Valid Inequalities and Facets of the Capacitated Plant Location Problem

by

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Abstract

Recently, several successful applications of strong cutting plane methods to combinatorial optimization problems have renewed interest in cutting plane methods, and polyhedral characterizations, of integer programming problems. In this paper, we investigate the polyhedral structure of the capacitated plant location problem. Our purpose is to identify facets and valid inequalities for a wide range of capacitated fixed charge problems that contain this prototype problem as a substructure.

The first part of the paper introduces a family of facets for a version of the capacitated plant location problem with constant capacity K for all plants. These facet inequalities depend on K and thus differ fundamentally from the valid inequalities for the uncapacitated version of the problem.

We also introduce a second formulation for a model with indivisible customer demand and show that it is equivalent to a vertex packing problem on a derived graph. We identify facets and valid inequalities for this version of the problem by applying known results for the vertex packing polytope.

1 Introduction

The plant location problem arises in a variety of settings, ranging from telecommunications to transportation and production planning, and constitutes a major area of study in operations research. The problem is usually formulated as an integer, or mixed integer, programming problem. In this paper, we focus on versions of this problem with capacity restrictions. Our interest is in studying the polyhedral structure of these problems.

Recently, there has been a resurgence of interest in cutting plane algorithms for integer programs and combinatorial optimization problems. Papers by Crowder, Johnson and Padberg [1983], Van Roy and Wolsey [1983], and Martin and Schrage [1985]are notable examples. Researchers have devised effective cutting plane methods for a variety of problems that utilize characterizations of the polyhedral structure of the underlying problem. (See, for example Padberg and Hong [1980], Grötschel, Jünger and Reinelt [1984].) Computational studies have shown that the polyhedral characterization of a substructure of a complex problem also *tightens* the problem formulation and yields successful cutting plane algorithms. (See Crowder, Johnson and Padberg [1983], Johnson, Kostreva and Suhl [1985], Eppen and Martin [1985].).

Our goal is to investigate the polyhedral structure of the capacitated plant location problem. We wish to identify valid inequalities that gives a tighter LP-relaxation of the problem, and, in particular, facets of the convex hull of feasible solutions which are the tightest possible inequalities. Valid inequalities and facets are the foundation stones upon which strong cutting plane algorithms are built. The purpose of this research is to identify the building blocks for use in developing cutting plane methods for a wide range of capacitated fixed charge problems that contain the capacitated plant location problem as a substructure.

In this paper, we will describe different models for the capacitated plant location problem, and identify families of valid inequalities and facets for each model. Although researchers have devised many insightful algorithms and theoretical results concerning facility location problems, only recently have results on the polyhedral structure of these problems emerged. (See Cornuejols and Thizy [1982] and Cho, Johnson, Padberg and Rao [1983a,b].) Also, surprisingly little is known about the polyhedral structure of capacitated versions of these problems. The results presented in this paper are intended to be a first step in extending our knowledge in this area; we hope it will lead to new and effective cutting plane methods for these capacitated problems.

This paper is organized as follows.

Section Two contains a survey of the literature. We describe some application areas and outline both heuristic and exact solution approaches that have been used for the plant location problem.

Section Three introduces a model of the capacitated plant location problem; it is a direct extension of a well-known model of the uncapacitated plant location problem and has exactly the same set of variables. The results in this section show that the defining constraints of the problem describe facets. Section Four identifies a new family of *"residual capacity"* facets for the capacitated plant location problem. It also mentions how the facets identified in this analysis relate to some general results on integer programming polytopes.

Section Five analyzes conditions under which the family of facets introduced in Section Four remain facets for modified versions of the capacitated plant location problem.

Section Six introduces a second model of the capacitated plant location problem and shows that it is equivalent to a vertex packing problem on a derived graph. This model differs from the one introduced in Section Three since it requires that all of a customer's demand be served from a single plant; on the other hand, the model allows plant capacities to vary by location. After stating some results on the facets of the vertex packing problem, we describe how these results can be applied to the capacitated plant location problem to identify several families of valid inequalities and facets. The last part of this section compares the two formulations of the capacitated plant location problem stated in Sections Three and Six.

Section Seven indicates several directions for future research.

2 Literature Review

Facility location problems have received widespread attention in the last two decades. There is a large body of research on just the uncapacitated plant location problem and its derivatives. This rich literature attests to the practical significance and theoretical interest of the problem.

2.1 Plant location

According to Krarup and Pruzan [1983], the problem usually referred to as the uncapacitated plant location problem was first formulated independently by Balinski and Wolfe [1963] (see Balinski [1964]), Kuehn and Hamburger [1963], Manne [1964] and Stollsteimier [1963]. Manne's work was on plant location and gave the general problem its current name. Balinski and Kuehn and Hamburger discussed this problem in the context of warehouse location.

The plant location problem has been used in a variety of application areas beyond the scope of distribution planning. Cornuejols, Fisher and Nemhauser [1977a] discuss an application in financial planning; in this setting, the facilities represent bank accounts and the objective is to maximize clearing times of cheques. Another area of application is in the design of telecommunications networks. The problem of access design, in which concentrators must be located to connect terminals to a central processor is often modelled and solved as a plant location problem (see Tanenbaum [1981]). Kochman and McCallum [1981] discuss a capacity expansion problem for transatlantic cables which they modelled as a plant location problem with some side constraints. For further discussions on telecommunications network design, see the papers by Kershenbaum and Boorstyn [1975] and Boorstyn and Frank [1977]. Dykstra and Riggs [1977] described an application in forestry; they modelled the design of a timber harvesting system as a hierarchical facility location problem. Other areas of application of facility location include machine scheduling and information retrieval (see Fisher and Hochbaum [1980]).

The early algorithms proposed for the uncapacitated plant location problem were mostly heuristics, the most well known being the ADD heuristic of Kuehn and

Hamburger [1963] that opens facilities one at a time until the marginal saving for opening an additional facility becomes negative. Feldman, Lehrer and Ray [1966] proposed a similar greedy heuristic, DROP, that initially opens all the facilities and then close them one at a time. Manne [1964] proposed a local search procedure that moves from one solution to the 'neighbouring' one that give the greatest decrease in cost. Two solutions are neighbours if some facility j is open in one solution and not the other while the status of all other facilities are identical. Kuehn and Hamburger [1963] also proposed an interchange heuristic (SHIFT). Cornuejols, Fisher and Nemhauser [1977a] studied the worst case behaviour of greedy heuristics.

The first exact solution method was proposed by Balinski and Wolfe [1963] who suggested a Benders' decomposition approach. Efroymson and Ray [1966] applied a branch-and-bound scheme using a weak formulation of the problem. Branch-and-bound solution methods have since been further refined by the introduction of clever branching rules. See the papers by Sa [1969], Davis and Ray [1969], Khumawala [1972] , Akinc and Khumawala [1977], and Nauss [1978]. Spielberg [1969] proposed a direct search scheme.

Erlenkotter [1978] suggested a dual ascent procedure, DUALOC, which appears to be the most successful currently available solution method for uncapacitated problems. Bilde and Krarup [1977]suggested the same approach. Guignard and Spielberg [1979] proposed a direct dual approach. Nauss [1978] and Christofides and Beasley [1983] also used dual-based methods, incorporating subgradient optimization for Lagrangian relaxation in the context of a branch-and-bound scheme.

Cornuejols, Nemhauser and Wolsey [1983] give a general survey of the uncapacitated plant location problem. Magnanti and Wong [1985] discuss decomposition methods and modelling issues in depth. Wong [1985] provides an annotated bibliography of facility location problems.

Most solution methods for the capacitated plant location problem are adaptations of the algorithms for the uncapacitated problem. Jacobsen [1983] generalized the ADD, DROP and SHIFT heuristics of Kuehn and Hamburger to the capacitated plant location problem. He also proposed two heuristics that were adapted from the Alternate-Location-Allocation heuristic (Rapp [1962]) and the Vertex Substitution Method (Teitz and Bart [1968]) for p-median problems. Branch-and-bound procedures suggested for the capacitated problem include those suggested by Davis and Ray [1969], Sa [1969], Ellwein and Gray [1971], Akinc and Khumawala [1977] and Nauss [1978].

Geoffrion and McBride [1978] suggested a Lagrangian relaxation approach for the capacitated plant location problem. Other dual-based methods have been suggested by Guignard and Spielberg [1979], Van Roy and Erlenkotter [1982] and Christofides and Beasley [1983]. Bitran, Chandru, Sempolinski and Shapiro [1981] proposed an inverse optimization approach using both Lagrangian and group theoretic techniques. Ross and Soland [1977] modelled facility location problems as generalized assignment problems and applied a branch-and-bound algorithm. Van Roy [1986] suggested a cross-decomposition approach for the problem.

Computational results on the uncapacitated plant location problem have been very impressive. One explanation is that integer solutions are often obtained while solving the linear programming relaxation of the problem. Part of the motivation for this research is to develop formulations for the capacitated plant location problem that provide tight LP-relaxations. This research focuses on investigating the facets of the underlying polytope and identifying valid inequalities.

Compared to the work devoted to algorithmic developments, few papers are devoted to characterizing the structure of the feasible solutions of the plant location problem. Cornuejols, Fisher and Nemhauser [1977b] and Guignard [1980] characterized the fractional solutions of the LP-relaxation of the uncapacitated plant location problem. Both papers also proposed valid inequalities for the problem. Results concerning the facets of the uncapacitated plant location problem are summarized in three excellent papers by Cornuejols and Thizy [1982] and Cho, Johnson, Padberg and Rao [1983a,b]. As far as we know, no results on the facets of the capacitated plant location problem has been published.

2.2 Strong Cutting Plane Algorithms

For references on polyhedral combinatorics, we refer the reader to the excellent annotated bibliography by Grõtschel [1985].

Cutting planes were used in the solution of a travelling salesman problem by Dantzig, Fulkerson and Johnson [1954]. More recently, Padberg and Hong [1980] and Crowder and Padberg [1980] reported very successful implementations of cutting plane algorithms using facet defining inequalities for the travelling salesman problem. Crowder, Johnson and Padberg [1983] obtained equally impressive results by using facet-based inequalities for single constraints in the solution of some sparse zero-one problems. Grötschel, Jünger and Reinelt [1984,1985] developed a *strong* cutting plane method (i.e. one using facet defining inequalities) for the linear ordering problem. Use of strong cutting plane methods for strategic planning was reported by Johnson, Kostreva and Suhl [1985] and for vehicle routing by Laporte, Mercure and Nobert [1986].

As far as we know, no strong cutting plane procedure has been developed for the facility location problem (either with or without facility capacities). The results in this paper provide the ingredients for developing strong cutting plane algorithms. We are currently working on such an algorithm and hope to report on our computational experience in the near future.

3 The Capacitated Plant Location Problem

3.1 Terminology

We will first formulate the capacitated plant location problem as a mixed-integer program. The variables in the problem are

$$y_j = \begin{cases} 1 & \text{if plant } j \text{ is open} \\ 0 & \text{otherwise} \end{cases}$$

and $\quad x_{ij}$ = fraction of the demand of customer i supplied by plant j

and the constraints are

$$(PI): \quad \begin{array}{rll} x_{ij} & \leq y_j & \forall\, i \in \mathcal{M}, \forall\, j \in \mathcal{N} \quad (1) \\ \sum_{j \in \mathcal{N}} x_{ij} & \leq 1 & \forall\, i \in \mathcal{M} \quad (2) \\ \sum_{i \in \mathcal{M}} d_i\, x_{ij} & \leq K y_j & \forall\, j \in \mathcal{N} \quad (3) \\ x_{ij} & \geq 0 & \forall\, i \in \mathcal{M} \quad (4) \\ y_j & \leq 1 & \forall\, j \in \mathcal{N} \quad (5) \\ y_j & \textit{integer} & \forall\, j \in \mathcal{N} \quad (6) \end{array}$$

where

$$\begin{array}{rll} \mathcal{M} = & \{1, 2, \ldots, M\} & \text{is the set of customers,} \\ \mathcal{N} = & \{1, 2, \ldots, N\} & \text{is the set of plants,} \end{array}$$

and

$$d_i = \text{demand of customer } i.$$

Without loss of generality, we assume that $d_i > 0$ for all i. Let $D_{\mathcal{M}}$ denote the total demand, i.e.,

$$D_{\mathcal{M}} = \sum_{i \in \mathcal{M}} d_i.$$

In this model, we assume that all plants have the same capacity K. If $K \geq D_{\mathcal{M}}$, then constraint (3) is redundant and (PI) becomes an uncapacitated plant location problem. If $NK < D_M$, then the problem is infeasible. In order to rule out cases not of interest in our development, we make the assumption that $\frac{D_M}{N} \leq K < D_M$. For simplicity, we also assume that K and all the d_i's are integer-valued.

