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Valence Bond Solids for Quantum Computation

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Cluster states are entangled multipartite states which enable to do universal quantum computation with local measurements only. We show that these states have a very simple interpretation in terms of valence bond solids, which allows to understand their entanglement properties in a transparent way. This allows to bridge the gap between the differences of the measurement-based proposals for quantum computing, and we will discuss several features and possible extensions.

The concept of teleportation [1] plays a crucial role in the understanding of entangled quantum systems. It does not only allow us to use entangled states as perfect quantum channels, but also to implement non-local unitary operations [2]. Based on this idea it was shown that universal quantum computation can be achieved if one can prepare a separable initial state and implement joint two-qubit measurements [3, 4, 5, 6]. In the same spirit, but somehow orthogonal to these schemes, Raussendorf and Briegel [7] showed that universal quantum computation is possible by doing local measurements on the qubits in a highly entangled so–called cluster state [8]. These studies highlighted the central role of entanglement for quantum computation [9, 10]. However, the structure of general multiparticle entanglement is, for the moment being, still very poorly understood, and it is somehow mysterious that the cluster states enable for universal quantum computation. In this note, we show that the structure of entanglement in cluster states is particularly simple and can be well understood by looking at it as a so–called valence bond solid with only nearest-neighbor bonds. This enables us to show that the one-way computer [7] essentially works in an equivalent way as the other measurement-based proposals for quantum computation.

Let us start by recalling how universal quantum computation can be performed using measurements only on a collection of maximally entangled states of 2 qubits. Schematically, we will present the logical qubits as the leftmost particles. Logical gates can be implemented by introducing extra pairs of maximally entangled states $|H\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$ of two qubits (every other maximally entangled would also be good), and doing Bell measurements on halves of these $|H\rangle$ and the logical qubits. The new logical qubits are now in the other parts of the maximally entangled state. It is well known that a universal set of gates [26] is given by arbitrary local unitary transformations and the phase gate $U_{ph} = |00\rangle\langle00| + |01\rangle\langle01| + |10\rangle\langle10| - |11\rangle\langle11|$. A local unitary transformation U on qubit i can be implemented by doing a Bell measurement in the basis

$$|\alpha\rangle = (U^\dagger\sigma_\alpha \otimes \mathbb{1})|H\rangle, \quad \alpha = 0, 1, 2, 3 \qquad (1)$$

where σ_α denote the Pauli matrices (including $\sigma_0 = \mathbb{1}$); see Figure 1A. This implements the unitary gate $\sigma_\alpha U$
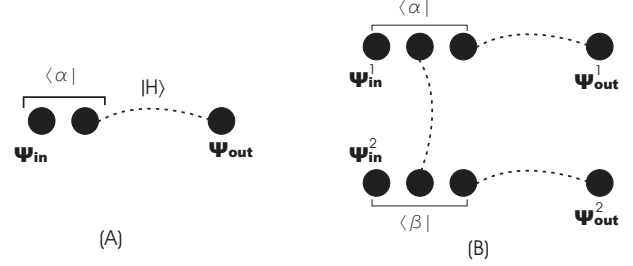

FIG. 1: (A) Implementation of a 1-qubit gate by measuring in the 2-qubit basis $|\alpha\rangle$ (1). The edges connected by the line denote the maximally entangled state $|H\rangle$. (B) Implementation of a 2-qubit gate by 3-qubit measurements in the basis $|\alpha\rangle$ and $|\beta\rangle$ (2).

which is U up to an extra multiplication with a Pauli operator (α is conditioned by the measurement outcome); this extra Pauli operator however does not harm [27]. Similarly, the phase gate U_{ph} can be implemented by adding three extra pairs of maximally entangled states $|H\rangle$ as depicted in Figure 1B. Suppose two three-qubit measurements are done (see Fig. 1B) in the complete bases

$$\{|\alpha\rangle\} = \{|\beta\rangle\} = (\sigma_x)^i \otimes (\sigma_x)^j \otimes \mathbb{1}(|0\rangle|0\rangle|0\rangle \pm |1\rangle|1\rangle|1\rangle) \qquad (2)$$

with $i, j \in \{0, 1\}$ and $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. It can readily be checked that this implements the gate $(H \otimes H)U_{ph}$ with H the Hadamard gate $H = |+\rangle\langle0| + |-\rangle\langle1|$, up to a harmless extra multiplication with Pauli operators. Together with the possibility of implementing local unitaries, this proves that universal quantum computation can be done by doing only Bell measurements on two or three qubits.

