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Valuation Theory on Finite Dimensional Division Algebras

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to P. Ribenboim with respect and affection

Valuation theory has classically meant the study of valuations on a commutative field. Such valuation theory has flourished for many decades, nourished by its connections with number theory and algebraic geometry. But there is also a noncommutative side of the subject in the study of valuations and valuation rings on division rings. This aspect has blossomed only in the last twenty odd years, and it is not so well known as its commutative counterpart. We give in this paper a survey of valuations and valuation theory for division rings finite dimensional over their centers. We describe the associated theory and also some of the most significant constructions that have been given as applications.

The earliest use of valuations on noncommutative division rings, to my knowledge, was by Hasse in his work in [Ha] in 1931 on orders over central simple algebras over p-adic fields. In the '40's and '50's there was a little further work with valuations on division algebras over fields with complete discrete valuations. Also, there was some discussion of valuations on division algebras in Schilling's work [Schi_1], [Schi_2], mostly observing that some results about valuations remain valid without assuming that the ambient field be commutative. One can speculate that there was little attention to valuations on division algebras because too often for the division algebras of interest there was no apparent valuation available. Additionally, it was not fully clear what to take as the definition of a valuation on a division ring, since the concepts of valuation and valuation ring are not equivalent in the noncommutative setting.

It was not until the late 1970's and the 1980's that valuation theory on division algebras began to develop substantially. This was due in large measure to the realization that some of the major constructions of counterexamples in the 1970's could best be understood using valuation theory. This applies to Amitsur's construction of noncrossed product division algebras and also to the constructions of division algebras with nontrivial SK_1 by Platonov and by Yanchevskiĭ. It began to

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be recognized that while valuations on division algebras might be relatively rare, when a valuation is present it can often be used to get more detailed arithmetic information on a division ring than is readily available by other means. Indeed, the greatest success of valuation theory on division algebras has been in the constuction of specific examples of division algebras with interesting properties. This has made it possible to settle a number of open questions about division algebras. In pursuing these constructions, it became evident that there needed to be a more systematic development of the theory of valuations on division algebras, and this was carried out. Special attention was given to the case of Henselian valuations, since it was known that a Henselian valuation on the center of a finite dimensional division algebra D always extends (uniquely) to a valuation on D.

Another significant strand in noncommutative valuation theory has been the study of Dubrovin valuation rings, which began with Dubrovin's pioneering work in the late '70's and '80's. Dubrovin introduced a more general concept of a valuation ring for simple Artinian rings than what had been considered previously, and showed many significant and nontrivial properties about them. His rings exist much more abundantly than the classical valuation rings, but still have enough uniqueness to carry significant arithmetic information. Results on Dubrovin valuation rings have also been very useful in proving some results about the more classical valuation rings.

Here is a brief overview of this paper. In §1 we describe some of the different notions of valuation rings for a division ring. Each of these is a reasonable generalization of the valuation rings of a field. Thereafter until §9 we will work with the most restrictive of the definitions (at the cost of having fewer examples), namely with invariant valuation rings, which are those for which there is a valuation satisfying the usual axioms. In §2 we describe some of the structure associated to a valuation on a finite dimensional division algebra D. There is of course a value group Γ_D and a valuation ring V_D and residue division ring $\overline{D}$, but there are also connections between Γ_D and the center of $\overline{D}$ and the roots of unity in $\overline{D}$. In §3 we fix a Henselian valuation on a field F and describe some of the special kinds of valued division algebras over F. There is a nice homological interpretation for the division algebras split by the maximal unramified extension of F with respect to v. Also, when D is tame over F, there are decomposition results allowing D to be described in terms of division algebras with simpler structure. Particular attention is given in §4 to division algebras tame and totally ramified over the center. Such division algebras are particularly easy to work with and have been used in some major constructions, yet were not studied systematically until the 1980's. In §§5–7 we describe some of the most important constructions that have been carried out using valued division algebras. These include (in §5) noncrossed product algebras and (in §6) division algebras D in which there are elements of reduced norm 1 not lying in the commutator group of D^*, i.e., $SK_1(D)$ is nontrivial. In §7 we give brief discussions of some of the other constructions which have been carried out using valuations on division algebras. The very short §8 has a few remarks on valuations in connection with orderings on finite dimensional division algebras. In §9 we briefly discuss total valuation rings, which are a more general class than the invariant valuation rings associated with valuations on division algebras. Finally, §10 treats Dubrovin valuation rings, which are more general still than total valuation rings. We record some of the highlights of the extensive and rich (but difficult) theory that has developed for these rings. In writing this survey, I have restricted attention to valuations and valuation rings only on division algebras finite dimensional over their centers. This is the area where the results have been most extensive, and for which, to date, there have been the most applications. (It is also the area that I have the most knowledge of.) As the reader can see there is plenty to be said in just the finite-dimensional case! With the hope of making this survey accessible to a wider audience, I have at various points recalled some well-known facts on commutative valuation theory, and on division algebras. Experts can easily enough skip over those points with which they are already familiar. Proofs, except some very short ones, are mostly omitted, but I have tried to give specific references where proofs can be found. There are many examples. The bibliography at the end is quite extensive, but certainly still not complete. Several of the papers listed in the bibliography are not specifically referred to.

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We now mention some of the terminology that will be used throughout the paper:

(0.1) If R is a ring (always assumed to have a 1), we write $Z(R)$ for the center of R; $M_n(R)$ for the $n \times n$ matrices over R; R^* for the group of units of R; and $J(R)$ for the Jacobson radical of R. "Field" always means a commutative field. By a central simple algebra over a field F, we mean a simple F-algebra A such that $Z(A) = F$ and $\dim_F(A) < \infty$. We write $[A:F]$ for $\dim_F(A)$. By Wedderburn's theorem, for such A we have $A \cong M_n(D)$, where D is a division algebra over F (i.e., $D^* = D - \{0\}$); this D is unique up to isomorphism. We call n the matrix size of A and D the underlying division algebra of A. The degree of A is $\deg(A) = \sqrt{[A:F]}$, a positive integer; the (Schur) index of A is $\operatorname{ind}(A) = \deg(D) = \deg(A)/n$. The exponent of A, $\exp(A)$, is the order of the class $[A]$ of A in the Brauer group $\operatorname{Br}(F)$ of F. For central simple algebras A and B over F, we write $A \sim B$ if A and B are Brauer equivalent, i.e., $[A] = [B]$ in $\operatorname{Br}(F)$. For any field $L \supseteq F$ let $\operatorname{Br}(L/F)$ denote the kernel of the scalar extension map $\operatorname{Br}(F) \to \operatorname{Br}(L)$. We write $\mathcal{D}(F)$ for the collection of division algebras D with center F such that $[D:F] < \infty$.

(0.2) We will see that the roots of unity are significant in noncommutative valuation theory. If F is a field, we write $\mu(F)$ for the group of roots of unity in F. For any natural number n, we write $\mu_n \subseteq F$ to say that F contains n n-th roots of unity (so $\operatorname{char}(F) \nmid n$). When this occurs, we write $\mu_n(F)$ for the group of n-th roots of unity in F, and $\mu_n^*(F)$ for the set of primitive n-th roots of unity in F.

(0.3) If n is a positive integer and F is a field with $\mu_n \subseteq F$, let $\omega \in \mu_n^*(F)$. For any $a, b \in F^*$, we write $A_\omega(a,b;F)$ for the "symbol algebra" which is the F-algebra with generators i, j and relations $i^n = a$, $j^n = b$, and $ij = \omega ji$. We call i and j the standard generators of $A_\omega(a,b;F)$. It is well known (see, e.g., [D$_2$, p. 78, Th. 1]) that $A_\omega(a,b;F)$ is a central simple F-algebra of degree n, with F-base $\{i^k j^\ell \mid 0 \le k, \ell \le n-1\}$. If $F \subseteq K$ is a Galois extension of fields (i.e., algebraic, normal, and separable, but not necessarily of finite degree) we write $\mathcal{G}(K/F)$ for the Galois group of K over F, a profinite group. If $[K:F] = n < \infty$ and $\mathcal{G}(K/F)$ is cyclic, with σ a generator, and if $b \in F^*$, we write $(K/F, \sigma, b)$ for the cyclic F-algebra generated by K and x with relations $xc = \sigma(c)x$ for all $c \in K$ and $x^n = b$. It is well known (see, e.g., [D$_2$, p. 49, Lemma 6]) that $(K/F, \sigma, b) = \bigoplus_{i=0}^{n-1} Kx^i$ (K-vector space direct sum), and that it is a central simple F-algebra of degree n.

1 Different kinds of noncommutative valuation rings

When we consider valuation theory of fields, there are at least three different but equivalent ways of formulating the foundations of the subject. Given a field F, one can consider

(i) a *valuation* on F, which is a function $v : F^* \to \Gamma$ satisfying appropriate axioms, where Γ is a totally ordered abelian group;

(ii) a *valuation ring* of F, which is a ring V with quotient field F, such that for every $a \in F^*$ we have $a \in V$ or $a^{-1} \in V$;

(iii) a *place* on F, which is a function $\lambda : F \to L \cup \infty$ satisfying appropriate axioms, where L is a field.

Our subject here is valuation theory on noncommutative division algebras finite dimensional over their centers. An immediate challenge arises because the three formulations listed above are no longer equivalent in the noncommutative setting. Let us look more closely at how each concept can be defined for a division ring D.

(i) A *valuation* on D is a function $v : D^* \to \Gamma$, where Γ is a totally ordered group (written additively), such that for all $a, b \in D^*$

(a) $v(ab) = v(a) + v(b)$;

(b) $v(a + b) \geq \min(v(a), v(b))$, whenever $b \neq -a$.

To any valuation v there is an associated ring $V = \{a \in D^* \mid v(a) \geq 0\} \cup \{0\}$, which satisfies the property that $a \in V$ or $a^{-1} \in V$, for each $a \in D^*$. But further, because v is a homomorphism on D^* and Γ is totally ordered, one finds that $aVa^{-1} \subseteq V$ for all $a \in D^*$. A subring W of D such that for all $a \in D^*$ we have $a \in W$ or $a^{-1} \in W$ and also $aWa^{-1} \subseteq W$ is called an *invariant valuation ring* of D—invariant because of the invariance under inner automorphisms. Given any such W, we have $aW = Wa$ of each $a \in D^*$; hence the set of principal fractional left (= right) ideals $\Gamma = \{aW \mid a \in D^*\}$ is a group, with operation $(aW) \cdot (bW) = abW$, which is totally ordered by reverse inclusion: $aW \leq bW$ iff $aW \supseteq bW$. (We could also describe Γ as D^*/W^*.) The natural map $w : D^* \to \Gamma$ given by $a \mapsto aW$ is clearly a valuation on D with valuation ring W. Thus, the study of valuations on D satisfying axioms (a) and (b) above is equivalent to the study of invariant valuation rings of D.

(ii) A *total valuation ring* of the division algebra D is a subring T of D such that for each $a \in D^*$ we have $a \in T$ or $a^{-1} \in T$. With a total valuation ring T, the left ideals (resp. right ideals) are linearly ordered by inclusion, but the left ideals will not coincide with the right ideals unless T is actually invariant. There do exist division algebras finite dimensional over their centers with total valuation rings which are not invariant—see Ex. 9.1 below. Such total valuation rings do not have a valuation as described above, but Mathiak has shown (see [Mat, p. 5]) that they have a valuation-like function whose image is a totally ordered set which is not a group.

(iii) If A is a simple Artinian ring, a subring B of A is called a *Dubrovin valuation ring* of A if B has an ideal J such that B/J is a simple Artinian ring, and for each $a \in A - B$ there are $b, b' \in B$, such that $ab \in B - J$ and $b'a \in B - J$. One might think of the map $\lambda : A \to B/J \cup \infty$, given by $b \mapsto b + J$ for $b \in B$, and $a \mapsto \infty$ for $a \in A - B$, as a place in the category of simple Artinian rings. This notion of a place is not the only one (see [vG] for others), but it has been by far the most successful one for noncommutative valuation theory. Some of the extensive theory of such valuation rings is described in §10 below. Clearly every total valuation ring is a Dubrovin valuation ring. But Dubrovin valuation rings comprise a much wider class of rings. For example, if $D \in \mathcal{D}(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers, then for every discrete valuation ring V of $\mathbb{Q}$ there is a Dubrovin valuation ring B of D with $B \cap \mathbb{Q} = V$; indeed B has these properties iff B is a maximal order of V in D (see the comments after Th. 10.2 below). Such a B is often not unique in D, but is unique up to conjugacy (see Th. 10.3). However, there are at most finitely many discrete valuation rings V of $\mathbb{Q}$ (and perhaps none at all, see Ex. 2.4(ii)) such that there is a total valuation ring T of D with $T \cap \mathbb{Q} = V$. When this occurs, T is actually an invariant valuation ring (since V has Krull dimension 1, see the comments at the end of §9) and there is only one such T; in fact $T = \{d \in D \mid d \text{ is integral over } V\}$. But, such a T exists iff $D \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_V$ is a division ring, by the complete version of Th. 2.3, where $\widehat{\mathbb{Q}}_V$ is the completion of $\mathbb{Q}$ with respect to the valuation of V.

There are still other notions of valuations or places for division rings, such as the matrix valuations of Mahdavi-Hezavehi (see, [MH_1], [MH_2]) and the places considered in [vG], but they have not been so well adapted to finite dimensional division algebras, and we will not pursue them here.

Each of the three types of valuation rings described above on a finite dimensional division algebra D is a plausible generalization of the idea of a valuation ring of a field. Each arises naturally in certain contexts. We will focus here primarily on the first type, the invariant valuation rings arising from valuations on D, since these are the ones that have to date been studied the most extensively and have had the most applications. But we will discuss the other two types in §§9 and 10. To simplify the terminology, henceforward (until §9) when we say "valuation ring" without further qualification, it is understood that we mean an *invariant* valuation ring.

2 Valuations and (invariant) valuation rings

Let us now take a closer look at valuations on division rings and their associated (invariant) valuation rings. Let D be a division ring, and let $F = Z(D)$, the center of D. We assume throughout that $[D:F] < \infty$; thus $D \in \mathcal{D}(F)$ in the notation of the Introduction. Let Γ be a totally ordered group, written additively, and let $v : D^* \to \Gamma$ be a valuation, which as noted in §1 is a function satisfying, for all $a, b \in D^*$,

$$\begin{aligned} &\text{(i)} \quad v(ab) = v(a) + v(b), \\ &\text{(ii)} \quad v(a+b) \geq \min(v(a), v(b)) \quad \text{whenever } a + b \neq 0. \end{aligned} \tag{2.1}$$

Associated to v is its valuation ring

$$V_D = \{a \in D^* \mid v(a) \geq 0\} \cup \{0\}; \tag{2.2}$$

the unique maximal left (and maximal right) ideal of V_D,

$$M_D = \{a \in D^* \mid v(a) > 0\} \cup \{0\}; \tag{2.3}$$

the group of valuation units,

$$U_D = \{a \in D^* \mid v(a) = 0\}; \tag{2.4}$$

the residue division algebra,

$$\overline{D} = V_D / M_D; \tag{2.5}$$

and the value group

$$\Gamma_D = \mathrm{im}(v) \subseteq \Gamma. \tag{2.6}$$

It is convenient to index these objects by D, since we will usually be considering only one valuation on D at a time. As we noted in §1, V_D is an invariant valuation ring of D in that for every $a \in D^*$ we have $a \in V_D$ or $a^{-1} \in V_D$, and also $aV_Da^{-1} = V_D$. Consequently, the left ideals of V_D are linearly ordered by inclusion, and every left ideal is a right ideal, and vice versa. Since M_D is a maximal one-sided as well as two-sided ideal of V_D, the residue ring $\overline{D}$ is a division ring. Also, though we did not assume Γ abelian, it is not hard to deduce from $[D:F] < \infty$ that Γ_D is abelian (cf. [W$_1$]). When we restrict v to any F-subalgebra E of D, we obtain a valuation $v|_E$ on the division ring E. The objects for $v|_E$ corresponding to those in (2.2)–(2.6) for v are denoted V_E, M_E, U_E, $\overline{E}$, Γ_E.

Before examining the structure of a valued division ring, let us consider the question of existence. For $D \in \mathcal{D}(F)$, any valuation on D restricts to a valuation on F. But frequently one starts out with a valuation w on F and asks whether w can be extended to a valuation on D. It is well known (see, e.g., [E, p. 62, Cor. 9.7]) that for any field $K \supseteq F$, w has at least one and often many different extensions to valuations on K. (We consider the extensions different when their valuation rings are different.) However, in the noncommutative setting, the story is very different.

