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# VALENCY SEVEN SYMMETRIC GRAPHS OF ORDER $2 p q$ 

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#### Abstract

A graph is said to be symmetric if its automorphism group acts transitively on its arcs. In this paper, all connected valency seven symmetric graphs of order $2 p q$ are classified, where $p, q$ are distinct primes. It follows from the classification that there is a unique connected valency seven symmetric graph of order $4 p$, and that for odd primes $p$ and $q$, there is an infinite family of connected valency seven one-regular graphs of order $2 p q$ with solvable automorphism groups, and there are four sporadic ones with nonsolvable automorphism groups, which is $1,2,3$-arc transitive, respectively. In particular, one of the four sporadic ones is primitive, and the other two of the four sporadic ones are bi-primitive.


Keywords: arc-transitive graph; symmetric graph; $s$-regular graph
MSC 2010: 05C25, 20B25

## 1. Introduction

For a finite, simple and undirected graph $X$, let $V(X), E(X), A(X)$ and $\operatorname{Aut}(X)$ denote the vertex set, edge set, arc set and full automorphism group of $X$, respectively. Note that an arc is an ordered edge, that is, an ordered pair of adjacent vertices. For $u, v \in V(X),\{u, v\}$ denotes the edge incident to $u$ and $v$ in $X$. An $s$-arc in a graph $X$ for some nonnegative integer $s$ is an ordered ( $s+1$ )-tuple $\left(v_{0}, v_{1}, \ldots, v_{s}\right)$ of $s+1$ vertices such that $\left(v_{i-1}, v_{i}\right) \in A(X)$ for $1 \leqslant i \leqslant s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$. For a subgroup $G$ of the automorphism group $\operatorname{Aut}(X)$ of a graph $X$, the graph $X$ is said to be $(G, s)$-arc-transitive or $(G, s)$-regular if $G$ acts transitively or regularly on the set of $s$-arcs of $X$, and $(G, s)$-transitive if $G$ acts transitively on the set of $s$-arcs but not on the set of $(s+1)$-arcs of $X$. A graph $X$ is said to be $s$-arc-transitive, $s$-regular or $s$-transitive if it is (Aut $(X), s)$-arc-transitive,

[^0](Aut $(X), s)$-regular or $(\operatorname{Aut}(X), s)$-transitive. In particular, 0 -arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A graph is said to be primitive if its automorphism group is primitive on the vertex set, and a graph is said to be bi-primitive if it is a bipartite graph with bi-parts $\Delta_{1}, \Delta_{2}$, and the setwise stabilizer of its automorphism group is primitive on both $\Delta_{1}$ and $\Delta_{2}$. Throughout this paper, we will denote by $\mathbb{Z}_{n}$ the cyclic group of order $n$, by $\mathbb{Z}_{n}^{*}$ the multiplicative group of units modulo $n$, by $D_{2 n}$ the dihedral group of order $2 n$, by $F_{n}$ the Frobenius group of order $n$, and by $A_{n}$ and $S_{n}$ the alternating group and the symmetric group of degree $n$, respectively.

It is well known that a graph $\Gamma$ is $G$-arc-transitive if and only if $G$ is vertextransitive and the vertex stabilizer $G_{v}$ of $v \in V(\Gamma)$ in $G$ is transitive on $N_{\Gamma}(v)$. Hence the structure of the vertex stabilizer of $G_{v}$ plays an important role in the study of ( $G, s$ )-transitive graphs. For example, benefitted from Djoković and Miller [4] result about the vertex stabilizer of cubic symmetric graphs, lots of works about classifications of cubic symmetric graphs were obtained by many authors (see [7], [8], [9], [23], [24]). Due to the vertex stabilizers given in [27], symmetric tetravalent graphs have also been studied extensively in the literature (see [11], [12], [22], [32], [34]). Simlarly, Guo and Feng [14] determined structure of vertex stabilixers of pentavalent symmetric graphs, some works about classifications of pentavalent symmetric graphs were also obtained (see [6], [14], [17], [18], [26]). Naturally, the next step is to characterize valency seven symmetric graphs. Recently, Guo et al. [15] gave the structure of vertex stabilizers of valency seven symmetric graphs, and this encourages us to consider some work on valency seven symmetric graphs. In [16], Guo et al. classified valency seven symmetric graphs of order $4 p$, and in [25], Pan et al. classified primevalent symmetric graphs of square-free order. But, we obtain this result for valency seven symmetric graphs of order $2 p q$ independently. Let $p, q$ be two distinct primes. In this paper, we classify valency seven symmetric graphs of order $2 p q$.

## 2. Preliminaries

Let $X$ be a graph, and $N$ a subgroup of $\operatorname{Aut}(X)$. Denote by $X_{N}$ the quotient graph corresponding to the orbits of $N$, that is the graph having the orbits of $N$ as vertices with two orbits adjacent in $X_{N}$ if there is an edge in $X$ between those orbits. In view of [20], Theorem 9, we have the following proposition.

Proposition 2.1. Let $X$ be a connected symmetric graph of prime valency $p$ and $G$ an $s$-arc-transitive subgroup of $\operatorname{Aut}(X)$ for some $s \geqslant 1$. If a normal subgroup $N$ of $G$ has more than two orbits on $V(X)$ then $X_{N}$ is also a symmetric graph
of valency $p$ and $N$ is the kernel of the action of $G$ on the set of orbits of $N$. Moreover, $N$ is semiregular on $V(X)$ and $G / N$ is an $s$-arc-transitive subgroup of $\operatorname{Aut}\left(X_{N}\right)$.

By Guo [15], we have the following statement.

Proposition 2.2. Let $X$ be a connected $(G, s)$-transitive graph of valency seven for some $G \leqslant \operatorname{Aut}(X)$ and $s \geqslant 1$. Let $v \in V(X)$. Then $s \leqslant 3$ and one of the following holds:
(1) If $G_{v}$ is soluble, then $\left|G_{v}\right| \mid 2^{2} \cdot 3^{2} \cdot 7$. Further, the triple $\left(s, G_{v},\left|G_{v}\right|\right)$ lies in the following table:

| $s=1$ |  | $s=2$ |  | $s=3$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $G_{v}$ | Order | $G_{v}$ | Order | $G_{v}$ | Order |
| $\mathbb{Z}_{7}$ | 7 | $F_{42}$ | $2 \cdot 3 \cdot 7$ | $F_{42} \times \mathbb{Z}_{6}$ | $2^{2} \cdot 3^{2} \cdot 7$ |
| $D_{14}$ | $2 \cdot 7$ | $F_{42} \times \mathbb{Z}_{2}$ | $2^{2} \cdot 3 \cdot 7$ |  |  |
| $F_{21}$ | $3 \cdot 7$ | $F_{42} \times \mathbb{Z}_{3}$ | $2 \cdot 3^{2} \cdot 7$ |  |  |
| $D_{28}$ | $2^{2} \cdot 7$ |  |  |  |  |
| $F_{21} \times \mathbb{Z}_{3}$ | $3^{2} \cdot 7$ |  |  |  |  |

(2) If $G_{v}$ is insoluble, then $s \geqslant 2$ and $\left|G_{v}\right| \mid 2^{24} \cdot 3^{4} \cdot 5^{2} \cdot 7$. Further, the triple $\left(s, G_{v},\left|G_{v}\right|\right)$ lies in the following table:

| $s=2$ |  | $s=3$ |  |
| :---: | :---: | :---: | :---: |
| $G_{v}$ | Order | $G_{v}$ | Order |
| $\operatorname{PSL}(3,2)$ | $2^{3} \cdot 3 \cdot 7$ | $\operatorname{PSL}(3,2) \times S_{4}$ | $2^{6} \cdot 3^{2} \cdot 7$ |
| $A_{7}$ | $2^{3} \cdot 3^{2} \cdot 5 \cdot 7$ | $A_{7} \times A_{6}$ | $2^{6} \cdot 3^{4} \cdot 5^{2} \cdot 7$ |
| $S_{7}$ | $2^{4} \cdot 3^{2} \cdot 5 \cdot 7$ | $S_{7} \times S_{6}$ | $2^{8} \cdot 3^{4} \cdot 5^{2} \cdot 7$ |
| $\mathbb{Z}_{2}^{3} \times \mathrm{SL}(3,2)$ | $2^{6} \cdot 3 \cdot 7$ | $\left(A_{7} \times A_{6}\right) \rtimes \mathbb{Z}_{2}$ | $2^{7} \cdot 3^{4} \cdot 5^{2} \cdot 7$ |
| $\mathbb{Z}_{2}^{4} \times \mathrm{SL}(3,2)$ | $2^{7} \cdot 3 \cdot 7$ | $\mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ | $2^{10} \cdot 3^{2} \cdot 7$ |
|  |  | $\left(\left[2^{20}\right] \rtimes(\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))\right.$ | $2^{24} \cdot 3^{2} \cdot 7$ |

From [3], pages 12-14, 3-prime factor simple groups can be found. And by [13], pages 134-136, one can obtain the following proposition by checking the orders of nonabelian simple groups.

