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Validation of CMM form and tolerance assessment software

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Abstract

While the verification of the performance of coordinate measuring machines (CMMs) is self-evidently important, the point coordinates they provide are usually the input to form and tolerance assessment software that calculate associated geometric features, such as a best fit cylinder to the data. For the measurement result to be reliable, it is also necessary to ensure that the calculations performed by this software are fit for purpose. The forthcoming standard ISO 10360-6:2001 [1] specifies the procedure by which Gaussian (least-squares) form assessment software should be tested.

Many tolerance assessment problems relate to Chebyshev (minimum-zone) fitting criteria in which the maximum error is minimized. These criteria lead to nonlinearly constrained optimization problems that are difficult to solve reliably if appropriate algorithms are not employed. Many existing and proposed algorithms fail on seemingly simple data sets.

In this paper we examine the problem of how to generate reference data sets and corresponding reference results for testing software for computing Gaussian and Chebyshev associated features. We indicate how to generate data sets for circles and cylinders, including data sets to help expose the deficiencies of inadequate software.

1 Introduction

This paper is concerned with the generation and use of reference data sets and corresponding reference results for testing geometric tolerance assessment software used by CMMs. Such software is a key component of computer-aided inspection in manufacturing. In order to verify that a manufactured artefact is within tolerance a CMM is used to collect a set of data

(x_i, y_i, z_i) representing points lying on the surface of the artefact.[1] The tolerance assessment software is used to analyse these data points with respect to the design and tolerance specification for the manufactured artefact to determine if the artefact is within tolerance.

It is clearly important that such software gives correct information and any serious quality management system will demand that its reliability can be verified in a practical manner. Numerical software with a well defined computational aim can be tested using the following general approach [3, 4]:

I Determine reference data sets (appropriate for the computational aim) and corresponding reference results;

II Apply the software under test to the reference data sets to produce test results;

III Compare the test results with the reference results.

However, the approach poses two fundamental questions. Firstly, how do we determine correct reference results for data sets? Secondly, how do we compare test results with reference results in a meaningful way? The viability of the approach depends on satisfactory answers to these questions and no formal software testing will be worthwhile unless these questions have been addressed.

The standard ISO 10360-6:2001 [1] defines a method (along the above lines) for testing software used to compute Gaussian associated features from coordinate measurements. The geometric features covered are the line (in two and three dimensions), the plane, the circle (in two and three dimensions), the sphere, the cylinder, the cone, and the torus. To answer the first question, the Standard proposes the use of either reference software or data generator software, but is not concerned with how such software might be implemented. To answer the second question, the standard specifies the use of performance values that measure the difference between reference parameter values and test parameter values for the four classes of parameters used (location, orientation, size and angle).

In this paper we are concerned with generators of reference data sets and corresponding results for testing tolerance assessment software. We address first the computations covered by ISO 10360-6:2001 and, thereafter, more general tolerance assessment problems including Chebyshev (or minimum zone) fitting criteria.

2 Computational aim of tolerance assessment software

Geometric tolerance assessment software can be classified into two broad types: those that are based on Gaussian and those on Chebyshev formula-

[1] Because the tip of the CMM probe has a finite radius the data values recorded in fact lie on a surface offset from the real surface. Compensation for this effect is possible [2], but is not considered here.

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tions. From an end-user perspective, both types perform roughly the same task – find the best-fit associated feature to data – but the optimisation technology employed is considerably different. This difference will have some bearing on the methods we employ to test such software. However, certain commonalities are present.

The computational aim of most assessment software used in coordinate metrology (and other areas) can be formulated in terms of an optimisation problem. This is illustrated below using the examples of the Gaussian and Chebyshev best-fit circles.