Theorem 2.1 *Let F be a field, and let $D \in \mathcal{D}(F)$. If w is a valuation on F, then w extends to a valuation on D iff w has a* unique *extension to each (commutative) field L with $F \subseteq L \subseteq D$. Hence, w has at most one extension to a valuation on D (but perhaps none at all).*

Indeed, when a valuation v on D exists, extending w on F, it satisfies the formula, for all $a \in D^*$

$$v(a) \;=\; \tfrac{1}{\sqrt{[D:F]}}\, w(\mathrm{Nrd}(a)) \in \Delta_F, \tag{2.7}$$

where $\mathrm{Nrd} : D \to F$ is the reduced norm (cf. [Re, p. 116]), and $\Delta_F = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$, which is the divisible hull of the torsion-free abelian group Γ_F. (We identify Γ_F with its isomorphic image $\mathbb{Z} \otimes_{\mathbb{Z}} \Gamma_F$ in Δ_F, then $\Delta_F = \mathbb{Q}\,\Gamma_F$. Recall that the total ordering on Γ_F has a unique extension to Δ_F.) Th. 2.1 and formula (2.7) were first proved by Ershov in [Er$_3$], and were proved independently later in [W$_1$]. It follows easily from Th. 2.1 that when w on F extends to a valuation on D, then $V_D = \{a \in D \mid a \text{ is integral over } V_F\}$. It is quite an unusual occurrence in a noncommutative division ring for the set of elements integral over a subring of the center to form a ring. (But this also occurs with total valuation rings—see Th. 9.2 below.)

Th. 2.1 points up a significant difference in flavor between commutative and noncommutative valuation theory. It shows that valuations on division algebras are far rarer than valuations on fields. This is likely a major reason why noncommutative valuation theory developed much later than the commutative theory—in many cases, a valuation on the center of a division algebra of interest simply does not extend to the division algebra.

But there is another viewpoint that Th. 2.1 suggests: While a valuation v on a division algebra D may be a rather rare occurrence, the presence of v and its associated valuation ring is a significant feature in the structure of D, and we can use the valuation to gain much information about arithmetic properties of D that might be very difficult to get at in any other way. Notably, one of the most basic ways of studying a division algebra is in terms of its commutative subfields. But it is usually enormously difficult to describe or classify the subfields of a given division algebra. A valuation on D imposes a major constraint on its possible subfields, and this is sometimes sufficient to determine all the subfields of D (see, e.g., Th. 3.3 and Th. 4.5(c) below).

While valuation theory is not available for studying all finite-dimensional division algebras, it has led to the construction of many division algebras with a rich but very well-understandable structure. This has led to the construction of examples that have helped to settle major open questions about finite-dimensional division algebras. We will describe some such constructions in §§5–7 below.

Th. 2.1 does not provide an easily testable criterion for the existence of a valuation on a field F to a given $D \in \mathcal{D}(F)$, but it does serve to highlight an important class of valuations on F for which the extension criterion will always be satisfied, namely Henselian valuations. Recall that a valuation w on a field K is *Henselian* just when Hensel's Lemma holds for w, i.e., for every monic polynomial $f \in V_K[x]$, if its image $\overline{f} \in \overline{K}[x]$ has a factorization $\overline{f} = \widetilde{g}\widetilde{h}$ on $\overline{K}[x]$ with $\widetilde{g}, \widetilde{h}$ monic and $\gcd(\widetilde{g}, \widetilde{h}) = 1$, then there exist monic $g, h \in V_K[x]$ with $f = gh$, $\overline{g} = \widetilde{g}$, and $\overline{h} = \widetilde{h}$. There are several other equivalent characterizations of Henselian valuations (see [R_3] for a very nice discussion of this); the one most relevant here is:

$$\text{A valuation } w \text{ on a field } K \text{ is Henselian iff } w \text{ has a unique extension to each field } L \supseteq K \text{ with } L \text{ algebraic over } K. \tag{2.8}$$

(See [R_3, Th. 3] or [E, Cor. 16.6] for a proof of (2.8).) The following corollary is immediate from Th. 2.1 in light of (2.8):

Corollary 2.2 *If a valuation w on a field K is Henselian, then w has a (unique) extension to each division algebra finite-dimensional over K.*

Cor. 2.2 was known long before Th. 2.1, tracing back at least to the work of Schilling in [Schi_2, p. 53, Th. 9].

The original and best-known examples of Henselian valuations are the complete and discrete rank 1 valuations.* We will give further examples of Henselian valuations in §3 below. The first use of valuations on noncommutative division algebras seems to be in Hasse's work [Ha] on maximal orders in central simple algebras over p-adic fields. When a field F has a complete discrete valuation w, then for any $D \in \mathcal{D}(F)$, the unique maximal order in D over V_F is precisely the valuation ring V_D.

For an arbitrary valuation w on a field F, there is a well-defined *Henselization* (w^h, F^h) of (w, F). The field F^h is an algebraic extension of F (usually of infinite degree), and w^h is a Henselian valuation on F^h with $w^h|_F = w$. Moreover, w^h is an immediate extension of w, i.e., $\overline{F^h} = \overline{F}$ and $\Gamma_{F^h} = \Gamma_F$. ("The" Henselization can be constructed as follows, see [E, p. 132, Cor. 17.12]: Let $\widetilde{w}$ be any extension of w to the separable closure F_{sep} of F, and let F^h be the decomposition field of $\widetilde{w}$ over w (i.e., the fixed field of subgroup $\{\sigma \mid \widetilde{w} \circ \sigma = \widetilde{w}\}$ of the Galois group $\mathcal{G}(F_{\text{sep}}/F)$) and let $w^h = \widetilde{w}|_{F^h}$. The Henselization depends on the choice of $\widetilde{w}$, but is unique up to

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*Recall that the rank of a valuation w on a field F is defined to be the Krull dimension of its valuation ring V_F. So, w has rank 1 iff M_F is the only nonzero prime ideal of V_F, iff Γ_F is isomorphic to a subgroup of the additive group of $\mathbb{R}$. The valuation w is discrete rank 1 just when Γ_F maps to a discrete subgroup of $\mathbb{R}$, i.e., just when $\Gamma_F \cong \mathbb{Z}$.

isomorphism.) The Henselization often plays the rôle for general valuations that the completion plays for valuations of rank 1. When w has rank 1, a Henselization F^h is obtainable as the separable closure of F in the completion $\widehat{F}$ of F with respect to w; then $w^h = \widehat{w}|_{F^h}$ (cf. [E, p. 135, Th. 17.18]). There is a criterion for extendability of a valuation from the center in terms of the Henselization:

Theorem 2.3 *Let F^h be the Henselization of a valuation w on a field F, and let $D \in \mathcal{D}(F)$. Then, w extends to a valuation on D iff $D \otimes_F F^h$ is a division ring. When this occurs, $\overline{D \otimes_F F^h} = \overline{D}$ and $\Gamma_{D\otimes_F F^h} = \Gamma_D$, and the valuation on D is the restriction of the valuation on $D \otimes_F F^h$ extending w^h on F^h.*

Th. 2.3 was first proved by Morandi in [M_1, Th. 2]. Another proof can be found in [Er_4, Prop. 3], and yet another, more ring theoretic, in [MMU_2, p. 43, Cor. 8.5]. Th. 2.3 holds also when w has rank 1 with F^h replaced by the completion $\widehat{F}$. The rank 1 version of Th. 2.3 using the completion was proved earlier by Cohn [C_1, Th. 1]. This theorem makes possible a useful approach to proving properties of valued division algebras, namely first to prove the property when the valuation on the center is Henselian.

Examples 2.4 (i) For any prime number p, let w_p denote the p-adic discrete valuation on the rational numbers $\mathbb{Q}$, whose valuation ring is the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$, and let $\widehat{\mathbb{Q}}_p$ be the p-adic local field, which is the completion of w_p on $\mathbb{Q}$. Let A be the quaternion algebra $A = \big(\frac{-1,-11}{\mathbb{Q}}\big)$ ($= A_{-1}(-1,-11;\mathbb{Q})$ in the notation of (0.3)). Then $A \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_{11}$ is a division algebra (as -1 is not a square mod 11), while $A \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_p$ is split for $p \neq 11$. Hence, w_{11} extends to a valuation on A (and one can check that $\Gamma_A = \frac{1}{2}\mathbb{Z}$ when we take $\Gamma_{\mathbb{Q}} = \mathbb{Z}$ for w_{11}), and $\overline{A} = \mathbb{Z}/11\mathbb{Z}(\sqrt{-1})$. Moreover, this is the only valuation on A, since the p-adic valuations are the only valuations on $\mathbb{Q}$, and for $p \neq 11$ we have that w_p does not extend to A.

(ii) Let $\omega \in \mu_9^*(\mathbb{C})$, i.e., ω is a primitive 9-th root of unity in $\mathbb{C}^*$, and let $L = \mathbb{Q}(\omega + \omega^{-1})$, which is a cyclic Galois field extension of $\mathbb{Q}$ of degree 3, say with σ a generator of $\mathcal{G}(L/\mathbb{Q})$. Let B be the cyclic algebra $B = (L/\mathbb{Q}, \sigma, 5)$ (see (0.3) for the notation). Then, $B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_5$ is a division ring, since $[L \cdot \widehat{\mathbb{Q}}_5 : \widehat{\mathbb{Q}}_5] = 3$ (as $[\widehat{\mathbb{Q}}_5(\omega) : \widehat{\mathbb{Q}}_5] = [\mathbb{Q}(\omega) : \mathbb{Q}\,] = 6$) and 5 is not a norm from $L\cdot\widehat{\mathbb{Q}}_5$ to $\widehat{\mathbb{Q}}_5$ (as $w_5(5) = 1$ and $\Gamma_{L\cdot\widehat{\mathbb{Q}}_5} = \Gamma_{\widehat{\mathbb{Q}}_5} = \mathbb{Z}$). (Or, invoke Ex. 2.7 below.) But, $B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_{11}$ is split (as $\sqrt[3]{5} \in \widehat{\mathbb{Q}}_{11}$ by Hensel's Lemma). Now, let $D = A \otimes_{\mathbb{Q}} B$, with the A of part (i). So, D is a division ring, since this is true for A and B, which have relatively prime degrees. But $D \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_p$ is not a division ring for any prime p. Thus, there is no valuation on D at all.

(iii) For any noncommutative $D \in \mathcal{D}(\mathbb{Q})$, it is known from class-field theory that $D \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_p$ is split for all but finitely many primes p (see [Pi, p. 358, Prop.]). So there are at most finitely many valuations on D. Of course the same is true if we replace the ground field $\mathbb{Q}$ by any algebraic number field or by any algebraic function field in one variable over a finite field (cf. [Re, Th. 25.7]).

(iv) Let F be any field, and let $D \in \mathcal{D}(F)$. Consider the division ring $D(t) = D \otimes_F F(t) \in \mathcal{D}(F(t))$ where t is transcendental over F. For each monic irreducible $f \in F[t]$ there is the f-adic discrete rank 1 valuation v_f on $F(t)$ with valuation ring $F[t]_{(f)}$; let $\widehat{F(t)}_f$ be the completion of $F(t)$ with respect to v_f. Then, one can see that v_f extends to a valuation on $D(t)$ iff $D(t) \otimes_{F(t)} \widehat{F(t)}_f$ is a division ring (by the complete version of Th. 2.3), iff $D \otimes_F (F[t]/(f))$ is a division ring (see Prop. 2.8 below for "if"). When this occurs, $\overline{D(t)} \cong D \otimes_F (F[t]/(f))$ and $\Gamma_{D(t)} = \Gamma_{F(t)} = \mathbb{Z}$. This holds, for example, whenever $\gcd(\deg(D), \deg(f)) = 1$.

Further constructions of valued division algebras will be given at the end of this section.

Now, let us return to the general setting of a division algebra $D \in \mathcal{D}(F)$, with a valuation v on D and associated structures as given in (2.2)–(2.6). Let E be an F-subalgebra of D and consider the relationship between v on D and $v|_E$ on E. Since clearly $M_D \cap V_E = M_E$, the canonical mapping $\overline{E} \to \overline{D}$ is injective, and we view $\overline{E}$ as a subalgebra of $\overline{D}$. Then, $[\overline{D} : \overline{E}]$ (the left, and right, dimension of $\overline{D}$ as an $\overline{E}$-vector space) is the *residue degree* of D over E with respect to v. Also, Γ_E is a subgroup of Γ_D, and the group index $|\Gamma_D : \Gamma_E| = |\Gamma_D/\Gamma_E|$ is the *ramification index* of D over E. The same argument as in the commutative case yields the "fundamental inequality,"

$$[\overline{D} : \overline{E}]|\Gamma_D : \Gamma_E| \;\le\; [D : E] \;<\; \infty, \tag{2.9}$$

so both the ramification index and the residue degree are finite. When we take $E = F$, the inequality takes a more exact form given by an Ostrowski-type "defect theorem," which says that if we define the *defect* of D, $\delta(D)$, by

$$[D : F] \;=\; [\overline{D} : \overline{F}]\, |\Gamma_D : \Gamma_F|\, \delta(D)\,, \tag{2.10}$$

then, $\delta(D)$ is a positive integer and

$$\delta(D) \;=\; \overline{\rho}^{\,c}\,, \tag{2.11}$$

where c is a nonnegative integer and $\overline{\rho}$ is the "characteristic exponent" of $\overline{F}$, i.e., $\overline{\rho} = p$ if $\operatorname{char}(\overline{F}) = p > 0$ and $\overline{\rho} = 1$ if $\operatorname{char}(\overline{F}) = 0$. Formula (2.11) was proved by Draxl in [D_3, Th. 2] if $v|_F$ is Henselian and deduced in general by Morandi in [M_1, Th. 3], by invoking Th. 2.3 above. The valuation v is said to be "defectless" if $\delta(D) = 1$. Clearly, v is defectless if $\operatorname{char}(\overline{F}) = 0$. It is also known that if $v|_F$ is discrete rank 1, then v is defectless; see [M_6, p. 359] for a short proof of this using Formaneck's theorem; a different proof is given in [TY_2, Prop. 2.2]. However, if $\operatorname{char}(\overline{F}) = p > 0$ and $v|_F$ is not discrete rank 1, then the defect can take any value up to the p part of $[D : F]$. See [TY_2] and [M_6] for studies of the possible values of $\delta(D)$ and examples of defective division algebras.

Ostrowski's original defect formula was given in [O, p. 355, Satz IV] for a finite degree (commutative) field extension of a field with rank 1 Henselian valuation.

While we have stated the defect formula (2.11) for $F = Z(D)$, observe that the formula holds also when F is replaced by any F-subalgebra E of D. This follows easily from the fundamental inequality (2.9) together with the defect equality for D over F and for E over $Z(E)$ together with the corresponding equality in the commutative case for $Z(E)$ over F (see [E, p. 170, Th. 20.21]).

For our valued division algebra $D \in \mathcal{D}(F)$, we define the *relative value group* of the valuation v on D to be

$$\Lambda_D \;=\; \Gamma_D/\Gamma_F. \tag{2.12}$$

This finite abelian group can provide information on the subfields of D (see, e.g., Th. 4.5(c) below). It is also related to the center of $\overline{D}$. Since $F = Z(D)$, clearly $Z(\overline{D})$ contains $\overline{F}$, but the inclusion is often strict. However, $Z(\overline{D})$ cannot be an arbitrary extension field of $\overline{F}$. For, given any $a \in D^*$, conjugation by a maps V_D

to itself, so its unique maximal ideal M_D also goes to itself, yielding an induced automorphism of $\overline{D}$ hence of $Z(\overline{D})$. Conjugation by any element of U_D or of F^* is trivial on $Z(\overline{D})$. Thus, there is a well-defined induced group homomorphism

$$\theta_D : \Lambda_D \to \mathcal{G}(Z(\overline{D})/\overline{F}) \quad \text{given by} \quad v(a) + \Gamma_F \mapsto (\overline{z} \mapsto \overline{aza^{-1}}). \tag{2.13}$$

Proposition 2.5 *For any valued field $D \in \mathcal{D}(F)$, the field $Z(\overline{D})$ is normal over $\overline{F}$, and the map θ_D of* (2.13) *is surjective. Hence, if $Z(\overline{D})$ is separable over $\overline{F}$ then $Z(\overline{D})$ is abelian Galois over $\overline{F}$.*

See [JW$_2$, Prop. 1.7] for a proof of Prop. 2.5. It follows, for example, that when $v|_F$ is discrete rank 1, then the separable closure of $\overline{F}$ in $Z(\overline{D})$ is cyclic Galois over $\overline{F}$. Prop. 2.5 shows that we have a short exact sequence

$$0 \longrightarrow \ker(\theta_D) \longrightarrow \Lambda_D \longrightarrow \mathcal{G}(Z(\overline{D})/\overline{F}) \longrightarrow 0\,. \tag{2.14}$$

There is also some further structure to $\ker(\theta_D)$, at least away from $\operatorname{char}(\overline{F})$. For this, let $\ker(\theta_D)'$ denote the prime-to-$\operatorname{char}(\overline{F})$ part of $\ker(\theta_D)$. By this we mean that $\ker(\theta_D)' = \ker(\theta_D)$ if $\operatorname{char}(\overline{F}) = 0$. But, if $\operatorname{char}(\overline{F}) = p > 0$, then $\ker(\theta_D)'$ is the subgroup of the abelian torsion group $\ker(\theta_D)$ consisting of all the elements of order prime to p.

Proposition 2.6 *There is a nondegenerate symplectic pairing on* $\ker(\theta_D)'$ *with values in the group of roots of unity $\mu(\overline{F})$. Hence,* $\ker(\theta_D)' \cong A \times A$ *for some abelian group A, and $\overline{F}$ contains a primitive e-th root of unity, where* $e = \exp(\ker(\theta_D)')$.