Notice that the constraints (2) of (PI) are inequalities. Thus, our model allows solutions in which some customers' demand may not be fully met. In Section 5.1, we will examine a variant of the capacitated plant location problem in which the demand of all the customers must be met in full.

We identify each feasible solution of (PI) with a point in $\mathbf{R}^{MN+N}$. Let $\mathcal{F}_{PI}$ be the set of feasible points for the problem (PI), and let $\mathcal{P}$ denote the convex hull of $\mathcal{F}_{PI}$, i.e.,

$$\mathcal{P} = conv\left\{(x,y) \in \mathbf{R}^{MN+N} \mid (x,y) \text{ satisfies constraints } (1)-(6)\right\}.$$

An inequality

$$\alpha_1 x + \alpha_2 y \leq \alpha_0 \qquad\qquad \alpha_1 \in \mathbf{R}^{MN},\ \alpha_2 \in \mathbf{R}^{N},\ \alpha_0 \in \mathbf{R} \tag{$\dagger$}$$

is a *valid inequality* for $\mathcal{P}$ if it is satisfied by all the points in $\mathcal{F}_{PI}$ (and hence in $\mathcal{P}$). The inequality defines an *improper face* of $\mathcal{P}$ if it is satisfied as an *equality* by all the points of $\mathcal{F}_{PI}$ (and $\mathcal{P}$).

Let $\alpha = (\alpha_1, \alpha_2, \alpha_0)$ and let

$$\mathcal{P}^{\alpha} \quad = \quad \{(x,y) \mid (x,y) \in \mathcal{P} \text{ and } \alpha_1 x + \alpha_2 y = \alpha_0\}.$$

These points are those in $\mathcal{P}$ that satisfy ($\dagger$) as an equality. If $\mathcal{P}^{\alpha}$ is not empty, then the intersection of $\mathcal{P}$ and the hyperplane defined by $\alpha_1 x + \alpha_2 y = \alpha_0$ is a *face* of the polyhedron $\mathcal{P}$. $\mathcal{P}^{\alpha}$ describes a k-dimensional face if it contains exactly $k+1$ affinely independent points of $\mathcal{P}$. For example, an extreme point of $\mathcal{P}$ is a 0-dimensional face. A face of dimension $(\dim \mathcal{P}) - 1$ is called a *facet*.

When $\mathcal{P}$ is full dimensional,[1] the inequality defining a facet of $\mathcal{P}$ is unique up to positive scaling. We will use the term *facet* to refer to both the physical face of the polyhedron and the inequality defining it.

We are interested in identifying the facets of the polyhedron $\mathcal{P}$.

[1] $\mathcal{P}$ is full dimensional if the affine subspace generated by it is of dimension $MN + N$. When $\mathcal{P}$ is not full dimensional, the defining inequality for a facet of $\mathcal{P}$ is not unique.

3.2 Trivial Facets

We first note that the polyhedron $\mathcal{P}$ is full-dimensional. For each j, we can construct a point in $\mathcal{F}_{PI}$ in which *either* y_j is set to 1 while all other variables are set to 0, *or* each x_{ij} is set to $\min\{1, \frac{K}{d_i}\}$ and the corresponding $y_j = 1$ while all other variables are set to 0. Since $(x, y) = (0, 0)$ is also in $\mathcal{P}$, we can see that $\mathcal{P}$ spans an $(MN+N)$-dimensional space.

Theorem 1
(i) For every $i \in \mathcal{M}$, if $\lceil \frac{d_i}{K} \rceil < N$, then $\sum_{j \in \mathcal{N}} x_{ij} \leq 1$ is a facet of $\mathcal{P}$.
(ii) For every $i \in \mathcal{M}$ and $j \in \mathcal{N}$, if $d_i \leq K$, then $x_{ij} \leq y_j$ is a facet of $\mathcal{P}$.
(iii) For every $i \in \mathcal{M}$, $j \in \mathcal{N}$, $x_{ij} \geq 0$ is a facet of $\mathcal{P}$.
(iv) For every $j \in \mathcal{N}$, $y_j \leq 1$ is a facet of $\mathcal{P}$.
(v) For every $j \in \mathcal{N}$, $\sum_{i \in \mathcal{M}} d_i x_{ij} \leq K y_j$ is a facet of $\mathcal{P}$.

This theorem asserts that the inequality constraints (1) - (5) of the capacitated plant location problem define facets of $\mathcal{P}$. Since the facets (1) - (5) correspond to the constraints of the standard formulation of the capacitated plant location problem, they are called the *trivial* facets of $\mathcal{P}$.

By definition, (1) - (5) must be valid inequalities. In order to show that a valid equality is a facet of $\mathcal{P}$, it is sufficient to exhibit $MN + N$ affinely independent points in $\mathcal{P}$ that satisfy the inequality as an *equality*. We will use this approach to prove that inequalities (1), (2), (4) and (5) are facets. The proof that (3) is a facet of $\mathcal{P}$ will require a different approach which we will develop in more detail in the next Section. The proof of Theorem 1 is long but relatively straightforward and will be deferred to Appendix A.

4 A New Family of Facets for the Capacitated Plant Location Problem

In this subsection, we introduce a new family of facets for the capacitated plant location problem. This family contains exponentially many facets. Since (PI) is NP-hard, a 'simple' complete characterization of the convex hull of its feasible solutions is unobtainable unless $P = co\text{-}NP$. (See Karp and Papadimitriou [1982] and Grōtschel, Lovász and Schrijver [1981].) Our goal is to identify potentially effective cuts for use in a cutting-plane method for this problem and related capacitated fixed charge problems.

Consider the family of inequalities

$$\sum_{i\in I}\sum_{j\in J} d_i x_{ij} - r\sum_{j\in J} y_j \ \leq\ D - r\lceil \frac{D}{K} \rceil \qquad (RC)$$

where $I \subseteq \mathcal{M}, J \subseteq \mathcal{N}, D = \sum_{i\in I} d_i$ and $r = D \pmod K$ with $1 \leq\ r\ \leq\ K$.[2]

This is a family of valid inequalities, most of which are facets, for the capacitated plant location problem (PI). Notice that $\lceil \frac{D}{K} \rceil$ is the minimum number of plants required to supply the customers in I. If D is a multiple of K, then all $\lceil \frac{D}{K} \rceil$ plants must produce to capacity. Otherwise, if $\lceil \frac{D}{K} \rceil - 1$ plants produce to capacity, then r is the residual demand that the last plant must satisfy. For this reason, we refer to the inequalities (RC) as the ***residual capacity*** inequalities (or, for convenience, the r-inequalities). In the next subsection, we will discuss how the residual capacity inequalities are derived and provide some intuition on why they define facets.

4.1 Why the residual capacity inequalities are valid

The residual inequalities focus on a subset of the customers and plants and ensures that the supply to these customers from these plants satisfies the capacity restrictions. Let us consider the aggregate variables, x and y, defined as follows:

[2] In this paper, we define $r = K$ instead of $r = 0$ if D is a multiple of K. This choice differs slightly from the convention in the literature.

$$x = \sum_{i \in I} \sum_{j \in J} d_i x_{ij} = \text{aggregate supply from plants in } J \text{ to customers in } I,$$
$$y = \sum_{j \in J} y_j \quad = \text{total number of open plants in } J.$$

In terms of the aggregate variables x and y, the capacity constraints (3) and the demand constraints (2) imply

$$x \leq K\,y \tag{7}$$

and

$$x \leq D \tag{8}$$

respectively. However, because y is an integer variable, the convex hull of the feasible solutions is strictly smaller than the region defined by the two inequality constraints (7) and (8) when $r \neq K$. See Figure 1. The figure suggests that the

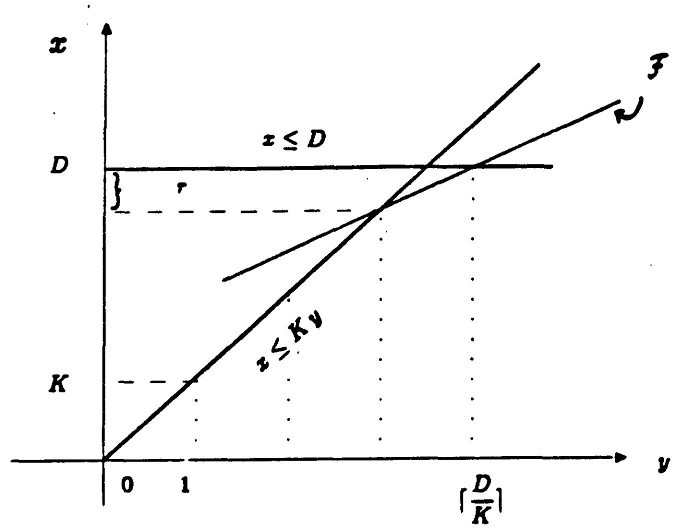

Figure 1: Feasible Region of CPLP in aggregate (y, x) space.

'facet' that needs to be added is the one marked $\mathcal{F}$, which has slope $= r$ and passes through the point $(y, x) = (\lceil \frac{D}{K} \rceil, D)$. The inequality describing this 'facet' is

$$\frac{x - D}{y - \lceil \frac{D}{K} \rceil} \leq r \tag{9}$$

or

$$x - ry \leq D - r\lceil \tfrac{D}{K} \rceil$$

which is exactly (RC) written in terms of the aggregate variables x and y. Figure 1 also indicates why the coefficient of the term $\sum_{j\in J} y_j - \lceil \frac{D}{K} \rceil$ in (RC) is r. If the coefficient is greater than r, then the inequality is not valid; if it is smaller, then the residual capacity inequality is a face but not a facet of $\mathcal{P}$.

4.2 When do the residual capacity inequalities define facets?

In this subsection, we will delineate the conditions under which the residual capacity inequalities define facets for $\mathcal{P}$.

Proposition 2 *(RC) is a valid inequality for (PI). Provided $|J| \geq 1$, it defines a face of $\mathcal{P}$ if and only if $|J| \geq \lfloor \frac{D}{K} \rfloor$.*

Proof. If $|J| = 0$, (RC) becomes $0 \leq 0$ which is vacuously valid.

For all feasible solutions of (PI), let

$$d = \sum_{i\in I}\sum_{j\in J} d_i x_{ij} = \text{the demand of customers in } I \text{ that is satisfied by the set of plants } J. \tag{10}$$

Since each plant has capacity K, we must have

$$\sum_{j\in J} y_j \geq \lceil \frac{d}{K} \rceil \tag{11}$$

and therefore

$$\sum_{i\in I}\sum_{j\in J} d_i\, x_{ij} - r \sum_{j\in J} y_j \leq d - r\lceil \frac{d}{K} \rceil, \tag{12}$$

which is an aggregation of (10) and (11). The maximum of the righthand side of (12) over the interval $0 \leq d \leq D$ is attained when $d = D$ (see Figure 2), so the r-inequality (RC) is satisfied by all feasible solutions to (PI) and is a valid inequality for $\mathcal{P}$.

Moreover, for any choice of I and J, provided $|J| \geq \lfloor \frac{D}{K} \rfloor$, it is clear that we can find a solution point in $\mathcal{P}$ that satisfies (RC) as an equality. (For example, given any ordering of the customers in I and plants in J, we can assign the customers in order to the first plant until it reaches capacity; subsequent customers' demand is then assigned to the second plant, and so on.) If $|J| \geq \lceil \frac{D}{K} \rceil$, then all the customer demand can be assigned; If $|J| = \lfloor \frac{D}{K} \rfloor < \lceil \frac{D}{K} \rceil$, then all plants in J are used and $\sum_{i \in I} d_i x_{ij} = D - r$. In both cases, the r-inequality (RC) is satisfied as an equality. Therefore, the residual capacity inequalities (RC) define a family of *faces* for $\mathcal{P}$. When $|J| < \lfloor \frac{D}{K} \rfloor$, the capacity constraints (3) forces the lefthand side of (RC) to be strictly less than its righthand side; in this case, the inequality (RC) is valid but not a face. ∎

We would like to determine when the residual capacity inequality (RC) defines a facet for $\mathcal{P}$.