Proposition 2.3. Let $p, q$ be distinct odd primes, and let $G$ be a nonabelian simple group of order $|G|=2^{i} \cdot 3^{j} \cdot 5^{k} \cdot 7 \cdot p \cdot q$ with $1 \leqslant i \leqslant 26,0 \leqslant j \leqslant 4,0 \leqslant k \leqslant 2$ and $7||G|$. Then $G$ has 3 -prime factor, 4-prime factor, 5 -prime factor or 6 -prime factor, and is one of the groups in Table 1.

| G | Order | G | Order |
| :---: | :---: | :---: | :---: |
| $\operatorname{PSL}(2,7)$ | $2^{3} \cdot 3 \cdot 7$ | $A_{11}$ | $2^{7} \cdot 3^{4} \cdot 5^{2} \cdot 7 \cdot 11$ |
| $\operatorname{PSL}(2,8)$ | $2^{3} \cdot 3^{2} \cdot 7$ | $A_{12}$ | $2^{9} \cdot 3^{5} \cdot 5^{2} \cdot 7 \cdot 11$ |
| $\operatorname{PSU}(3,3)$ | $2^{5} \cdot 3^{3} \cdot 7$ | $M_{22}$ | $2^{7} \cdot 3^{2} \cdot 5 \cdot 7 \cdot 11$ |
| $A_{7}$ | $2^{3} \cdot 3^{2} \cdot 5 \cdot 7$ | $H S$ | $2^{9} \cdot 3^{2} \cdot 5^{3} \cdot 7 \cdot 11$ |
| $A_{8}$ | $2^{6} \cdot 3^{2} \cdot 5 \cdot 7$ | $\operatorname{PSL}\left(2,2^{6}\right)$ | $2^{6} \cdot 3^{2} \cdot 5 \cdot 7 \cdot 13$ |
| $A_{9}$ | $2^{6} \cdot 3^{4} \cdot 5 \cdot 7$ | $\operatorname{PSL}\left(2,2^{9}\right)$ | $2^{9} \cdot 3^{3} \cdot 7 \cdot 19 \cdot 73$ |
| $A_{10}$ | $2^{7} \cdot 3^{4} \cdot 5^{2} \cdot 7$ | $\operatorname{PSL}\left(2,5^{3}\right)$ | $2^{2} \cdot 3^{2} \cdot 5^{3} \cdot 7 \cdot 31$ |
| $\operatorname{PSL}(2,13)$ | $2^{2} \cdot 3 \cdot 7 \cdot 13$ | $\operatorname{PSL}(4,4)$ | $2^{12} \cdot 3^{4} \cdot 5^{2} \cdot 7 \cdot 17$ |
| $\operatorname{PSL}(2,27)$ | $2^{2} \cdot 3^{3} \cdot 7 \cdot 13$ | $\operatorname{PSL}(5,2)$ | $2^{10} \cdot 3^{2} \cdot 5 \cdot 7 \cdot 31$ |
| $\operatorname{PSL}(3,4)$ | $2^{6} \cdot 3^{2} \cdot 5 \cdot 7$ | $\operatorname{PSp}(4,8)$ | $2^{12} \cdot 3^{4} \cdot 5 \cdot 7^{2} \cdot 13$ |
| $\operatorname{PSL}(3,8)$ | $2^{9} \cdot 3^{2} \cdot 7^{2} \cdot 73$ | ${ }^{2} D_{4}(2)$ | $2^{12} \cdot 3^{4} \cdot 5 \cdot 7 \cdot 17$ |
| $\operatorname{PSU}(3,5)$ | $2^{4} \cdot 3^{2} \cdot 5^{3} \cdot 7$ | $G_{2}(4)$ | $2^{12} \cdot 3^{3} \cdot 5^{2} \cdot 7 \cdot 13$ |
| $\operatorname{PSU}(3,8)$ | $2^{9} \cdot 3^{4} \cdot 7 \cdot 19$ | $G_{2}(8)$ | $2^{18} \cdot 3^{5} \cdot 7^{2} \cdot 19 \cdot 73$ |
| $J_{2}$ | $2^{7} \cdot 3^{3} \cdot 5^{2} \cdot 7$ | $M_{23}$ | $2^{7} \cdot 3^{2} \cdot 5 \cdot 7 \cdot 11 \cdot 23$ |
| $S z(8)$ | $2^{6} \cdot 5 \cdot 7 \cdot 13$ | $M_{24}$ | $2^{10} \cdot 3^{3} \cdot 5 \cdot 7 \cdot 11 \cdot 23$ |
| $D_{4}(2)$ | $2^{12} \cdot 3^{5} \cdot 5^{2} \cdot 7$ | $J_{1}$ | $2^{3} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ |
| ${ }^{3} D_{4}(2)$ | $2^{12} \cdot 3^{4} \cdot 7^{2} \cdot 13$ | $\operatorname{PSL}(3,16)$ | $2^{12} \cdot 3^{2} \cdot 5^{2} \cdot 7 \cdot 13 \cdot 17$ |
| $\operatorname{PSp}(6,2)$ | $2^{9} \cdot 3^{4} \cdot 5 \cdot 7$ | $\operatorname{PSL}(2, t)$ | $t= \pm 1(\bmod 7)$ and $t>13$ |
| $\operatorname{PSp}(8,2)$ | $2^{16} \cdot 3^{5} \cdot 5^{2} \cdot 7$ |  |  |
| $\operatorname{PSL}(2,49)$ | $2^{4} \cdot 3 \cdot 5^{2} \cdot 7^{2}$ |  |  |

Table 1. Nonabelian simple $\{2,3,5,7, p, q\}$-groups.

Proof. Clearly, we have

$$
\begin{equation*}
2^{27} \nmid G\left|, 3^{6} \nmid\right| G\left|, 5^{4} \nmid\right| G\left|, 7^{3} \nmid\right| G|, 7||G|, t^{2} \nmid|G| \tag{2.1}
\end{equation*}
$$

where $t \in\{q, p\}$ and $t \geqslant 11$.
From [3], pages 12-14, 3-prime factor simple groups can be found. If $7||G|$, one has $G \cong \operatorname{PSL}(2,7), \operatorname{PSL}(2,8)$ or $\operatorname{PSU}(3,3)$. Specially, if $7^{2}| | G\left|, 3^{5}\right||G|$ or $5^{3}| | G \mid$, then $|G|$ has at most five prime divisors. By [31], page 3, each finite nonabelian simple group is isomorphic to $A_{n}$ with $n \geqslant 5$, one of 26 sporadic simple groups, or a classical group or an exceptional group of Lie type. For the orders of these simple groups, one can see [13], Table 2.4, pages 134-136, and for more details, see [31], Sections 3, 4, 5 .

For $A_{n}$ with $n \geqslant 5$, since $3^{6} \nmid|G|$ and $7\left||G|\right.$, we have $G \cong A_{7}, A_{8}, A_{9}, A_{10}, A_{11}$ or $A_{12}$. For the 26 sporadic simple groups, by equation (2.1) we have $G \cong M_{22}, M_{23}$, $M_{24}, J_{1}, J_{2}, H S$.

For the groups of Lie type, since each odd prime divisor of $|G|$ has power at most 5 , by [13], Table 2.4, pages $134-136, G \cong D_{4}(2),{ }^{2} D_{4}(2),{ }^{3} D_{4}(2), \operatorname{PSL}(n, t)$ with $n \geqslant 2$, $\operatorname{PSU}(n, t)$ with $n \geqslant 3, \operatorname{PSp}(2 n, t)$ with $n \geqslant 2$, or $S z\left(2^{2 n+1}\right)$ with $n \geqslant 1$, where $t$ is a prime power.

Let $G \cong \operatorname{PSL}(n, t)$. Then $|G|=(n, t-1)^{-1} t^{n(n-1) / 2} \prod_{i=2}^{n}\left(t^{i}-1\right)$. First assume $n \geqslant 3$. Then $n(n-1) / 2 \geqslant 3$, and by equation (2.1), we have $n=3$ and $t=3,5$ or $2^{i}$ with $i<9, n=4$ and $t=2^{i}$ with $i<5$, or $n=5$ and $t=2^{i}$ with $i<3$. For each case, by checking orders with equation (2.1) again, we have $G \cong \operatorname{PSL}(3,4)$, $\operatorname{PSL}(3,8), \operatorname{PSL}(3,16), \operatorname{PSL}(4,2)\left(\cong A_{8}\right), \operatorname{PSL}(4,4)$ or $\operatorname{PSL}(5,2)$. Now assume $n=2$. Then $|G|=(2, t-1)^{-1} t\left(t^{2}-1\right)$. If $t=2^{i}$ then $i \leqslant 26$ by equation (2.1). Similarly, if $t=3^{i}$ then $i \leqslant 5$; if $t=5^{i}$ then $i \leqslant 3$; if $t=7^{i}$ then $i \leqslant 2$; if $t=s^{i}$ with $s>7$ and $s \in\{q, p\}$ then $i=1$. For each case, checking the orders of $\operatorname{PSL}(2, t)$ again, we have $G \cong \operatorname{PSL}\left(2,2^{6}\right), \operatorname{PSL}\left(2,2^{9}\right), \operatorname{PSL}(2,27), \operatorname{PSL}(2,125), \operatorname{PSL}(2,49)$, or $\operatorname{PSL}(2, t)$ with some prime $t \geqslant 13$ and $t \in\{q, p\}$.

Let $G \cong \operatorname{PSU}(n, t)$ with $n \geqslant 3$. Then

$$
|G|=(n, t+1)^{-1} t^{n(n-1) / 2} \prod_{i=2}^{n}\left(t^{i}-(-1)^{i}\right)
$$

Since $n(n-1) / 2 \geqslant 3$, we have $n=3$ and $t=3,5$ or $t=2^{i}$ with $i<9, n=4$ and $t=2^{i}$ with $i<5$, or $n=5$ and $t=2^{i}$ with $i<3$ by equation (2.1). Hence, $G \cong \operatorname{PSU}(3,8), \operatorname{PSU}(3,5)$. For the other two infinite families $\operatorname{PSp}(2 n, t)$ of order $(n, t-1)^{-1} t^{n^{2}} \prod_{i=2}^{n}\left(t^{2 i}-1\right)$ with $n \geqslant 2$ and $S z\left(2^{2 n+1}\right)$ of order $2^{4 n+2}\left(2^{4 n+2}+1\right) \times$ ( $2^{2 n+1}-1$ ) with $n \geqslant 1$, one can similarly obtain that $G \cong \operatorname{PSp}(4,8), S z(8)$.

From [30], page 417, we have the following proposition.