See [JW$_2$, Th. 1.10, Remark 1.13, Remark 1.14] for a proof of Prop. 2.6. We will see in §4 below how the pairing on $\ker(\theta_D)$ arises.

We will look more closely at the structure of valued division algebras when $v|_F$ is Henselian in the next section. But before turning to that, we give a couple of basic constructions of valued division algebras which can be used to build numerous further examples of such algebras. The spirit of both Ex. 2.7 and Prop. 2.8 is that if we have "sufficiently separate" valuations on two different pieces of a division algebra, then we can combine the valuations on the pieces to obtain a valuation on the whole of the division algebra.

Example 2.7 Let L be a cyclic Galois extension of a field F, and let σ be a generator of $\mathcal{G}(L/F)$. Let w be a valuation on L such that $w \circ \sigma = w$ (i.e., w is the unique extension of $w|_F$ to L). Let $n = [L:F]$. Suppose there is $b \in F^*$ such that the image of $w(b)$ has order n in $\Gamma_L/n\Gamma_L$. Then w extends (uniquely) to a valuation v on the cyclic division algebra $C = (L/F, \sigma, b)$ (see notation (0.3)). If y is the standard generator of C, such that $y^n = b$ and $y\ell y^{-1} = \sigma(\ell)$ for all $\ell \in L$, then v is given by, for $\ell_i \in L$, not all equal to zero,

$$v\big(\sum_{i=0}^{n-1} \ell_i y^i\big) \;=\; \min_{0 \le i \le n-1} \big\{ w(\ell_i) + \tfrac{i}{n} w(b) \mid \ell_i \neq 0 \big\}\,. \tag{2.15}$$

We have $\overline{C} = \overline{L}$ and $\Gamma_C = \Gamma_L + \langle \frac{1}{n} w(b) \rangle$ in the divisible hull Δ_L of Γ_L. Here, $\overline{L}$ is a field cyclic Galois over $\overline{F}$ with $\mathcal{G}(\overline{L}/\overline{F})$ generated by the image $\overline{\sigma}$ of σ. The map θ_C of (2.13) sends $\frac{1}{n} w(b) + \Gamma_F$ to $\overline{\sigma}$, and $\ker(\theta_C) = \big(\big\langle \frac{[\overline{L}\,:\,\overline{F}]}{n} w(b) \big\rangle + \Gamma_L\big)/\Gamma_F$.

We indicate how the assertions in Ex. 2.7 can be verified, since the argument is a prototype of an approach that is often used in building valuations on division algebras. We need to see that C is a division algebra and that formula (2.15) gives a valuation on C. In any case formula (2.15) does define a function $v : C-\{0\} \to \frac{1}{n}\Gamma_L$, and it is easy to check that

$$v(a+b) \geq \min(v(a), v(b)), \quad \text{for all } a, b \in C-\{0\}, \text{ with } b \neq -a, \tag{2.16}$$

since this is true for w on L. Since $v(a) = v(-a)$, it follows as usual from (2.16) that

$$v(a+b) \;=\; \min(v(a), v(b)), \quad \text{whenever } v(b) \neq v(a). \tag{2.17}$$

A short calculation shows that for any $\ell, m \in L^*$ and any $i, j \in \mathbb{Z}$,

$$v((\ell y^i)(my^j)) \;=\; v(\ell y^i) + v(my^j), \tag{2.18}$$

since $\ell y^i m y^j = \ell\sigma^i(m)y^{i+j}$ and $w(\sigma^i(m)) = w(m)$. From (2.18) and (2.16), it follows easily that for any $a, b \in C$ with $ab \neq 0$,

$$v(ab) \geq v(a) + v(b). \tag{2.19}$$

Note further that for any nonzero $a = \sum\limits_{i=0}^{n-1} \ell_i y^i \in C$ (with $\ell_i \in L$), we have $v(a) = \min\{v(\ell_i y^i) \mid \ell_i \neq 0\}$, and each nonzero summand $\ell_i y^i$ has a different value because the values of the y^i are distinct modulo Γ_L. Call the summand $\ell_j y^j$ of least value the "leading term" of a. So, $a = \ell_j y^j + a'$, where $v(a) = v(\ell_j y^j)$ and $v(a') > v(a)$ or $a' = 0$. Now take a second nonzero element $b = \sum\limits_{i=0}^{n-1} m_i y^i$ of C, say with leading term $m_k y^k$; so $b = m_k y^k + b'$ with $v(b) = v(m_k y^k)$ and $v(b') > v(b)$ or $b' = 0$. Then,

$$ab \;=\; \ell_j y^j m_k y^k + r,$$

where r is a sum of terms each of which is zero or by (2.19) has value strictly larger than $v(\ell_j y^j m_k y^k)$. So, by (2.16), if $r \neq 0$, then $v(r) > v(\ell_j y^j m_k y^k)$. Consequently, $r \neq -\ell_j y^j m_k y^k$, which shows that $ab \neq 0$. Thus, the central simple F-algebra C has no zero divisors, showing that it is a division ring. Furthermore, by (2.17) and (2.18), we have

$$v(ab) \;=\; v(\ell_j y^j m_k y^k + r) \;=\; v(\ell_j y^j m_k y^k) \;=\; v(a) + v(b),$$

proving, with (2.16), that v is a valuation on C. The other assertions about v in Ex. 2.7 now follow at once since it is easy to determine V_C and M_C from (2.15).

Proposition 2.8 *Let F be a field, let $D_1 \in \mathcal{D}(F)$, and let D_2 be a division ring with $Z(D_2) \supseteq F$. Suppose there are valuations w_1 on D_1 and w_2 on D_2 such that $w_1|_F = w_2|_F$ and w_1 is defectless. Suppose also that $\overline{D_1} \otimes_{\overline{F}} \overline{D_2}$ is a division ring and that $\Gamma_{D_1} \cap \Gamma_{D_2} = \Gamma_F$. Then, $D_1 \otimes_F D_2$ is a division ring (with center $Z(D_2)$) and there is a unique valuation v on $D_1 \otimes_F D_2$ such that $v|_{D_i} = w_i$ for $i = 1, 2$. Also, $\overline{D_1 \otimes_F D_2} \cong \overline{D_1} \otimes_{\overline{F}} \overline{D_2}$ and $\Gamma_{D_1 \otimes_F D_2} = \Gamma_{D_1} + \Gamma_{D_2}$ (in the divisible hull of Γ_{D_2}).*

Prop. 2.8 appears in the paper [M$_1$], where a full proof can be found, see [M$_1$, Th. 1]. The proof is along the same general lines as for Ex. 2.7 above, but there are considerable technical complications. We here merely describe the valuation on $D_1 \otimes_F D_2$ in Prop. 2.8. For this, choose any $b_1, \ldots, b_n \in U_{D_1}$ which map to an $\overline{F}$-base of $\overline{D_1}$, and choose any $c_1, \ldots, c_m \in D_1^*$ whose values are a set of coset representatives of Γ_{D_1}/Γ_F. Then, $\{c_i b_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is an F-base of

D_1, since w_1 is defectless. Every element a of $D_1 \otimes_F D_2$ is expressible uniquely as $a = \sum_{i=1}^{m} \sum_{j=1}^{n} c_i b_j \otimes \ell_{ij}$, with the $\ell_{ij} \in D_2$. Then v is given by (for $a \neq 0$)

$$v(a) \ = \ \min\{w_1(c_i) + w_2(\ell_{ij}) \mid \ell_{ij} \neq 0\}.$$

Note that in Prop. 2.8 the division algebra D_2 need not be finite-dimensional over F, and need not be noncommutative. For example, we could take D_2 to be the Henselization F^h of F with respect to $w_1|_F$. Since $\overline{F^h} = \overline{F}$ and $\Gamma_{F^h} = \Gamma_F$, Prop. 2.8 shows that whenever the valuation on D_1 is defectless over F, we have that $D_1 \otimes_F F^h$ is a division ring. This gives a rather easy way of seeing the nontrivial implication in Th. 2.3 above in the defectless case.

A generalization of Prop. 2.8 relaxing the requirement that $\overline{D_1} \otimes_{\overline{F}} \overline{D_2}$ be a division ring is given in Ex. 10.7 below.

The following corollary follows immediately from Prop. 2.8 and the defect formula (2.11):

Corollary 2.9 *Let F be a field with a valuation w, and let $D_1, D_2 \in \mathcal{D}(F)$ with $\gcd(\deg(D_1), \deg(D_2)) = 1$. If w extends to D_1 and to D_2, then w extends to $D_1 \otimes_F D_2$, with $\overline{D_1 \otimes_F D_2} \cong \overline{D_1} \otimes_{\overline{F}} \overline{D_2}$ and $\Gamma_{D_1 \otimes_F D_2} = \Gamma_{D_1} + \Gamma_{D_2}$.*

3 Division algebras over Henselian fields

Let F be a field with a Henselian valuation v. Then, as noted in §2, since v has a unique extension to each field algebraic over F, v also has a unique extension to each division algebra finite dimensional over F. We will describe in this section some of the particular types of valued division algebras over F, and their special properties. We also describe portions of the Brauer group of F. Indeed there is a canonical filtration on $\mathrm{Br}(F)$ induced by v:

$$\mathrm{IBr}(F) \subseteq \mathrm{SBr}(F) \subseteq \mathrm{TBr}(F) \subseteq \mathrm{Br}(F).$$

We will consider each of these pieces of $\mathrm{Br}(F)$ in turn—see (3.6), (3.8), and (3.18) below for the definitions. Most of the results given in this section about division algebras over F and $\mathrm{Br}(F)$ can be found, with proofs, in [JW_2]. Some things in this section appear also in [PY_4] and [PY_5], which the reader might wish to consult for a somewhat different perspective.

Before looking at division algebras, let us recall some of the basic examples of Henselian valued fields, and a few of their basic properties. Of course, the classical example is that a complete discrete rank 1 valuation on a field is Henselian. This includes the non-Archimedean local fields of number theory, for which Hensel proved Hensel's Lemma. Another basic example is the Laurent series field $F((x)) = \Big\{ \sum_{i=k}^{\infty} a_i x^i \mid k \in \mathbb{Z}, \text{all } a_i \in F \Big\}$ in an indeterminate x over a ground field F. It is well-known that $F((x))$ is complete with respect to the discrete rank 1 x-adic valuation v given by $v\Big(\sum_{i=k}^{\infty} a_i x^i \Big) = \min\{ i \mid a_i \neq 0\}$. Indeed, $F((x))$ is the completion of the rational function field $F(x)$ with respect to the x-adic valuation whose valuation ring is the localization $F[x]_{(x)} = \{f/g \mid f, g \in F[x],\ g(0) \neq 0\}$. For the valuation on $F((x))$ the associated valuation ring $V_{F((x))}$ is the formal power series ring $F[[x]]$, the residue field is $\overline{F((x))} \cong F$, and the value group is $\Gamma_{F((x))} = \mathbb{Z}$.

There is a generalization of $F((x))$ which we will see frequently below: For any natural number n, let $x_1, \ldots, x_n$ be n independent indeterminates over a field F, and let $F((x_1))\cdots((x_n))$ be the *n-fold iterated Laurent series field* over F. This can be defined inductively by setting $F_0 = F$, $F_1 = F_0((x_1)), \ldots, F_i = F_{i-1}((x_i)), \ldots$; then

$$F((x_1))\cdots((x_n)) \;=\; F_n. \tag{3.1}$$

We can describe the elements of F_n as suitable Laurent series in the x_i: On $\prod_{i=1}^{n}\mathbb{Z}$ define a total ordering by

$$\begin{array}{l}(i_1,\ldots,i_n) < (j_1,\ldots,j_n) \text{ just when there is a } k \text{ such that}\\ i_k < j_k \text{ and } i_\ell = j_\ell \text{ for all } \ell > k.\end{array} \tag{3.2}$$

This is the right-to-left lexicographic ordering on $\mathbb{Z}^n$. Then,

$\}.(3.4)$

well-ordered subset of $(\mathbb{Z}^n, \leq)$

$\}.$

Note that $F((x_1))\cdots((x_n))$ is *not* the quotient field of the iterated formal power series ring $F[[x_1,\ldots,x_n]]$, and that $F((x_1))\cdots((x_n))$ is not symmetric in the order of the x_i. For example, $F((x_1))((x_2))$ contains $\sum_{i=0}^{\infty} x_1^{-i}x_2^{i}$, but not $\sum_{i=0}^{\infty} x_1^{i}x_2^{-i}$. There is a valuation v on $F((x_1))\cdots((x_n))$ with value group $\mathbb{Z}^n$, ordered as in (3.2), given by

$$v\Big(\sum_{i_1}\cdots\sum_{i_n} c_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}\Big) \;=\; \min\{(i_1,\ldots,i_n) \mid c_{i_1\cdots i_n} \;\neq\; 0\}. \tag{3.5}$$

So, v has rank n, with $\Gamma_{F((x_1))\cdots((x_n))} = \mathbb{Z}^n$ and $\overline{F((x_1))\cdots((x_n))} = F$. We will call v the *standard valuation* on $F((x_1))\cdots((x_n))$. The important fact we need is that this standard valuation is Henselian, as can be seen very easily by viewing v in terms of composite valuations. This is worth recalling briefly:

Let w be a valuation on a field K with valuation ring W and residue field $\overline{W}$, and let $\pi : W \to \overline{W}$ be the canonical epimorphism. Let y be a valuation on $\overline{W}$ with valuation ring Y and residue field $\overline{Y}$. Then, let $V = \pi^{-1}(Y) \subseteq K$. It is easy to see that V is a valuation ring with quotient field K; the valuation v associated to V is called the *composite valuation* of w and y. (More accurately, the place associated to v is the composition of the places associated to w and y.) Note that if $P = \ker(\pi)$, the maximal ideal of W, then P is a prime ideal of V, with $V/P \cong Y$ and localization $V_P = W$. Also, the prime ideals of V are the prime ideals of W together with the $\pi^{-1}(Q)$, where Q is a prime ideal of Y. So, the Krull dimension of V is the sum of that W and that of Y. Also, the residue field $\overline{V} \cong \overline{Y}$, and for the value groups we have a canonical short exact sequence

$$0 \longrightarrow \Gamma_y \longrightarrow \Gamma_v \longrightarrow \Gamma_w \longrightarrow 0\,.$$

Most relevantly here, there is also good behavior regarding the Henselian property:

Proposition 3.1 *For a composite valuation v of w and y as above, v is Henselian iff w and y are each Henselian.*

See, e.g., [R_2, p. 211, Prop. 10] for a proof of Prop. 3.1. Indeed, this follows easily from the characterization of Henselian valuations by the uniqueness of their extensions to algebraic field extensions of the ground field.

Now, returning to the iterated Laurent series field $F((x_1))\cdots((x_n)) = F_n$, where $F_0 = F$ and $F_i = F_{i-1}((x_i))$. We have the usual complete discrete rank 1 (so Henselian) x_1-adic valuation on $F_1 = F((x_1))$. On $F_2 = F_1((x_2))$ we have the complete discrete and rank 1 x_2-adic valuation whose residue field is F_1. We compose this valuation with the x_1-adic valuation on F_1, to obtain the standard valuation on $F_2 = F((x_1))((x_2))$ as described in (3.5) with $n = 2$. It is Henselian by Prop. 3.1. The standard valuation on $F((x_1))\cdots((x_n))$ is obtained by iterating this process, so it too is Henselian.

One can also obtain a Henselian valued field by Henselization, starting out from any valuation on a field (cf. §2). But for most of the specific examples we consider in the rest of this paper, it will suffice to consider Henselian valued fields like $F((x_1))\cdots((x_n))$. However, it is important to allow $n > 1$, since some interesting phenomena only show up when the value group has rational rank at least 2—for example, the tame and totally ramified division algebras studied in §4 below.)

Now, let us consider any field F with a Henselian valuation v. We recall some of the basic facts and terminology about extension fields of F. This will set the stage for the appropriate analogues for division rings. Recall that a finite degree field extension L of F is defined to be *unramified over* F (with respect to the unique extension of v to L) if $[\overline{L} : \overline{F}] = [L : F]$ and $\overline{L}$ is separable over $\overline{F}$ (hence L is separable over F). If L is algebraic over F of infinite degree, then L is defined to be unramified over F if for each field L_0 with $F \subseteq L_0 \subseteq L$ and $[L_0 : F] < \infty$ we have L_0 is unramified over F. It is well known that within a fixed algebraic closure F_{alg} of F there is a unique maximal unramified extension F_{nr} of F; for every field L with $F \subseteq L \subseteq F_{\text{alg}}$, L is unramified over F iff $L \subseteq F_{\text{nr}}$. Indeed, F_{nr} is the inertia field of the separable closure F_{sep} of F over F (with respect to the unique extension of v to F_{sep}). Recall further that F_{nr} is a Galois extension of F (typically of infinite degree) with $\overline{F_{\text{nr}}} \cong \overline{F}_{\text{sep}}$; also, $\Gamma_{F_{\text{nr}}} = \Gamma_F$. Each F-automorphism σ of F_{nr} maps $V_{F_{\text{nr}}}$ to itself (as $V_{F_{\text{nr}}}$ is the unique extension of V_F to F_{nr}); hence, σ induces an $\overline{F}$-automorphism $\overline{\sigma} : \overline{F_{\text{nr}}} \to \overline{F_{\text{nr}}}$. The mapping $\sigma \mapsto \overline{\sigma}$ is a continuous isomorphism $\mathcal{G}(F_{\text{nr}}/F) \to G_{\overline{F}}$, where $G_{\overline{F}} = \mathcal{G}(\overline{F}_{\text{sep}}/\overline{F})$ is the absolute Galois group of $\overline{F}$. Likewise, the map $L \mapsto \overline{L}$ gives a one-to-one degree-preserving and inclusion-preserving correspondence between the unramified field extensions L of F in F_{alg} and the separable algebraic extensions $\overline{L}$ of $\overline{F}$ in $\overline{F}_{\text{sep}}$. We call L the *inertial lift* of $\overline{L}$ over F. If a field K is any finite degree extension of F and $\widetilde{L}$ is the separable closure of $\overline{F}$ in $\overline{K}$, then K contains a copy of the inertial lift of $\widetilde{L}$ over F. The assertions in this paragraph are all well known, and are given in or easily deducible from the results in [E, §19].