Let us summarize the three ingredients needed for being able to implement quantum computing along the lines sketched: 1/ it must be possible to create ancillary singlets; 2/ two- and three-qubit measurements of the form (1,2) can be implemented between halves of these extra singlets and the logical qubits.

Inspired by the AKLT-valence bond solids (VBS) [11, 12, 13], playing a central role in condensed matter physics, it is now interesting to investigate whether it is possible to interpret the two (or three) qubits on
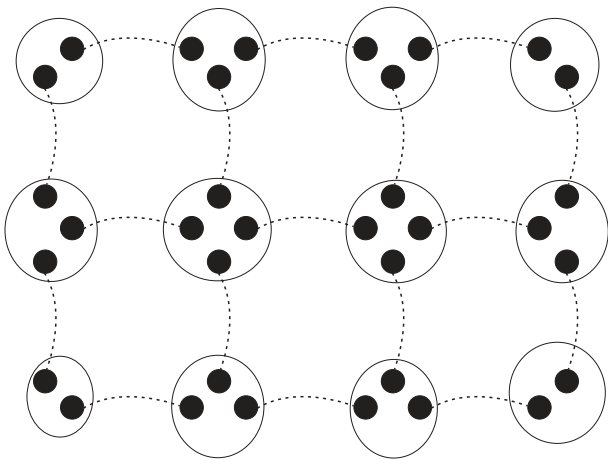
FIG. 2: Representation of a Valence Bond Solid. The edges connected with a dotted line denote a singlet, while the circles denote a projection P of all qubits inside it with Hilbert space $H_2^{\otimes n}$ to a single qubit $\tilde{H}_2$. In the present paper, P is always of the form $P = |\tilde{0}\rangle\langle 00\ldots 0| + |\tilde{1}\rangle\langle 11\ldots 1|$.

FIG. 3: Implementing a global unitary transformation on qubits 1 and 2 by doing local projections P_1 and P_2 on them and a maximally entangled state $|H\rangle$.

which the measurements have to be implemented as virtual qubits representing one physical qubit. As will become clear, this will exactly lead to the concept of the one-way computer [7]. As depicted in Figure 2, VBS are constructed by distributing singlets $|H\rangle$ made from virtual qubits between different sites, followed by a local projection of the virtual qubits on a smaller dimensional subspace $\tilde{H}$ encoding the physical qubit. In the case of the 2-D lattice in Figure 2 e.g., the 4 qubits can be projected on a one qubit subspace by the operator $P_4 = |\tilde{0}\rangle\langle 0000| + |\tilde{1}\rangle\langle 1111|$ (here 4 specifies that there are 4 virtual qubits; P_n is defined as $P_n = |\tilde{0}\rangle\langle 00\ldots 0| + |\tilde{1}\rangle\langle 11\ldots 1|$ with n virtual qubits). The idea is thus to interpret one of the virtual qubits as the logical one, and the other virtual ones as the ones enabling teleportation.

Let us see whether the two listed requirements for measurement based quantum computation can be fulfilled.

1/ Consider the VBS of Figure 2. To implement a virtual 2-qubit (3-qubit) measurement, we want that the physical qubit is only made up of 2 (3) virtual qubits (on which we want to implement a Bell measurement), and hence that 2 (1) virtual qubit(s) effectively disappear. Suppose the physical qubit (Hilbert space $\tilde{H}$) is obtained by projecting the virtual ones by the projector $P_4 = |\tilde{0}\rangle\langle 0000| + |\tilde{1}\rangle\langle 1111|$. Measuring 2 (1) neighboring physical qubit(s) in the $|\tilde{0}\rangle, |\tilde{1}\rangle$-basis effectively replaces 2 (1) virtual qubit(s) with $|+\rangle$ or $|-\rangle$ depending on the measurement outcome. But it holds that $P_4|+\rangle = P_3 \equiv |\tilde{0}\rangle\langle 000| + |\tilde{1}\rangle\langle 111|$ and $P_3|-\rangle = \tilde{\sigma}_z P_3$, and similarly $P_3|+\rangle = P_2$. Therefore the bonds in a VBS can be broken at will by measuring neighboring physical qubits, the virtual qubits effectively disappear and the projector P_n changes into P_{n-1}.