Let us first explore the question "When is (RC) satisfied as an equality?". Consider the function

$$f(d) = \min \left\{ \sum_{j \in J} y_j \mid \sum_{i \in I} \sum_{j \in J} d_i x_{ij} = d, \text{ and } (x, y) \text{ is feasible for } (PI) \right\}.$$

Clearly, $f(d) = \lceil \frac{d}{K} \rceil$. Define $g(d) = d - r f(d) = d - r \lceil \frac{d}{K} \rceil$. The function $g(d)$ for $\{d \mid 0 \leq d \leq D\}$ is plotted in Figure 2.

We can see that $g(d)$ attains its maximum value of $D - r \lceil \frac{D}{K} \rceil$ at $d = D$ or $d = D - r$, whence $\sum_{j \in J} y_j = \lceil \frac{D}{K} \rceil$ or $\lceil \frac{D}{K} \rceil - 1$ respectively. The following proposition follows directly from this observation.

Proposition 3 *If* $|J| = \lfloor \frac{D}{K} \rfloor$, *then* (RC) *is a face but not a facet.*

Proof. Let

$$\mathcal{P}^{RC} = \{(x, y) \in \mathbf{R}^{MN+N} \mid (x, y) \in \mathcal{P} \text{ and satisfies } (RC) \text{ as an equality}\}. \quad (13)$$

If the residual capacity inequality (RC) were a facet, then the dimension of $\mathcal{P}^{RC}$ must be $MN + N - 1$. When $|J| = \lfloor \frac{D}{K} \rfloor$, then all the points in $\mathcal{P}^{RC}$ must have

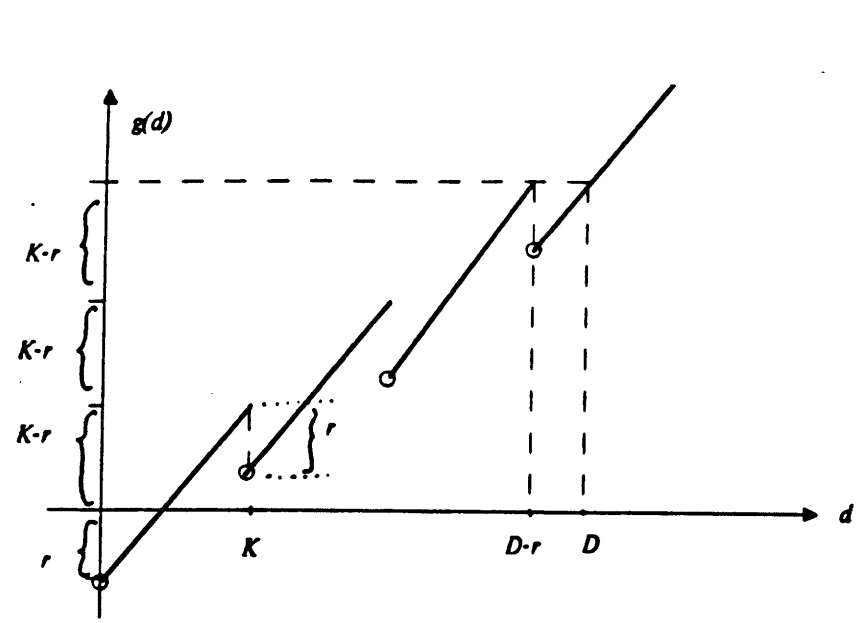

Figure 2: Graph of $g(d)$.

$y_j = 1$ for all $j \in J$. the dimension of $\mathcal{P}^{RC}$ is at most [3] $(MN+N)-(|J|+1)$ and the residual capacity inequality (RC) cannot be a facet. ∎

If $D = qK$ for some q (i.e., when $r = K$), then the residual capacity inequality (RC) becomes

$$\sum_{i\in I}\sum_{j\in J} d_i x_{ij} - K\sum_{j\in J} y_j \le D - K\lceil\frac{D}{K}\rceil = 0.$$

This inequality is a positive linear combination of the following valid inequalities:

$$\sum_{i\in I} d_i x_{ij} \quad \le \quad K y_j \qquad \forall j \in J.$$

Thus, when $D = qK$, the r-inequality (RC) cannot be a facet of $\mathcal{P}$.

The only remaining case to consider is when $D = qK + r$ for some $q \ge 0$ with $1 \le r \le K-1$ and $|J| \ge \lceil\frac{D}{K}\rceil$.

Theorem 4 *When $r = D \pmod K$ satisfies $1 \le r \le K-1$, then, provided $|J| \ge \lceil\frac{D}{K}\rceil$, the residual capacity inequality (RC) is a facet of $\mathcal{P}$.*

[3]Although $\mathcal{P}$ is full-dimensional, its intersection with an m-dimensional affine subspace may be of dimension $< m$. For example, consider the intersection of the n-dimensional unit cube with the $(n-1)$-dimensional affine subspace $x_1 + x_2 + \ldots + x_n = n$. It is a single point and has dimension 0.

Proof. By Proposition 2, we know that (RC) is a face of $\mathcal{P}$. To show that (RC) is a facet, we need to prove the following:

1. (RC) is not an improper face.
2. The face $\mathcal{P}^{RC}$, as defined by (13), is of dimension $MN + N - 1$.

If $\mathcal{P}^{RC}$ were a lower-dimensional face but not a facet, then we could find another valid inequality that is also *tight* (i.e., satisfied as an equality) for *all* the points in $\mathcal{P}^{RC}$. Thus, to establish the second of these conditions, it is sufficient to prove that (RC) is the only valid inequality that is tight for all the points in $\mathcal{P}^{RC}$. (Since $\mathcal{P}$ is full-dimensional, the facet is uniquely defined by (RC).)

Claim 1: The r-inequality (RC) is not an improper face of $\mathcal{P}$.

Proof of Claim 1: Consider the point in $\mathcal{P}$ defined as follows:

$$\begin{cases} y_j = 1 \qquad \forall j \in J \\ \text{all other variables set to zero.} \end{cases}$$

Substituting these values into (RC), we get

$$-r|J| \;<\; 0 \;\leq\; D - r\lceil \tfrac{D}{K} \rceil.$$

Therefore, the residual capacity inequality (RC) is not satisfied as an equality by all points in $\mathcal{P}$ and so it is not an improper face of $\mathcal{P}$.

Let

$$\mathcal{P}^{RC} = \{(x, y) \mid (x, y) \in \mathcal{P} \text{ and } (x, y) \text{ satisfies } (RC) \text{ as an equality } \}$$

as defined in (13). $\mathcal{P}^{RC}$ is not empty since we have already shown that the r-inequality (RC) is a face.

Claim 2: $\dim \mathcal{P}^{RC} \;=\; \dim \mathcal{P} - 1 = MN + N - 1$.

Proof of Claim 2: To prove this assertion, we need to show that no other valid inequality is satisfied as an equality by all points in $\mathcal{P}^{RC}$.

Suppose that $\alpha x + \beta y \le \alpha_0$ is a non-trivial valid inequality for (PI) and

$$\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}} \alpha_{ij}x_{ij} \; + \; \sum_{j\in\mathcal{N}} \beta_j y_j \;=\; \alpha_0 \tag{$\ddagger$}$$

holds for all $(x, y) \in \mathcal{P}^{RC}$. We will show that ($\ddagger$) is a multiple of

$$\sum_{i\in I}\sum_{j\in J} d_i\, x_{ij} - r\sum_{j\in J} y_j \;=\; D - r\lceil\frac{D}{K}\rceil. \tag{$RC_=$}$$

We will do so by substituting the coordinate values of the points in $\mathcal{P}^{RC}$ into ($\ddagger$). By comparing coefficients of the resulting equations, we will show that ($\ddagger$) is identical to $(RC_=)$ up to a multiplicative constant.

For any $j_1 \in J$, choose $J_1 \subseteq J$ satisfying

$$|J_1| \;=\; \lceil\frac{D}{K}\rceil \quad \text{and} \quad j_1 \in J_1.$$

Let (x^1, y^1) be the solution defined by

$$y_j^1 = \begin{cases} 1 & \text{if } j \in J_1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and

$$\begin{aligned} \sum_{j\in J_1} x_{ij}^1 &= 1 & \forall i \in I \\ x_{ij}^1 &= 0 & \forall i \notin I, \forall j \in \mathcal{N} \\ \sum_{i\in I} d_i\, x_{ij_1}^1 &= r \end{aligned} \tag{15}$$

This choice is always possible since $\sum_{i\in I} d_i = K|J_1| - (K - r)$. Moreover, it is clear that $(x^1, y^1) \in \mathcal{P}^{RC}$. The set J_1 corresponds to the set of open plants. The x_{ij}'s represent an assignment of customers so that plant j_1 is not at capacity, producing only r units.

Next consider any $i_1 \notin I$. We perturb the solution (x^1, y^1) so that plant j_1 supplies customer i_1 as well. Let $\delta = \min\{1, \frac{K-r}{d_{i_1}}\}$. Let (x^2, y^2) be defined as in (x^1, y^1) except $x_{i_1 j_1}^2 = \delta$. Then $d_{i_1}x_{i_1j_1}^2 + \sum_{i\in I} d_i x_{i_1j_1}^2 \le K$ and (x^2, y^2) is also in $\mathcal{P}^{RC}$.

Evaluating the two solutions in (‡) and comparing the resultant expressions, we see that $\delta \alpha_{i_1 j_1} = 0$. Since $\delta > 0$ and the indices were chosen arbitrarily, we get

$$\alpha_{ij} = 0 \qquad \forall i \notin I, j \in J. \tag{16}$$

Next define (x^3, y^3) as in (x^2, y^2) except that $y_{j_3} = 1$ for some $j_3 \in \mathcal{N} \setminus J$. Define (x^4, y^4) as in (x^3, y^3) except the demand for customer $i_1 \notin I$ is assigned to plant j_3 instead, i.e.,

$$y_j^4 = y_j^3 \qquad \forall j$$

$$x_{ij}^4 = \begin{cases} x_{i_1 j_1}^3 & \text{if } i = i_1,\ j = j_3 \\ 0 & \text{if } i = i_1,\ j = j_1 \\ x_{ij}^3 & \text{otherwise.} \end{cases}$$

Comparing (x^3, y^3) and (x^4, y^4), we see that $\alpha_{i_1 j_3} = 0$. Again since the indices are arbitrary, we have

$$\alpha_{ij} = 0 \qquad \forall i \notin I, \quad j \notin J. \tag{17}$$

Now, consider again (x^2, y^2) and (x^3, y^3). Substituting their values into (‡) and comparing, we see that we must have $\beta_{j_3} = 0$, and so, we can conclude that

$$\beta_j = 0 \qquad \forall j \notin J. \tag{18}$$

Next consider (x^5, y^5) in P^{RC} satisfying $\sum_{j \in J} x_{i_5 j}^5 < 1$ for some $i_5 \in I$, and $y_{j_5}^5 = 0$ for some $j_5 \notin J$. This assignment is clearly possible. Now modify (x^5, y^5) so that customer i_5 is assigned to plant $j_5 \notin J$, i.e.,

$$y_j^6 = \begin{cases} 1 & \text{if } j = j_5 \\ y_j^5 & \text{otherwise} \end{cases}$$

$$x_{ij}^6 = \begin{cases} \min\{1, \frac{K}{d_{i_5}}\} & \text{if } i = i_5,\ j = j_5 \\ x_{ij}^5 & \text{otherwise.} \end{cases}$$

From these solutions, we see that $\alpha_{i_5 j_5} = 0$, and similarly

$$\alpha_{ij} = 0 \qquad \forall i \in I, \qquad j \notin J. \tag{19}$$

At this point, we have shown that the coefficients α_{ij}'s are zero when either $j \notin J$ or $i \notin I$, and the β_j's are zero when $j \notin J$.