Proposition 2.4. Let $p$ be a prime, and $q=p^{n} \geqslant 5$. Then a maximal subgroup of $\operatorname{PSL}(2, q)$ is isomorphic to one of the following groups:
(1) $D_{2(q-1) / d}$, where $d=(2, q-1)$ and $q \neq 5,7,9,11$;
(2) $D_{2(q+1) / d}$, where $d=(2, q-1)$ and $q \neq 7,9$;
(3) $\mathbb{Z}_{q} \rtimes \mathbb{Z}_{(q-1) / d}$;
(4) $A_{4}$, when $q=p=5$, or $q=p \equiv 3,13,27,37(\bmod 40)$;
(5) $S_{4}$, when $q=p \equiv \pm 1(\bmod 8)$;
(6) $A_{5}$, when $q=p \equiv \pm 1(\bmod 5)$, or $q=p^{2} \equiv-1(\bmod 5)$ with $p$ an odd prime;
(7) $\operatorname{PSL}(2, r)$, when $q=r^{m}$ with $m$ an odd prime;
(8) $\operatorname{PGL}(2, r)$, when $q=r^{2}$.

To extract a classification of connected valency seven symmetric graphs of order $2 p$ for a prime $p$ from Cheng and Oxley [2], we introduce the graphs $G(2 p, r)$. Let $V$ and $V^{\prime}$ be two disjoint copies of $\mathbb{Z}_{p}$, say $V=\{0,1, \ldots, p-1\}$ and $V^{\prime}=$ $\left\{0^{\prime}, 1^{\prime}, \ldots,(p-1)^{\prime}\right\}$. Let $r$ be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of $Z_{p}^{*}$ of order $r$. Define the graph $G(2 p, r)$ to have vertex set $V \cup V^{\prime}$ and edge set $\left\{x y^{\prime}: x-y \in H(p, r)\right\}$.

Proposition 2.5. Let $p$ be a prime, and let $X$ be a connected valency seven symmetric graph of order $2 p$. Then one of the following situations occurs:
(1) $X \cong K_{7,7}$, the complete bipartite graph of order 14, and $\operatorname{Aut}\left(K_{7,7}\right)=\left(S_{7} \times\right.$ $\left.S_{7}\right) \rtimes \mathbb{Z}_{2}$;
(2) $X \cong G(2 p, 7)$ with $p \equiv 1(\bmod 7)$, and $\operatorname{Aut}(G(2 p, 7))=\left(\mathbb{Z}_{p} \rtimes \mathbb{Z}_{7}\right) \rtimes \mathbb{Z}_{2}$.

Finally, we introduce the so called Cayley graph. For a finite group $G$ and a subset $S$ of $G$ such that $S=S^{-1}$ and $1 \notin S$, the Cayley graph $\operatorname{Cay}(G, S)$ on $G$ with respect to $S$ is defined to have vertex set $G$ and edge set $\{\{g, s g\}: g \in G, s \in S\}$. Given $g \in G$, right multiplication $x \mapsto x g$ (for $x \in G$ ) is a permutation $R(g)$ on $G$, and the homomorphism from $G$ to $\operatorname{Sym}(G)$ taking each $g$ to $R(g)$ is called the right regular representation of $G$. The image $R(G)=\{R(g): g \in G\}$ of $G$ is a regular permutation group on $G$, and is isomorphic to $G$, which can therefore be regarded as a subgroup of the automorphism group $\operatorname{Aut}(\operatorname{Cay}(G, S))$. In particular, the Cayley graph Cay $(G, S)$ is vertex-transitive. Moreover, the group $\operatorname{Aut}(G, S)=\left\{\alpha \in \operatorname{Aut}(G): S^{\alpha}=S\right\}$ is a subgroup of $\operatorname{Aut}(\operatorname{Cay}(G, S))$, indeed of the stabilizer $\operatorname{Aut}(\operatorname{Cay}(G, S))_{1}$ of the vertex 1. Also, a Cayley graph $\operatorname{Cay}(G, S)$ is said to be normal if $R(G)$ is normal in $\operatorname{Aut}(\operatorname{Cay}(G, S))$. By [33], Propositions 1.3 and 1.5, a Cayley graph $\operatorname{Cay}(G, S)$ is normal if and only if $\operatorname{Aut}(\operatorname{Cay}(G, S))_{1}=\operatorname{Aut}(G, S)$, or equivalently, if and only if $\operatorname{Aut}(\operatorname{Cay}(G, S))$ is isomorphic to the semidirect product $R(G) \rtimes \operatorname{Aut}(G, S)$.

Now we introduce an infinite family of one-regular Cayley graphs on the dihedral group $D_{2 n}=\left\langle a, b \mid a^{n}=b^{2}=1, b^{-1} a b=a^{-1}\right\rangle$. Let $m$ and $l$ be integers such that $l^{6}+l^{5}+l^{4}+l^{3}+l^{2}+l+1 \equiv 0(\bmod m)$. Define

$$
\begin{equation*}
\mathcal{C} \mathcal{D}_{2 m}^{l}=\operatorname{Cay}\left(D_{2 m}, S\right) \tag{2.2}
\end{equation*}
$$

where $S=\left\{b, a b, a^{l+1} b, a^{l^{2}+l+1} b, a^{l^{3}+l^{2}+l+1} b, a^{l^{4}+l^{3}+l^{2}+l+1} b, a^{l^{5}+l^{4}+l^{3}+l^{2}+l+1} b\right\}$.
By [10], we have the following propositions.
Proposition 2.6 ([10], Theorem 3.5). Let $n$ be a square-free integer and $X$ a connected valency seven one-regular graph of order $n$. Then $n=2 \cdot 7^{t} \cdot p_{1} p_{2} \ldots p_{s}$, where $t \leqslant 1, s \geqslant 1$, and $p_{i}$ 's are distinct primes such that $7 \mid\left(p_{i}-1\right)$. Furthermore, $X$ is isomorphic to one of $\mathcal{C D}{ }_{n}^{l}$ and there are exactly $6^{s-1}$ nonisomorphic such graphs of order $n$.

Now we introduce the so called coset graph (see [22], [28]) constructed from a finite group $G$ relative to a subgroup $H$ of $G$ and a union $D$ of some double cosets of $H$ in $G$ such that $D^{-1}=D$. Denote by $H_{G}$ the largest normal subgroup of $G$ in $H$. The coset graph $\operatorname{Cos}(G, H, D)$ of $G$ with respect to $H$ and $D$ is defined to have vertex set $[G: H]$, the set of right cosets of $H$ in $G$, and edge set $\{\{H g, H d g\}: g \in$ $G, d \in D\}$. The action of $G$ on $V(\operatorname{Cos}(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is faithful if and only if $H_{G}=1$. Furthermore, $\operatorname{Aut}(G, H, D)=\left\{\alpha \in \operatorname{Aut}(G): H^{\alpha}=H, D^{\alpha}=D\right\}$ induces a group of automorphisms, which lies in the stabilizer of $H$ in $\operatorname{Aut}(\operatorname{Cos}(G, H, D))$. Clearly, $\operatorname{Cos}(G, H, D) \cong \operatorname{Cos}\left(G, H^{\alpha}, D^{\alpha}\right)$ for every $\alpha \in \operatorname{Aut}(G)$. Note that the concept of a coset graph is equivalent to the concept of an orbital graph (see [29]). Conversely, by [28] we have the following statement.

Proposition 2.7. Let $X$ be a graph and let $G$ be a vertex-transitive subgroup of $\operatorname{Aut}(X)$. Then $X$ is isomorphic to a coset graph $\operatorname{Cos}(G, H, D)$, where $H=G_{u}$ is the stabilizer of $u \in V(X)$ in $G$ and $D$ consists of all elements of $G$ which map $u$ to one of its neighbors. Further,
(1) $X$ is connected if and only if $D$ generates the group $G$;
(2) $X$ is $G$-arc-transitive if and only if $D$ is a single double coset. In particular, if $g \in G$ interchanges $u$ and one of its neighbors, then $g^{2} \in H$ and $D=H g H$;
(3) the valency of $X$ is equal to $|D| /|H|=\left|H: H \cap H^{g}\right|$.

## 3. Constructions

In this section, we construct valency seven symmetric graphs of order $2 p q$, where $p$ and $q$ are distinct primes.

Example 3.1. Let $G$ be a subgroup of $S_{14}$ such that $G \cong \operatorname{PSL}(2,13)$, and $G$ contains the following elements:

$$
\begin{aligned}
a & =(1,12)(2,6)(3,13)(4,7)(8,9)(10,11), \\
b & =(1,12,2,10,14,11,6)(3,9,5,8,13,4,7), \\
g_{2} & =(1,6)(2,4)(3,8)(5,7)(9,10)(13,14), \\
g_{2} & =(1,8)(3,5)(4,12)(6,7)(9,10)(11,13) .
\end{aligned}
$$

By Magma [1], $G=\left\langle a, b, g_{i}\right\rangle$ for each $1 \leqslant i \leqslant 2$ and $H=\langle a, b\rangle$. Define the following coset graphs:

$$
\mathcal{C}_{78}^{i}=\operatorname{Cos}\left(G, H, H g_{i} H\right), \quad 1 \leqslant i \leqslant 2 .
$$

Again by Magma [1], the two coset graphs $\mathcal{C}_{78}^{i}(i=1,2)$ are pairwise nonisomorphic connected valency seven 1-transitive graphs of order 78 with $\operatorname{Aut}\left(\mathcal{C}_{78}^{1}\right)=\operatorname{PSL}(2,13)$ and $\operatorname{Aut}\left(\mathcal{C}_{78}^{2}\right)=\operatorname{PGL}(2,13)$.

Lemma 3.2. Each connected valency seven symmetric graph $X$ of order 78 admitting $\operatorname{PSL}(2,13)$ as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{78}^{i}$ $(i=1,2)$. Furthermore, $X$ is 1-transitive and $\operatorname{Aut}(X) \cong \operatorname{PSL}(2,13)$ or $\operatorname{PGL}(2,13)$.