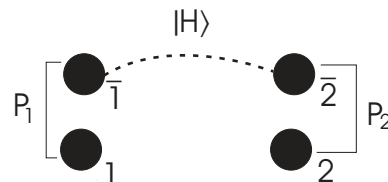2/ Let us now investigate whether virtual measurements such as (1,meas2) can be implemented by local measurements on physical qubits. This will be possible if the projector P_n has full support on a subset of rays corresponding to these multiqubit measurements. Clearly, this is the case for the phase gate if the projector is P_3, as a measurement in the physical basis $|\tilde{0}\rangle \pm |\tilde{1}\rangle$ effectively corresponds to the measurement of the virtual qubits in the basis $|000\rangle \pm |111\rangle$ which belongs to the optimal basis (2). In the case of the single qubit gates, things are a bit more subtle. Suppose the projector is P_2. Only measurements in bases of the form $|\tilde{0}\rangle \pm \exp(i\xi)|\tilde{1}\rangle$ correspond to Bell measurements, and its effect is to implement the unitary

$$U = (\sigma_x)^k \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \exp(-i\xi) \\ 1 & -\exp(-i\xi) \end{pmatrix}$$

on the (virtual) logical qubit ($k = 0, 1$ depending on the measurement outcome). It can however easily be checked that a sequence of 4 such unitaries can generate every specified unitary $\in SU(2)$ [28] (this again holds up to left multiplication with Pauli matrices). Therefore the second requirement is also fulfilled.

By measuring the qubits from left to right on a lattice, the (virtual) logical qubits travel from left to right, yielding quantum computation using single-qubit measurements only. This shows that 2-D valence bond solids can be used to do a quantum computation. This model of computation exactly coincides with the one-way computer of Rausendorf and Briegel; indeed, we will next show that the cluster state is exactly the VBS used in the above construction.

The cluster states are a subset of the so–called stabilizer states [14], which are defined by specifying a complete set of commuting observables O_i, where each O_i is a tensor product of the Pauli matrices $\sigma^0 = \mathbb{1}_2, \sigma^x, \sigma^y, \sigma^z$. The stabilizer states are the common eigenstates of these operators. Let us show that any stabilizer state can be interpreted as a valence bond state. Stabilizer states can efficiently be prepared from a completely separable state by applying appropriate 2-qubit unitary operations to it (see e.g. [15]). The reason that stabilizer states are very simple and manageable to work with is due to the fact all these 2-qubit unitary operations can be chosen to commute with each other. This can readily be seen by looking at the normal form for stabilizer states [16], and then making use of properties of Pauli operators such as $U(\sigma^x \otimes \sigma^z)U^\dagger = \sigma^x \otimes I$ with $U = \text{diag}[1, 1, 1, -1]$ to

diagonalize all operators O_i. The trick is now to implement these commuting 2-qubit unitary transformations by a teleportation-like principle that consists of adding virtual singlets, and then doing appropriate projections [2, 17]. More specifically, consider the two qubits 1 and 2 in Figure 3; an extra singlet $|H\rangle_{\bar{1}\bar{2}}$ is added, and then any unitary transformation between 1 and 2 can be simulated by projecting the two-qubit spaces labelled by $1, \bar{1}$ $(2, \bar{2})$ onto the qubits 1 (2 with appropriate projectors P_1 $(P_2$ $(P_1, P_2$ are 2×4 matrices). Iterating this scheme, one readily sees that every stabilizer state can be interpreted as a VBS, possibly with bonds extending over all sites [29]. In the case of the cluster states however, only unitaries between the nearest neighbors have to be implemented, and hence a simple VBS as depicted in Figure 2 is obtained.