Next, for any $\{j_1, j_7\} \subseteq J$, choose $J_1 \subseteq J$ so that

$$|J_1| = \lceil \frac{D}{K} \rceil \quad \text{and } \{j_1, j_7\} \subseteq J_1.$$

Assign (x^1, y^1) according to (14) and (15) as before. Thus plant j_1 is supplying only r units. For any $i_7 \in I$, we can assume without loss of generality that the assignment satisfies

$$x^1_{i_7 j_7} > 0.$$

Now we consider modifying the solution by re-assigning customer i_7 to plant j_1. Let $\delta = \min\{\frac{K-r}{d_{i_7}}, x^1_{i_7 j_7}\}$. Define (x^7, y^7) by

$$y^7_j = y^1_j \qquad \forall j \in J$$

$$x^7_{ij} = \begin{cases} x^1_{i_7 j_1} + \delta & \text{if } i = i_7 \text{ and } j = j_1 \\ x^1_{i_7 j_7} - \delta & \text{if } i = i_7 \text{ and } j = j_7 \\ x^1_{ij} & \text{otherwise.} \end{cases} \tag{20}$$

It is easy to check that both of these points are in P^{RC}. Substituting their values into (‡) and subtracting, we see that

$$\alpha_{i_7 j_1}\,\delta = \alpha_{i_7 j_7}\,\delta.$$

Since i_7, j_1 and j_7 are arbitrarily chosen, we can conclude that

$$\alpha_{ij} = \alpha_i \qquad \forall\, i \in I,\ j \in J. \tag{21}$$

Next, for any $j_8, j_9 \in J$, choose $J_1 \subseteq J$ satisfying

$$j_8 \in J_1, \quad j_9 \notin J_1, \quad \text{and } |J_1| = \lceil \frac{D}{K} \rceil$$

Define (x^1, y^1) according to (14) and (15) as before. Define $(x^8, y^8) \in P^{RC}$ by

$$y_j^8 = \begin{cases} 1 & \text{if } j = j_9 \\ 0 & \text{if } j = j_8 \\ y_j^1 & \text{otherwise} \end{cases}$$

$$x_{ij}^8 = \begin{cases} x_{ij_8}^1 & \text{if } j = j_9 \\ x_{ij_9}^1 & \text{if } j = j_8 \\ x_{ij}^1 & \text{otherwise,} \end{cases}$$

which represents swapping the assignments for plant j_8 and j_9.

Because of equation (21), the first term on the lefthand side of (‡) is identical for (x^1, y^1) and (x^8, y^8). Hence, substituting (x^1, y^1) and (x^8, y^8) into (‡) and subtracting, we obtain

$$\beta_{j_8} = \beta_{j_9}.$$

which implies that

$$\beta_j = \beta \qquad \forall\, j \in J \tag{22}$$

since j_8 and j_9 were chosen arbitrarily.

Our proof will be complete if we can show that the α_i's and β satisfy the appropriate algebraic relationship.

Consider any $i_8, i_9 \in I$ and any $j_1, j_{10} \in J$. Choose J_1 so that $\{j_1, j_{10}\} \subseteq J_1$. Define (x^1, y^1) by (14) and (15) as before. We can assume that the assignment of (x^1, y^1) satisfies

$$\begin{aligned} x_{i_8 j_1}^1 &> 0 \\ x_{i_9 j_{10}}^1 &> 0. \end{aligned}$$

We can check that $\mathcal{P}^{RC}$ indeed contains such a solution.

Define (x^9, y^9) as follows:

$$y_j^9 = \begin{cases} 0 & \text{if } j = j_1 \\ y_j^1 & \text{otherwise} \end{cases}$$

$$x_{ij}^9 = \begin{cases} 0 & \text{if } j = j_1 \\ x_{ij}^1 & \text{otherwise.} \end{cases}$$

(x^1, y^1) and (x^9, y^9) differ only because the solution corresponding to (x^9, y^9) opens one fewer plant and supplies r fewer units from the plants in J.

Substituting (x^1, y^1) into (‡) and using equations (21) and (22), we find that

$$\sum_{\substack{i \in I \\ j \in J_1}} \alpha_i x^1_{ij} \; + \; \beta \lceil \frac{D}{K} \rceil \; = \; \alpha_0.$$

Substituting (x^9, y^9) into (‡) , we obtain

$$\sum_{\substack{i \in I \\ j \in J_1}} \alpha_i x^9_{ij} \; + \; \beta \, (\lceil \frac{D}{K} \rceil - 1) \; = \; \alpha_0$$

or

$$\sum_{\substack{i \in I \\ j \in J_1}} \alpha_i x^1_{ij} - \sum_{i \in I} \alpha_t x^1_{ij_1} \; + \; \beta \, (\lceil \frac{D}{K} \rceil - 1) \; = \; \alpha_0.$$

Therefore, we must have

$$\sum_{i \in I} \alpha_i x^1_{ij_1} = -\beta. \tag{23}$$

Let $\delta \; = \; \min\{d_{i_8} x^1_{i_8 j_1}, d_{i_9} x^1_{i_9 j_{10}}, d_{i_8}(1 - x^1_{i_8 j_{10}}), d_{i_9}(1 - x^1_{i_9 j_1})\}$. Define (x^{10}, y^{10}) by modifying (x^1, y^1) as follows:

$$\begin{aligned}
x^{10}_{i_8 j_1} &= x^1_{i_8 j_1} - \frac{\delta}{d_{i_8}} \\
x^{10}_{i_8 j_{10}} &= x^1_{i_8 j_{10}} + \frac{\delta}{d_{i_8}} \\
x^{10}_{i_9 j_1} &= x^1_{i_9 j_1} + \frac{\delta}{d_{i_9}} \\
x^{10}_{i_9 j_{10}} &= x^1_{i_9 j_{10}} - \frac{\delta}{d_{i_9}}.
\end{aligned}$$

This solution shifs the demand for customer i_8 from plant j_1 to j_{10} and shifts the demand for customer i_9 from plant j_{10} to j_1.

Define (x^{11}, y^{11}) by modifying (x^{10}, y^{10}) as follows:

$$y_j^{11} = \begin{cases} 0 & \text{if } j = j_1 \\ y_j^{10} & \text{otherwise} \end{cases}$$

$$x_{ij}^{11} = \begin{cases} 0 & \text{if } j = j_1 \\ x_{ij}^{10} & \text{otherwise,} \end{cases}$$

and repeating the previous argument on (x^{10}, y^{10}) and (x^{11}, y^{11}), we find that

$$\sum_{i \in I} \alpha_i x_{ij_1}^{10} = -\beta \tag{24}$$

or

$$\sum_{i \in I} \alpha_i x_{ij_1}^{1} - \alpha_{i_8} \frac{\delta}{d_{i_8}} + \alpha_{i_9} \frac{\delta}{d_{i_9}} = -\beta. \tag{25}$$

Comparing equations (23) and (25), we see that

$$\frac{\alpha_{i_8}}{d_{i_8}} = \frac{\alpha_{i_9}}{d_{i_9}}$$

and since the indices are arbitrarily chosen, we have

$$\alpha_i = \alpha d_i \qquad \forall i \in I$$

for some α. Moreover, our choice of (x^1, y^1) ensures that

$$\sum_{i \in I} d_i \, x_{ij_1}^1 = r$$

so substituting into (23), we get

$$\beta = -r\alpha. \tag{26}$$

Thus, (‡) is equivalent to

$$\alpha \left(\sum_{i \in I} \sum_{j \in J} d_i x_{ij} - r \sum_{j \in J} y_j \right) = \alpha_0. \tag{27}$$

Since (‡) is non-trivial, $\alpha \neq 0$, and so we must have $\frac{\alpha_0}{\alpha} = D - r\lceil \frac{D}{K} \rceil$, since $(RC_=)$ holds for all points in $\mathcal{P}^{RC}$. Thus we have shown that (‡) is α times $(RC_=)$.

At this point, we have established that the residual capacity inequality (RC) is a facet for $\mathcal{P}$ when $1 \leq r \leq K-1$. ∎

4.3 Relation to Chvatal's Result

Chvatal [1973] showed that all facets of the integer programming polytope

$$\mathcal{Q} = conv\{x \mid Ax \leq b,\ x \quad integer\},$$

where A and b are integer-valued, can be constructed recursively using the following argument:

> Let $a^1x \leq b^1,\ a^2x \leq b^2,\ \ldots,\ a^kx \leq b^k$ be valid inequalities for $\mathcal{Q}$. Selecting any set of values $\mu^i > 0$ so that
>
> $$\sum_{i=1}^{k} \mu^i a^i \text{ is an integer vector,}$$
>
> then
>
> $$\left(\sum_{i=1}^{k} \mu^i a^i\right) x \leq b^0$$
>
> is valid for all $b^0 \geq \lfloor \sum_{i=1}^{k} \mu^i b^i \rfloor$.

Thus, new valid inequalities can be generated by taking positive linear combinations of previously known valid inequalities and "rounding down" the righthand side. Moreover, any facet of $\mathcal{Q}$ can be generated by a sequence of combinations and roundings. However, the linear combinations of a *fixed* set of inequalities might not generate the complete characterization of the convex hull — the procedure must be applied recursively to valid inequalities generated at a previous level. (The next level of inequalities is constructed by applying the previous argument to all possible linear combinations of inequalities in the current level.)

The fact that the residual capacity facets identified in Section 4 contain the term $\lceil \frac{D}{K} \rceil$ suggests that there may be a simply construction via Chvatal's procedure to generate the r-inequalities (RC). (Notice that Chvatal's construction applies to *integer* programs; therefore, our comments in this Section applies to the version of the capacitated plant location problem where the x_{ij}'s are also integer variables.)

When $r = 1$, the inequality (RC) can indeed be directly constructed *in one step* using the following original constraints with the corresponding multipliers indicated on the left:

$$
\begin{array}{lrcll}
d_i(1 - \frac{1}{K}) : & \sum_{i \in I} \sum_{j \in J} x_{ij} & \leq & 1 & \forall i \in I \\
\frac{1}{K} : & \sum_{i \in I} d_i \, x_{ij} \; - \; K y_j & \leq & 0 & \forall j \in J
\end{array}
$$

When $r > 1$, the analogous Chvatal construction generates

$$\sum_{i \in I} \sum_{j \in J} d_i \, x_{ij} \; - \; r \sum_{j \in J} y_j \; \leq \; D - \lceil \frac{rD}{K} \rceil$$

which is not as tight a constraint as the residual capacity inequality (RC). The direct Chvatal construction for (RC) does not appear to be obvious.

5 Other Versions of the Capacitated Plant Location Problem

In this section, we will examine some variations in the model of the capacitated plant location problem and investigate whether the inequalities introduced in the previous section define facets for these modified problems.

5.1 The Capacitated Plant Location Problem with Equality Demand Constraints

In some applications, the demand of the customers must be met in full. Thus, instead of (PI), we may consider a modified version of the capacitated plant location problem that replaces the constraint

$$\sum_{j \in \mathcal{N}} x_{ij} \leq 1 \qquad \forall\, i \in \mathcal{M} \tag{2}$$

by

$$\sum_{j \in \mathcal{N}} x_{ij} = 1 \qquad \forall\, i \in \mathcal{M}. \tag{28}$$

Let us call this version of the capacitated plant location problem (PE), and the corresponding convex hull of solutions $\mathcal{PE}$. (28) defines M non-redundant equality constraints of (PE), so $\mathcal{PE}$ cannot be full-dimensional. In fact, provided $N > \lceil \frac{D_{\mathcal{M}}}{K} \rceil$, $\mathcal{PE}$ is of dimension $MN + N - M$.

Since (PI) can be viewed as a relaxation of the problem (PE), the family of residual capacity inequalities, introduced in Section 4, must also be valid for (PE).

As in the case for (PI), the following is true for (PE):

1. if $|J| < \lfloor \frac{D}{K} \rfloor$, the r-inequality (RC) is valid but not a face, and
2. if $|J| = \lfloor \frac{D}{K} \rfloor$, (RC) is a face but not a facet.

Thus, in order for the residual capacity inequality (RC) to be a facet, we must have

$$|J| > \lfloor \frac{D}{K} \rfloor. \tag{29}$$

The intuition is that we must have enough 'degrees of freedom' in the set

$\mathcal{PE}^{RC}$ = the set of feasible solutions of (PE) that satisfies the residual capacity inequality (RC) as an equality.

Moreover, the solutions in $\mathcal{PE}^{RC}$ correspond to assignments with at least $|J| - \lceil \frac{D}{K} \rceil$ plants in J closed. However, because of constraint (28), at most $N - \lceil \frac{D_{\mathcal{M}}}{K} \rceil$ plants out of the total of N plants can be closed. Therefore, we must have

$$|J| - \lceil \frac{D}{K} \rceil < N - \lceil \frac{D_{\mathcal{M}}}{K} \rceil \tag{30}$$

for the residual capacity inequality (RC) to be a facet of $\mathcal{PE}$.