Pro of. Let $G=\operatorname{PSL}(2,13)$. As $X$ is a $G$-arc-transitive graph of order 78, one has $\left|G_{v}\right|=14$ for any vertex $v \in V(X)$, and by Proposition 2.4, we have $H=G_{v} \cong D_{14}$. The simplicity of $G$ and the maximality of $H$ imply that $H=N_{G}(H)$. Take an involution $x$ in $H$, and set $\langle x\rangle=L$. Since $G$ has one conjugacy class of involutions, by Proposition 2.4, $N_{G}(L)=D_{12}$. Clearly, $H \cap H^{g}=L$ and $N_{H}(L) \cong L$. Thus, there exists an involution $g$ such that $g \in N_{G}(L)$ and $g \notin N_{H}(L)$. Furthermore, $g \notin H$, $|H g H| /|H|=7$ and $\langle H, g\rangle=G$. This implies that $\operatorname{Cos}(G, H, H g H)$ is a connected valency seven symmetric graph of order 78 .

Let $X$ be a connected valency seven symmetric graph of order 78 admitting $G=\operatorname{PSL}(2,13)$ as an arc-transitive automorphism group. Note that $G_{v} \cong D_{14}$ for any $v \in V(X)$. To complete the proof, it suffices to show that $X \cong \operatorname{Cos}(G, H, H g H)$. By Magma [1], $G$ has one conjugacy class of $D_{14}$ and since $G_{v}$ has seven subgroups isomorphic to $\mathbb{Z}_{2}$, each of the subgroups fixes a vertex adjacent to $v$. By Proposition 2.7, one may assume that $X=\operatorname{Cos}(G, H, H f H)$ such that $H \cap H^{f}=L$ and $f \in N_{G}(L)$. By [5], Theorem 2.1, $f$ can be chosen to be a 2 -element, and hence $f$ is an involution in $N_{G}(L) \cong D_{12}$. By the connectivity of $X, f \notin N_{H}(L) \cong \mathbb{Z}_{2}$. Thus, $f$ has six choices and by Magma [1], the six coset graphs $\operatorname{Cos}(G, H, H f H)$ corresponding to the six involutions have two nonisomorphism classes. It follows that $X=\operatorname{Cos}(G, H, H f H) \cong \mathcal{C}_{78}^{1}$ or $\mathcal{C}_{78}^{2}$, as required.

Example 3.3. Let $G=S_{8}$. Then $G$ has a subgroup $H \cong \mathbb{Z}_{2}^{3} \rtimes \operatorname{SL}(3,2)$ and an involution $g$ such that $|H g H| /|H|=7$ and $\langle H, g\rangle=G$. The coset graph $\operatorname{Cos}(G, H, H g H)$ is denoted by $\mathcal{C}_{30}$.

Lemma 3.4. Each connected valency seven symmetric graph $X$ of order 30 admitting $S_{8}$ as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{30}$. Furthermore, $X$ is 2-transitive and $\operatorname{Aut}(X) \cong S_{8}$.

Proof. Let $G=S_{8}$. Clearly, $G$ has a maximal subgroup $T \cong A_{8}$ containing a maximal subgroup $H$ such that $H \cong \mathbb{Z}_{2}^{3} \rtimes \mathrm{SL}(3,2)$. Let $L=\mathbb{Z}_{2}^{3} \rtimes S_{4}$ be a subgroup of $H$. By Magma [1], $N_{G}(L)=L \cdot \mathbb{Z}_{2}$ and $N_{T}(L)=L$, and by [19], one has $N_{G}(L)=S_{2}$ 亿 $S_{4}$. Let $g \in N_{G}(L) \backslash L$ be an involution. Then $N_{G}(L)=L \cup L g$, $L=H \cap H^{g},|H g H| /|H|=7$ and $\langle H, g\rangle=G$. It follows that the coset graph
$\operatorname{Cos}(G, H, H g H)$ is a connected valency seven symmetric graph of order 30. (Note that $H$ has yet another conjugacy class of order $\left|\mathbb{Z}_{2}^{3} \rtimes S_{4}\right|$, which is not isomorphic to $\mathbb{Z}_{2}^{3} \rtimes S_{4}$. By Magma [1], $N_{G}(L)=L$, no graph arises.)

Let $X$ be a connected valency seven symmetric graph of order 30 admitting $G=S_{8}$ as an arc-transitive automorphism group. Then $G_{v} \cong \mathbb{Z}_{2}^{3} \rtimes \mathrm{SL}(3,2)$ for any $v \in V(X)$. To complete the proof, it suffices to show that $X \cong \operatorname{Cos}(G, H, H g H)$. Since $G$ has one conjugacy class of $\mathbb{Z}_{2}^{3} \rtimes \operatorname{SL}(3,2)$ and $\mathbb{Z}_{2}^{3} \rtimes \mathrm{SL}(3,2)$ has seven subgroups isomorphic to $\mathbb{Z}_{2}^{3} \rtimes S_{4}$, by Proposition 2.7, one may assume that $X=\operatorname{Cos}(G, H, H f H)$ such that $H \cap H^{f}=L$ and $f \in N_{G}(L)$. Since $N_{G}(L)=L \cup L g$, one has $f=l g$ for some $l \in L$. It follows that $H f H=H g H$, that is, $X=\operatorname{Cos}(G, H, H f H) \cong \operatorname{Cos}(G, H, H g H)$. By Magma [1], $\operatorname{Aut}(X)=G$.

Example 3.5. Let $G=\operatorname{Aut}(\operatorname{PSL}(5,2))=\operatorname{PSL}(5,2) . \mathbb{Z}_{2}$. Then $G$ has a subgroup $H \cong \mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ and an involution $g$ such that $|H g H| /|H|=7$ and $\langle H, g\rangle=G$. The coset graph $\operatorname{Cos}(G, H, H g H)$ is denoted by $\mathcal{C}_{310}$.

Lemma 3.6. Each connected valency seven symmetric graph $X$ of order 310 admitting $\operatorname{Aut}(\mathrm{PSL}(5,2))$ as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{310}$. Furthermore, $X$ is 3-transitive and $\operatorname{Aut}(X) \cong \operatorname{Aut}(\operatorname{PSL}(5,2))$.
$\operatorname{Proof}$. By Atlas [3], $\operatorname{Aut}(\operatorname{PSL}(5,2))=\operatorname{PSL}(5,2) . \mathbb{Z}_{2}$. Let $G=\operatorname{Aut}(\operatorname{PSL}(5,2))$. Clearly, $G$ has an index two maximal subgroup $T \cong \operatorname{PSL}(5,2)$ containing a maximal subgroup $H$ such that $H \cong \mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$. Let $L=\mathbb{Z}_{2}^{6} \rtimes\left(\mathrm{SL}(2,2) \times S_{4}\right)$ be a subgroup of $H$. By Magma [1], $N_{G}(L)=L . \mathbb{Z}_{2}$ and $N_{T}(L)=L$. Let $g \in N_{G}(L) \backslash L$ be an involution. Then $N_{G}(L)=L \cup L g, L=H \cap H^{g},|H g H| /|H|=7$ and $\langle H, g\rangle=G$. It follows that the coset graph $\operatorname{Cos}(G, H, H g H)$ is a connected valency seven symmetric graph of order 310. (Note that $\mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times \mathrm{SL}(3,2)$ ) has yet another conjugacy class of order $\left|\mathbb{Z}_{2}^{6} \rtimes\left(\mathrm{SL}(2,2) \times S_{4}\right)\right|$, which is not isomorphic to $\mathbb{Z}_{2}^{6} \rtimes\left(\operatorname{SL}(2,2) \times S_{4}\right)$. By Magma [1], $N_{G}(L)=L$, no graph arises.)

Let $X$ be a connected valency seven symmetric graph of order 310 admitting $G$ as an arc-transitive automorphism group. Then $G_{v} \cong \mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ for any $v \in V(X)$. To complete the proof, it suffices to show that $X \cong \operatorname{Cos}(G, H, H g H)$. Since $T$ has two conjugacy classes of maximal parabolic subgroups $\mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times$ $\mathrm{SL}(3,2)$ ), and has a graph automorphism $g$, which is of order $2, g$ fuses the two conjugacy classes of maximal parabolic subgroups. By Magma [1], $\mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times$ $\mathrm{SL}(3,2))$ has a conjugacy class of $\mathbb{Z}_{2}^{6} \rtimes\left(\mathrm{SL}(2,2) \times S_{4}\right)$. By Proposition 2.7, one may assume that $X=\operatorname{Cos}(G, H, H f H)$ so that $H \cap H^{f}=L$ and $f \in N_{G}(L)$. Since $N_{G}(L)=L \cup L g$, one has $f=l g$ for some $l \in L$. It follows that $H f H=H g H$, that is, $X=\operatorname{Cos}(G, H, H f H) \cong \operatorname{Cos}(G, H, H g H)$. By Magma [1], $\operatorname{Aut}(X)=G$.

## 4. MAIN RESULTS

In this section, we classify valency seven symmetric graphs of order $2 p q$ for $p$ and $q$ primes. First, we consider valency seven symmetric graphs of order $4 p$, where $p$ is a prime.

Theorem 4.1. Let $p$ be a prime. Then $X$ is a connected valency seven symmetric graph of order $4 p$ if and only if $X \cong K_{8}$, a complete graph of order 8 .

Proof. For $p=2, K_{8}$ is a unique symmetric graph of valency seven. For $p=3$, by [21], there is no symmetric graph of valency seven. Thus, in what follows, we assume that $p \geqslant 5$. Let $A=\operatorname{Aut}(X)$ and $v \in V(X)$. By Guo [15], $\left|A_{v}\right| \mid 2^{24} \cdot 3^{4} \cdot 5^{2} \cdot 7$, and hence $|A| \mid 2^{s} \cdot 3^{t} \cdot 5^{r} \cdot 7 \cdot p$ with $2 \leqslant s \leqslant 26,0 \leqslant t \leqslant 4$ and $0 \leqslant r \leqslant 2$. We divide our discussion into the following two cases. Let $N$ be a minimal normal subgroup of $A$.

Assume that $N$ is solvable. Then $N$ is elementary abelian. By Proposition 2.1, $N$ is semiregular on $V(X)$, and the quotient graph $X_{N}$ of $X$ relative to the orbits of $N$ has valency seven. Since $|V(X)|=4 p, A$ has no normal subgroup of order 4 or $p$.