Gaussian best-fit circle. A circle in the plane can be described by its centre coordinates (x_0, y_0) and its radius r_0. The distance d from a point $\mathbf{x} = (x, y)$ to a circle defined by parameters $\mathbf{a} = (x_0, y_0, r_0)$ is given by

$$d = d(\mathbf{x}; \mathbf{a}) = \left[(x - x_0)^2 + (y - y_0)^2\right]^{1/2} - r_0. \quad (1)$$

Given a set of data points $X = \{\mathbf{x}_i : i = 1, \ldots, m\}$, the Gaussian (least-squares) associated circle [5] is found by solving the optimisation problem

$$\min_{\mathbf{a}} \sum_{i=1}^{m} d^2(\mathbf{x}_i, \mathbf{a}). \quad (2)$$

Chebyshev best-fit circle. The Chebyshev or minimum zone circle to data points X is defined as the solution to the problem

$$\min_{\mathbf{a}} \max_{i=1,\ldots,m} |d(\mathbf{x}_i, \mathbf{a})|.$$

By introducing the parameter $s = \max_i |d(\mathbf{x}_i, \mathbf{a})|$, this problem can be reformulated as

$$\min_{\mathbf{a}, s} s \quad \text{subject to} \quad -s \leq d(\mathbf{x}_i, \mathbf{a}) \leq s, \quad i = 1, \ldots, m, \quad (3)$$

i.e., as a constrained optimisation problem in terms of parameters $\mathbf{a}$ and s.[2]

3 Generation of reference data and results

In this section we describe methods for generating reference data and results "simultaneously", independently of reference software; see also [4, 7, 8].

Consider the Gaussian optimisation problem (2). It is possible to derive *sufficiency conditions* involving the data X and parameters $\mathbf{a}$ which when satisfied by $\mathbf{a}^*$ will guarantee that $\mathbf{a}^*$ is a local minimum for the optimisation problem. Algorithms for solving (2) usually function by starting

[2]Minimum circumscribed and maximum inscribed problems can similarly be formulated as constrained optimisation problems [6], but are not considered here.

from an initial estimate of the solution and then making systematic progress until these conditions are satisfied. The reference data generation problem is in some sense dual to the optimisation problem: given "solution" parameters $\mathbf{a}^*$ find data X such that the sufficiency conditions are satisfied. The reference data generation problem for the Chebyshev problem is treated similarly by considering sufficiency conditions for a solution to the optimisation problem (3).

An important aspect of the reference data generation problem is that while the sufficiency conditions are usually over-determined with respect to the optimisation parameters they are usually under-determined with respect to the data points X. For this reason, it is generally much easier to solve the data generation problem.

3.1 Data generation for Gaussian best-fit features

3.1.1 Sufficiency conditions

Suppose $\mathbf{a} \mapsto \mathcal{S}(\mathbf{a})$ parametrizes a geometric feature and $d = d(\mathbf{x}; \mathbf{a})$ is the distance from a point $\mathbf{x}$ to the feature $\mathcal{S}(\mathbf{a})$. The sufficiency conditions for $\mathbf{a}^*$ to be the Gaussian best-fit feature for a data set $X = \{\mathbf{x}_i\}$ are that (i) $J^{\mathrm{T}}\mathbf{d} = 0$, where J is the *Jacobian* matrix of partial derivatives defined by

$$J_{ij} = \frac{\partial d}{\partial a_j}(\mathbf{x}_i, \mathbf{a}^*),$$

and the ith element of $\mathbf{d}$ is $d(\mathbf{x}_i, \mathbf{a}^*)$, and (ii) the *Hessian* matrix carrying second derivative information is strictly positive definite [9].

The sufficiency conditions correspond to the (familiar) conditions of zero first derivative and positive second derivative that characterize a minimum of a function, but are here expressed in terms of the component distance functions $d(\mathbf{x}_i, \mathbf{a}^*)$ that define the Gaussian (least-squares) error measure in (2). The first of the conditions states that $\mathbf{d}$ lies in the null space of J^{T}, i.e., in the vector space $\{\mathbf{z} : J^{\mathrm{T}}\mathbf{z} = 0\}$. This observation underpins the *null space method* for generating reference data that has been applied at NPL to test a wide range of least-squares software [4, 10]. The method is illustrated below using the example of the Gaussian best-fit circle.