In fact, the two conditions (29) and (30) are sufficient. The proof that (RC) is a facet for $\mathcal{PE}$ under these two conditions parallels that of Theorem 4. In this case, however, we must be more careful in our choice of feasible points used for comparing coefficients, since we must ensure that all such points satisfy constraint (28) of (PE).

The problem (PE) requires that the demand of every customer be completely met and each plant can supply only K units, Therefore, the number of plants that are opened must be at least $\lceil \frac{D_{\mathcal{M}}}{K} \rceil$, where $D_{\mathcal{M}} = \sum_{i \in \mathcal{M}} d_i$. Thus,

$$\sum_{j \in \mathcal{N}} y_j \geq \lceil \frac{D_{\mathcal{M}}}{K} \rceil \tag{31}$$

is a valid inequality for (PE). In fact, if $\lceil \frac{D_{\mathcal{M}}}{K} \rceil \neq \frac{D_{\mathcal{M}}}{K}$, then (31) is a facet of $\mathcal{PE}$.

5.2 The Capacitated Plant Location Problem with Indivisible Demand

In certain applications of the plant location problem, demand for a customer must be supplied from a single plant. Thus, in addition to constraints (1) – (6), we also impose the constraint

$$x_{ij} \qquad integer. \tag{32}$$

This model is often used in applications for which the assignment cost is not proportional to demand, but represents a 'fixed' cost in establishing the link between the customer and the plant. (In telecommunications applications, the assignment cost reflects the cost of building a cable connection between a household and a switching facility and does not depend on the volume of traffic generated by the household.)

When the x_{ij} variables are restricted to be 0-1 variables, the capacitated plant location problem becomes a "harder" problem. The residual capacity facets for (PI), introduced in Section 4, are valid inequalities for this modified problem, but they may no longer be facets or faces. The following example illustrate this point.

Example

Consider a capacitated plant location problem with $\mathcal{M} = \{1,2,3\}$ and $\mathcal{N} = \{1,2,3\}$ and with

$$d_1 = 3,\ d_2 = 4,\ d_3 = 4 \quad \text{and} \quad K = 6.$$

The residual capacity inequality (RC) with $I = \mathcal{M}$ and $J = \mathcal{N}$ is

$$3x_{11} + 3x_{12} + 4x_{21} + 4x_{22} + 4x_{31} + 4x_{32} - 5(y_1 + y_2 + y_3) \leq 11 - 5 \times 2 = 1 \tag{33}$$

which is valid but not binding. There is no feasible solutions to the set of constraints $(1), \ldots, (6)$ and (32) that satisfies (33) as an equality.

On the other hand, if the demand of all the customers are equal, say, $d_i = d$ for all i, the capacitated plant location problem with indivisible demand is equivalent to the following problem:

$$
(PI'): \quad
\begin{array}{rlll}
x_{ij} & \leq y_j & \forall i \in \mathcal{M}, \forall j \in \mathcal{N} & (34) \\
\sum_{j \in \mathcal{N}} x_{ij} & \leq 1 & \forall i \in \mathcal{M} & (35) \\
\sum_{i \in \mathcal{M}} x_{ij} & \leq K' y_j & \forall j \in \mathcal{N} & (36) \\
x_{ij} & \geq 0 & \forall i \in \mathcal{M} & (37) \\
0 \leq y_j & \leq 1 & \forall j \in \mathcal{N} & (38) \\
x_{ij}, y_j & \text{integer} & \forall i \in \mathcal{M}, \forall j \in \mathcal{N} & (39)
\end{array}
$$

where $K' = \lfloor \frac{K}{d} \rfloor$. The residual capacity inequalities (RC) for the problem (PI') is of the following form:

$$\sum_{i \in I} \sum_{j \in J} x_{ij} - r' \sum_{j \in J} y_j \leq |I| - r' \lceil \frac{|I|}{K'} \rceil \tag{40}$$

with $r' = |I| \pmod{K'}$.[4]

The valid inequalities (40) are in fact *facets* for the problem (PI'). Thus, when the customer demand is equal to one unit for all customers, (RC) define facets for the version of the capacitated plant location problem when the x_{ij}'s are constrained to be integers as well as for the version when the x_{ij}'s are allowed to be fractional. The proof that the residual capacity inequalities (40) define facets for (PI') is entirely analogous to the proof of Theorem 4. (All the feasible points $(x^1, y^1), \ldots, (x^{11}, y^{11})$ used in the proof can be chosen to be integer-valued.)

In the next Section, we study a different formulation of the capacitated plant location problem for the case when the demand for all customers are equal to one and each customer can be supplied from only one plant. This formulation also allows the capacity to be different for different plants. We will show that this problem is equivalent to a vertex packing problem for a particular graph and derive facets for this formulation using graph-theoretic results.

---

[4]By our definition, $1 \leq r' \leq K'$.

6 A Vertex Packing Formulation of the Capacitated Plant Location Problem

Suppose the demand of each customer is the same and that the demand of a single customer cannot be split between two plants. Then, without loss of generality, we can assume that the capacity of each plant is an integral multiple of the demand. For simplicity, we can take the demand of each customer to be one unit and the capacity of each plant to be integer-valued.

An alternative way of modelling the capacity restriction is by viewing the plant as a collection of plant-units. Each customer is assigned, not to a plant, but to a particular plant-unit. Each plant-unit can serve only one customer and is unavailable unless the plant is open. (We are tacitly assuming that opening a plant incurs a fixed cost irrespective of the number of plant-units actually assigned to customers.)

This model leads to a formulation of the capacitated plant location problem with the variables

$$x_{ijk} = \begin{cases} 1 & \text{if customer } i \text{ is assigned to unit } k \text{ of plant } j \\ 0 & \text{otherwise} \end{cases}$$

and $\quad y_j = \begin{cases} 1 & \text{if plant } j \text{ is open} \\ 0 & \text{otherwise} \end{cases}$

and the following constraints:

$$(PV): \quad \begin{array}{rlll} \sum_{j \in \mathcal{N}} \sum_{k \in K_j} x_{ijk} & \leq 1 & \forall\, i \in \mathcal{M} & (41) \\ \sum_{i \in \mathcal{M}} x_{ijk} & \leq y_j & \forall\, j \in \mathcal{N},\ k \in K_j & (42) \\ \sum_{k \in K_j} x_{ijk} & \leq y_j & \forall\, i \in \mathcal{M},\ \forall\, j \in \mathcal{N} & (43) \\ y_j & \in \{0,1\} & \forall\, j \in \mathcal{N} & (44) \\ x_{ijk} & \in \{0,1\} & \forall\, i \in \mathcal{M},\ \forall\, j \in \mathcal{N},\ \forall k \in K_j & (45) \end{array}$$

where

$$\mathcal{M} = \{1,2,\ldots,M\} \text{ and } \mathcal{N} = \{1,2,\ldots,N\} \text{ and } K_j = \{1,2,\ldots,K_j\}.$$

Since x_{ijk}'s are integer variables, the constraints (41) forces each customers to be assigned to at most one plant-unit. Constraints (42) are the capacity constraints for each plant-unit; only one customer can be assigned to each plant-unit and customers can be assigned to those plants that are designated open. Constraints (43) stipulates that a customer is assigned to only one plant-unit in any given plant.

For this model, we allow the capacity of each plant to be different, and denote the capacity of plant j by K_j. Let $\mathcal{PV}$ denote the convex hull of the solution points of (PV).

6.1 A Vertex Packing Formulation

Substituting the complement $\overline{y_j} = 1 - y_j$ of the variables y_j into the constraints of (PV) gives the following formulation:

$$(\overline{PV}): \quad \begin{array}{rcll} \sum_{j \in \mathcal{N}} \sum_{k \in K_j} x_{ijk} & \leq & 1 & \forall\, i \in \mathcal{M} \quad (46) \\ \sum_{i \in \mathcal{M}} x_{ijk} + \overline{y_j} & \leq & 1 & \forall\, j \in \mathcal{N},\ k \in K_j \quad (47) \\ \sum_{k \in K_j} x_{ijk} + \overline{y_j} & \leq & 1 & \forall\, i \in \mathcal{M},\ \forall\, j \in \mathcal{N} \quad (48) \\ x_{ijk},\ \overline{y_j} & \in & \{0,1\} & \forall\, i \in \mathcal{M},\ \forall\, j \in \mathcal{N},\ \forall k \in K_j. \quad (49) \end{array}$$

Examining these constraints, we see that $(\overline{PV})$ is a set packing problem. The solution set to this problem is equivalent to the collection of independent sets of vertices of a related graph, called the *intersection graph* which we will define next. This observation will permit us to used results that have been developed for vertex packing problems to identify facets for problem (PV).

The intersection graph $G_{PV} = (V, E)$ for problem $(\overline{\mathrm{PV}})$ has the following structure:

- There is a vertex for every variable in $(\overline{\mathrm{PV}})$.
- There is an edge linking vertices (x_{ijk}) and $(x_{i'j'k'})$ if and only if
  1. $i = i'$, or

2. $j = j'$ and $k = k'$.

- There is an edge linking vertices (x_{ijk}) and $(\overline{y_{j'}})$ if and only if $j = j'$.

The number of vertices in the intersection graph is $|V| = M\left(\sum_{j \in \mathcal{N}} K_j\right) + N$. There is a one-to-one correspondence between the integer solutions of $(\overline{PV})$ and the independent vertex sets of the intersection graph. By considering the facets and valid inequalities of the vertex packing polytope, we can identify the facets and valid inequalities of the problem (PV).

As an example, the intersection graph G_{PV} derived from the problem $(\overline{PV})$ for $M = 3$, $N = 2$ and $K_j = 2$ for all $j \in \mathcal{N}$ is depicted in Figure 3.

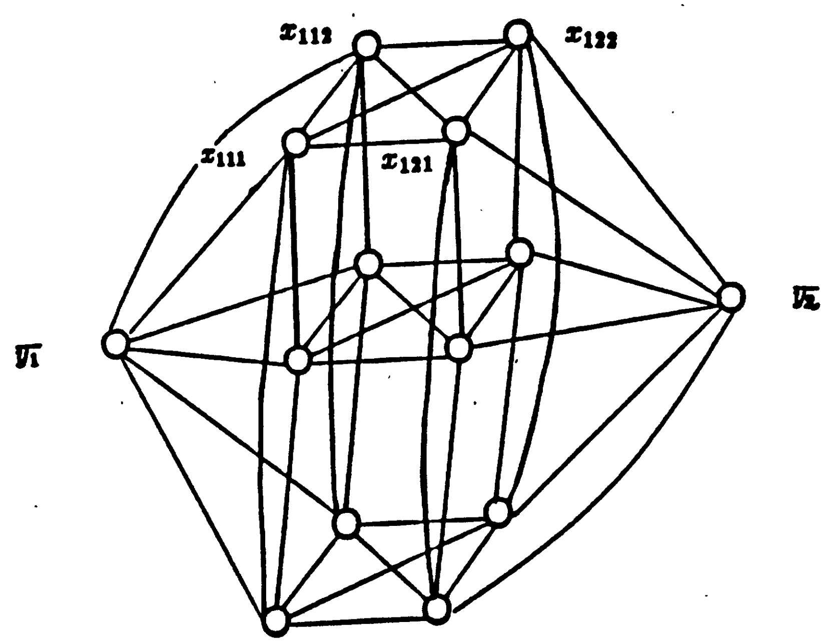

Figure 3: A Sample Intersection Graph.

The vertices corresponding to the assignment variables x_{ijk} form a three-dimensional grid. Each horizontal layer corresponds to a customer i. Each vertical layer corresponds to a plant j and each column corresponds to a plant-unit indexed by jk.

In the next section, we summarize relevant results concerning the vertex packing problem. These results will be applied to the intersection graph derived from the problem $(\overline{PV})$ in subsequent sections to identify valid inequalities and facets of $\mathcal{PV}$.

6.2 Facet Producing Subgraphs

Let $G = (V, E)$ be a graph. Let $x \in \mathbf{R}^{|V|}$ be the characteristic vector of a set $V' \subseteq V$.

An *independent set* of G is a subset of vertices $I \subseteq V$ having the property that no edge links any two vertices of I. Let

$$\mathcal{P}^G = conv\left\{x \in \mathbf{R}^{|V|} \mid x \text{ is the characteristic vector of an independent set of } G\right\}.$$

We call $\mathcal{P}^G$ the vertex packing polytope.