It follows that $N \cong \mathbb{Z}_{2}$, forcing that $N \leqslant Z(A)$, the center of $A$. By Proposition $2.1, X_{N}$ is a connected valency seven symmetric graph of order $2 p$ with $A / N$ as an arc-transitive subgroup of $\operatorname{Aut}\left(X_{N}\right)$. By Proposition 2.5 , either $X_{N} \cong K_{7,7}$ or $X_{N} \cong G(2 p, 7)$ with $7 \mid p-1$. Take a minimal normal subgroup of $A / N$, say $M / N$. Let $X_{N} \cong K_{7,7}$. Clearly, $p=7$. Suppose that $M / N$ is solvable. Then $M / N \cong \mathbb{Z}_{2}, \mathbb{Z}_{7}$ or $\mathbb{Z}_{7}^{2}$. If $M / N \cong \mathbb{Z}_{2}$ or $\mathbb{Z}_{7}$, then $A$ has a normal subgroup of order 4 or 7 because $N \cong \mathbb{Z}_{2}$, a contradiction. If $M / N \cong \mathbb{Z}_{7}^{2}$ then $M \cong \mathbb{Z}_{2} \times \mathbb{Z}_{7}^{2}$. It is easy to see that $M$ has two orbits on $V(X)$, and since $M$ is abelian and $M_{v} \cong \mathbb{Z}_{7}$, one has $X \cong 2 K_{7,7}$, a union of two copies of $K_{7,7}$, which contradicts the connectivity of $X$. Suppose that $M / N$ is nonsolvable. Then $M / N \cong A_{7}$ or $A_{7} \times A_{7}$. Obviously, $M / N$ has two orbits on $V\left(X_{N}\right)$. Since $(M / N)_{u} \unlhd(A / N)_{u}$ for any $u \in X_{N}$, by the primitivity of $(A / N)_{u}$ on the neighborhood of $u$ one has $7\left|\left|(M / N)_{u}\right|\right.$, implying that 49$||M / N|$. Thus, $M / N \cong A_{7} \times A_{7}$. Let $B / N \cong A_{7}$ and $B / N \unlhd M / N$. Similarly, $B / N$ has two orbits on $V\left(X_{N}\right)$ and $7\left|\left|(B / N)_{w}\right|\right.$. Thus, 49$||B / N|$, a contradiction. Let $X_{N} \cong G(2 p, 7)$ with $7 \mid p-1$. Then a normal Sylow $p$-subgroup of $\operatorname{Aut}\left(X_{N}\right)$ must be $P N / N$ because each Sylow $p$-subgroup of $A / N$ is a Sylow $p$-subgroup of $\operatorname{Aut}\left(X_{N}\right)$. It follows that $P \unlhd A$ because $P$ is characteristic in $P N$, which is impossible because $A$ has no normal subgroup of order $p$.

If $A$ has a solvable nontrivial normal subgroup, then $A$ has a solvable minimal normal subgroup isomorphic to $\mathbb{Z}_{2}, \mathbb{Z}_{2}^{2}$ or $\mathbb{Z}_{p}$, which is impossible by the above
argument. Thus, in what follows we assume that $A$ has no solvable nontrivial normal subgroups.

Now assume that $N$ is nonsolvable. Then $N \cong T^{m}$, where $T$ is a nonabelian simple group. By Proposition 2.1, $N$ has at most two orbits on $V(X)$. Then $|N|$ is divisible by $2 p \cdot 7$, and since $|N| \mid 2^{s} \cdot 3^{t} \cdot 5^{r} \cdot 7 \cdot p$ with $1 \leqslant s \leqslant 26,0 \leqslant t \leqslant 4$ and $0 \leqslant r \leqslant 2$. One has $N=T$ except $p=7$.

If $p=5$, then $|N|$ is a factor of $2^{26} \cdot 3^{4} \cdot 5^{3} \cdot 7$ and $|N|$ is divisible by $2 \cdot 5 \cdot 7$. By Table 1,

$$
\begin{equation*}
N \cong A_{7}, A_{8}, A_{9}, A_{10}, \operatorname{PSL}(3,4), \operatorname{PSU}(3,5), J_{2}, \operatorname{PSp}(6,2) . \tag{4.1}
\end{equation*}
$$

For $p=7$, one has $7^{2}| | N \mid$ and $|N| \mid 2^{26} \cdot 3^{4} \cdot 5^{2} \cdot 7^{2}$. If $N \cong T^{2}$, then $T \cong$ $\operatorname{PSL}(3,4), \operatorname{PSL}(2,7), A_{7}, A_{8}, \operatorname{PSL}(2,8)$ by Table 1. Clearly, $N$ has a normal subgroup isomorphic to $T$, say $S$. Since $S \unlhd N$, one has $7\left|\left|S_{v}\right|\right.$ and $S$ has an orbit of length 7,14 or 28 , implying that $49||S|$, a contradiction. Thus, $N=T$. In this case, $|N|$ has at most four primes, and $7^{2}| | N \mid$. Again by Table 1, one has

$$
\begin{equation*}
N \cong \operatorname{PSL}(2,49) . \tag{4.2}
\end{equation*}
$$

Let $p>7$. We first consider $N \cong \operatorname{PSL}(2, p)$, the infinite family listed in Table 1 . By the subgroup structure of $\operatorname{PSL}(2, p)$, one has $N_{v}$ is solvable and $\left|N_{v}\right| \mid 2^{2} \cdot 3^{2} \cdot 7$, and $5 \nmid\left|N_{v}\right|$. Then $|N|$ is a factor of $2^{4} \cdot 3^{2} \cdot 7 \cdot p$ and $|N|$ is divisible by $2 \cdot p \cdot 7$. Hence $|N|=|\operatorname{PSL}(2, p)|=\frac{1}{2} p(p-1)(p+1)$ and $\left(\frac{1}{2} p(p+1), \frac{1}{2}(p-1)\right)=1$. If $7 \mid p-1$, then $p+1=2^{i} \cdot 3^{j}$, where $1 \leqslant i \leqslant 4,0 \leqslant j \leqslant 2$. It follows that $p=71$. Similarly, if $7 \mid p+1$, then $p=13$. Combining with Table $1, N$ is one of the following:
(4.3) $\operatorname{PSL}(2,13), \operatorname{PSL}(2,71), \operatorname{PSL}(2,27), \operatorname{PSU}(3,8), S z(8), A_{11}, \mathrm{M}_{22}, \operatorname{PSL}\left(2,2^{6}\right)$,

$$
\begin{equation*}
\operatorname{PSL}(4,4), \operatorname{PSL}(5,2),{ }^{2} D_{4}(2), G_{2}(4) . \tag{4.4}
\end{equation*}
$$

Since $N$ is nonsolvable, $N$ has at most two orbits. We may assume that $N$ is a group listed in (4.1)-(4.4). Let $N$ be transitive on $V(X)$. By Proposition 2.7, $X \cong \operatorname{Cos}(N, H, H a H)$, where $H=N_{v}, a \in N \backslash H$ and $a^{2} \in H$. By the Atlas [3], $N=A_{7}(p=5)$ has no subgroup of order $|H|=|N| /|V(X)|$. Thus, $N \neq A_{7}$. Similarly, $N \neq \operatorname{PSL}\left(2,2^{6}\right)(p=13)$. For $N=A_{8}(p=5),|N| /|V(X)|$ is not the order of the vertex stabilizer by Proposition 2.2, a contradiction. It follows that $N \neq A_{8}$. Similarly, $N \neq A_{9}(p=5), A_{10}(p=5), \operatorname{PSL}(3,4)(p=5), \operatorname{PSU}(3,5)$ $(p=5), J_{2}(p=5), \operatorname{PSp}(6,2)(p=5), \operatorname{PSL}(2,49), \operatorname{PSL}(2,13)(p=13), \operatorname{PSL}(2,71)$ $(p=71), \operatorname{PSL}(2,27)(p=13), \operatorname{PSU}(3,8)(p=19), S z(8)(p=13), A_{11}(p=11), M_{22}$ $(p=11), \operatorname{PSL}(4,4)(p=17), \operatorname{PSL}(5,2)(p=31),{ }^{2} D_{4}(2)(p=17), G_{2}(4)(p=13)$.

Let $N$ have two orbits on $V(X)$. Then $|H|=|N| / \frac{1}{2}|V(X)|$. For $N=A_{7}(p=5)$, by Proposition 2.2, $|N| / \frac{1}{2}|V(X)|$ is not the order of the vertex stabilizer, a contradiction. It follows that $N \neq \mathrm{A}_{7}$. Similarly, $N \neq A_{9}(p=5), A_{10}(p=5)$, $\operatorname{PSL}(3,4)(p=5), \operatorname{PSU}(3,5)(p=5), J_{2}(p=5), \operatorname{PSp}(6,2)(p=5), \operatorname{PSL}(2,49)$, $\operatorname{PSL}(2,13)(p=13), \operatorname{PSL}(2,71)(p=71), \operatorname{PSL}(2,27)(p=13), \operatorname{PSU}(3,8)(p=19)$, $S z(8)(p=13), M_{22}(p=11), \operatorname{PSL}(4,4)(p=17), \operatorname{PSL}(5,2)(p=31),{ }^{2} D_{4}(2)$ $(p=17), G_{2}(4)(p=13)$. By the Atlas [3], $N=A_{8}$ has no subgroup of order $|H|=|N| / \frac{1}{2}|V(X)|$. Thus, $N \neq \mathrm{A}_{8}(p=5)$. Similarly, $N \neq \operatorname{PSL}\left(2,2^{6}\right)(p=13)$. For $N=A_{11}$, one has $|H|=2^{6} \cdot 3^{4} \cdot 5^{2} \cdot 7$. By Proposition $2.2, H \cong A_{7} \times A_{6}$, and by [19], $A_{11}$ has no subgroup which is isomorphic to $\mathrm{A}_{7} \times \mathrm{A}_{6}$, a contradiction. This completes the proof.