There is a corresponding notion of unramified division algebras. For our Henselian valued field F, let D be a division algebra finite-dimensional over F (we allow $Z(D) \supsetneqq F$). We say that D is *unramified* or *inertial* over F if, with respect to the unique extension of v to D, we have $[\overline{D} : \overline{F}] = [D : F]$ and $Z(\overline{D})$ is separable over $\overline{F}$. The fundamental inequality (2.9) then shows that $\Gamma_D = \Gamma_F$. Moreover, it is known that $\overline{Z(D)} = Z(\overline{D})$ (cf. [JW_2, Lemma 2.2]). We have the following characterizations of inertial division algebras with center F. See [JW_2, Ex. 2.4(ii), Prop. 2.5] for a proof:

Theorem 3.2 *Let F be a Henselian valued field, and let $D \in \mathcal{D}(F)$. Then the following are equivalent:*

(i) *D is inertial over F.*
(ii) *$[\overline{D} : \overline{F}] = [D : F]$ and $Z(\overline{D}) = \overline{F}$.*
(iii) *V_D is an Azumaya algebra over V_F.*
(iv) *There is an Azumaya algebra A over V_F with $A \otimes_{V_F} F \cong M_n(D)$ for some n.*

We can now define the inertial part of the Brauer group of F: Set

$$\mathrm{IBr}(F) \;=\; \{\,[D] \in \mathrm{Br}(F) \mid D \in \mathcal{D}(F) \text{ and } D \text{ is inertial over } F\}. \tag{3.6}$$

Note that $\mathrm{IBr}(F)$ is a subgroup of $\mathrm{Br}(F)$. Indeed, Th. 3.2 (iv) shows that $\mathrm{IBr}(F)$ is the image of the canonical group homomorphism $\alpha : \mathrm{Br}(V_F) \to \mathrm{Br}(F)$ given by $[A] \mapsto [A \otimes_{V_F} F]$. This map α is known to be injective (this is true even if the valuation ring V_F is not Henselian, see [JW_2, Prop. 2.5] or [Sa_5, Lemma 1.2]). But further, because our V_F is Henselian, it is known that the canonical map $\beta : \mathrm{Br}(V_F) \to \mathrm{Br}(\overline{F})$ given by $[A] \mapsto [A \otimes_{V_F} \overline{F}]$ is an isomorphism (cf. [JW_2, pp. 140–141]). This was proved by Azumaya in [Az, Th. 31]. (Azumaya's argument for the surjectivity of β is similar to the one given by Nakayama in the complete discrete case in [Na, Satz 1].) Thus, by using the map $\beta \circ \alpha^{-1}$, we have

$$\mathrm{IBr}(F) \;\cong\; \mathrm{Br}(\overline{F}) \quad \text{by the map } [D] \mapsto [\overline{D}], \tag{3.7}$$

an index-preserving group isomorphism. Moreover, if K is any field containing F, and K has a Henselian valuation w with $w|_F = v$ (e.g., if $[K : F] < \infty$), then the scalar extension map $\mathrm{Br}(F) \to \mathrm{Br}(K)$ sends $\mathrm{IBr}(F)$ into $\mathrm{IBr}(K)$. This is clear from Th. 3.2 (iv), since $\mathrm{Br}(V_F)$ maps to $\mathrm{Br}(V_K)$.

Unramified division algebras are thus completely determined by their residue division algebras. But it is important to know further that they exist inside other division algebras whenever possible, just as in the commutative case:

Theorem 3.3 *Let F be a Henselian valued field, and let $\widetilde{E}$ be a division algebra finite dimensional over $\overline{F}$ with $Z(\widetilde{E})$ separable over $\overline{F}$. Then,*

(a) *there is a unique up to isomorphism division algebra E over F such that E is unramified over F and $\overline{E} \cong \widetilde{E}$. This E is called the* inertial lift *of $\widetilde{E}$ over F.*
(b) *If B is any division algebra finite-dimensional over F and $\widetilde{E}$ is an $\overline{F}$-subalgebra of $\overline{B}$, then B contains a copy of the inertial lift of $\widetilde{E}$.*

See [JW_2, Th. 2.8(a), Th. 2.9] for a proof of Th. 3.3. Part (a) actually follows easily from the isomorphism (3.7) above, but part (b) takes more work to prove.

For our Henselian field F, we have seen that the inertial part of $\mathrm{Br}(F)$ is completely determined by $\overline{F}$. But $\mathrm{IBr}(F)$ is usually only a small piece of $\mathrm{Br}(F)$. In the rest of $\mathrm{Br}(F)$ there is a significant interplay between $\overline{F}$ and Γ_F. This shows up already in the algebras split by the maximal unramified extension F_{nr} of F, which we now consider. If F_{nr} splits $[D] \in \mathrm{Br}(F)$, we say that D is *inertially split*. Let

$$\mathrm{SBr}(F) \;=\; \{\,[D] \in \mathrm{Br}(F) \mid D \text{ is inertially split}\} \;=\; \mathrm{Br}(F_{\mathrm{nr}}/F). \tag{3.8}$$

Clearly $\mathrm{SBr}(F)$ is a subgroup of $\mathrm{Br}(F)$. Moreover, $\mathrm{IBr}(F) \subseteq \mathrm{SBr}(F)$. For, if $D \in \mathcal{D}(F)$ and D is unramified over F, then D contains as a maximal subfield the inertial lift L of a maximal subfield $\widetilde{L}$ of $\overline{D}$ such that $\widetilde{L}$ is separable over $\overline{F}$; since $L \subseteq F_{\mathrm{nr}}$ and L splits D, F_{nr} splits D.

We have the following characterizations of the division algebras in $\mathrm{SBr}(F)$ (see [JW$_2$, Lemma 5.1]):

Theorem 3.4 *Let F be a Henselian valued field, and let $D \in \mathcal{D}(F)$. Then the following are equivalent:*

(i) *D is inertially split.*
(ii) *D has a maximal subfield L with L unramified over F.*
(iii) *$Z(\overline{D})$ is separable (hence abelian Galois by Prop. 2.5) over $\overline{F}$, the map $\theta_D : \Gamma_D/\Gamma_F \to \mathcal{G}(Z(\overline{D})/\overline{F})$ of* (2.13) *is an isomorphism, and $[D:F] = [\overline{D}:\overline{F}]|\Gamma_D:\Gamma_F|$.*

There is a homological characterization of $\mathrm{SBr}(F)$, which was first given by Scharlau in [Sch]. It is analogous to the classical homological description of the Brauer group of a field with complete discrete rank 1 valuation, which is very well presented in [S, Ch. XII, §3]. For the general Henselian case, let $G = \mathcal{G}(F_{\mathrm{nr}}/F) \cong G_{\overline{F}}$, the absolute Galois group of $\overline{F}$, a profinite group. There is a short exact sequence of discrete G-modules given by the valuation v (since $\Gamma_{F_{\mathrm{nr}}} = \Gamma_F$)

$$1 \longrightarrow U_{F_{\mathrm{nr}}} \longrightarrow F_{\mathrm{nr}}^* \xrightarrow{\ v\ } \Gamma_F \longrightarrow 0\,, \tag{3.9}$$

which leads to the following part of the long exact sequence of continuous cohomology groups:

$$H_c^1(G,\Gamma_F) \longrightarrow H_c^2(G,U_{F_{\mathrm{nr}}}) \xrightarrow{\ \gamma'\ } H_c^2(G,F_{\mathrm{nr}}^*) \xrightarrow{\ \varepsilon'\ } H_c^2(G,\Gamma_F)\,. \tag{3.10}$$

(The subscript c indicates continuous cohomology.) Each of the terms in (3.10) has a nice interpretation, which we indicate here; see [JW$_2$, pp. 155–158] for proofs. Since G acts trivially on Γ_F, we have $H_c^1(G,\Gamma_F) = \mathrm{Hom}_c(G,\Gamma_F)$ (continuous homomorphisms), and this group is trivial, since G is profinite and Γ_F is torsion-free. Because F is Henselian, it is known that the map $H_c^2(G,U_{F_{\mathrm{nr}}}) \to H_c^2(G,\overline{F_{\mathrm{nr}}}^*)$ induced by the canonical projection $U_{F_{\mathrm{nr}}} \to \overline{F_{\mathrm{nr}}}^*$ is an isomorphism. Hence, as $\overline{F_{\mathrm{nr}}} \cong \overline{F}_{\mathrm{sep}}$, we have, using (3.7), $H_c^2(G,U_{F_{\mathrm{nr}}}) \cong \mathrm{Br}(\overline{F}) \cong \mathrm{IBr}(F)$. For the next term in (3.10) we have the standard isomorphism $H_c^2(G,F_{\mathrm{nr}}^*) \cong \mathrm{Br}(F_{\mathrm{nr}}/F) = \mathrm{SBr}(F)$. Let $\Delta_F = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$, the divisible hull of Γ_F. Since Δ_F is uniquely divisible (so $H_c^i(G,\Delta_F) = 0$ for all $i > 0$), $H_c^2(G,\Gamma_F) \cong H_c^1(G,\Delta_F/\Gamma_F) = \mathrm{Hom}_c(G,\Delta_F/\Gamma_F)$. Furthermore, explicit constructions (see the remark after Ex. 3.6 below) show that the map ε' of (3.10) is onto. Thus, (3.10) translates to the short exact sequence

$$0 \longrightarrow \mathrm{IBr}(F) \xrightarrow{\ \gamma\ } \mathrm{SBr}(F) \xrightarrow{\ \varepsilon\ } \mathrm{Hom}_c(G,\Delta_F/\Gamma_F) \longrightarrow 0\,, \tag{3.11}$$

where γ coincides with the inclusion map. A striking interpretation of the map ε was given in [JW$_2$]:

Theorem 3.5 *For a Henselian field F, and any inertially split $D \in \mathcal{D}(F)$, let $h_D = \varepsilon([D]) \in \mathrm{Hom}_c(G,\Delta_F/\Gamma_F)$, where ε is the map of* (3.11)*. Then, $\mathrm{im}(h_D) = \Gamma_D/\Gamma_F$; the fixed field of $\ker(h_D)$ is $Z(\overline{D})$; and the isomorphism $\mathcal{G}(Z(\overline{D})/\overline{F}) \longrightarrow \Gamma_D/\Gamma_F$ induced by h_D is the inverse of the map θ_D of* (2.13) *induced by conjugation by elements of D.*

See [JW$_2$, Th. 5.6(b)] for a proof of Th. 3.5. It is rather remarkable that the map h_D obtained cohomologically has such a direct interpretation within the underlying division algebra D. For, a cocycle describing $[D]$ yields a crossed product algebra which is matrices of some size over D, in which it is often difficult to detect D itself. Note that by using Th. 3.5 we can easily compute $Z(\overline{E})$, Γ_E, and θ_E whenever E is the underlying division algebra of a central simple algebra obtained from an inertially split division algebra over F by scalar extension, or obtained as a tensor product of inertially split division algebras.

When Γ_F is a free abelian group the short exact sequence (3.9) is split exact, so (3.11) is also split exact, though there is no canonical splitting map. In particular, when the Henselian valuation v on F is discrete rank 1 (so $\Gamma_F \cong \mathbb{Z}$) then the last term in the exact sequence (3.9) is the continuous character group $X(G) = \mathrm{Hom}_c(G, \mathbb{Q}/\mathbb{Z})$. When, further, $\overline{F}$ is perfect, it is known that $\mathrm{SBr}(F)$ is the full Brauer group $\mathrm{Br}(F)$. Hence, we recover the isomorphism

$$\mathrm{Br}(F) \;\cong\; \mathrm{Br}(\overline{F}) \oplus X(G_{\overline{F}}), \tag{3.12}$$

which was originally proved by Witt in [Wi] in the complete case. The isomorphism in (3.12) is not canonical, since it depends on the choice of a uniformizing parameter of V_F, which determines the splitting map in (3.9). (For the equality $\mathrm{SBr}(F) = \mathrm{Br}(F)$ for v discrete rank 1 Henselian with $\overline{F}$ perfect, see [S, Ch. XII, §1, Th. 1; §2, Prop. 2] for the case when v is complete. In the case that v is not complete, then Cohn's complete version of Th. 2.3 above [C_1, Th. 1] shows that $\mathrm{Br}(F)$ maps injectively into $\mathrm{Br}(\widehat{F})$, where $\widehat{F}$ is the completion of F; also $\mathrm{SBr}(F)$ maps onto $\mathrm{SBr}(\widehat{F})$ by (3.11), so $\mathrm{SBr}(F) = \mathrm{Br}(F)$ since this holds for $\widehat{F}$.)

Example 3.6 Given a Henselian valued field F, let $\widetilde{L_1}, \ldots, \widetilde{L_k}$ be cyclic Galois extension fields of $\overline{F}$ such that $\widetilde{L_1} \otimes_{\overline{F}} \cdots \otimes_{\overline{F}} \widetilde{L_k}$ is a field. Say $[\widetilde{L_i} : \overline{F}] = n_i$. Let L_i be the inertial lift of $\widetilde{L_i}$ over F, so each L_i is cyclic Galois over F, say $\mathcal{G}(L_i/F) = \langle \sigma_i \rangle$, and $L_1 \otimes_F \cdots \otimes_F L_k$ is a field. Let $n = \mathrm{lcm}(n_1, \ldots, n_k)$, and suppose there are $a_1, \ldots, a_k \in F^*$ such that the images of $\{\frac{n}{n_1} v(a_1), \ldots, \frac{n}{n_k} v(a_k)\}$ generate a subgroup of $\Gamma_F / n\Gamma_F$ of (maximum possible) order $n_1 \cdots n_k$. Let

$$N \;=\; (L_1/F, \sigma_1, a_1) \otimes_F \cdots \otimes_F (L_k/F, \sigma_k, a_k). \tag{3.13}$$

Then, N is an inertially split division algebra over F with $\overline{N} = \widetilde{L_1} \otimes_{\overline{F}} \cdots \otimes_{\overline{F}} \widetilde{L_k}$ and $\Gamma_N = \langle \frac{1}{n_1} v(a_1), \ldots, \frac{1}{n_k} v(a_k) \rangle + \Gamma_F$, so that $\Gamma_N / \Gamma_F \cong \prod_{i=1}^{k} (\mathbb{Z}/n_i\,\mathbb{Z})$. For, each $D_i = (L_i/F, \sigma_i, a_i)$ is a division algebra by Ex. 2.7, with $\overline{D_i} = \widetilde{L_i}$ and $\Gamma_{D_i} = \langle \frac{1}{n_i} v(a_i) \rangle + \Gamma_F$; then the assertions about N follow by repeated application of Prop. 2.8. This N is inertially split since it contains the maximal subfield $L_1 \otimes_F \cdots \otimes_F L_k$ which is unramified over F. Note that $\deg(N) = [\overline{N} : \overline{F}] = |\Gamma_N : \Gamma_F| = n_1 \cdots n_k$, while $\exp(N) = \exp(\Gamma_N/\Gamma_F) = \mathrm{lcm}(n_1, \ldots, n_k) = n$. For, clearly $N^{\otimes n}$ is split, since each $D_i^{\otimes n}$ is split. But, Th. 3.5 shows that $n = \exp(\Gamma_N/\Gamma_F) = \exp(\varepsilon[N]) \;\big|\; \exp(N)$.

The division algebras constructed in Ex. 3.6 were dubbed "nicely semiramified" in [JW_2, §4], where some other characterizations of such algebras are given. Note that for any $h \in \mathrm{Hom}_c(G_{\overline{F}}, \Delta_F/\Gamma_F)$ it is easy to construct an N as in Ex. 3.6 with θ_N the inverse of the isomorphism $G_{\overline{F}} / \ker(h) \to \mathrm{im}(h)$ induced by h; then, for the map ε of (3.11), Th. 3.5 shows that $\varepsilon([N]) = h$. This verifies the surjectivity of ε. Nicely semiramified division algebras are a part of a useful (though noncanonical) decomposition of any inertially split division algebra, which is described in the next theorem. For part (e), we will need the Dec group introduced by Tignol in [T_1].