The next theorem identifies facets of $\mathcal{P}^G$.

Theorem 5 (Padberg [1973]) *For any $C \subseteq V$, the inequality $\sum_{j \in C} x_j \leq 1$ is a facet of $\mathcal{P}^G$ if and only if C is a clique of G. (A clique of G is a maximal complete subgraph of G.)*

The node induced subgraph $G' = (V', E')$ is the subgraph of G such that $V' \subseteq V$ and

$$E' = \{(i, j) \mid i \in V', j \in V' \text{ and } (i, j) \in E\}$$

Consider the vertex packing polytope of G':

$$\mathcal{P}^{G'} = conv\{x \mid x \in \mathbf{R}^{|V'|}, x \text{ defines an independent set in } G'\}.$$

This polytope is equivalent to

$$conv\left\{x \in \mathbf{R}^{|V|} \mid x \text{ defines an independent set in } G \text{ and } x_j = 0 \text{ for all } j \notin V'\right\}.$$

Thus we can see that the vertex packing polytope of the subgraph G' is the intersection of the vertex packing polytope of G with the subspace of $\mathbf{R}^{|V|}$ spanned by the variables $\{x_j \mid j \in V'\}$.

Theorem 6 (Nemhauser & Trotter [1974]) *Suppose*

$$\sum_{j \in V'} \pi_j x_j \leq \pi_0$$

is a facet for $P^{G'}$ with $\pi_j \geq 0$ and $\pi_0 > 0$. The the inequality can be lifted to give a facet for P^G, i.e., there exists β_j with $0 \leq \beta_j \leq \pi_0$ so that

$$\sum_{j \in V'} \pi_j x_j + \sum_{j \in V \setminus V'} \beta_j x_j \leq \pi_0$$

is a facet of P^G. The β_j's are not necessarily unique.

The lifting can be done by a sequential lifting procedure that computes the β_j's one at a time by solving a related optimization problem. (See Padberg [1973].) Because of the power of the lifting procedure, facets of the vertex packing polytope can be readily constructed once we have identified subgraphs of the intersection graph that have special structure and are so-called facet producing.

The next theorem identifies a class of facet producing graphs.

Theorem 7 (Padberg [1973]) *If $G' = (V', E')$ is an odd hole (i.e., an odd cycle without chords) then*

$$\sum_{j \in V'} x_j \leq \frac{|V'| - 1}{2}$$

is a facet of $P^{G'}$.

Using Theorems 5 and 7, we can readily generate two classes of facets by identifying the cliques and odd holes of the intersection graph. The inequlities corresponding to odd holes can be *'lifted'* to give facets, whereas every clique of the graph provides a facet inequality directly. We will now return to examine G_{PV}, the intersection graph for the problem $(\overline{PV})$ and investigate what the cliques and odd holes of this graph look like and identify the facets and valid inequalities generated by these subgraphs.

6.3 Cliques of G_{PV}

There are three types of cliques, C^1, C^2 and C^3, of the intersection graph G_{PV}:

1. $C_i^1 = \{x_{i'jk} \mid i' = i\}$, for each $i \in \mathcal{M}$,
2. $C_{jk}^2 = \{x_{ij'k'} \mid (j',k') = (j,k)\} \cup \{y_j\}$, for each $j \in \mathcal{N}$ and each $k \in K_j$,
3. $C_{ij}^3 = \{x_{i'j'k} \mid (i',j') = (i,j)\} \cup \{y_j\}$, for each $i \in \mathcal{M}$ and each $j \in \mathcal{N}$.

It is possible to verify that these are the only types of cliques for this graph. Figure 4 illustrates the three types of cliques for the example in Figure 3.

These three types of cliques give rise to three families of facets for (PV), namely,

$$C_i^1 : \quad \sum_{j\in\mathcal{N}} \sum_{k\in K_j} x_{ijk} \le 1 \quad \forall i \in \mathcal{M} \tag{50}$$

$$C_{jk}^2 : \quad \sum_{i\in\mathcal{M}} x_{ijk} \le y_j \quad \forall j \in \mathcal{N},\ \forall k \in K_j \tag{51}$$

$$C_{ij}^3 : \quad \sum_{k\in K_j} x_{ijk} \le y_j \quad \forall i \in \mathcal{M},\ \forall j \in \mathcal{N}. \tag{52}$$

These families of facets are exactly constraints (41), (42) and (43) of the problem (PV). Hence we have shown that some of the original constraints of the problem (PV) define facets.

6.4 Odd Holes of G_{PV}

The odd holes of G_{PV} can be characterized by a zero-one matrix A whose rows and columns are indexed by $I \subseteq \mathcal{M}$ and $\mathcal{J}$ with

$$\mathcal{J} \subseteq J \times \bigcup_{j\in J} K_j \;=\; \{(j,k) \mid j \in J,\, k \in K_j\} \qquad \text{with } J \subseteq \mathcal{N},\ |J| = |I| - 1$$

having the following properties:

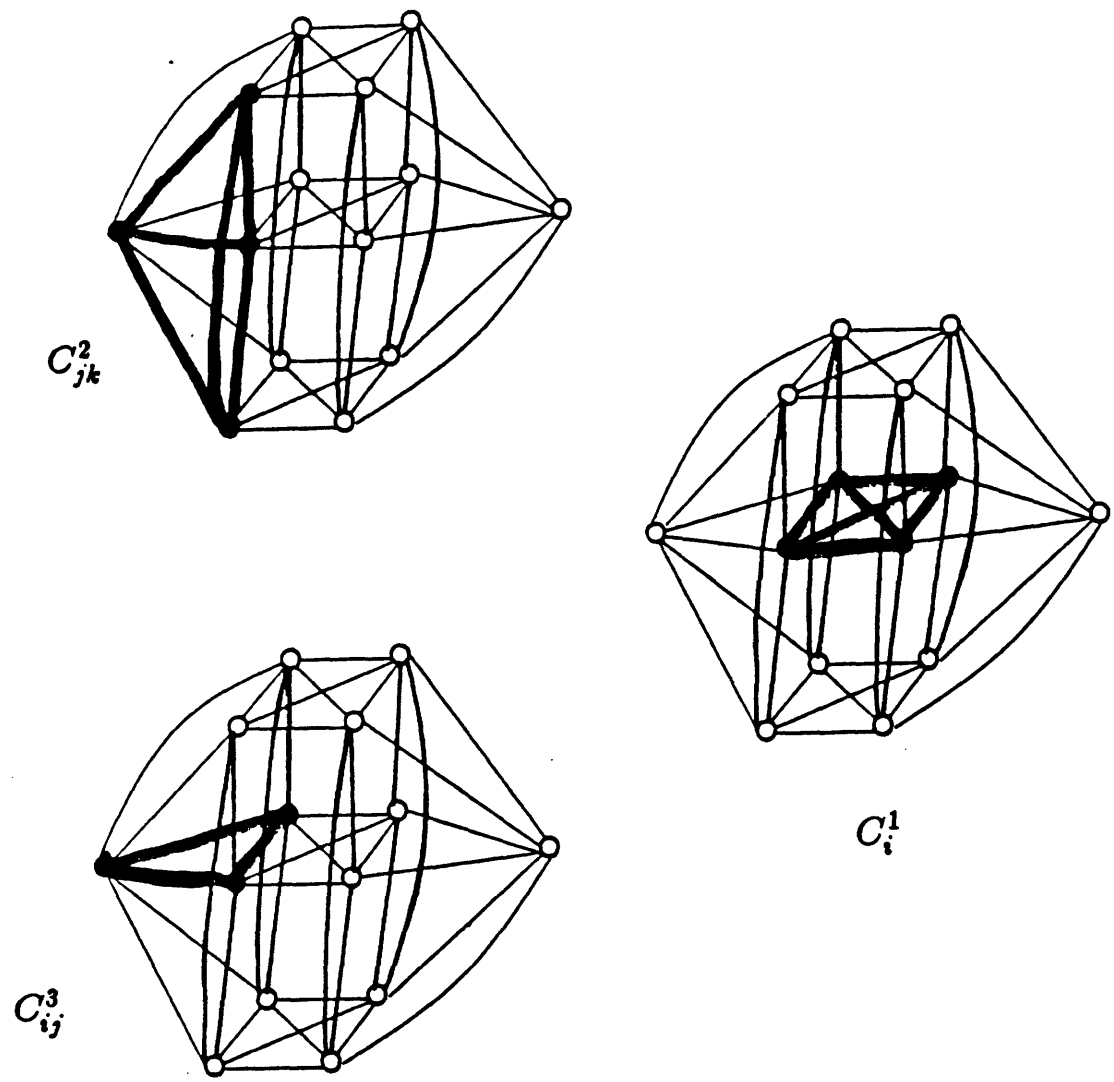

Figure 4: Cliques of the Intersection Graph G_{PV}.

1. Each row of A has exactly two 1's,

2. Each column of A has either one or two 1's,

3. $|J| - |J_2|$ is odd where J_2 =set of columns with two 1's.

The set of vertices of G_{PV} corresponding to the set of variables

$$\{x_{ijk} | i \in I, (j,k) \in J,\ a_{i,jk} = 1\} \ \cup\ \{y_j | j \in J \setminus J_2\}$$

form an odd hole. An odd hole for our example is shown in Figure 5.

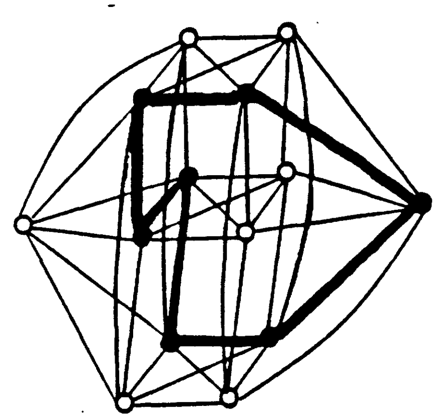

Figure 5: An Odd Hole of the Intersection Graph.

Since the x_{ijk} nodes appear in pairs, the smallest odd hole is of size 5.

The corresponding valid inequality generated by this odd hole is

$$\sum_{\substack{i\in I\\(j,k)\in J}} a_{i,jk}x_{ijk} + \sum_{j\in J\setminus J_2} \overline{y_j} \le \frac{2|I| + |(J\setminus J_2)| - 1}{2} = |I| + \frac{|(J\setminus J_2)| - 1}{2}$$

or equivalently

$$\sum_{\substack{i\in I\\(j,k)\in J}} a_{i,jk}x_{ijk} - \sum_{j\in J\setminus J_2} y_j \le |I| + \frac{(|J\setminus J_2|) - 1}{2} - |(J\setminus J_2)| = |I| - \lceil\frac{|(J\setminus J_2)|}{2}\rceil.$$

Notice that if $J_2 = \emptyset$, then this inequality is a direct extension of the valid inequality generated by odd holes for the intersection graph of the uncapacitated plant location problem as introduced by Padberg [1973].

6.5 Other Facet Producing Subgraphs

Other facet producing graphs that have been identified includes webs and anti-webs. (See Trotter [1975].)

A *web* $W(n,k)$ is a graph on n nodes $\{1,2,\ldots,n\}$ with the property that(i,j) is an edge if and only if $j = i+k, i+k+1, \ldots, i+n-k$ with the sums computed modulo n. Webs subsume both the classes of cliques and odd holes since $W(n,1)$ is a clique and $W(2k+1,k)$ is an odd hole. A web is facet producing if N and k are relatively prime (See Trotter [1975]).

An *anti-web* is the complement graph of a web. Thus, $G' = \overline{W}(n,k)$ is an anti-web if $G' = (V', E')$ with $V' = \{1,2,\ldots,n\}$ and

$$\begin{array}{lll} \exists \text{ edge } (i,j) & \text{if and only if} & j = i+1, i+2, \ldots, i+k-1 \pmod{n} \\ & \text{or} & j = i-1, i-2, \ldots, i-k+1 \pmod{n}. \end{array}$$

Anti-webs are facet producing if and only if $k\lfloor \frac{n}{k} \rfloor + 1 = n$ (See Trotter [1975]).

By examining the adjacency relationship of the graph G_{PV}, it is possible to show that, except for cliques and odd holes, G_{PV} does not contain any facet-producing webs or anti-webs. Thus, no new facet for $\mathcal{P}^G$ is generated by these structures.