Theorem 4.2. Let $X$ be a connected valency seven symmetric graph of order $2 p q$, where $p>q$ are odd primes. Then $X$ is 1-, 2- or 3-transitive. Furthermore, one of the following situations occurs:
(1) $X$ is 1-transitive, and $X \cong \mathcal{C}_{78}^{i}(i=1,2)$ with $\operatorname{Aut}\left(\mathcal{C}_{78}^{1}\right) \cong \operatorname{PSL}(2,13)$ and $\operatorname{Aut}\left(\mathcal{C}_{78}^{2}\right) \cong \operatorname{PGL}(2,13)$, or $X \cong \mathcal{C D}_{2 p q}^{l}$ (defined in equation (2.2)) with $\operatorname{Aut}(X) \cong D_{2 p q} \rtimes \mathbb{Z}_{7}$ for some $l$ satisfying $l^{6}+l^{5}+l^{4}+l^{3}+l^{2}+l \equiv 0$ $(\bmod p q)$-the number of pairwise nonisomorphic such graphs of order $2 p q$ is

$$
f(p, q)= \begin{cases}1, & q=7 \text { and } 7 \mid p-1 \\ 6, & 7 \mid q-1 \text { and } 7 \mid p-1 \\ 0, & \text { otherwise }\end{cases}
$$

(2) $X$ is 2-transitive, and $X \cong \mathcal{C}_{30}$ is a vertex bi-primitive graph with $\operatorname{Aut}(X) \cong S_{8}$.
(3) $X$ is 3-transitive, and $X \cong \mathcal{C}_{310}$ is a vertex bi-primitive graph with $\operatorname{Aut}(X) \cong$ $\operatorname{PSL}(5,2) \cdot \mathbb{Z}_{2}$.

Proof. Let $A=\operatorname{Aut}(X)$ and $v \in V(X)$. By Guo [15], $\left|A_{v}\right| \mid 2^{24} \cdot 3^{2} \cdot 5^{2} \cdot 7$, and hence $|A|=2^{s} \cdot 3^{t} \cdot 5^{r} \cdot 7 \cdot q \cdot p$ with $1 \leqslant s \leqslant 25,0 \leqslant s \leqslant 4$ and $0 \leqslant r \leqslant 2$. We first prove a claim.

Claim: If $A$ has a normal subgroup of order $q$ then $X \cong \mathcal{C D}_{2 p q}^{l}$.
Let $Q$ be a normal subgroup of $A$ of order $q$. By Proposition 2.1, $Q$ is semiregular on $V(X)$ and the quotient graph $X_{Q}$ of $X$ relative to $Q$ is a symmetric graph of order $2 p$ and valency seven with $A / Q$ as an arc-transitive subgroup of $\operatorname{Aut}\left(X_{Q}\right)$. By Proposition 2.5, one has $X_{Q} \cong K_{7,7}$ or $X_{Q} \cong G(2 p, 7)$ with $7 \mid p-1$.

Suppose that $X_{Q} \cong K_{7,7}$. Then $p=7$ and $q=3$ or 5 . Take a minimal normal subgroup of $A / Q$, say $M / Q$. Assume that $M / Q$ is nonsolvable. Then $M / Q \cong A_{7}$ or $A_{7} \times A_{7}$ because $A / Q \leqslant \operatorname{Aut}\left(K_{7,7}\right) \cong\left(S_{7} \times S_{7}\right) \rtimes \mathbb{Z}_{2}$. Obviously, $M / Q$ has two orbits
on $V\left(X_{Q}\right)$ and $7\left|\left|(M / Q)_{w}\right|\right.$ for any $w \in V\left(X_{Q}\right)$, implying that 49$||M / Q|$. Thus, $M / Q \cong \mathrm{~A}_{7} \times \mathrm{A}_{7}$. Let $B / Q \cong \mathrm{~A}_{7}$ and $B / Q \unlhd M / Q$. Similarly, $B / Q$ has two orbits on $V\left(X_{Q}\right)$ and $7\left|\left|(B / Q)_{w}\right|\right.$. Thus, 49$||B / Q|$, a contradiction. Now assume that $M / Q$ is solvable. Then $M / Q \cong \mathbb{Z}_{2}, \mathbb{Z}_{7}$ or $\mathbb{Z}_{7}^{2}$. If $M / Q \cong \mathbb{Z}_{2}$ then $X_{M}$ is a symmetric graph of order $p$ and valency seven, a contradiction. If $M / Q \cong \mathbb{Z}_{7}$ then $M \cong \mathbb{Z}_{21}$ or $\mathbb{Z}_{35}$ and $M$ has two orbits on $V(X)$, implying that $X$ is a bipartite graph. Let $R \leqslant M$ and $R \cong \mathbb{Z}_{7}$. Then $R \triangleleft A$, and since $R \leqslant M$, the quotient graph $X_{R}$ is bipartite and of valency seven. However, $\left|X_{R}\right|=6$ or 10 , a contradiction. If $M / Q \cong \mathbb{Z}_{7}^{2}$ then $M \cong Q \times \mathbb{Z}_{7}^{2}$ because $Q \cong \mathbb{Z}_{3}$ or $\mathbb{Z}_{5}$. Since $M$ is abelian and $M_{v} \cong \mathbb{Z}_{7}$, one has $X \cong 3 K_{7,7}$ or $5 K_{7,7}$, which contradicts the connectivity of $X$.

Thus, $X_{Q} \cong G(2 p, 7)$ with $7 \mid p-1$. By Proposition $2.5, X_{Q}$ is valency seven and 1-regular graph of order $2 p$. Since $A / Q$ is arc-transitive on $X_{Q}$, one has $A / Q=\operatorname{Aut}\left(X_{Q}\right)$ and $X$ is a valency seven and 1-regular graph of order $2 p q$. By Proposition 2.6, $X \cong \mathcal{C} \mathcal{D}_{2 p q}^{l}$. This completes the proof of Claim.

If $A$ has a normal subgroup of order 2 , then the quotient graph has valency seven and odd order $p q$, a contradiction.

Let $A$ have a normal subgroup $P$ of order $p$. By Proposition 2.5, $X_{P} \cong K_{7,7}$ or $G(2 q, 7)$ with $7 \mid q-1$. Let $C:=C_{A}(P)$. Clearly, $P \leqslant C$. If $P=C$ then $A / P \leqslant \operatorname{Aut}(P) \cong \mathbb{Z}_{p-1}$, implying that $A$ is abelian. It follows that $A$ is regular on $V(X)$, which contradicts the fact that $X$ is symmetric. Hence, $P<C$. Take a minimal normal subgroup of $A / P$, say $M / P$, in $C / P$. Suppose that $M / P$ is solvable. By Proposition 2.1, $M / P$ is semiregular on $V\left(X_{P}\right)$. Then $M / P \cong \mathbb{Z}_{2}$ or $\mathbb{Z}_{q}$, which implies that $A$ has a normal subgroup of order 2 or $q$ respectively; we have done two cases. Thus, $M / P$ is nonsolvable, and hence $X_{P} \cong K_{7,7}$. Then $M / P \cong A_{7}$ or $A_{7} \times A_{7}$. Obviously, $M / P$ has two orbits on $V\left(X_{P}\right)$, and $7\left|\left|(M / P)_{u}\right|\right.$ for any $u \in V\left(X_{P}\right)$, implying that $49\left||M / P|\right.$. Thus, $M / P \cong A_{7} \times A_{7}$. Let $B / P \cong A_{7}$ and $B / P \unlhd M / P$. Similarly, $B / P$ has two orbits on $V\left(X_{P}\right)$ and $7\left|\left|(B / P)_{u}\right|\right.$. Thus, $49||B / P|$, a contradiction.

If $A$ has a solvable nontrivial normal subgroup, then $A$ has a solvable minimal normal subgroup isomorphic to $\mathbb{Z}_{2}, \mathbb{Z}_{p}$ or $\mathbb{Z}_{q}$, which was done by the above argument. Thus, in what follows we assume that $A$ has no solvable nontrivial normal subgroups.

Let $N$ be a minimal normal subgroup of $A$. Then $N \cong T^{m}$, where $T$ is a nonabelian simple group. By Proposition 2.1, $N$ has at most two orbits on $V(X)$. Since $p q \cdot 7||N|$ and $|N|\left||A|=2^{s} \cdot 3^{t} \cdot 5^{r} \cdot 7 \cdot q \cdot p\right.$ with $1 \leqslant s \leqslant 25,0 \leqslant t \leqslant 4$ and $0 \leqslant r \leqslant 2$, one has $N=T$ except for $p=7$. Assuming that $p=7$, one has $q=3$ or 5 . Hence $7^{2}| | N \mid$ and $|N| \mid 2^{25} \cdot 3^{5} \cdot 5^{3} \cdot 7^{2}$. If $N \cong T^{2}$, then $T \cong \operatorname{PSL}(2,7), \operatorname{PSL}(2,8), \operatorname{PSL}(3,4), A_{7}$, $A_{8}$ by Table 1. Clearly, $N$ has a normal subgroup isomorphic to $T$, say $S$. Since $S \unlhd N$, one has $7\left|\left|S_{v}\right|\right.$ and $S$ has an orbit of length $7,7 q$ or $14 q$, implying that $49||S|$, a contradiction. Thus, $N=T$. In this case, $| N \mid$ has at most four primes
$\{2,3,5,7\}$, and $7^{2}| | N \mid$. Again by Table 1 , one has

$$
\begin{equation*}
N \cong \operatorname{PSL}(2,49) \tag{4.5}
\end{equation*}
$$

Next, we assume that $p \neq 7$ and $N=T$. We first consider $N \cong \operatorname{PSL}(2, p)(p>7)$, the infinite family listed in Table 1. By the subgroup structure of $\operatorname{PSL}(2, p)$, one has $N_{v}$ is solvable, and by Proposition 2.2, $\left|N_{v}\right| \mid 2^{2} \cdot 3^{2} \cdot 7$ and $5 \nmid\left|N_{v}\right|$. Thus $|N| \mid 2^{3} \cdot 3^{2} \cdot 7 \cdot q \cdot p$, implying that $N$ is at most five-prime factor simple group. Hence $|N|=|\operatorname{PSL}(2, p)|=\frac{1}{2} p(p-1)(p+1)$ and $\left(\frac{1}{2}(p+1), \frac{1}{2}(p-1)\right)=1$. For $q \leqslant 7$, if $7 \left\lvert\, \frac{1}{2}(p+1)\right.$, then $p-1=2^{i} \cdot 3^{j} \cdot q$, where $1 \leqslant i \leqslant 3,0 \leqslant j \leqslant 2$. It follows that $p=13,41,181$. Since $\operatorname{PSL}(2,181)$ is a six-prime factor simple group, one has $p=13,41$. Similarly, if $7 \left\lvert\, \frac{1}{2}(p-1)\right.$, then $p=29,71$. For $p>q>7$, if $q \left\lvert\, \frac{1}{2}(p+1)\right.$, then $p-1=2^{i} \cdot 3^{j} \cdot 7$, where $1 \leqslant i \leqslant 3,0 \leqslant j \leqslant 2$. It follows that $p=43,127$. Since $2^{7}| | \operatorname{PSL}(2,127) \mid$, a contradiction. Thus $p=43$. Similarly, if $q \left\lvert\, \frac{1}{2}(p-1)\right.$, then $p=83,97,251,503$. Hence $2^{5}| | \operatorname{PSL}(2,97) \mid$ and $5^{3}| | \operatorname{PSL}(2,251) \mid$, a contradiction. Thus $p=83,503$.