3.1.2 Gaussian best-fit circle

Let $\mathbf{a} = (x_0, y_0, r_0)$ and $\mathbf{a}^* = (0, 0, r_0^*)$. Given a set of polar angles $\theta_i, i = 1, \ldots, m$, with $m > 3$, define

$$\mathbf{x}_i^* = r_0^*(\cos\theta_i, \sin\theta_i),$$

and

$$\mathbf{x}_i = (r_0^* + \delta_i)(\cos\theta_i, \sin\theta_i).$$

The points $X^* = \{\mathbf{x}_i^*\}$ lie on the circle specified by the parameters $\mathbf{a}^*$, and the points $X = \{\mathbf{x}_i\}$ at distances δ_i from this circle. Clearly, the solution

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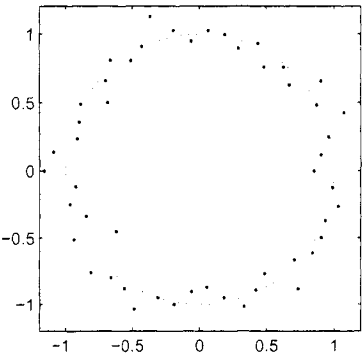

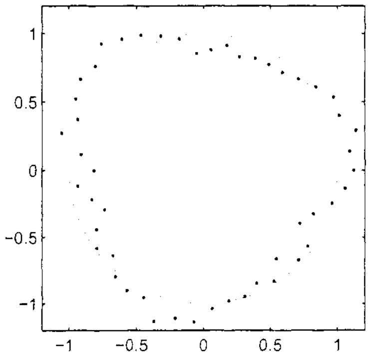

Figure 1: Reference data sets for the Gaussian best-fit circle.

to the Gaussian best-fit circle problem for the points X^* is $\mathbf{a}^*$. The data generation problem is to determine values for δ_i such that the sufficiency conditions are satisfied.

Now, using (1) and the above definitions, it follows that the elements of the Jacobian matrix J are given by

$$\frac{\partial d_i}{\partial x_0} = -\cos\theta_i, \quad \frac{\partial d_i}{\partial y_0} = -\sin\theta_i, \quad \frac{\partial d_i}{\partial r_0} = -1.$$

Consequently, the first of the sufficiency conditions takes the form of the equations

$$\sum_{i=1}^{m} \delta_i \cos\theta_i = \sum_{i=1}^{m} \delta_i \sin\theta_i = \sum_{i=1}^{m} \delta_i = 0.$$

The equations are under-determined because there are only 3 equations in the m unknowns $\delta_i, i = 1, \ldots, m$. This property may be exploited to generate many reference data sets having the same reference solution $\mathbf{a}^*$ and to generate data sets possessing prescribed properties. Having obtained a solution to these equations it is necessary to use the second of the sufficiency conditions to verify that $\mathbf{a}^*$ defines a minimum of the Gaussian best-fitting problem.

Figure 1 shows two reference data sets for the Gaussian best-fit circle problem which have the same solution $x_0^* = 0$, $y_0^* = 0$ and $r_0^* = 1$. The second of the data sets is constructed such that the residual deviations for the solution circle exhibit a "three-lobed" pattern, thus mimicking a commonly-observed behaviour.

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3.2 Data generation for Chebyshev best-fit features

3.2.1 Sufficiency conditions

It is convenient to write the Chebyshev optimisation problem (3) in the general form

$$\min_{\mathbf{b}} F(\mathbf{b}) \quad \text{subject to} \quad c_i^+(\mathbf{b}) \geq 0 \ \text{ and } \ c_i^-(\mathbf{b}) \geq 0, \quad i = 1, \ldots, m, \qquad (4)$$

where $\mathbf{b} = (\mathbf{a}, s)$, $F(\mathbf{b}) = s$, $c_i^+(\mathbf{b}) = s - d(\mathbf{x}_i, \mathbf{a})$, and $c_i^-(\mathbf{b}) = s + d(\mathbf{x}_i, \mathbf{a})$.

The sufficiency conditions for $\mathbf{b}^*$ to be a local solution for (4) are (i) *feasibility*: all the constraints are satisfied, i.e., $c_i^+(\mathbf{b}^*) \geq 0$ and $c_i^-(\mathbf{b}^*) \geq 0$; (ii) the *Kuhn-Tucker equations* are satisfied, i.e., if $J^0 = \{j : c_j^+(\mathbf{b}^*) = 0\}$ and $K^0 = \{k : c_k^-(\mathbf{b}^*) = 0\}$ specify the indices of the *active* constraints, then

$$\nabla_{\mathbf{b}} F(\mathbf{b}^*) = \sum_{j \in J^0} \lambda_j \nabla_{\mathbf{b}} c_j^+ + \sum_{k \in K^0} \mu_k \nabla_{\mathbf{b}} c_k^-, \qquad (5)$$

with *Lagrange multipliers* $\lambda_j > 0$, $j \in J^0$, and $\mu_k > 0$, $k \in K^0$, and (iii) a Hessian-type matrix constructed from the matrices of second derivatives of the active constraint functions is strictly positive definite [11].