For $L \subseteq K$ fields with K abelian Galois over L and $[K:L] < \infty$,

$$\mathrm{Dec}(K/L) \;=\; \sum_{cyclic} \mathrm{Br}(C/L) \subseteq \mathrm{Br}(K/L) \tag{3.14}$$

where the sum ranges over all fields C, with $L \subseteq C \subseteq K$ and C cyclic Galois over L. Then, $\mathrm{Dec}(K/L)$ is the subgroup of L-algebras that "decompose according to K." That is, we can express $K = C_1 \otimes_L \cdots \otimes_L C_k$ with each C_i cyclic Galois over L; then for any $[E] \in \mathrm{Br}(L)$, we have $[E] \in \mathrm{Dec}(K/L)$ iff E is Brauer equivalent to some $(C_1/L, \sigma_1, a_1) \otimes_L \cdots \otimes_L (C_k/L, \sigma_k, a_k)$.

Theorem 3.7 *Let F be a Henselian valued field, let $I, N \in \mathcal{D}(F)$ with N nicely semiramified as in Ex. 3.6, and I unramified over F. Let D be the underlying division algebra of $I \otimes_F N$. Then,*

(a) *$Z(\overline{D}) = \overline{N}$, and $\overline{D}$ is the underlying division algebra of $\overline{I} \otimes_{\overline{F}} \overline{N}$.*
(b) *$\Gamma_D = \Gamma_N$ and $\theta_D = \theta_N$.*
(c) *$\exp(D) = \mathrm{lcm}(\exp(\Gamma_N/\Gamma_F), \exp(\overline{I}))$.*
(d) *$\mathrm{ind}(D) = \mathrm{ind}(\overline{I} \otimes_{\overline{F}} \overline{N}) \cdot |\Gamma_N/\Gamma_F|$.*
(e) *D is nicely semiramified iff $\overline{I} \in \mathrm{Dec}(\overline{N}/\overline{F})$.*
(f) *If $S \in \mathcal{D}(F)$ is any inertial split division algebra, then there is an unramified $I \in \mathcal{D}(F)$ and nicely semiramified $N \in \mathcal{D}(F)$ such that $S \sim I \otimes_F N$ in $\mathrm{Br}(F)$. Hence, $\exp(\overline{S}) \;\big|\; \exp(S)$ and $\exp(\Gamma_S/\Gamma_F) \;\big|\; \exp(S)$.*

See [JW$_2$, Lemma 5.14, Th. 5.15] for a proof of Th. 3.7.

Inertially split division algebras have been used in a number of significant constructions, as we will see in §§5–7 below. Here is another example of such algebras, the "Mal′cev-Neumann" algebras studied by Tignol and Amitsur in [TA$_1$] and [T$_5$].

Example 3.8 Let $F_0 \subseteq L$ be fields with L abelian Galois over F_0, and let $n = [L:F_0] < \infty$. Let $G = \mathcal{G}(L/F_0)$, let $f : G \times G \to L^*$ be any normalized 2-cocycle, and let $\varepsilon : \mathbb{Z}^k \to G$ be any group epimorphism. The corresponding Mal′cev-Neumann algebra D_f is defined by $D_f = L((y_1))\cdots((y_k))$ as an abelian group, but with multiplication in D_f being defined on monomials by, for $\ell, m \in L$,

$$\begin{aligned}(\ell y_1^{r_1}\cdots y_k^{r_k})&(m y_1^{s_1}\cdots y_k^{s_k}) \\ &= \ell\varepsilon(r_1,\ldots,r_k)(m) f\big(\varepsilon(r_1,\ldots,r_k),\varepsilon(s_1,\ldots,s_k)\big) y_1^{r_1+s_1}\cdots y_k^{r_k+s_k};\end{aligned} \tag{3.15}$$

this multiplication is then extended in the obvious way to all of D_f. The standard valuation v for $L((y_1))\cdots((y_k))$ as in (3.5) above is also a valuation for D_f, and a leading monomial argument like that for Ex. 2.7 above shows that D_f has no zero divisors. Since D_f is finite-dimensional over its central subfield $F_1 = F_0((y_1^n))\cdots((y_k^n))$, it follows that D_f is a division ring. Let $F = Z(D_f) \supseteq F_1$. So,

$$\Big\}.(3.17)$$

whenever $(i_1,\ldots,i_k) \notin \ker(\varepsilon)$

$$\Big\}.$$

Since $v|_{F_1}$ is the standard Henselian valuation and $[F:F_1] < \infty$, we have $v|_F$ is Henselian. Note that $\overline{D_f} = L$ and $\Gamma_{D_f} = \mathbb{Z}^k$, while $\overline{F} = F_0$ and $\Gamma_F = \ker(\varepsilon)$, so

$[\overline{D_f} : \overline{F}] = |\Gamma_{D_f} : \Gamma_F| = n$. Also, $\theta_{D_f} : \Gamma_{D_f}/\Gamma_F \to \mathcal{G}(L/F_0)$ is the isomorphism induced by ε. Since D_f contains the maximal subfield $L \cdot F$ which is unramified over F, D_f is inertially split. The center F of D_f is determined by ε and not affected by f. If we fix the map ϵ, there is a whole family of division algebras D_f over the same F, depending on the choice of f. In [T_5, Th. 2.3], Tignol showed that for any cocycle f, $D_f \sim I_f \otimes_F D_1$ in $\mathrm{Br}(F)$, where D_1 is the Mal′cev-Neumann division algebra built as above from the trivial cocycle 1, and I_f is the inertial lift over F of the division algebra B_f over F_0 which corresponds to the class of f in $H^2(G, L^*) \cong \mathrm{Br}(L/F_0)$. Since it is easy to check that D_1 is nicely semiramified, this decomposition of D_f is an example of the $I \otimes N$ decomposition in Th. 3.7(f). Although (for fixed ε) all the D_f have the same center, the same residue field, and the same value group, Tignol and Amitsur showed in [TA_1] and [T_5] that the subfields of D_f depend very much on the choice of f. Their work on the subfields of D_f was abstracted and put in a general valuation-theoretic framework by Morandi and Sethuraman in [MS_3].

For another perspective on Mal′cev-Neumann division algebras, and a more general construction of them, see [L_2, pp. 241–246].

When the value group of a Henselian field F is sufficiently big, there is a larger part of $\mathrm{Br}(F)$ beyond $\mathrm{SBr}(F)$ which is amenable to analysis using the valuation. This consists of the tame division algebras, which we consider next.

First, we recall the commutative analogue. A finite degree extension field L of a field F with Henselian valuation v is said to be *tamely ramified* over F (with respect to the unique extension of v to L) if $\mathrm{char}(\overline{F}) = 0$ or $\mathrm{char}(\overline{F}) = p \neq 0$ and $\overline{L}$ is separable over $\overline{F}$, $p \nmid |\Gamma_L : \Gamma_F|$, and $[L : F] = [\overline{L} : \overline{F}]|\Gamma_L : \Gamma_F|$. Such an L is necessarily separable over F. If L is algebraic over F and $[L : F] = \infty$, then L is tamely ramified over F if each finite degree subextension of F in L is tamely ramified over F. There is a unique maximal tamely ramified extension F_{tr} of F in F_{sep}, which is the compositum of all the finite-degree tamely ramified extensions of F. We have $\overline{F_{\mathrm{tr}}} = \overline{F}_{\mathrm{sep}}$ and, for $\Delta_F = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$, we have $\Gamma_{F_{\mathrm{tr}}}/\Gamma_F = (\Delta_F/\Gamma_F)'$, by which we mean the prime-to-p subgroup of the torsion abelian group Δ_F/Γ_F if $\mathrm{char}(\overline{F}) = p \neq 0$, and all of Δ_F/Γ_F if $\mathrm{char}(\overline{F}) = 0$. The field F_{tr} is a Galois extension of F, usually of infinite degree, and F_{tr} contains the maximal unramified extension F_{nr} of F. Indeed, if we consider the Galois extension F_{sep} of F and the unique extension of v to F_{sep} (again denoted v), then F_{tr} is the ramification field of v for F_{sep} over F (while F_{nr} is the inertia field, and F itself is the decomposition field, as v is Henselian). F_{tr} is an abelian Galois extension of F_{nr}, with $\mathcal{G}(F_{\mathrm{tr}}/F_{\mathrm{nr}}) \cong \mathrm{Hom}\big((\Delta_F/\Gamma_F)', \mu(\overline{F})\big)$, where the isomorphism is given by $\sigma \mapsto f_\sigma$, where for $a \in F_{\mathrm{tr}}$, $f_\sigma(v(a) + \Gamma_F) = \overline{\sigma(a)a^{-1}} \in \mu(\overline{F})$. When $\mathrm{char}(\overline{F}) = p \neq 0$, $\mathcal{G}(F_{\mathrm{sep}}/F_{\mathrm{tr}})$ is the unique p-Sylow subgroup of the profinite group $\mathcal{G}(F_{\mathrm{sep}}/F_{\mathrm{nr}})$. When $\mathrm{char}(\overline{F}) = 0$, $F_{\mathrm{tr}} = F_{\mathrm{sep}}$. See [E, §20] for proofs of the assertions in this paragraph, all of which are well known.

The noncommutative version of a tamely ramified extension is given by the following:

Proposition 3.9 *Let F be a Henselian valued field, and let $D \in \mathcal{D}(F)$. Suppose* $\mathrm{char}(\overline{F}) = p \neq 0$. *Then the following are equivalent:*

(i) F_{tr} *splits D.*
(ii) F_{nr} *splits the p-primary component of D.*

(iii) *$Z(\overline{D})$ is separable over $\overline{F}$, $p \nmid |\ker(\theta_D)|$, where θ_D is the map of* (2.13), *and $[D:F] = [\overline{D}:\overline{F}]|\Gamma_D : \Gamma_F|$.*
(iv) *D has a maximal subfield which is tamely ramified over F.*

See [JW$_2$, Lemma 6.1] and [HW$_2$, Prop. 4.3] for a proof of Prop. 3.9. See also [D$_3$, Th. 3] for (i) $\Rightarrow$ (iv). For $D \in \mathcal{D}(F)$ with F Henselian, we say that D is *tame* if $\mathrm{char}(\overline{F}) = 0$ or if $\mathrm{char}(\overline{F}) = p \neq 0$ and D satisfies the equivalent conditions of Prop. 3.9. The tame part of the Brauer group of F is defined to be

$$\mathrm{TBr}(F) \;=\; \{\,[D] \bigm| D \in \mathcal{D}(F) \text{ and } D \text{ is tame}\} = \mathrm{Br}(F_{\mathrm{tr}}/F). \tag{3.18}$$

Note that $\mathrm{TBr}(F)$ is a subgroup of $\mathrm{Br}(F)$ containing $\mathrm{SBr}(F)$ and containing the entire q-primary component of $\mathrm{Br}(F)$ for each prime $q \neq \mathrm{char}(\overline{F})$. In particular, if $\mathrm{char}(\overline{F}) = 0$, then $\mathrm{TBr}(F) = \mathrm{Br}(F)$.

A special type of tame division algebra is of particular interest. Still assuming that F is Henselian, we say that $T \in \mathcal{D}(F)$ is *tame and totally ramified* (abbreviated TTR) if $|\Gamma_T : \Gamma_F| = [T:F]$ and $\mathrm{char}(\overline{F}) \nmid [T:F]$. We will devote all of §4 below to TTR division algebras. For the present, we will focus on how arbitrary tame division algebras are related to TTR division algebras.

Theorem 3.10 *Let F be a Henselian valued field, and let $D \in \mathcal{D}(F)$ with D tame.*

(a) *There are $S, T \in \mathcal{D}(F)$ with S inertially split and T TTR such that $D \sim S \otimes_F T$ in $\mathrm{Br}(F)$.*
(b) *For any S and T as in part (a), we have $\Gamma_D = \Gamma_S + \Gamma_T$, $\ker(\theta_D) = \Gamma_T/\Gamma_F$, and $Z(\overline{D})$ is the fixed field of $\theta_S\big((\Gamma_S \cap \Gamma_T)/\Gamma_F\big)$.*

See [JW$_2$, Lemma 6.2, Th. 6.3] for a proof of Th. 3.10. Note that the decomposition $D \sim S \otimes_F T$ is not canonical, and in general it is not possible to find such S and T with $D \cong S \otimes_F T$. Some consequences of this theorem are given in [JW$_2$, §6]. We mention just one: Suppose F is Henselian and $D \in \mathcal{D}(F)$ with D tame. Let $D^{\otimes m}$ denote the underlying division algebra of $D \otimes_F \cdots \otimes_F D$ (m times). Then, (see [JW$_2$, Prop. 6.9]),

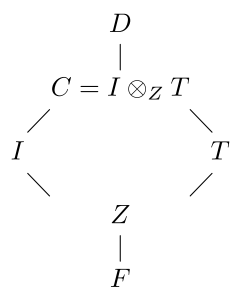
$$\Gamma_{D^{\otimes m}} \;=\; m\Gamma_D + \Gamma_F. \tag{3.19}$$

The proof of (3.19) depends on writing $D \sim S \otimes_F T$ as above and invoking the good information about $T^{\otimes m}$ we have because T is a tensor product of cyclic algebras (see Th. 4.5(a) below), and about $S^{\otimes m}$, by virtue of Th. 3.5. It follows from (3.19) that $\exp(\Gamma_D/\Gamma_F) \bigm| \exp(D)$ for D tame, as had been proved earlier in [PY$_3$, (3.19)]. However, this divisibility does not hold in general if D is not tame—see [JW$_2$, Ex. 7.5] for a counterexample.

If the Henselian valuation v on F has rank 1 (as in the case of a complete discrete valuation) then there are no nonsplit TTR division algebras over F—see Cor. 4.3 below. Therefore, for such valued fields, Th. 3.10 shows that $\mathrm{TBr}(F) = \mathrm{SBr}(F)$.

One of the complications in dealing with valued division algebras $D \in \mathcal{D}(F)$ is that $Z(\overline{D})$ may be strictly larger than $\overline{F}$. When the valuation on F is Henselian, and D is inertially split, we can at least say that $[\overline{D}] \in \mathrm{im}\big(\mathrm{Br}(\overline{F}) \to \mathrm{Br}(Z(\overline{D}))\big)$, as is clear from Th. 3.5. However, this does not always hold when D is tame—see [JW$_2$, Ex. 7.1] for a counterexample. Nonetheless, it was shown in [W$_2$] that if $\overline{F}$ contains enough roots of unity, then for any tame D it is true that $[\overline{D}]$ lies in the image of $\mathrm{Br}(\overline{F})$.

While the $D \sim S \otimes_F T$ decomposition for D tame, and the $S \sim I \otimes_F N$ decomposition of an inertially split S are not canonical, there are certain canonical pieces of a tame D, which are unique up to isomorphism and give some sense of how such a D is built up. For this, take any tame $D \in \mathcal{D}(F)$ with F Henselian. Let I be an inertial lift of $\overline{D}$ over F, such that $I \subseteq D$. Such an I exists and is unique up to isomorphism by Th. 3.3. Let $Z = Z(I)$, which is an inertial lift of $Z(\overline{D})$ over F, and let $C = C_D(Z)$, the centralizer of Z in D. Let $T = C_C(I)$. Then $C \cong I \otimes_Z T$ by the Double Centralizer Theorem. Since C and I are tame, so is T. We have $\overline{T} \subseteq \overline{D} = \overline{I}$, but $\overline{T} \cap \overline{I} = \overline{Z}$ by Th. 3.3(b) since T and I can can contain no proper field extension of Z in common. Hence, T is TTR (with center Z). Easy calculations show: $\overline{Z} = Z(\overline{D})$ and $\Gamma_Z = \Gamma_F$, so Z is unramified abelian Galois over F with $\mathcal{G}(Z/F) \cong \mathcal{G}(\overline{Z}/\overline{F}) = \operatorname{im}(\theta_D)$; $\overline{I} = \overline{D}$ and $\Gamma_I = \Gamma_F$; $\overline{T} = \overline{Z}$ and $\Gamma_T/\Gamma_F = \ker(\theta_D)$; $\overline{C} = \overline{D}$ and $\Gamma_C/\Gamma_F = \Gamma_T/\Gamma_F = \ker(\theta_D)$, so D is totally ramified over C and $[D:C] = |\Gamma_D : \Gamma_C| = [Z:F]$, with $\Gamma_D/\Gamma_C \cong \mathcal{G}(Z(\overline{D})/\overline{F})$ via θ_D. The Double Centralizer Theorem shows that there is a one-to-one inclusion-reversing correspondence between the fields Y such that $F \subseteq Y \subseteq Z$ and the division algebras E such that $C \subseteq E \subseteq D$, given by $Y \leftrightarrow E$ just when $E = C_D(Y)$ (then also $Y = C_D(E)$).



