\bar{\tau}(=\tau-i)$. Note the line $z=0$ corresponds to the event horizon. The relations between the retarded null times u,τ(or $\bar{\tau}$) and U are given by the continuity of the metric at the boundary of each region.

As the metric can be put into double null form for whole spacetime, we can apply the general scheme to obtain two-dimensional energy-momentum tensor developed in Ref. 4). We shall choose the initial positive frequency modes as $\exp(i\omega u)$ and $\exp(i\omega v)$. By the coordinate transformation, we can obtain the energy-momentum tensor of the created particle in each region.

In region I, we have

$$T_{\mu\nu}=0. \quad (\mu,\ \nu=u,\ v) \tag{9}$$

In region II, we have[5)]

$$24\pi T_{\tau\tau}=A(\tau,\ v)-A(\tau,\ -v_0), \tag{10}$$

$$24\pi T_{vv}=\frac{\beta x}{8r^4(2\beta x-1)^3}\{\beta x^3(12\beta^2x^2-6\beta x+1)+4x(2\beta^2x^2-2\beta x+1)z+4(1-\beta x)z\}, \tag{11}$$

$$24\pi T_{v\tau}=\frac{\pi\beta z x^2}{r^4(1-2\beta x)}(\sqrt{z}J_1+x\sqrt{\beta}J_0)^2, \tag{12}$$

where

$$A(\tau,\ v)=\frac{\pi^2\beta}{2r^4}(\sqrt{z}J_1+x\sqrt{\beta}J_0)^4\{\beta x^4+4x^2(\beta^2x^2-\beta x+1)z+4x(\beta x-1)z^2+4z^3\}, \tag{13}$$

and the arguments of the Bessel functions are $2\sqrt{\beta z}$. The energy-momentum tensor is found to be finite except at the singularity $r=0$(note $1-2\beta x>0$ in region II). The coordinate τ behaves badly as $z\to 0$ (event horizon), but the examination of the energy-momentum tensor components in the Kruskal type $(e^{\pi\tau},\ v)$ coordinate system which is regular on the event horizon shows that they are finite there. In region III, we have

$$T_{vv}=T_{vU}=0, \tag{14}$$

$$96\pi T_{UU}=\frac{\beta}{2r_1^4[J_1(2\sqrt{\beta z_0})]^4z_0^2}\cdot\{\sqrt{z_1}J_1(2\sqrt{\beta z_1})+x_1\sqrt{\beta}J_0(2\sqrt{\beta z_1})\}^4 \times\{\beta x_1^4+4x_1^2(\beta^2x_1^2-\beta x_1+1)z_1+4x_1(\beta x_1-1)z_1^2+4z_1^3\}-\frac{2\beta}{z_0}, \tag{15}$$

where $z_0=z(v=0)$, $r_1=r(v=-v_0)$, $z_1=z(v=-v_0)$ and $x_1=x(v=-v_0)$. The equation $\tau=$constant gives the following equation:

$$\frac{\sqrt{z_1}J_1(2\sqrt{\beta z_1})+x_1\sqrt{\beta}J_0(2\sqrt{\beta z_1})}{J_1(2\sqrt{\beta z_0})}=\frac{\sqrt{z_1}N_1(2\sqrt{\beta z_1})+x_1\sqrt{\beta}N_0(2\sqrt{\beta z_1})}{N_1(2\sqrt{\beta z_0})}. \tag{16}$$

If the condition $\beta z_0\ll 1$, $\beta z_1\ll 1$ is satisfied, from Eq. (16) and the relation $z_0=-U/2$, we can obtain the expression for T_{UU} for sufficiently small U. It is finally given by

$$24\pi T_{UU}\sim\frac{2}{\beta^2U^4}+\frac{\beta}{U}. \tag{17}$$

Then we can see that the energy flux of the created particle is quartically divergent on the Cauchy horizon $U=0$. This should be compared with the fact that the divergence is quadratic when the mass disappears with the condition $\dot{M}=$finite<0.[2)]

Though our model is one possible example in which mass disappears as $\dot{M}=0$, the above result is enough to show that the divergent situation on the Cauchy horizon is not altered by simply requiring the mass to disappear with the condition $\dot{M}=0$, which is contradictory to Hiscock's conclusion.[2)] This is because his conclusion is based on the perturbation of the outgoing null geodesic equation around that on the flat spacetime (such a perturbation is not valid near the singularity). Considering that for the case $M\sim v^2$ particle creation becomes more divergent compared with the case $M\propto -v$, we want to claim that any model in which mass disappears with the condition $\dot{M}=0$ creates infinite flux on the Cauchy horizon.

Next, let us consider the opposite case, i.e., a model in which mass disappears with the condition $\dot{M}\to-\infty$. Suppose the mass in region II is given by

$$m(v)=\frac{1}{2}\beta w^{\alpha}(1+\alpha\beta w^{\alpha-1}), \tag{18}$$

where $w\equiv -v$, $\beta>0$ and $1/2<\alpha<1$. The outgoing null geodesic satisfies the following equation:

$$\left(y-\frac{w}{2}+z\right)\frac{dz}{dy}+z=0, \tag{19}$$

where $y=w/2+\beta w^{\alpha}$ and $z=r-\beta w^{\alpha}$. In this case the line $z=0$ is the event horizon as before. The solution of Eq. (19) for small w is given by

$$z=-y+\sqrt{y^2+2D}+O(\beta^{-1/\alpha}y^{1/\alpha}), \tag{20}$$

where D is a constant. If we continue the outgoing null geodesics from region I to III, we can obtain T_{UU} in region III as before. The result for small v_0 is

$$24\pi T_{UU}\sim\frac{3y_1^2}{8D}\frac{1}{y_1^2+2D}-\frac{3y_1^4}{16D}\frac{1}{(y_1^2+2D)^2}, \tag{21}$$

where $y_1=y(v=-v_0)$. If we take the limit $D\to 0$ and use the relation $z(v=0)=\sqrt{2D}=-U/2$, we have

$$24\pi T_{UU}\sim\frac{3}{2U^2} \tag{22}$$

near the Cauchy horizon. Then this model also creates a divergent flux on the Cauchy horizon.

As a model for evaporating black hole should create finite energy flux by the energy conservation,[6)] it seems that the imploding Vaidya spacetime cannot represent the model. Though there exists a possibility that the use of two-dimensional energy-momentum tensor calculation gives rise to the divergence, it seems that the full four-dimensional calculation would not give qualitatively different results as have been shown in the various works.[4),7)~9)] We must perhaps consider an effect of outgoing created particles. This may be modeled by using exploding Vaidya spacetime in region III. There is another possibility that we must always consider the effects of the ingoing and outgoing created particles at the same time and the use of the Vaidya spacetime is unsuitable for the model of the evaporating black holes.

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