The sufficiency conditions have one very noticeable feature in that only the active constraints feature significantly. To generate data sets with a known local solution it is sufficient to generate (a small number of) points which satisfy the optimality conditions and then introduce other data points which satisfy the feasibility constraints. The sufficiency conditions, and their use as the basis of generating reference data sets, are illustrated below using the examples of the Chebyshev circle and cylinder problems.

3.2.2 Chebyshev circle

Let $\mathbf{b} = (x_0, y_0, r_0, s)$ and $\mathbf{b}^* = (0, 0, r_0^*, s^*)$. Define points $X = \{\mathbf{x}_i : i = 1, \ldots, m\}$ by their polar coordinates r_i and θ_i to satisfy

$$r_0^* - s^* \leq r_i \leq r_0^* + s^*, \quad i = 1, \ldots, m,$$

and

$$r_j = r_0^* + s^*, \quad j \in J_0, \quad r_k = r_0^* - s^*, \quad k \in K_0.$$

Choosing the values r_i, $i = 1, \ldots, m$, in this way ensures that the constraints are satisfied for $\mathbf{b}^*$, and that J_0 and K_0 define the indices of the active constraints. Geometrically, the points $\mathbf{x}_i$ lie between two concentric circles of radii $r_0^* - s^*$ and $r_0^* + s^*$, and J_0 and K_0 define the indices of the points lying on these circles.

Now,

$$\begin{aligned} \nabla_{\mathbf{b}} F(\mathbf{b}^*) &= (0, 0, 0, 1)^{\mathrm{T}}, \\ \nabla_{\mathbf{b}} c_j^+(\mathbf{b}^*) &= (\cos\theta_j, \sin\theta_j, 1, 1)^{\mathrm{T}}, \\ \nabla_{\mathbf{b}} c_k^-(\mathbf{b}^*) &= (-\cos\theta_k, -\sin\theta_k, -1, 1)^{\mathrm{T}}. \end{aligned}$$

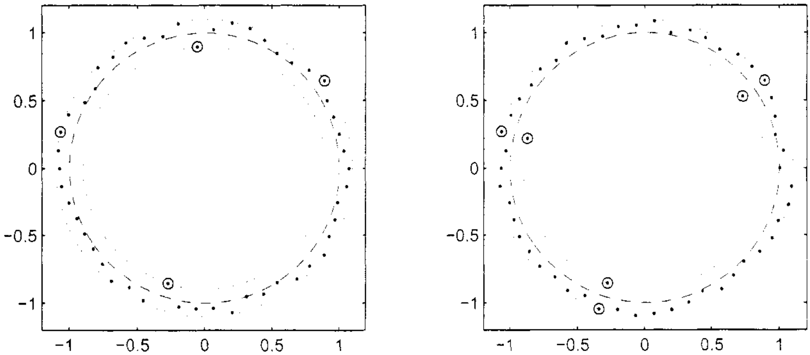

Figure 2: Reference data sets for the Chebyshev best-fit circle.

It follows (after some simplification) that the Kuhn-Tucker equations (5) take the form

$$\sum_{j\in J_0} \lambda_j \cos\theta_j = \sum_{k\in K_0} \mu_k \cos\theta_k, \qquad \sum_{j\in J_0} \lambda_j \sin\theta_j = \sum_{k\in K_0} \mu_k \sin\theta_k,$$

and

$$\sum_{j\in J_0} \lambda_j = \sum_{k\in K_0} \mu_k = 1,$$

with

$$\lambda_j > 0,\ j \in J_0, \quad \mu_k > 0,\ k \in K_0.$$

These equations tell us how the polar angles θ_j, $j \in J_0$, and θ_k, $k \in K_0$, must be chosen if the sufficiency conditions are to be satisfied and the solution to the Chebyshev best-fit circle problem for the data X is to be $\mathbf{b}^*$. One way for the equations to hold is if there are two points on each of the concentric circles (identified by indices J_0 and K_0) which when radially projected onto a common concentric circle define chords that intersect internally.