As an example, let us explicitly construct the valence bond states corresponding to arbitrary cluster states [30]. To each cluster state, one can associate a graph parameterized by its adjacency matrix Γ. The number of qubits on each site in the VBS is of course equal to the number of bonds of the given site, and is equal to the number of vertices emanating from a given physical qubit. The bonds are maximally entangled states $|H\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$, and the projectors on each site are all of the form $P = |\tilde{0}\rangle\langle 00 \ldots 0| + |\tilde{1}\rangle\langle 11 \ldots 1|$. This simple construction describes all possible cluster states.

This VBS interpretation of cluster states makes their nice and appealing properties very explicit. The fact that e.g. a singlet can be created between two arbitrary qubits by doing appropriate local measurement on the other ones can readily be understood by the concept of entanglement swapping [20]. The entropy of a block of spins can readily be seen to be given by the number of bonds emanating from it (i.e. proportional to the area of the surface of the block). The fact that the sensitivity to noise of a cluster state does not scale with the number of (physical) qubits [21], is of due to the fact that it is effectively made up by *local* singlet pairs. This insight also enables to construct distillation protocols for cluster states by translating bipartite distillation protocols to the valence bond picture [22].

On the other hand, the description of valence bond states in terms of stabilizer states is also interesting from the point a view of condensed matter theory. It is e.g. well-known that operations of the Clifford group acting on a stabilizer state can easily be simulated efficiently classically. This implies that evolutions generated by the Clifford group on VBS-states can be simulated efficiently, and correlation functions of products of Pauli operators can easily calculated. On the other hand, the possibility to do quantum computation with VBS using local measurements only proves that the complexity-class for calculating general expectation values on 2-D VBS is the same as the complexity class of quantum computing.

The present study also opens the question whether there exist ground states of (gapped) Hamiltonians involving only 2-body short-range interactions on a lattice that would enable to implement the presented measurement scheme (this is not the case for cluster states). Such 2-D Valence Bond Solids indeed exist for higher spins (e.g. spin $3/2$), and it is trivial do devise a toy model for which this holds. Consider e.g. a hexagonal lattice with spin-$7/2$ particles at each vertex. To each particle corresponds a 8-dimensional Hilbert space, which we can interpret as a system of 3 qubits. We associate each outgoing edge to one of these qubits, and associate the Hamiltonian $\vec{S}\vec{S} + 3\mathbb{1}$ to two of these qubits connected by an edge. One readily sees that the ground state on such a hexagonal lattice with this 2-body local Hamiltonian will be unique, and that the teleportation scheme can be implemented perfectly on it. Note that the cluster state is very similar to that construction, but there the 3 qubits are interpreted as virtual qubits and a smart projection was used to reduce the dimension of the effective Hilbert space.

More interestingly, the trick used to implement 2-qubit unitary gates by introducing a virtual singlet followed by a projection - this is the way cluster states can be generated from completely separable ones - can also be extended to the case where the unitaries do not commute with each other. Indeed, the cluster state can be made in the lab if an Ising interaction can be implemented on neighboring qubits [23]. However, in some experimental setups, it is not always possible to implement such commuting gates, as is the case e.g. for quantum dots [24]: here one is essentially restricted to implement 2-qubit gates generated by the Heisenberg interaction, which certainly do not commute when acting on neighboring spins. However, if one can apply these unitary gates sequentially (i.e. one has control over the sites on which one implements the gate), then it is also possible to construct valence bond solids that are suitable for quantum computation.

The present results also prove that the valence bond solid picture is very useful for understanding multipartite entanglement. Indeed, VBS are particularly interesting from the point a view of quantum information theory, as the simple and elegant tools developed for bipartite quantum systems can be applied to it (see e.g. [13]). Moreover, one can readily see that the VBS form a dense subset of all possible quantum states if one allows the bonds to extend beyond nearest neighbors, if the singlets are replaced by higher dimensional maximally entangled states and if the projectors can be chosen arbitrary (e.g. in the case of 3 qubits, every state can be made by considering 2 singlets and projecting 2 qubits of them onto a qubit space [25]). It would be very interesting to develop a general theory of multiparticle entanglement based on this VBS-picture, where one could construct entanglement measures that quantify the valence bond resources needed to describe the state. This will be reported elsewhere.