For $q=3$, one has $3 \cdot 7 \cdot p||N|$ and $| N\left|\mid 2^{25} \cdot 3^{5} \cdot 5^{2} \cdot 7 \cdot p\right.$. By Table $1, N$ is one of the following groups:

$$
\begin{align*}
& A_{7}, A_{8}, A_{9}, A_{10}, \operatorname{PSL}(2,27), \operatorname{PSL}(3,4), \operatorname{PSU}(3,5), \operatorname{PSU}(3,8), J_{2},  \tag{4.6}\\
& D_{4}(2), \operatorname{PSp}(6,2), \operatorname{PSp}(8,2), A_{11}, A_{12}, M_{22}, \operatorname{PSL}\left(2,2^{6}\right), \operatorname{PSL}(4,4),  \tag{4.7}\\
& \quad \operatorname{PSL}(4,4), \operatorname{PSL}(5,2),{ }^{2} D_{4}(2), G_{2}(4), \operatorname{PSL}(2, p)(p=13,127) . \tag{4.8}
\end{align*}
$$

For $q=5$, one has $5 \cdot 7 \cdot p||N|$ and $| N\left|\mid 2^{25} \cdot 3^{4} \cdot 5^{3} \cdot 7 \cdot p\right.$. By Table $1, N$ is one of the following groups:

$$
\begin{gather*}
S z(8), A_{11}, M_{22}, H S, \operatorname{PSL}\left(2,2^{6}\right), \operatorname{PSL}\left(2,5^{3}\right), \operatorname{PSL}(5,2), \operatorname{PSL}(4,4),  \tag{4.9}\\
{ }^{2} D_{4}(2), G_{2}(4), \operatorname{PSL}(2, p)(p=29,41,71) \tag{4.10}
\end{gather*}
$$

For $q \geqslant 7$, one has $7 \cdot q \cdot p||N|$ and $| N\left|\mid 2^{25} \cdot 3^{4} \cdot 5^{2} \cdot 7 \cdot q \cdot p\right.$. By Table $1, N$ is one of the following groups:

$$
\begin{gather*}
\operatorname{PSL}(3,8),{ }^{3} D_{4}(2), \operatorname{PSL}\left(2,2^{9}\right), \operatorname{PSp}(4,8), M_{23}, M_{24}, J_{1}, \operatorname{PSL}(3,16),  \tag{4.11}\\
\operatorname{PSL}(2, p)(p=43,83,503) . \tag{4.12}
\end{gather*}
$$

We may assume that $N$ is a group listed in (4.5)-(4.12). Let $G \leqslant A$ be a transitive subgroup of $X$. By Proposition 2.7, $X \cong \operatorname{Cos}(G, H, H g H)$, where $H=G_{v}, g \in G \backslash H$, and $g^{2} \in H$, implying that $a$ normalizes $R=H \cap H^{g}$, that is, $g \in N_{G}(R) \backslash H$. Recall that $N$ has at most two orbits on $V(X)$. First let $N$ be transitive on $V(X)$. Take $G=N$.

If $N=A_{7}(q=3, p=5)$, then $N_{v}$ has order $|N| /|V(X)|=2^{2} \cdot 3 \cdot 7$. However, $A_{7}$ has no subgroups of order $2^{2} \cdot 3 \cdot 7$ by Atlas [3]. Thus, $N \neq A_{7}$. Similarly, $N \neq \operatorname{PSL}(2,127)$ $(q=3, p=127), \operatorname{PSL}(2,29)(q=5, p=29), \operatorname{PSL}(2,41)(q=5, p=41), \operatorname{PSL}(2,71)$ $(q=5, p=71)$ and $\operatorname{PSL}(2,503)(q=251, p=503)$ by Proposition 2.4. If $N=A_{9}$ $(q=3, p=5), A_{10}(q=3, p=5), \operatorname{PSU}(3,8)(q=3, p=19), \mathrm{A}_{11}(q=3, p=11)$ or ${ }^{2} D_{4}(2)(q=3, p=17)$, then $3^{3} \|\left|N_{v}\right|$ and $3^{4} \nmid\left|N_{v}\right|$. By Proposition 2.2, it is not possible. If $N=A_{8}(q=3, p=5)$, then $\left|N_{v}\right|=2^{5} \cdot 3 \cdot 7$. By Proposition 2.2, there exists a vertex stabilizer whose order is $2^{5} \cdot 3 \cdot 7$, a contradiction. Similarly, $N \neq \operatorname{PSL}(2,49)(q=3, p=7), \operatorname{PSL}(2,49)(q=5, p=7), \operatorname{PSL}(2,27)(q=3, p=13)$, $\operatorname{PSL}(3,4)(q=3, p=5), \operatorname{PSU}(3,5)(q=3, p=5), J_{2}(q=3, p=5), \operatorname{PSp}(6,2)$ $(q=3, p=5), \operatorname{PSp}(8,2)(q=3, p=5), A_{11}(p=5, q=11), M_{22}(q=3, p=11)$, $\operatorname{PSL}\left(2,2^{6}\right)(q=3, p=13), \operatorname{PSL}\left(2,2^{6}\right)(q=5, p=13), \operatorname{PSL}(4,4)(q=3, p=17)$, $\operatorname{PSL}(4,4)(q=5, p=17), \operatorname{PSL}(5,2)(q=3, p=31), \operatorname{PSL}(5,2)(q=5, p=31)$, $D_{4}(2)(q=3, p=5), H S(q=5, p=11), \operatorname{PSL}\left(2,5^{3}\right)(q=5, p=31),{ }^{2} D_{4}(2)(q=5$, $p=17), G_{2}(4)(q=3, p=13), G_{2}(4)(q=5, p=13), S z(8)(q=5, p=13)$, $\operatorname{PSL}(2,71)(q=3, p=71), \operatorname{PSL}(3,8)(q=7, p=73),{ }^{3} D_{4}(2)(q=7, p=17)$, $\operatorname{PSL}\left(2,2^{9}\right)(q=19, p=73), \operatorname{PSp}(4,8)(q=7, p=13), M_{23}(q=11, p=23), M_{24}$ $(q=11, p=23), J_{1}(q=11, p=19)$ and $N \neq \operatorname{PSL}(3,16)(q=13, p=17)$.

Suppose that $N=A_{12}(q=3, p=11)$. Then $\left|N_{v}\right|=|N| /|V(X)|=2^{8} \cdot 3^{4} \cdot 5^{2} \cdot 7$. By Proposition 2.2, $N_{v} \cong S_{7} \times S_{6}$. By [19], one concludes that $N$ has no subgroup which is isomorphic to $S_{7} \times S_{6}$, a contradiction.

Suppose that $N=\operatorname{PSL}(2,13)(q=3, p=13)$. Then $\left|N_{v}\right|=2 \cdot 7$. By Proposition 2.2, $N_{v} \cong D_{14}$, and by Proposition 2.4, $N_{v}$ is a maximal subgroup. By Example 3.1, $X \cong \mathcal{C}_{78}^{1}$ or $\mathcal{C}_{78}^{2}$.

Suppose that $N=\operatorname{PSL}(2,43)(q=11, p=43)$. Then $\left|N_{v}\right|=2 \cdot 3 \cdot 7$. By Proposition $2.2, N_{v} \cong F_{42}$, and by Proposition 2.4, one concludes that $N$ has a unique conjugacy class $D_{42}$ which has order 42 . Clearly, it is isomorphic to $F_{42}$, a contradiction.

Suppose that $N=\operatorname{PSL}(2,83)(q=41, p=83)$. Then $\left|N_{v}\right|=2 \cdot 3 \cdot 7$. By Proposition $2.2, N_{v} \cong F_{42}$. By Proposition 2.4, one concludes that $N$ has a unique maximal subgroup conjugacy class $D_{84}$ which contains subgroups of order 42. Clearly, the subgroups of order 42 of $D_{84}$ are isomorphic to $D_{42}$ or $\mathbb{Z}_{42}$. They are not isomorphic to $F_{42}$, a contradiction.

Suppose that $N=M_{22}(q=5, p=11)$. Then $\left|N_{v}\right|=2^{6} \cdot 3^{2} \cdot 7$. By Atlas [3], the unique maximal subgroup class of $M_{22}$ which has order divided by $2^{6} \cdot 3^{2} \cdot 7$ is $L_{3}(4)$; again by Atlas [3], $L_{3}(4)$ has no subgroup of order $2^{6} \cdot 3^{2} \cdot 7$, a contradiction.