However, there are also other possibilities, e.g., if there are three points on each circle, lying on three radial lines. Such an arrangement also satisfies the sufficiency conditions and hence defines a local solution, although there is no subset of four points satisfying the characterization given above. An "exchange-type" algorithm for solving this problem that operates by seeking a set of four points that satisfy the sufficiency conditions for a solution may cycle when it encounters such a situation.

Figure 2 shows two reference data sets for the Chebyshev best-fit circle problem which have the same solution $x_0^* = 0$, $y_0^* = 0$, $r_0^* = 1$ and $s^* = 0.1$. In each case the points defining the active constraints are highlighted. For the first data set there are four such points, and for the second six points.

3.2.3 Chebyshev cylinder

Define points $X = \{\mathbf{x}_i : i = 1, \ldots, m\}$ by their cylindrical polar coordinates r_i, θ_i and z_i. For the Chebyshev cylinder, the optimality conditions corresponding to (5) for a solution cylinder in standard position, i.e., with its axis coincident with the z-axis, are those for the Chebyshev circle given before together with the additional conditions

$$\sum_{j \in J_0} \lambda_j z_j \cos\theta_j = \sum_{k \in K_0} \mu_k z_k \cos\theta_k, \quad \sum_{j \in J_0} \lambda_j z_j \sin\theta_j = \sum_{k \in K_0} \mu_k z_k \sin\theta_k. \tag{6}$$

We can use the (geometrical) characterization of a solution to the Chebyshev circle problem to determine θ_j and λ_j, $j \in J_0$, and θ_k and μ_k, $k \in K_0$, to satisfy the conditions for the Chebyshev circle. In this way the $x-$ and $y-$ components of the points defining the active constraints are determined. The $z-$components are chosen to satisfy the conditions (6) by solving a linear system of equations derived from those conditions.

Having determined a set of contacting points in this way it is then necessary to verify the second order optimality conditions to ensure the solution cylinder corresponds to a local minimum. Further data points are added subject to the constraint that they lie on or between the cylinders of radius $r_0^* \pm s^*$. Finally, the solution cylinder and data points are transformed (rotated to adjust the orientation of the cylinder axis and translated to adjust its position) to generate data and a corresponding solution in general, rather than standard, position.

This procedure can be applied to produce reference data sets for the Chebyshev cylinder with four or more contacting points at the solution. Table 1 contains a reference data set for the Chebyshev cylinder problem. The solution for the data set is a cylinder in standard position of radius $r_0^* = 100$. The first five points define the active constraints at the solution.

4 Concluding remarks

We have considered methods for generating reference data sets and corresponding reference results for testing Gaussian and Chebyshev feature fitting software. Such software plays an important part of computer-aided inspection in manufacturing and, consequently, it is essential that the numerical correctness of such software is verified. The method of data generation advocated in this work is based on examining the (sufficiency) conditions for a solution to the problem considered, and determining data to satisfy the conditions for a solution specified *a priori*. The approach has the advantages that it does not require the development of reference software, allows many data sets with the same solution to be generated easily, and permits data sets with specified properties to be produced. The approach has been illustrated for Gaussian and Chebyshev circle and cylinder problems.

i	x_i	y_i	z_i
1	-0.5729898924454593	-100.0983600394290	30.56373320610054
2	99.93524267611404	-5.740842382977500	-11.47288911815088
3	2.067975691341978	100.0786364642326	-38.51138609240303
4	73.37777456764940	-67.79168237696440	1.456636722411527
5	65.99221271537206	75.00025240576920	-26.31733206298247
6	-50.77237299225759	-86.08674501655730	-67.25866707485226
7	83.59457524517181	54.71596641722667	-99.43039920404216
8	-89.68897347648601	-44.30625130963349	15.82090089703252
9	50.71407491536649	-86.22792904622092	68.38301331354209
10	4.656525747369551	-99.97855791177561	-53.63577066834554

Table 1: Reference data set for the Chebyshev cylinder problem. The solution is a cylinder with its axis coincident with the z−axis and radius 100. The first five points define the active constraints at the solution.

METROS (the METROlogy Software environment) [12] is a system aimed at providing metrologists with access to software appropriate to their needs. Reference data sets and corresponding results for the problems considered here will subsequently be incorporated in METROS.

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