6.6 Comparison of the Two Formulations of the Capacitated Plant Location Problem

In Section 3 and this section, we have given two different formulations of the capacitated plant location problem. These two formulations are related in that there is a natural surjection from the set of feasible solutions of (PV) to the set of feasible solutions of (PI'). Setting

$$x_{ij} = \sum_{k \in K_j} x_{ijk}$$

in any solution of (PV) yields a feasible solution to (PI'). On the other hand, given any solution to (PI'), we can construct at least one solution to (PV) with the same set of open plants and having $x_{ij} = 1$ in (PI') whenever $x_{ijk} = 1$ in (PV).

Moreover, equating x_{ij} in (PI') with $\sum_{k \in K_j} x_{ijk}$ in (PV), we see that

$$x_{ij} \le y_j \qquad \Longleftrightarrow \qquad \sum_{k \in K_j} x_{ijk} \le y_j$$

$$\sum_{j \in N} x_{ij} \le 1 \qquad \Longleftrightarrow \qquad \sum_{j \in N} \sum_{k \in K_j} x_{ijk} \le 1$$

indicating that some constraints of the two formulations are 'equivalent', namely, constraints (34) and (41), and (35) and (42). However, constraint (43) cannot be transformed into any equation of (PI'). In fact, summing the constraints (43) over k for a particular j, we get

$$\sum_{k \in K_j} \left(\sum_{i \in M} x_{ijk} \le y_j \right) \Leftrightarrow \sum_{i \in M} \sum_{k \in K_j} x_{ijk} \le K_j y_j \qquad \Longleftrightarrow \qquad \sum_{i \in M} x_{ij} \le K_j y_j$$

and we can view constraint (36) of (PI') as an aggregate version of the constraints (43) of (PV). For two integer programming formulations with the same set of variables, it is well-known that disaggregate constraints are preferable in that they give tighter LP-relaxations which lead to improved algorithmic performance. However, in this case, we are comparing two integer programs with different sets of variables and the relative merits of the two formulations are not as clear cut. The problem (PI') has $MN + N$ variables whereas problem (PV) has $M(\sum_{j=1}^{N} K_j) + N$, considerably more.

From this discussion, we can see that the formulations (PI') and (PV) are completely equivalent as integer programs. In fact, even their linear-programming relaxations are equivalent via the following correspondence:

For any solution of the LP-relaxation of (PV), setting

$$x_{ij} = \sum_{k \in K_j} x_{ijk}$$

gives a solution to the LP-relaxation of (PI').

Moreover,

for any solution of the LP-relaxation of (PI'), setting

$$x_{ijk} = \frac{1}{K_j} x_{ij}$$

gives a solution to the LP-relaxation of (PV).

It would be interesting to compare the two formulations in more detail, either theoretically or computationally. It is often the case that the same underlying problem can be modelled by two different integer programs and choosing the right model is still very much an art. Such a study may help in clarifying some issues in problem formulation.

7 Conclusion

In this paper, we have presented different formulations of the capacitated plant location problem and identified valid inequalities and facets for each formulation. These results are intended as a first step in the investigation of the polyhedral structure of capacitated facility location, and more generally, fixed charge problems.

The motivation for this work is the search for effective strong cutting plane methods for these problems. It is important that computational studies be conducted to assess if the facets identified in this paper are *useful* as cuts in a cutting plane algorithm. Additional research in the development of good separation heuristics are also necessary in building an effective cutting plane algorithm for the capacitated plant location problem.

Another avenue of research is the generalization of the facets introduced in Section 4 to other capacitated fixed-charge problems. The plant location problem can be interpreted as a network design problem, as can many other canonical fixed charge problems in the area of production/operations management. It is possible that the facets introduced in this paper can be generalized to these other problems by exploiting the common network design framework.

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Appendix A

In this appendix, we provide the proof of Theorem **1** stated in Section 3.2.

Proposition 8 *For every* $i \in \mathcal{M}$, *if* $\lceil \frac{d_i}{K} \rceil < N$, *then* $\sum_{j \in \mathcal{N}} x_{ij} \leq 1$ *is a facet of* $\mathcal{P}$.

Proof. Before we launch into the proof, we will first explain why the proviso $\lceil \frac{d_i}{K} \rceil < N$ is necessary. By our assumption that $\lceil \frac{D_{\mathcal{M}}}{K} \rceil \leq N$, we know that $\lceil \frac{d_i}{K} \rceil \leq N$. If $\lceil \frac{d_i}{K} \rceil = N$, then whenever

$$\sum_{j \in \mathcal{N}} x_{ij} = 1 \tag{53}$$

we must also have

$$y_j = 1 \qquad \forall j \in \mathcal{N}. \tag{54}$$

Now (54) and (53) form a linear independent set of equations. Hence, the set

$$\left\{ (x,y) \mid (x,y) \in \mathcal{P}, \sum_{j \in \mathcal{N}} x_{ij} = 1 \right\} \tag{55}$$

is of dimension at most $(MN + N) - (N + 1) = MN - 1$ and cannot contain $MN + N$ affinely independent points. Therefore, $\sum_{j \in \mathcal{N}} x_{ij} \leq 1$ is a face but not a facet if $\lceil \frac{d_i}{K} \rceil = N$.

To prove the proposition, we will construct $MN + N$ points that are in $\mathcal{P}$ and that satisfy (53) and show that they are affinely independent. We will prove the case when $i = 1$. By re-indexing the variables, we see that the same proof will apply for all $i \in \mathcal{M}$ for which the proviso holds.

Let s be the minimum number of plants needed to fully supply the demand of customer 1, i.e,

$$s = \lceil \frac{d_1}{K} \rceil$$

and define

$$\star = \frac{d_1}{K} + 1 - \lceil \frac{d_1}{K} \rceil.$$

Define the first $s+1$ feasible points as follows. For $m = 1, 2, \ldots, s+1$, let

$$x_{ij}^m = \begin{cases} 0 & \text{if } i = 1 \text{ and } j = m \\ \star & \text{if } i = 1 \text{ and } j = m-1 \pmod{(s+1)}^5 \\ 1 & \text{if } j \neq m \text{ or } m-1 \pmod{(s+1)} \text{ and } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y_j^m = \begin{cases} 1 & \text{if } j = 1, 2, \ldots, s+1 \\ 0 & \text{otherwise.} \end{cases}$$

Define the next $s+1$ solution points as follows. For $n = 1, \ldots, s+1$, let

$$x_{ij}^n = \begin{cases} 0 & \text{if } i = 1 \text{ and } j = n \\ \star & \text{if } i = 1 \text{ and } j = n-1 \pmod{(s+1)} \\ 1 & \text{if } i = 1 \text{ and } j \neq n \text{ or } n-1 \pmod{(s+1)} \\ 0 & \text{otherwise} \end{cases}$$

$$y_j^n = \begin{cases} 0 & \text{if } j = n \\ 1 & \text{if } j = 1, \ldots, s+1 \text{ but } j \neq n \\ 0 & \text{otherwise.} \end{cases}$$

Define the next $N - (s+1)$ solution points as follows. For $n = s+2, \ldots, N$, let

$$x_{ij}^n = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 1, 2, \ldots, s-1 \\ \star & \text{if } i = 1 \text{ and } j = s \\ 0 & \text{otherwise} \end{cases}$$

$$y_j^n = \begin{cases} 1 & \text{if } j = n \\ 1 & \text{if } j = 1,\ 2, \ldots, s \\ 0 & \text{otherwise.} \end{cases}$$

Define the remaining $MN - (s+1)$ solution points, indexed by $m \in \mathcal{M}$ and $n \in \mathcal{N}$ (except for the pairs with $m = 1$ and $n = 1, \ldots, s+1$), as follows:

$$x_{ij}^{mn} = \begin{cases} \star & \text{if } i = m \text{ and } j = n \\ 1 & \text{if } i = 1 \text{ and } j = 1, 2, \ldots, s \\ 0 & \text{otherwise} \end{cases}$$

[5] When $m = 1$, we define $m-1 \pmod{(s+1)}$ to be $s+1$. This differs slightly from convention.

$$y_j^{mn} = \begin{cases} 1 & \text{if } j = n \text{ or } j = 1, 2, \ldots, s \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that these points are indeed points in the set described by (55). Letting the vectors representing these feasible points be the rows of a matrix, we obtain the matrix M as shown in Figure 6.

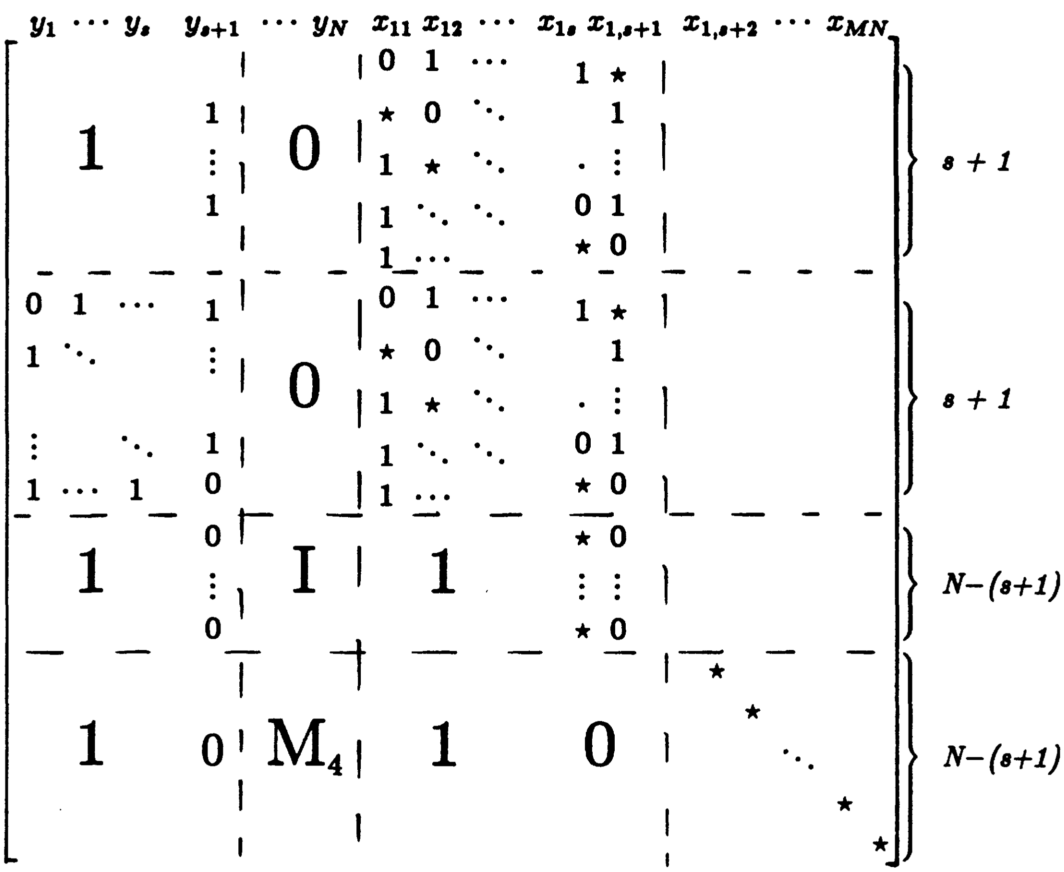

Figure 6: Affinely Independent Points on the facet $\sum_{j \in \mathcal{N}} x_{ij} \leq 1$.

We want to show that the rows of this matrix are affinely independent. Let μ be a vector of multipliers satisfying

$$\mu M = 0 \qquad \text{and} \qquad \mu \mathbf{1} = 0$$

where $\mathbf{1}$ is a vector of one's. We want to show that $\mu = 0$.

Notice that the matrix M can be partitioned as follows:

$$M = \left(\begin{array}{c|c|c|c} \mathbf{1} & \mathbf{0} & C_1 & \mathbf{0} \\ \hline C_0 & \mathbf{0} & C_1 & \mathbf{0} \\ \hline M_1 & I & M_2 & \mathbf{0} \\ \hline M_3 & M_4 & M_5 & D \end{array}\right)$$

where I is an identity matrix, $\mathbf{0}$ is a matrix of zeroes, $\mathbf{1}$ is a matrix of one's, D is a diagonal matrix, C_1 is a cyclic matrix, C_0 is a matrix with zeroes on its diagonal and one's elsewhere, and the M_i's are general matrices.