Now let $N$ have two orbits on $V(X)$. If $N \neq A_{7}$, then $N_{v}$ has order $|N| / \frac{1}{2}|V(X)|=$ $2^{3} \cdot 3 \cdot 7$. However, $A_{7}$ has no subgroups of order $2^{3} \cdot 3 \cdot 7$ by Atlas [3]. Thus, $N \neq A_{7}$. Similarly, $N \neq \operatorname{PSL}(2,13)(q=3, p=13), \operatorname{PSL}(2,27)(q=3, p=13), \operatorname{PSL}(2,127)$
$(q=3, p=127), \operatorname{PSL}(2,29)(q=5, p=29), \operatorname{PSL}(2,41)(q=5, p=41), \operatorname{PSL}(2,71)$ $(q=5, p=71)$ and $\operatorname{PSL}(2,43)(q=11, p=43)$ by Proposition 2.4. If $N=A_{9}$ $(q=3, p=5), A_{10}(q=3, p=5), A_{11}(q=3, p=11), \operatorname{PSU}(3,8)(q=3, p=19)$ or ${ }^{2} D_{4}(2)(q=3, p=17)$, then $3^{3} \|\left|N_{v}\right|$. By Proposition 2.2 , this is not possible. If $N=A_{12}(q=3, p=11)$, then $\left|N_{v}\right|=2^{9} \cdot 3^{4} \cdot 5^{2} \cdot 7$. By Proposition 2.2 , there exists no vertex stabilizer whose order is $2^{9} \cdot 3^{4} \cdot 5^{2} \cdot 7$, a contradiction. Similarly, $N \neq \operatorname{PSL}(2,49)(q=3, p=7), \operatorname{PSL}(2,49)(q=5, p=7), \operatorname{PSL}(2,503)(q=251$, $p=503), \operatorname{PSL}(3,4)(q=3, p=5), \operatorname{PSU}(3,5)(q=3, p=5), J_{2}(q=3, p=5)$, $\operatorname{PSp}(6,2)(q=3, p=5), \operatorname{PSp}(8,2)(q=3, p=5), S z(8)(q=5, p=13), A_{11}$ $(p=5, q=11), M_{22}(q=3, p=11), M_{22}(q=5, p=11), \operatorname{PSL}\left(2,2^{6}\right)(q=3$, $p=13), \operatorname{PSL}\left(2,2^{6}\right)(q=5, p=13), \operatorname{PSL}(4,4)(q=3, p=17), \operatorname{PSL}(4,4)(q=5$, $p=17), \operatorname{PSL}(5,2)(q=3, p=31), H S(q=5, p=11),{ }^{2} D_{4}(2)(q=5, p=17)$, $D_{4}(2)(q=3, p=5), G_{2}(4)(q=3, p=13), G_{2}(4)(q=5, p=13), \operatorname{PSL}\left(2,5^{3}\right)$ $(q=5, p=31), \operatorname{PSL}(2,71)(q=3, p=71), \operatorname{PSL}(2,71)(q=5, p=71), \operatorname{PSL}(3,8)$ $(q=7, p=73),{ }^{3} D_{4}(2)(q=7, p=17), \operatorname{PSp}(4,8)(q=7, p=13), \operatorname{PSL}\left(2,2^{9}\right)$ $(q=19, p=73), M_{23}(q=11, p=23), M_{24}(q=11, p=23), J_{1}(q=11, p=19)$ and $N \neq \operatorname{PSL}(3,16)(q=13, p=17)$.

Suppose that $N=\operatorname{PSL}(2,83)(q=41, p=83)$. Then $\left|N_{v}\right|=2^{2} \cdot 3 \cdot 7$. By Proposition $2.2, N_{v} \cong F_{42} \times \mathbb{Z}_{2}$. By Proposition 2.4 , one concludes that $N$ has a unique conjugacy class $D_{84}$ which has order 84 . Clearly, it is isomorphic to $F_{42} \times \mathbb{Z}_{2}$, a contradiction.

Suppose that $N=A_{8}(q=3, p=5)$. Then $\left|N_{v}\right|=|N| / \frac{1}{2}|V(X)|=2^{6} \cdot 3 \cdot 7$. By Proposition 2.2, $N_{v} \cong \mathbb{Z}_{2}^{3} \times \mathrm{SL}(3,2)$. In this case, $N$ has two orbits on $V(X)$, and $N_{v} \cap N_{v}^{g} \cong \mathbb{Z}_{2}^{3} \rtimes S_{4}$. Let $C=C_{A}(N)$. Since $N$ is simple, $C \cap N=1$ and $C N=$ $C \times N \unlhd A$. Since $A / C N \lesssim \operatorname{Out}(N)$, we have $A=(C \times N) \cdot O$ with $O \lesssim \operatorname{Out}(N)$, where $\operatorname{Out}(N)$ is the outer automorphism group of $N$. Hence $|A| \mid 2^{25} \cdot 3^{5} \cdot 5^{3} \cdot 7$ and $|N|=2^{6} \cdot 3^{2} \cdot 5 \cdot 7$. Then $|C| \mid 2^{19} \cdot 3^{3} \cdot 5^{2}$. If $C$ is insolvable, by [3], pages $12-14$, then $C$ has a minimal normal insolvable subgroup $M \cong A_{5}, A_{6}$ or $A_{5}^{2}$. Then $N M \unlhd C N$ has at most two orbits on $V(X)$. Then $\left|(M N)_{v}\right|=|M N| /|V(X)|=2^{7} \cdot 3^{2} \cdot 5 \cdot 7$ or $|M N| / \frac{1}{2}|V(X)|=2^{8} \cdot 3^{2} \cdot 5 \cdot 7$ for $M \cong A_{5}$. By Proposition 2.2 , there exists no vertex stabilizer whose order is $\left|(M N)_{v}\right|$, a contradiction. For $M \cong A_{6}$ or $A_{5}^{2}$, one has $3^{3}| |(M N)_{v} \mid$. By Proposition 2.2 , this is a contradiction. Thus, $C$ is solvable. Clearly, $C$ is not semiregular on $V(X)$. If it were, $X_{C}$ would be a connected valency seven graph of order $2 p q /|C|$, yielding that $2 \nmid|C|$. Furthermost, $C \nsubseteq \mathbb{Z}_{3}$ or $\mathbb{Z}_{5}$, because there is no connected valency seven symmetric graph of order 6 or 10 . If $C$ has at most two orbits on $V(X)$, then $|C|=15$ or 30 . Let $R \cong \mathbb{Z}_{5} \leqslant C$. Then $R \triangleleft A$, and then $X_{C}$ is a connected valency seven graph of order 6 , a contradiction. Thus, $C=1$ and $A \leqslant \operatorname{Aut}\left(A_{8}\right)$. Further, $A=S_{8}$. By Example 3.3, $X \cong \mathcal{C}_{30}$. Hence
$N_{v}$ is a maximal subgroup of $N$, and $N$ has two orbits on $V(X)$. Then $X$ is a vertex bi-primitive 3 -arc transitive graph.

Suppose that $N=\operatorname{PSL}(5,2)(q=5, p=31)$. Then $\left|N_{v}\right|=2^{10} \cdot 3^{2} \cdot 7$. By Proposition 2.2, $N_{v} \cong \mathbb{Z}_{2}^{6} \rtimes(\mathrm{SL}(2,2) \times \operatorname{SL}(3,2))$. Let $C=C_{A}(N)$. Similarly to the above proof, one has $A=(C \times N) . O$ with $O \lesssim \operatorname{Out}(N)$, where $\operatorname{Out}(N)$ is the outer automorphism group of $N$. Hence $|A| \mid 2^{25} \cdot 3^{4} \cdot 5^{3} \cdot 7 \cdot 31$ and $|N|=2^{10} \cdot 3^{2} \cdot 5 \cdot 7 \cdot 31$. Then $|C| \mid 2^{15} \cdot 3^{2} \cdot 5^{2}$. If $C$ is insolvable, by [3], pages $12-14$, then $C$ has a minimal normal insolvable subgroup $M \cong A_{5}, A_{6}$ or $A_{5}^{2}$. Then $N M \unlhd C N$ has at most two orbits on $V(X)$. For $M \cong A_{5}$, one has $3^{3}| |(M N)_{v} \mid$. By Proposition 2.2, this is a contradiction. Thus, $M \nsubseteq A_{5}$. If $M \cong A_{6}$, then $\left|(M N)_{v}\right|=|M N| /|V(X)|=$ $2^{13} \cdot 3^{4} \cdot 5 \cdot 7$ or $|M N| / \frac{1}{2}|V(X)|=2^{12} \cdot 3^{4} \cdot 5 \cdot 7$. By Proposition 2.2 , there exists no vertex stabilizer whose order is $\left|(M N)_{v}\right|$, a contradiction. Similarly, $M \not \equiv A_{5}^{2}$. Thus, $C$ is solvable. Clearly, $C$ is not semiregular on $V(X)$. If it were, $X_{C}$ would be a connected valency seven graph of order $2 p q /|C|$, yielding that $2 \nmid|C|$. Furthermost, $C \nsubseteq \mathbb{Z}_{31}$ because there is no connected valency seven symmetric graph of order 10 . If $C \cong \mathbb{Z}_{5}$, by Proposition 2.5 , there is no connected valency seven symmetric graph of order 62 because $7 \nmid p-1$ with $p=31$. Thus $C$ has at most two orbits on $V(X)$, then $|C|=5 p$ or $10 p$. Let $R \cong \mathbb{Z}_{p}<C$. Then $R \triangleleft A$, and then $X_{R}$ is a connected valency seven graph of order 10, a contradiction. Thus, $C=1$ and $A \leqslant \operatorname{Aut}(N)$. Further, $A \cong \operatorname{Aut}\left(\operatorname{PSL}(5,2) \cdot \mathbb{Z}_{2}\right.$ because $\operatorname{Out}(N)=\mathbb{Z}_{2}$. By Example 3.5, $X \cong \mathcal{C}_{310}$. Hence $N_{v}$ is a maximal subgroup of $N$, and $N$ has two orbits on $V(X)$. Then $X$ is a vertex bi-primitive 3 -arc transitive graph. This completes the proof.

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