We have thus far considered only tame division algebras over a Henselian valued field F, since these are the best understood. There has been only a little work on wild (i.e., non-tame) division algebras over F. Partly this may be because there are no wild division algebras over the local fields of number theory (see (3.12) above). Additionally, computations with such algebras are quite difficult. But we do want to mention a few results on wild division algebras. In view of the comments after (3.18) above, $\operatorname{Br}(F)/\operatorname{TBr}(F)$ is generated by wild p-primary division algebras, where $p = \operatorname{char}(\overline{F}) > 0$.

Consider first a Henselian valuation v on F with $\Gamma_F = \mathbb{Z}$ and $\operatorname{char}(\overline{F}) = p > 0$. We noted above after (3.12) that if $\overline{F}$ is perfect, then $\operatorname{Br}(F) = \operatorname{SBr}(F)$, so there are no wild division algebras over F. But, when $\overline{F}$ is not perfect, $\operatorname{Br}(F)$ has a nontrivial wild part. For example, suppose $\operatorname{char}(F) = p$, let $\pi \in F$ with $v(\pi) = 1$, and let $L = F(s)$, where $s^p - s - \pi^{-1} = 0$. (That is, $L = F(\wp^{-1}(\pi^{-1}))$, where $\wp$ is the Artin-Schreier map given by $a \mapsto a^p - a$.) When we extend v to L, clearly $v(s) = -\frac{1}{p}$; so $L \neq F$, $[L:F] = |\Gamma_L : \Gamma_F| = p$, and L is cyclic Galois over F with $\mathcal{G}(L/F)$ generated by σ with $\sigma(s) = s + 1$. Whenever $\overline{F}$ is not perfect, there is $b \in F$ with $v(b) = 0$ such that $\overline{b} \notin \overline{F}^p$. Then it is easy to check that $D = (L/F, \sigma, b)$ is a division algebra over F (since b is not a norm from L), with $\overline{D} = \overline{F}(\sqrt[p]{\overline{b}})$ and $\Gamma_D = \Gamma_L = \frac{1}{p}\Gamma_F$; so D is wild. If, instead, $\operatorname{char}(F) = 0$, assume for simplicity that F contains a primitive p-th root of unity ω. Take $\pi, c \in F$ with $v(\pi) = 1$, $v(c) = 0$, and $\overline{c} \notin \overline{F}^p$; then let $L' = F(\sqrt[p]{1+\pi})$ and $D' = A_\omega(1+\pi, c; F)$. Let $m = 1 - r \in L$, where $r^p = 1 + \pi$ and $\overline{r} = 1$; then an examination of the equation $(1+m)^p = 1 + \pi$ shows that (as $v(m) > 0$ and $v(p) \geq v(\pi)$) $p\,v(m) = v(\pi)$, so $|\Gamma_{L'} : \Gamma_F| = [L':F] = p$. It follows that D' is a division ring (as c is not a norm from L'), with $\Gamma_{D'} = \Gamma_{L'} = \frac{1}{p}\Gamma_F$ and $\overline{D'} = \overline{F}(\sqrt[p]{\overline{c}})$, so D' is wild. Likewise, if

$[\overline{F} : \overline{F}^p] \geq p^2$, one can construct examples of wild division algebras $E \in \mathcal{D}(F)$ with $\overline{E}$ a field purely inseparable over $\overline{F}$ and $[\overline{E} : \overline{F}] = [E : F] = p^2$ (see [Sa$_4$, p. 1760, Ex.; p. 1765, Th. 2.13]).

If F is a field with complete discrete rank 1 valuation v with $\mathrm{char}(\overline{F}) = \mathrm{char}(F)$, then it is well known (see, e.g., [ZS, p. 307, Cor.]) that $F \cong \overline{F}((t))$, with v corresponding to the standard valuation on $\overline{F}((t))$. Yuan proved in [Yu, Th. 5.2] that in this situation there is a split exact sequence

$$0 \longrightarrow \mathrm{Br}(\overline{F}[t^{-1}]) \longrightarrow \mathrm{Br}(\overline{F}((t))) \longrightarrow X_{G_{\overline{F}}} \longrightarrow 0\,. \tag{3.20}$$

This reduces to (3.12) above when $\overline{F}$ is perfect, since then $\mathrm{Br}(\overline{F}[t^{-1}]) = \mathrm{Br}(\overline{F})$ (see, e.g., [OS, p. 100, Cor. 8.7]). For a field F with Henselian discrete rank 1 valuation there is a split exact sequence analogous to (3.20) whenever F contains a copy of $\overline{F}$. This follows from (3.20) by passing to the completion, just as in the comments after (3.12) above. Of course, this exact sequence can be difficult to work with, since it is not so easy to determine $\mathrm{Br}(\overline{F}[t^{-1}])$. An extensive analysis of the wild division algebras of degree p over a field F with complete discrete rank 1 valuation with $\mathrm{char}(\overline{F}) = p$ was given by Saltman in [Sa$_4$].

Moving beyond the discrete case, Tignol analyzed in [T$_8$] the defectless division algebras D of degree p over a field F with Henselian valuation v with $\mathrm{char}(F) = p > 0$. He worked with the *height* of D, defined by

$$h(D) = \min\{v(ab - ba) - v(a) - v(b) \mid a, b \in D^*\}\,.$$

(This invariant was originally defined by Saltman in [Sa$_4$] in the complete discrete of rank 1 case; Saltman called $h(D)$ the *level* of D.) Tignol showed, see [T$_8$, p. 2] that if $\mathrm{char}(F) = 0$, then $0 \leq h(D) \leq v(p)/(p-1)$, while if $\mathrm{char}(F) = p$, then $0 \leq h(D) < \infty$. He proved in [T$_8$, Th. 4.11] that D is a cyclic algebra except possibly in the following cases: (a) D is tame and inertial over F; (b) D is wild with $\Gamma_D = \Gamma_F$ (so $\overline{D}$ is a field purely inseparable over $\overline{F}$), and $\mathrm{char}(F) = 0$ and $h(D) = v(p)/(p-1)$. For these exceptional cases: in (a), he proved in [T$_9$, §2] that D is a cyclic algebra iff $\overline{D}$ is cyclic; in (b) he gave in [T$_8$, Th. 4.9] several conditions equivalent to D being a cyclic algebra. He showed further in [T$_7$, Cor. 3] that in case (b) D is cyclic if $p = 5$ and the Henselian field F contains a primitive fifth root of unity.

Aravire and Jacob considered in [AJ] central simple algebras A which are tensor products of cyclic algebras of degree p over a maximally complete (so Henselian) valued field F of characteristic $p > 0$ with $\overline{F}$ perfect. (A valued field F is *maximally complete* if there is no extension of the valuation to any field $L \supsetneqq F$ such that $\Gamma_L = \Gamma_F$ and $\overline{L} = \overline{F}$.) They showed that every such A is Brauer equivalent to a tensor product of t cyclic p-algebras in a somewhat complicated standard form, with $t \leq \dim_{\mathbb{Z}/p\mathbb{Z}}(\Gamma_F/p\Gamma_F)$; further, if $\dim_{\mathbb{Z}/p\mathbb{Z}}(\overline{F}/\wp(\overline{F})) < \dim_{\mathbb{Z}/p\mathbb{Z}}(\Gamma_F/p\Gamma_F)$, then $t < \dim_{\mathbb{Z}/p\mathbb{Z}}(\Gamma_F/p\Gamma_F)$. This applies to fields of the form $F = k((x_1))\cdots((x_n))$, where k is a perfect field with $\mathrm{char}(k) = p$, since it is known (see [Schi$_2$, p. 51, Cor. to Th. 8]) that such fields are maximally complete with respect to the standard Henselian valuation.

4 Tame totally ramified division algebras

Recall that a division algebra $D \in \mathcal{D}(F)$ with valuation v is said to be *tame and totally ramified* (TTR) (over its center F) if $|\Gamma_D : \Gamma_F| = [D : F]$ and $\mathrm{char}(\overline{F}) \nmid$

$[D : F]$. When $v|_F$ is Henselian, we will see that there is a very complete picture of such D and its subalgebras; this makes such division algebras very convenient for building examples with specified properties.

Before examining TTR division algebras, let us recall the basic facts about the corresponding type of field extensions. This information will be crucial for the examples in §5 below, and it also suggests what to expect with TTR division rings. For fields $F \subseteq K$ with $[K : F] < \infty$, and with a valuation v on K, we say that K is tame and totally ramified over F if $[K : F] = |\Gamma_K : \Gamma_F|$ and $\operatorname{char}(\overline{F}) \nmid [K : F]$. When this occurs, the fundamental inequality shows that $\overline{K} = \overline{F}$. Recall the conventions on roots of unity in (0.2) above.

Proposition 4.1 *Let K be a field with a valuation v which is tame and totally ramified over a subfield F.*

(a) *If K is a Galois over F, then there is a (well-defined) perfect pairing $\mathcal{G}(K/F) \times \Gamma_K/\Gamma_F \to \mu(\overline{F})$ given by $(\sigma, v(a) + \Gamma_F) \mapsto \overline{\sigma(a)a^{-1}}$. Hence, $\mu_e \subseteq \overline{F}$, where $e = \exp(\mathcal{G}(K/F))$, and also $\mathcal{G}(K/F) \cong \Gamma_K/\Gamma_F$ (not canonically).*

(b) *Suppose $v|_F$ is Henselian. Then K is a radical extension of F. More specifically, if $\Gamma_K/\Gamma_F \cong (\mathbb{Z}/r_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/r_k\mathbb{Z})$, then there are $a_1, \ldots, a_k \in F^*$ such that $K = F(\sqrt[r_1]{a_1}, \ldots, \sqrt[r_k]{a_k})$ and $v(\sqrt[r_j]{a_j}) + \Gamma_F$ maps to $(0, \ldots, 0, 1 + r_j\mathbb{Z}, 0, \ldots, 0)$ in $\prod_{i=1}^{k} \mathbb{Z}/r_i\,\mathbb{Z}$.*

(c) *Suppose $v|_F$ is Henselian. Then K is Galois over F iff $\mu_e \subseteq \overline{F}$, iff $\mu_e \subseteq F$, where $e = \exp(\Gamma_K/\Gamma_F)$.*

Let us sketch a proof of Prop. 4.1. First, for (b) pick any $c \in K^*$ with $v(c) + \Gamma_F$ mapping to $(0, \ldots, 1 + r_j\mathbb{Z}, 0, \ldots, 0)$ in the isomorphism $\Gamma_K/\Gamma_F \cong \prod_{i=1}^{k} \mathbb{Z}/r_i\,\mathbb{Z}$. Then, $v(c^{r_j}) \in \Gamma_F$, so $v(c^{r_j}) = v(b)$ for some $b \in F^*$. Since $\overline{K} = \overline{F}$ as K is totally ramified over F, there is $d \in F^*$ with $v(d) = 0$ and $\overline{d} = \overline{c^{r_j}/b}$ in $\overline{K}$. Let $u = c^{r_j}/bd \in U_K$. Since $\overline{u} = 1$ it follows by Hensel's Lemma applied over K, since $\operatorname{char}(\overline{K}) \nmid r_j$, that there is $y \in K^*$ with $y^{r_j} = u$. Then, set $a_j = bd \in F^*$. We have $(c/y)^{r_j} = c^{r_j}/u = a_j$, so $c/y = \sqrt[r_j]{a_j} \in K^*$ and $v(c/y) = v(c)$. Likewise, for each i there is $a_i \in F^*$ with some r_i-th root $\sqrt[r_i]{a_i} \in K^*$ with $v(\sqrt[r_i]{a_i}) + \Gamma_F$ mapping to $(0, \ldots, 0, 1 + r_i\,\mathbb{Z}, 0, \ldots, 0) \in \prod_{i=1}^{k} \mathbb{Z}/r_i\,\mathbb{Z}$. Since $[K : F] = |\Gamma_K : \Gamma_F| = |\Gamma_{F(\sqrt[r_1]{a_1}, \ldots, \sqrt[r_k]{a_k})} : \Gamma_F| \leq [F(\sqrt[r_1]{a_1}, \ldots, \sqrt[r_k]{a_k}) : F] \leq [K : F]$, we have $K = F(\sqrt[r_1]{a_1}, \ldots, \sqrt[r_k]{a_k})$ proving (b). Note that (c) follows immediately from (b) by Kummer theory and Hensel's Lemma. Observe that (a) follows easily from (b) and (c) when $v|_F$ is Henselian, by comparing the pairing of (a) with the perfect pairing of Kummer theory. Then (a) holds in general, since the pairing is unchanged in passage from F to a Henselization of F.

Returning now to division rings, let $D \in \mathcal{D}(F)$ be a valued division algebra such that D is TTR. There is a *canonical pairing* on $\Lambda_D = \Gamma_D/\Gamma_F$ which keeps track of the noncommutativity of D. The canonical pairing is the map

$$\beta_D : \Lambda_D \times \Lambda_D \to \overline{F}^* \text{ given by } (v(a) + \Gamma_F, v(b) + \Gamma_F) \mapsto \overline{aba^{-1}b^{-1}} \in \overline{F}. \qquad (4.1)$$

Proposition 4.2 *For any TTR valued division algebra $D \in \mathcal{D}(F)$ the pairing β_D of (4.1) is well-defined, $\mathbb{Z}$-bilinear, alternating, and nondegenerate, with $\operatorname{im}(\beta_D) = \mu_e(\overline{F})$, where $e = \exp(\Lambda_D)$.*

Everything in Prop. 4.2 is easy to prove except for the assertion that β_D is nondegenerate. (That β_D is alternating means that $\beta_D(\gamma,\gamma)=0$, for each $\gamma\in\Lambda_D$.) Since β_D is $\mathbb{Z}$-bilinear and Λ_D is a finite group, the image of β_D must consist of roots of unity in $\overline{F}$. For the nondegeneracy of β_D, see [TW, Prop. 3.1]. The proof given there is much like the one for Prop. 4.1(a) sketched above. The nondegeneracy of β_D when $v|_F$ is Henselian follows immediately from the cyclic algebra decomposition given in Th. 4.5(a) below.

Let A be a finite abelian group (written additively), and let C be a finite cyclic group (written multiplicatively) and let $f: A\times A\to C$ be a nondegenerate $\mathbb{Z}$-bilinear alternating pairing on A. Then, it is easy to prove that A has a "symplectic base" relative to f, i.e., there exist $a_1,b_1,\ldots,a_k,b_k\in A$ such that $A=\langle a_1\rangle\times\langle b_1\rangle\times\cdots\times\langle a_k\rangle\times\langle b_k\rangle$, with $f(a_i,a_j)=f(b_i,b_j)=1$ for all i,j, also $f(a_i,b_j)=1$ whenever $i\neq j$, and further, if we let $c_i=f(a_i,b_i)=f(b_i,a_i)^{-1}\in C$, then $\operatorname{ord}(c_i)=\operatorname{ord}(a_i)=\operatorname{ord}(b_i)$ for each i, where ord means the order of the element in its group. Furthermore, the a_i and b_i can be chosen so that $\operatorname{ord}(c_2)\bigm|\operatorname{ord}(c_1),\ldots,\operatorname{ord}(c_k)\bigm|\operatorname{ord}(c_{k-1})$. It then follows that $\operatorname{im}(f)=\langle c_1\rangle$, a subgroup of C of order equal to the exponent $\exp(A)$. Note also that the invariant factors of A come in pairs: $\operatorname{ord}(c_1)$, $\operatorname{ord}(c_1),\ldots,\operatorname{ord}(c_k),\operatorname{ord}(c_k)$. So, a finite abelian group A can admit such a nondegenerate pairing into a cyclic group C iff $A\cong B\times B$ for some group B and $\exp(A)\bigm| |C|$.

The nondegeneracy noted in Prop. 4.2 together with the observations in the preceding paragraph imply some significant constraints on the possible TTR division algebras over a valued field F. First, recall that the rational rank $\operatorname{rrk}(\Gamma_F)$ $(=\dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma_F))$ is an upper bound on the number of invariant factors of any finite homomorphic image of Γ_F (since $\dim_{\mathbb{Z}/p\mathbb{Z}}(\Gamma_F/p\Gamma_F)\leq\operatorname{rrk}(\Gamma_F)$ for every prime number p). Hence, $\operatorname{rrk}(\Gamma_F)$ is also an upper bound on the number of invariant factors of Γ_D/Γ_F for any valued division algebra $D\in\mathcal{D}(F)$. Thus, in particular,

Corollary 4.3 *If there is a TTR division algebra D over a field F with valuation v, and if $D\neq F$, then* $\operatorname{rrk}(\Gamma_F)\geq 2$. *Hence, if v on F is a discrete rank* 1 *valuation (i.e., $\Gamma_F\cong\mathbb{Z}$) then there is no such D.*

Cor. 4.3 perhaps explains why TTR valued division algebras were not studied any sooner than they were. For, the most frequently occurring valuations on the fields occurring in number theory and in algebraic geometry are discrete rank 1. Such valuations admit no proper TTR division algebras.

Notice another constraint imposed by the nondegeneracy of the canonical pairing: If $D\in\mathcal{D}(F)$ is a TTR valued division algebra, and if $e=\exp(\Gamma_D/\Gamma_F)$, then we must have $\mu_e\subseteq\overline{F}$.

Despite these remarks, examples of TTR valued division are easy to construct when Γ_F is sufficiently large and $\overline{F}$ contains enough roots of unity. We illustrate this with the next examples. See (0.2) and (0.3) above for our notation on roots of unity and on symbol algebras.