When restricted to the rightmost columns (corresponding to $\binom{0}{D}$), the condition $\mu M = 0$ implies that the multipliers for the rows for the submatrix $(M_3 \;\; M_4 \;\; M_5 \;\; D)$ have to be identically zero. Next, by considering the columns for $y_{s+2}, \ldots, y_N$, we see that the multipliers for the rows for the submatrix $(M_1 \;\; I \;\; M_2 \;\; 0)$ must also be zero. Thus, we have shown that all but the multipliers corresponding to the first $2(s+1)$ rows are identically zero. Now, in each of the columns corresponding to $y_1, \ldots, y_s + 1$, the elements in the first $2(s+1)$ rows are all 1's except for exactly one row. Each of the $(s+2)$-nd to the $2(s+1)$-th row contain exactly one of these zeroes, hence we can conclude that the multipliers for the $(s+2)$-nd to the $2(s+1)$-th row are zero.

What remains to be shown is that the multipliers for the submatrix $(1 \; 0 \; C_1 \; 0)$ are zero. We need only consider C_1. Let $\mu_1, \mu_2, \ldots, \mu_s$ and μ_{s+1} be the multipliers for these rows as indicated below:

$$C_1 = \begin{array}{cc} \begin{array}{cccccc} x_{11} & x_{12} & x_{13} & \cdots & x_{1s} & x_{1,s+1} \end{array} & \\ \left[\begin{array}{cccccc} 0 & 1 & 1 & \cdots & 1 & 0 \\ \star & 0 & 1 & \cdots & 1 & 1 \\ 1 & \star & 0 & \ddots & \cdots & 1 \\ \vdots & & \ddots & \ddots & & \vdots \\ 1 & \cdots & 1 & \star & 0 & 1 \\ 1 & \cdots & & 1 & \star & 0 \end{array}\right] & \begin{array}{c} \mu_1 \\ \mu_2 \\ \cdot \\ \vdots \\ \mu_s \\ \mu_{s+1} \end{array} \end{array}$$

Since the sum of the multipliers is zero, the first column shows that $\mu_1 = (\star - 1)\mu_2$. Similarly, column m shows that

$$\mu_m = (\star - 1)\mu_{m+1} \qquad \text{for } m = 1, 2, \ldots, s$$

The last column then indicates that $\mu_{s+1} = (\star - 1)\mu_1$. The only solution to this set of equations in $\mu_1, \ldots, \mu_{s+1}$ is

$$\mu_1 = \mu_2 = \cdots = \mu_{s+1} = 0,$$

and hence μ is identically zero.

Therefore, the points we constructed are affinely independent and the proof is complete. ∎

Proposition 9 *For every* $\mathrm{i} \in \mathcal{M}$ *and every* $\mathrm{j} \in \mathcal{N}$*, if* $d_{\mathrm{i}} \leq K$*, then* $x_{\mathrm{ij}} \leq y_{\mathrm{j}}$ *is a facet of* $\mathcal{P}$.

Proof. Define the first $N - 1$ solutions, indexed by $n \in \mathcal{N} - \{\mathrm{j}\}$, as follows:

$$x_{ij}^n = 0 \qquad \forall i \in \mathcal{M},\ j \in \mathcal{N}$$

$$y_j^n = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$

Let the solution point corresponding to $n = \mathrm{j}$ be $(0, 0)$.

Define the next MN solutions, indexed by $m \in \mathcal{M}$ and $n \in \mathcal{N}$ as follows.

If $n \neq \mathrm{j}$, then

$$x_{ij}^{mn} = \begin{cases} 1 & \text{if } i = m \text{ and } j = n \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j^{mn} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise,} \end{cases}$$

If $n = \mathrm{j}$, then

$$x_{ij}^{mn} = \begin{cases} 1 & \text{if } i = m \text{ and } j = n \\ 1 & \text{if } i = \mathrm{i} \text{ and } j = n \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j^{mn} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$

Displaying the solution points as the rows of a matrix and permuting the rows and columns of the matrix, we obtain a lower triangular matrix with 1's on the diagonal except in the j-th row (which is all zero). Hence the matrix has rank $MN + N - 1$. Thus the points exhibited above are affinely independent. ∎

Proposition 10 *For every* $\mathrm{i} \in \mathcal{M},\ \mathrm{j} \in \mathcal{N}$, $x_{\mathrm{ij}} \geq 0$ *is a facet of* $\mathcal{P}$.

Proof. Define N solution points as follows. For $n = 1, \ldots, N$, let

$$x_{ij}^{n} = 0 \qquad \forall i \in \mathcal{M},\ \forall j \in \mathcal{N}$$

$$y_j^n = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$

Define the next $MN-1$ solution points as follows. For $m = 1, \ldots, M$, $n = 1, \ldots, N$, except when $m = \mathrm{i}$ and $n = \mathrm{j}$ simultaneously, let

$$x_{ij}^{mn} = \begin{cases} \min\{1, \frac{K}{d_i}\} & \text{if } i = m \text{ and } j = n \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j^{mn} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let $(0,0)$ be the solution point corresponding to $m = \mathrm{i}$ and $n = \mathrm{j}$.

It is clear that these $MN + N$ solution points satisfy $x_{\mathrm{ij}} = 0$ and are affinely independent. They span an $(MN + N - 1)$-dimensional face of $\mathcal{P}$ and therefore, $x_{\mathrm{ij}} \geq 0$ is a facet of $\mathcal{P}$. Moreover, since i and j are arbitrary, by re-indexing the sets $\mathcal{M}$ and $\mathcal{N}$, we can apply the same argument for all other i's and j's. ∎

Proposition 11 *For every* $\mathrm{j} \in \mathcal{N}$, $y_{\mathrm{j}} \leq 1$ *is a facet.*

Proof. Define the first N solution points, indexed by $n = 1, \ldots, N$, as follows.

$$x_{ij}^{n} = 0 \qquad \forall i \in \mathcal{M},\ j \in \mathcal{N}$$

$$y_j^n = \begin{cases} 1 & \text{if } j = n \text{ or } j = \mathrm{j} \\ 0 & \text{otherwise.} \end{cases}$$

Define the next MN solutions, indexed by $m \in \mathcal{M}$ and $n \in \mathcal{N}$ as follows.

$$x_{ij}^{mn} = \begin{cases} \min\{1, \frac{K}{d_i}\} & \text{if } i = m \text{ and } j = n \\ 0 & \text{otherwise,} \end{cases}$$

$$y_j^{mn} = \begin{cases} 1 & \text{if } j = n \text{ or } j = \mathrm{j} \\ 0 & \text{otherwise.} \end{cases}$$

Consider the matrix whose rows correspond to these solution vectors. A permutation of the rows and columns of this matrix will produce a lower triangular matrix, with 1's on the diagonal, hence the matrix has full row rank. Thus the solution points corresponding to the rows are linearly independent. Hence, we have shown that

$$y_{\mathrm{j}} \leq 1$$

is a facet. Moreover, since j is arbitrary, we have covered all the cases. ∎

Proposition 12 *For every $j \in \mathcal{N}$, $\sum_{i \in \mathcal{M}} d_i x_{ij} \leq K y_j$ is a facet.*

Proof. We will first show that

$$\sum_{i \in \mathcal{M}} d_i x_{i1} \leq K y_1 \tag{56}$$

is a facet. By re-indexing the set $\mathcal{N}$, the same argument can be used for other values of j, thereby proving the proposition.

First, notice that (56) is not an improper face of $\mathcal{P}$ since there is a solution to (PI) where constraint (56) is not binding (e.g. let $y_1 = 1$, all other variables $= 0$). Let

$$\mathcal{P}^{\ddagger} = \{ (x,y) \mid (x,y) \in \mathcal{P} \text{ and } \sum_{i \in \mathcal{M}} x_{i1} = K y_1 \}.$$

Secondly, notice that $\mathcal{P}^{\ddagger}$ is not empty. Let

$$s = \operatorname*{argmin}_i \left\{ \sum_{u=1}^{i} d_u > K \right\}$$

then the solution defined by:

$$\begin{cases} y_1 = 1 \\ x_{t1} = \begin{cases} 1 & \text{if } 1 \le i \le s-1, \\ \dfrac{1}{d_s}\left(K - \sum_{u=1}^{s-1} d_u\right) & \text{if } i = s, \\ 0 & \text{otherwise,} \end{cases} \\ \text{All other variables are zero} \end{cases}$$

belongs to $\mathcal{P}^{\ddagger}$.

Suppose that for every point in $\mathcal{P}^{\ddagger}$,

$$\sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{N}} \alpha_{ij} x_{ij} + \sum_{j \in \mathcal{N}} \beta_j y_j = \alpha_0. \tag{$\ddagger$}$$

We want to show that ($\ddagger$) is a linear multiple of $\sum_{i \in \mathcal{M}} d_i x_{i1} = K y_1$.

Since $(0,0)$ is in $\mathcal{P}^{\ddagger}, \alpha_0$ must be zero. Next, consider the point in $\mathcal{P}^{\ddagger}$ defined by:

$$\begin{cases} y_j = 1 \\ \text{All other variables are zero} \end{cases}$$

for $j \neq 1$. The lefthand side of ($\ddagger$) is β_j , so we must have $\beta_j = 0$ for $j \in \mathcal{N}, j \neq 1$.

Next consider the point in $\mathcal{P}^{\ddagger}$ defined by:

$$\begin{cases} x_{ij} = \min\{1, \frac{K}{d_i}\} \\ y_j = 1 \\ \text{All other variables are zero} \end{cases}$$

for some $j \in \mathcal{N}, j \neq 1$. The lefthand side of ($\ddagger$) is $\alpha_{ij} x_{ij}$ (since β_j is zero), so we must have $\alpha_{ij} = 0$ (since $x_{ij} > 0$) for $j \neq 1$ and $i \in \mathcal{M}$.

Next define $(x^1, y^1) \in \mathcal{P}^{\ddagger}$ by:

$$(x^1, y^1): \begin{cases} x^1_{ij} = \begin{cases} 1 & \text{if } i = 1, 2, \ldots, s-1 \\ \dfrac{1}{d_s}\left(K - \sum_{u=1}^{s-1} d_u\right) & \text{if } i = s, \\ 0 & \text{otherwise} \end{cases} \\ y^1_1 = 1 \\ \text{All other variables are zero.} \end{cases}$$

Let $\delta = \min\{d_1, K - \sum_{u=1}^{s-1} d_u\} > 0$. Define $(x^2, y^2) \in \mathcal{P}^\ddagger$ as follows:

$$(x^2, y^2): \begin{cases} x_{ij}^2 = \begin{cases} x_{ij}^1 - \frac{\delta}{d_1} & \text{if } i = 1 \\ x_{ij}^1 + \frac{\delta}{d_s} & \text{if } i = s \\ x_{ij}^1 & \text{otherwise} \end{cases} \\ \\ y_1^2 = 1 \\ \text{All other variables are zero.} \end{cases}$$

The lefthand sides of (‡) for these two solution points differ by exactly $\delta(\frac{\alpha_{11}}{d_1} - \frac{\alpha_{s1}}{d_s})$. Hence we must have $\frac{\alpha_{11}}{d_1} - \frac{\alpha_{s1}}{d_s}$. Now, by re-indexing and considering other pairs of points in $\mathcal{P}^\ddagger$ that differ only in two x_{i1}'s, we can show that there is some $\alpha \neq 0$ such that

$$\alpha_{i1} = d_i \alpha \qquad \forall i \in \mathcal{M}.$$

At this point we have shown that (‡) is of the form

$$\alpha \sum_{i \in \mathcal{M}} d_i x_{i1} \; + \; \beta_1 y_1 \; = \; 0.$$

Substituting in the values for (x^1, y^1), we see that

$$K\alpha \; + \; \beta_1 = 0$$

so (‡) is α times $\sum_{i \in \mathcal{M}} d_i x_{i1} = K y_1$. Hence the only equation satisfied by all points in $\mathcal{P}^\ddagger$ is $\sum_{i \in \mathcal{M}} d_i x_{i1} = K y_1$.

This result proves that

$$\sum_{i \in \mathcal{M}} d_i x_{i1} \; \leq \; K y_1$$

is a facet of $\mathcal{P}$. By substituting another $j \in \mathcal{N}$ in place of the index 1, we can apply the same approach to complete the proof for the other cases of the proposition. ∎

Propositions (8) – (12) together give a proof of Theorem 1.