In conclusion, we have identified the entanglement properties of the cluster states that are responsible for

the possibility of universal quantum computation. The main insight was given by the fact that the structure of entanglement in these states is essentially bipartite and can be understood in terms of valence bonds. This allowed to prove the equivalence of the one-way computer with teleportation-based computation schemes, and to clarify the special features of the cluster states.

[1] C.H. Bennett et al., Phys. Rev. Lett. **70**, 1895 (1993).
[2] M. A. Nielsen and I. L. Chuang, Phys. Rev. Lett. **79**, 321 (1997).
[3] D. Gottesman and I. Chuang, Nature **402**, 390 (1999).
[4] E. Knill, R. Laflamme and G. Milburn, Nature **409**, 26 (2001).
[5] M.A. Nielsen, quant-ph/0108020.
[6] D. Leung, quant-ph/0111122 and quant-ph/0310189.
[7] R. Raussendorf and H.J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001); Quant. Inf. Comp. 6, 443 (2002).
[8] H. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001).
[9] R. Jozsa and N. Linden, quant-ph/0201143.
[10] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[11] I. Affleck, T. Kennedy, E.H. Lieb and H. Tasaki, Commun. Math. Phys. 115, 477 (1988); Phys. Rev. Lett. 59, 799-802 (1987)
[12] M. Fannes, B. Nachtergaele and R.F. Werner, Comm. Math. Phys. **144**, 443 (1992).
[13] F. Verstraete, M.A. Martín–Delgado and J.I. Cirac, quant-ph/0311087.
[14] D. Gottesman, Phys.Rev. A **54**, 1862 (1996).
[15] M. Nielsen and I. Chuang, Quantum computation and quantum information, Cambridge University Press, Cambridge, U.K (2000).
[16] D. Gottesman, Caltech Ph.D. Thesis, quant-ph/9705052.
[17] J.I. Cirac, W. Dür, B. Kraus and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).
[18] D. Schlingemann and R.F. Werner, Phys. Rev. A **65**, 012308 (2002).
[19] M. Hein, J. Eisert and H.J. Briegel, quant-ph/0307130;

M. Van den Nest, J. Dehaene and B. De Moor, quant-ph/0308151; J. Dehaene and B. De Moor, Phys. Rev. A **68**, 042318 (2003).
[20] M. Zukowski et al., Phys. Rev. Lett. **71**, 4287 (1993).
[21] W. Dür and H.-J. Briegel, quant-ph/0307180.
[22] W. Dür, H. Aschauer and H.-J. Briegel, Phys. Rev. Lett. **91**, 107903 (2003).
[23] O. Mandel et al., Nature **425**, 937 (2003).
[24] D. Loss and D.P. DiVincenzo, Phys. Rev. A **57**, 120 (1998).
[25] A. Miyake and F. Verstraete, quant-ph/0307067.
[26] Note that universal quantum computation can be achieved by only implementing gates between neighboring qubits.
[27] The extra left multiplication with Pauli operators does not harm: a later 1-qubit operation V can be chosen to be conditioned on the outcome (i.e. implementing $V\sigma_\alpha$ instead of V). Furthermore, right multiplication of the 2-qubit phase gate U_{ph} with Pauli operators is equivalent to left multiplication of it with different ones. Therefore the extra Pauli operators can be pushed through the quantum circuit without affecting the computation.
[28] See e.g. the way 1-qubit unitary gates are implemented in the one-way computer [7].
[29] It would be interesting in this respect to find a normal form for stabilizer states that minimizes this number of bonds. This however seems to be a difficult problem [19].
[30] The construction given also works for the set of so–called graph states [18], which is, up to local unitaries, equivalent to the set of stabilizer states.