Examples 4.4 (i) Let n be a natural number, let F be a field with valuation v, such that $\mu_n\subseteq F$ and $\operatorname{char}(\overline{F})\nmid n$. Let $a,b\in F^*$ such that the images of $v(a)$ and $v(b)$ generate a subgroup of order n^2 in $\Gamma_F/n\Gamma_F$. Choose some $\omega\in\mu_n^*(F)$, and let D be the symbol algebra $D=A_\omega(a,b;F)$. Then D is a division algebra, v extends to a valuation on D with respect to which D is TTR, and $\Gamma_D=\langle\frac{1}{n}v(a),\frac{1}{n}v(b)\rangle+\Gamma_F$, so $\Lambda_D\cong(\mathbb{Z}/n\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z})$. For the standard generators

i, j of the symbol algebra D, $v(i) = \frac{1}{n}v(a)$ and $v(j) = \frac{1}{n}v(b)$ in the divisible hull Δ_F of Γ_F. Moreover, $v(i) + \Gamma_F$ and $v(j) + \Gamma_F$ form a symplectic base for the canonical pairing β_D. The fact that D is a division ring and v extends to D is a special case of Ex. 2.7, when we take $L = F(i) = F(\sqrt[n]{a})$. For, v has a unique extension to the cyclic Galois extension L of F, with necessarily $\Gamma_L = \langle \frac{1}{n}v(a)\rangle + \Gamma_F$.

(ii) Let $F = F_0((x_1))\cdots((x_m))$ be the m-fold iterated Laurent power series field over a field F_0, and let v be the standard Henselian valuation on F (see (3.5)). Suppose that for $\ell \le m/2$, we have positive integers $n_1, \ldots, n_\ell$ such that each $\mu_{n_k} \subseteq F$, and let $\omega_k \in \mu^*_{n_k}(F)$. Let $D = \bigotimes_{k=1}^{\ell} A_{w_k}(x_{2k-1}, x_{2k}; F)$. Then D is a division algebra which is TTR with respect to the extension of v to D. For, part (i) shows that each $A_{\omega_k}(x_{2k-1}, x_{2k}; F)$ is a division algebra TTR with respect to its extension of v; if i_k, j_k are the standard generators of $A_{\omega_k}(x_{2k-1}, x_{2k}; F)$, then clearly $v(i_k) = (0, \ldots, 0, \frac{1}{n_k}, 0, \ldots, 0)$ and $v(j_k) = (0, \ldots, 0, \frac{1}{n_k}, 0, \ldots, 0)$ (with nonzero entries in the $(2k-1)$-st and $2k$-th positions respectively) and $A_{\omega_k}(x_{2k-1}, x_{2k}; F)$ has value group $\langle v(i_k), v(j_k)\rangle + \Gamma_F$. Then, repeated applications of Prop. 2.8 yield that D is a division algebra which is TTR. Observe that $\Gamma_D = \langle v(i_1), v(j_1), \ldots, v(i_\ell), v(j_\ell)\rangle + \Gamma_F$, hence $\Lambda_D \cong \prod_{k=1}^{\ell} (\mathbb{Z}/n_k\,\mathbb{Z}) \times (\mathbb{Z}/n_k\,\mathbb{Z})$. Also, $\{v(i_k) + \Gamma_F, v(j_k) + \Gamma_F \mid 1 \le k \le \ell\}$ is a symplectic base for the canonical pairing on Λ_D.

When D is TTR with respect to a Henselian valuation on its center, then the next theorem shows that the canonical pairing β_D carries enormous information about the structure of D. It also shows that Ex. 4.4(ii) is very typical of the Henselian situation. We will use the following terminology: A subgroup Λ' of Λ_D is said to be *totally isotropic* relative to β_D if $\beta_D(\alpha, \gamma) = 1$ for all $\alpha, \gamma \in \Lambda'$.

Theorem 4.5 *Let $D \in \mathcal{D}(F)$ be a division algebra with valuation v such that D is TTR and $v|_F$ is Henselian. Then,*

(a)
$$D \;\cong\; A_{\omega_1}(a_1, b_1; F) \otimes_F \cdots \otimes_F A_{\omega_\ell}(a_\ell, b_\ell; F)\,, \tag{4.2}$$
where $a_1, b_1, \ldots, a_\ell, b_\ell \in F^*$ and $\omega_1, \ldots, \omega_\ell$ are primitive n_1-st$, \ldots, n_\ell$-th roots of unity in F. If i_k, j_k are the standard generators of $A_{\omega_k}(a_k, b_k; F)$ then $v(i_k) = \frac{1}{n_k}v(a_k)$ and $v(j_k) = \frac{1}{n_k}v(b_k)$, and $\{v(i_k) + \Gamma_F, v(j_k) + \Gamma_F \mid 1 \le k \le \ell\}$ is a symplectic base for the canonical pairing β_D on Λ_D. Conversely, for every symplectic base of Λ_D with respect to β_D, there is a corresponding tensor product decomposition of D into symbol algebras as in (4.2).

(b) *For every field K with $F \subseteq K \subseteq D$, K is tame and totally ramified over F (with respect to the unique extension of $v|_F$ to K), and Γ_K/Γ_F is a totally isotropic subgroup of Λ_D with respect to β_D.*

(c) *There is a one-to-one correspondence between F-isomorphism classes of fields K with $F \subseteq K \subseteq D$ and totally isotropic subgroups of Λ_D with respect to β_D.*

(d) *There is a one-to-one correspondence between F-isomorphism classes of F-subalgebras of D and subgroups of Λ_D.*

See [D_3, Th. 1] for a proof of part (a) of Th. 4.5; part (b) is evident from the fundamental inequality; parts (c) and (d) are given in [TW, Th. 3.8]. Note that part (c) gives a classification of the isomorphism classes of subfields of D containing

F, and shows that there are only finitely many such isomorphism classes. It is a very unusual situation to be able to get such complete information about the subfields of a division algebra. Note that once we have a decomposition of D into a tensor product of symbol algebras, we can use the standard generators of the symbol algebras to obtain an "armature" $\mathcal{A}$ of D in the terminology of [TW, p. 229], i.e., $\mathcal{A}$ is an abelian subgroup of D^*/F^* such that $|\mathcal{A}| = [D:F]$ and D is generated as an F-vector space by inverse images of the elements of $\mathcal{A}$. (Armatures, defined a little differently, were introduced in [T_2, §1].) Clearly, $\mathcal{A}$ can be chosen so that $\mathcal{A} \cong \Lambda_D$ via the valuation map. Then for any subgroup Λ' of Λ_D, we can obtain a subalgebra B of D with $\Gamma_B/\Gamma_F \cong \Lambda'$ by taking B to be the subalgebra generated by inverse images in D^* of the elements of the subgroup of $\mathcal{A}$ isomorphic to Λ' in the isomorphism $\mathcal{A} \cong \Lambda_D$.

In the strictly Henselian case we can say even more:

Theorem 4.6 *Let F be a field with a strictly Henselian valuation v, i.e., v is Henselian and $\overline{F}$ is separably closed. Then,*

(a) *For every $D \in \mathcal{D}(F)$ with $\mathrm{char}(\overline{F}) \nmid [D:F]$, we have D is TTR with respect to the extension of v to D, and D is determined up to F-isomorphism by Γ_D and the canonical pairing β_D.*
(b) *An algebraic field extension L of F splits D iff L contains a maximal subfield of D.*

See [TW, Prop. 4.2, Cor. 4.6] for a proof of Th. 4.6. More is proved there as well: (i) If A is a tensor product of symbol algebras over a strictly Henselian valued field F and $\mathrm{char}(\overline{F}) \nmid \dim_F(A)$, then an algorithm is given in [TW, Th. 4.3] for computing the value group of the underlying division algebra D of A (i.e., the D such that $A \cong M_n(D)$) and the canonical pairing on Λ_D; thereby Th. 4.6(a) shows that D is fully determined. This is quite an unusual situation in studying central simple algebras, to be able to describe the underlying division algebra so explicitly. (ii) Suppose a field F has a strictly Henselian valuation v and $D \in \mathcal{D}(F)$ with $\mathrm{char}(\overline{F}) \nmid [D:F]$ (so D is TTR), and Γ_D and β_D are known, and L is any algebraic field extension of F; then the unique extension of v to L is also strictly Henselian. An algorithm is given in [TW, Prop. 4.5] in terms of Γ_D, β_D, and Γ_L for determining the value group and canonical pairing (hence the isomorphism class) of the underlying division algebra of $D \otimes_F L$.

Remark 4.7 Let us return to Ex. 4.4(ii), where $F = F_0((x_1))\cdots((x_m))$ and $D = \bigotimes_{k=1}^{\ell} A_{\omega_k}(x_{2k-1}, x_{2k}; F)$, where $\omega_k \in \mu^*_{n_k}(F_0)$. Suppose that $\ell \geq 2$ and that there is a prime number p dividing at least two of the n_k. Then, we can see easily that D is not a cyclic algebra. For, if D had a maximal subfield L cyclic Galois over F, then L would be totally and tamely ramified over F by the fundamental inequality, so $\mathcal{G}(L/F) \cong \Gamma_L/\Gamma_F \subseteq \Lambda_D$, by Prop. 4.1(a). However, since the invariant factors of Λ_D occur in pairs and at least four are multiples of p, Λ_D has no cyclic subgroups of order $\sqrt{|\Lambda_D|}$; so, there can be no such L. If we assume further that F_0 is separably closed, then Th. 4.6(b) shows that there is no cyclic Galois extension field of F which splits D.

5 Noncrossed product division algebras

Let K be a Galois extension of a field F with $[K : F] = n < \infty$, and let $G = \mathcal{G}(K/F)$. Recall that from any 2-cocycle $f \in Z^2(G, K^*)$ one can build a *crossed product algebra* $(K/F, G, f)$ as $\bigoplus_{\sigma \in G} K x_\sigma$ with multiplication given by

$$(c\, x_\sigma)(d\, x_\tau) \;=\; c\, \sigma(d) f(\sigma, \tau)\, x_{\sigma\tau}\, .$$

This $(K/F, G, f)$ is a central simple F-algebra of dimension n^2 over F, and it contains a copy of K (as Kx_{id}) as a maximal subfield. Conversely, one deduces from the Skolem-Noether theorem that if A is a central simple F-algebra of dimension n^2 and A contains a Galois extension field K' of F with $[K' : F] = n$, then $A \cong (K'/F, \mathcal{G}(K'/F), f)$ for some 2-cocycle f, whose cohomology class in $H^2(\mathcal{G}(K'/F), K'^*)$ is uniquely determined. The crossed product construction provides the isomorphism between the Brauer group $\operatorname{Br}(F)$ and the continuous cohomology group $H^2_c(G_F, F^*_{\text{sep}})$, where F_{sep} is the separable closure of F and $G_F = \mathcal{G}(F_{\text{sep}}/F)$. Besides this, knowing that a specific central simple algebra A is a crossed product gives a concrete description of the multiplication in A that can help us to understand A.

For several decades, the biggest open question in the theory of finite-dimensional division algebras was whether every such algebra D is a crossed product. Restated, the question was: Does every such D contain a maximal subfield which is Galois over the center of D? This possibility seemed plausible in light of Köthe's theorem [K] (or see, e.g., [Re, Th. 7.15(ii)]) which says that D has a maximal subfield which is separable over the center. Moreover, there was the great theorem of the 1930's, see [BHN], [AH], which says that every central simple algebra over an algebraic number field is a cyclic algebra (i.e., a crossed product with cyclic Galois group). Albert had given in [A$_2$] an example of a tensor product of two quaternion algebras which is not a cyclic algebra. (We have seen such an example in Remark 4.7 above.) But, that had not ruled out the possibility of crossed products with noncyclic Galois groups. Also, it had been proved by Wedderburn and Albert that every division algebra of degree $2, 3, 4, 6,$ or 12 is a crossed product (cf. [Ro, pp. 180–183]).

In 1972 Amitsur in [Am$_1$] finally settled the crossed product question which had been lingering since the 1930's, by producing counterexamples. We will describe Amitsur's noncrossed product construction both because it is a beautiful argument in its own right and because it can be better understood using valuation theory. (Noncommutative valuation theory was not invoked explicitly in Amitsur's construction, but is just under the surface there.) We will then describe some of the more recent noncrossed product constructions, most of which have depended heavily on valuation theory. Detailed accounts of Amitsur's approach can be found, e.g., in [Ja] or [Ro, pp. 175–196], or see Amitsur's original paper.

Amitsur's examples of noncrossed product division algebras are certain generic division algebras, which are defined as follows: Let F be any ground field. For positive integers n and k with $k \geq 2$, let $\{x_{ij}^{(\ell)} \mid 1 \leq i, j \leq n, 1 \leq \ell \leq k\}$ be n^2k commuting indeterminates over F, and let L be the rational function field $F(\{x_{ij}^{(\ell)}\})$. Let $X_1, \ldots, X_k \in M_n(L)$ be the *generic matrices* determined by the $x_{ij}^{(\ell)}$, i.e., X_ℓ has ij-entry $x_{ij}^{(\ell)}$. Let $A(F; n)$ be the generic matrix algebra over F of degree n in k generic matrices, i.e., the F-subalgebra of $M_n(L)$ generated by $X_1, \ldots, X_k$. (Of course, $A(F;n)$ depends on k as well as n and F. But, for everything we do here, the choice of k will be irrelevant, so long as $k \geq 2$.) It is known that $A(F;n)$ has no zero divisors, and that it satisfies the following important Specialization Property:

For any field $K \supseteq F$ and any central simple K-algebra B of dimension n^2 over K, and any $b_1, \ldots, b_k \in B$, there is a unique F-algebra homomorphism $\alpha : A(F;n) \to B$ with $\alpha(X_\ell) = b_\ell$ for $1 \leq \ell \leq k$. Moreover, if $|K| = \infty$, and if g is any nonzero element of $A(F;n)$, then there are $b_1, \ldots, b_k \in B$ so that in the map α sending $X_\ell \mapsto b_\ell$, we have $\alpha(g) \neq 0$. (5.1)

Since the generic matrix ring $A(F;n)$ is a prime ring satisfying all the identities of $n \times n$ matrices but not all those of $(n-1) \times (n-1)$ matrices, the theory of prime p.i.-rings shows that $A(F;n)$ has a classical ring of quotients which is a central simple algebra of dimension n^2 over its center. Furthermore, since $A(F;n)$ has no zero divisors it is known that its ring of quotients is a division ring. The *generic* (or *universal*) *division ring* $UD(F;n)$ of degree n over F is by definition the ring of quotients of $A(F;n)$. (Again, in all respects of interest here, it does not matter how many generic matrices we start with, so long as $k \geq 2$.) From the theory of prime p.i.-rings one knows that $UD(F;n)$ is the central localization of $A(F;n)$ obtained by inverting all the nonzero central elements of $A(F;n)$. Thus, the center of $UD(F;n)$ is the quotient field of the center of $A(F;n)$.

Let G be a finite group. We say that a central simple algebra A over a field F is a *crossed product with group* G if $A \cong (K/F, G', f)$ with Galois group $G' \cong G$. The key result Amitsur proved in order to show that certain $UD(F;n)$ are not crossed products is the following:

Theorem 5.1 *Let G be a finite group with $|G| = n$. Suppose $UD(F;n)$ is a crossed product with group G. Take any division algebra B of degree n (dimension n^2) over any field M containing F; then, B is also a crossed product with the same group G.*

This theorem is proved by showing that the property that D is a crossed product with group G can be coded up into a system of equations defined over a finitely generated central localization of $A(F;n)$, then using the Specialization Property (5.1) to see that the corresponding equations hold in B.

To see that $UD(F;n)$ is not a crossed product, it suffices by Th. 5.1 to produce division algebras B_1 and B_2 of degree n over fields M_1 and M_2 containing F such that B_1 and B_2 are crossed products but not crossed products with the same group. Here is where valuation theory is helpful. In fact, we have already seen such examples in the preceding sections. We have the following specific case of Ex. 4.4(ii) above:

Examples 5.2 Let p be a prime number, and let F be any field with $\mathrm{char}(F) \neq p$. Let $M = F(\omega)((x_1)) \cdots ((x_6))$, the 6-fold iterated Laurent power series field over $F(\omega)$, where ω is a primitive p^3-rd root of unity. We work with the standard Henselian valuation v on M, with $\Gamma_M = \mathbb{Z}^6$ and $\overline{M} = F(\omega)$. Let $\varepsilon = \omega^{p^2}$, a primitive p-th root of unity. Let $B_1 = A_\varepsilon(x_1, x_2; M) \otimes_M A_\varepsilon(x_3, x_4; M) \otimes_M A_\varepsilon(x_5, x_6; M)$ and let $B_2 = A_\omega(x_1, x_2; M)$. Then B_1 and B_2 are each division algebras of degree p^3 over M. However, B_1 is a crossed product only with the group $(\mathbb{Z}/p\,\mathbb{Z})^3$, while B_2 is a crossed product only with the groups $\mathbb{Z}/p^3\,\mathbb{Z}$ and $(\mathbb{Z}/p^2\,\mathbb{Z}) \times (\mathbb{Z}/p\,\mathbb{Z})$. For, Ex. 4.4(ii) shows that B_1 is a division algebra which is TTR with respect to the unique extension of v to B_1, with $\Gamma_{B_1}/\Gamma_M \cong (\mathbb{Z}/p\,\mathbb{Z})^6$.

If K_1 is any maximal subfield of B_1, then $[K_1 : M] = p^3$, and K_1 is tame and totally ramified over M by the fundamental inequality. So, if K_1 is Galois over M, then by Prop. 4.1(a) $\mathcal{G}(K_1/M) \cong \Gamma_{K_1}/\Gamma_M$ which is a subgroup of Γ_{B_1}/Γ_M, hence elementary abelian of order p^3. Likewise, B_2 is a division algebra over M which is TTR with $\Gamma_{B_2}/\Gamma_M \cong (\mathbb{Z}/p^3\mathbb{Z})^2$, so if K_2 is a Galois maximal subfield of B_2, then $\mathcal{G}(K_2/M) \cong \Gamma_{K_2}/\Gamma_M$ which is a subgroup of $(\mathbb{Z}/p^3\,\mathbb{Z})^2$ of order p^3, so not elementary abelian.

Since the set of Galois groups for maximal subfields of B_1 does not overlap that of B_2, Th. 5.1 yields:

Theorem 5.3 *If p is a prime number and F is a field with* $\operatorname{char}(F) \neq p$*, then $UD(F; p^3)$ is not a crossed product.*

An easy variant of the examples in 5.2 allows us to see that if n is any integer which is a multiple of p^3 and F is a field with $\operatorname{char}(F) \nmid n$, then $UD(F; n)$ is not a crossed product. For, if $n = p^r b$ with $r \geq 3$ and $p \nmid b$, we can use Ex. 4.4(ii) to construct division algebras B_1 and B_2 of degree n over a suitable Henselian valued field $M \supseteq F$, such that B_1 and B_2 are each TTR, with $\Gamma_{B_1}/\Gamma_M \cong (\mathbb{Z}/p\,\mathbb{Z})^{2r} \times (\mathbb{Z}/b\,\mathbb{Z})^2$ and $\Gamma_{B_2}/\Gamma_M \cong (\mathbb{Z}/p^r\,\mathbb{Z})^2 \times (\mathbb{Z}/b\,\mathbb{Z})^2$. Since these relative value groups have no common subgroups of order n, the argument in Ex. 5.2 shows that B_1 and B_2 have no common Galois groups of maximal subfields. Thus, Th. 5.1 shows that $UD(F; n)$ is not a crossed product.

Amitsur also showed that noncrossed products of degree p^2 exist over fields not containing a primitive p-th root of unity. The key added ingredient for the p^2 result is provided by the following example:

Example 5.4 Let p be a prime number, and let F be a field with $\operatorname{char}(F) \neq p$ and $\mu_p \not\subseteq F$, such that F has a cyclic Galois field extension L with $[L : F] = p^2$. Let $M = F((x))$, and let $N = L((x))$, which is cyclic Galois over M, with $\mathcal{G}(N/M) \cong \mathcal{G}(L/F)$. For any generator σ of $\mathcal{G}(N/M)$, let $D = (N/M, \sigma, x)$. Then D is a division algebra, and the standard complete discrete rank 1 valuation on M extends to D with $\overline{D} = L$ and $\Gamma_D = \frac{1}{p^2}\mathbb{Z}$ (see Ex. 2.7 above). We claim that if K is any maximal subfield of D which is Galois over M, then K is unramified over M, and $\mathcal{G}(K/M) \cong \mathbb{Z}/p^2\,\mathbb{Z}$. For, since $[K : M] = p^2$, K is a tame extension of the Henselian field M, so we know that if Y is the maximal unramified extension of M in K, then K is tame and totally ramified over Y. Because $[\overline{Y} : \overline{M}] = [Y : M]$ which is a power of p and $\overline{M} = F$, so that $\mu_p \not\subseteq \overline{M}$, we must also have $\mu_p \not\subseteq \overline{Y}$. But, K is tame and totally ramified over Y with $[K : Y] = p^a$ for $a = 0, 1$, or 2. But if $a \geq 1$, then $\mu_p \subseteq \overline{Y}$ by Prop. 4.1(a) above. We have ruled this out. Hence, $a = 0$, i.e., $Y = K$, so K is unramified over M. Since $\overline{K} \subseteq \overline{D} = L$ and $p^2 = [L : F] = [\overline{D} : \overline{M}]$ and also $p^2 = [K : M] = [\overline{K} : \overline{M}]$ we must have $\overline{K} = L$, so $\mathcal{G}(K/M) \cong \mathcal{G}(\overline{K}/\overline{M}) = \mathcal{G}(L/F) \cong \mathbb{Z}/p^2\,\mathbb{Z}$.

Since by using Ex. 4.4 we can construct an example of a division algebra of degree p^2 over $F(\mu_p)((x_1))\cdots((x_4))$ such that every Galois maximal subfield has group $\mathbb{Z}/p\,\mathbb{Z} \times \mathbb{Z}/p\,\mathbb{Z}$, Ex. 5.4 yields the following corollary to Th. 5.1. (The F in Th. 5.5 need not have any cyclic extensions of degree p^2. In that case, we apply Ex. 5.4 to a field F' containing F such that $\mu_p \not\subseteq F'$ and F' has a cyclic extension of degree p^2.)

Theorem 5.5 *Let p be a prime number, and let F be a field with* $\operatorname{char}(F) \neq p$ *such that $\mu_p \not\subseteq F$ (so p is odd). Then $UD(F;p^2)$ is not a crossed product.*

Examples 5.6 Let p be a prime number and let F be a field with $\operatorname{char}(F) \neq p$ and $\mu_p \not\subseteq F$. We now give some variations of the preceding examples to show that $UD(F;n)$ is not a crossed product for any multiple n of p^2. Say $n = p^a b$ with $a \geq 2$ and $p \nmid b$.

(i) Let F' be a field containing F such that F' has a cyclic Galois extension field L with $[L(\mu_p) : L] = [F(\mu_p) : F] > 1$. Let $\mathcal{G}(L/F') = \langle\sigma\rangle$. (For example, we could take L to be the rational function field $F(z_1, \ldots, z_n)$ and σ an automorphism of L such that $\sigma|_F = \mathrm{id}$ and σ permutes the z_i cyclically; then let F' be the fixed field of σ.) Let $F_1 = F'((x))$ and let $D_1 = (L((x))/F_1, \sigma, x)$. The complete discrete rank 1 x-adic valuation v on F_1 extends to the division algebra D_1 with $\overline{D_1} = L$ and $\Gamma_{D_1} = \frac{1}{n}\mathbb{Z}$, by Ex. 2.7. Let K_1 be a maximal subfield of D_1 with K_1 Galois over F_1, and let $G_1 = \mathcal{G}(K_1/F_1)$. Let I be the inertia subgroup and R the ramification subgroup of G_1 for the extension K_1/F_1 with respect to v, and let K_I be the fixed field of I, and K_R that of R. Then we have $R \subseteq I \subseteq G_1$, and I and R are each normal in G_1; also, K_I is the maximal unramified extension of F_1 in K_1, and K_R is the maximal tamely ramified extension, and R is the unique q-Sylow subgroup of I if $q = \operatorname{char}(F) \neq 0$ (while R is trivial if $\operatorname{char}(F) = 0$), see [E, Th. 20.18]. Since K_R is tame and totally ramified over K_I, but $\mu_p \not\subseteq \overline{K_I}$ as $\overline{K_I} \subseteq \overline{D_1} = L$, we have $p \nmid [K_R : K_I] = |I : R|$, by Prop. 4.1(a). Since also $p \nmid |R|$ as $|R|$ is a power of $\operatorname{char}(F)$ or $|R| = 1$, this shows $p \nmid |I|$, so $p^a \;\big|\; |G_1/I|$. But $G_1/I \cong \mathcal{G}(K_I/F_1) \cong \mathcal{G}(\overline{K_I}/\overline{F_1})$, so G_1/I is cyclic as L/F' is cyclic Galois. Hence, G_1 has a cyclic homomorphic image of order p^a. Therefore, every p-Sylow subgroup of G_1 is cyclic.

(ii) Let ω be a primitive p-th root of unity in F_{sep}. With the F' and L of part (i), let L' be the field with $F' \subseteq L' \subseteq L$ and $[L' : F'] = b$, and let $L_2 = L'(\omega)$. So, $[L_2 : F'(\omega)] = b$ and L_2 is cyclic Galois over $F'(\omega)$, say $\mathcal{G}(L_2/F'(\omega)) = \langle\tau\rangle$. Let $F_2 = F'(\omega)((x_1))\cdots((x_{2a}))((y))$, and let

$$D_2 \;=\; \Big[\bigotimes_{k=1}^{a} A_\omega(x_{2k-1}, x_{2k}; F_2)\Big] \otimes_{F_2} (L_2 \cdot F_2/F_2, \tau, y)\,.$$

Then, D_2 is division algebra by Ex. 4.4(ii), Ex. 2.7, and Prop. 2.8, and the standard Henselian valuation on F_2 extends to D_2 with $\overline{D_2} \cong L_2$ and $\Gamma_{D_2}/\Gamma_{F_2} \cong (\mathbb{Z}/p\,\mathbb{Z})^{2a} \times (\mathbb{Z}/b\,\mathbb{Z})$. Let K_2 be a maximal subfield of D_2 which is Galois over F_2, and let $G_2 = \mathcal{G}(K_2/F_2)$; let P be a p-Sylow subgroup of G_2, let E be the fixed field of P, and let C be the centralizer $C_{D_2}(E)$. So, $E \subseteq C$, and $[C : E] = p^{2a}$. We have $[\overline{C} : \overline{E}] \;\big|\; \gcd([C : E], [\overline{D_2} : \overline{F_2}]) = 1$; hence $\overline{C} = \overline{E}$. Therefore, C must be tame and totally ramified over its center E, by (2.11). Since Γ_C/Γ_E is a subquotient of $\Gamma_{D_2}/\Gamma_{F_2}$ of order p^{2a}, we must have $\Gamma_C/\Gamma_E \cong (\mathbb{Z}/p\,\mathbb{Z})^{2a}$. We have $E \subseteq K_2 \subseteq C$, and K_2 is Galois over E. Since K_2 must be tame and totally ramified over E, Prop. 4.1(a) shows that $P = \mathcal{G}(K_2/E) \cong \Gamma_{K_2}/\Gamma_E \subseteq \Gamma_C/\Gamma_E$; hence P must be elementary abelian. Because no G_1 in part (i) can be isomorphic to a G_2 here, since their p-Sylow subgroups cannot be isomorphic, Th. 5.1 shows that $UD(F;n)$ is not a crossed product. These examples were adapted from ones given in [Ri$_4$].

In his paper [Am$_1$], Amitsur proved specifically only that $UD(\mathbb{Q};n)$ is a noncrossed product for any positive integer n such that $8 \mid n$ or $p^2 \;\big|\; n$ for p an odd prime. His specific examples of crossed product algebras D with only certain groups appearing as Galois groups of maximal subfields were iterated twisted Laurent series division algebras. In his argument there appear certain triangular integer valued matrices whose diagonal entries are crucial. A close look at such D reveals that they have a natural valuation, with respect to which D is TTR, and the diagonal entries of the associated matrix are the invariant factors in a cyclic decomposition of the relative value group of D. Thus, noncommutative valuation theory is implicit, though not explicit, in Amitsur's paper.

Valuation theory does not work so well for division algebras of degree p^a where $p = \mathrm{char}(F)$. Nonetheless, Saltman proved in [Sa_2] that $UD(F;p^a)$ is a noncrossed product for $a \geq 3$ for any field F of characteristic p. He did this by proving that certain generic abelian crossed product algebras over a field of characteristic p (these algebras were defined in [AS]) for noncyclic abelian p-groups G have only G occurring as the Galois group of a maximal subfield.

The first examples of noncrossed products over fields of prime characteristic were given by Schacher and Small in [SS]. Papers [Am_2], [Ri_3] and [Ri_4] gave further examples of noncrossed products. All these noncrossed product division algebras were generic division algebras $UD(F;n)$; all were proved to be noncrossed products using Th. 5.1 or a generalization of Th. 5.1 noted in [FS], see also [Ri_4, Lemma 5], which says that if $UD(F;n)$ contains a Galois extension field of its center with Galois group G, then every division algebra of degree n over any field $M \supseteq F$ contains a field Galois over M with group G. Saltman showed in [Sa_1] how these generic noncrossed products could be used to build further noncrossed product algebras D where one could arrange that $\mathrm{ind}(D)$ exceeds $\exp(D)$. Here is an abstraction of the key step in Saltman's result, which appears in [JW_2, Th. 5.15(b)]:

Proposition 5.7 *Let D be an inertially split division algebra over a Henselian valued field F. If $\overline{D}$ is not a crossed product, then D is not a crossed product.*

This is proved by first showing that if K is a maximal subfield of D which is Galois over F, then $\overline{K} \cdot Z(\overline{D})$ is a maximal subfield of $\overline{D}$ which is normal over $Z(\overline{D})$. Then, one can invoke the following result of Saltman [Sa_1, Lemma 3], which is interesting in its own right, and not so easy to prove:

Proposition 5.8 *Let F be any field, and let $D \in \mathcal{D}(F)$. If D has a maximal subfield K which is normal over F, then D has a maximal subfield L which is Galois over F. (So D is a crossed product.)*

What Saltman showed in [Sa_1, Th. 2(d)] is that if $D \in \mathcal{D}(F)$ and if L is a cyclic Galois field extension of F such that $D \otimes_F L$ is a noncrossed product division algebra (e.g., $D = UD(F;n)$, so that $D \otimes_F L \cong UD(L;n)$, which we know is a noncrossed product for suitable n), then $E = D \otimes_F (L((x))/F((x)), \sigma, x)$ is a division algebra (see Th. 3.7(d) above). We have that E is inertially split with respect to the complete discrete rank 1 valuation on its center $F((x))$, and $\overline{E} \cong D \otimes_F L$ (see Th. 3.7(a)); so by Prop. 5.7, since $D \otimes_F L$ is not a crossed product, E is also not a crossed product. But by Th. 3.7, $\deg(E) = \deg(D) \cdot [L : F]$ while $\exp(E) = \exp(D)$ if $[L : F] \bigm| \deg(D)$. By iterating this process, one can obtain noncrossed product division algebras with the degree exceeding the exponent by as large an amount as one wishes.

All the noncrossed products we have described so far have been built from generic division algebras. But, once the valuation theory behind these constructions has been understood, another way of obtaining noncrossed products is suggested:

Instead of using two different valued division algebras to show that a generic division algebra is not a crossed product, find a division algebra $D \in \mathcal{D}(F)$ with two different valuations on it. Use the two valuations to obtain incompatible information about $\mathcal{G}(K/F)$ if K is a maximal subfield of D which is Galois over F. Hence, there can be no such K, and D is not a crossed product. A typical result in this vein is:

Proposition 5.9 *Let p be a prime number, and let $D \in \mathcal{D}(F)$ be a division algebra of degree p^3 with $\operatorname{char}(F) \neq p$, such that D has two different valuations v_1 and v_2, say with value groups $\Gamma_{D,i}$ for v_i and $\Gamma_{F,i}$ for $v_i|_F$, for $i = 1, 2$. Suppose D is TTR with respect to each v_i, and suppose $\Gamma_{D,1}/\Gamma_{F,1} \cong (\mathbb{Z}/p\,\mathbb{Z})^6$ while $\Gamma_{D,2}/\Gamma_{F,2} \cong (\mathbb{Z}/p^3\,\mathbb{Z})^2$. Then D is not a crossed product.*

For, if K is a maximal subfield of D which is Galois over F, then K is tame and totally ramified over F with respect to each $v_i|_K$. So, by Prop. 4.1(a), $\mathcal{G}(K/F) \cong \Gamma_{K,i}/\Gamma_{F,i}$ for $i = 1, 2$, and $\Gamma_{K,i}/\Gamma_{F,i}$ is a subgroup of order p^3 of $\Lambda_{D,i} = \Gamma_{D,i}/\Gamma_{F,i}$. But $\Lambda_{D,1}$ and $\Lambda_{D,2}$ have no common subgroups of order p^3, so there can be no such K.

But how can one construct such a D as in Prop. 5.9? It is certainly not a tensor product of cyclic algebras, since it is not a crossed product. An approach to this was found in [JW$_1$], where we started with a division algebra E_0 over a ground field F_0 such that E_0 is a tensor product of symbol algebras, and $\deg(E_0) = p^k$ for $k > 3$; we then extended scalars by some field $F \supseteq F_0$, so that $E_0 \otimes_{F_0} F \cong M_{p^{k-3}}(D)$, and D is the desired division algebra of degree p^3. Of course, it is often difficult (often *very* difficult) to get at the properties of the underlying division algebra of a central simple algebra. We were able to do so in this case by taking F to be an intersection of two Henselian valued fields and proving that a division algebra over F is determined by its scalar extensions to the Henselian fields. This was done by proving a "local-global" principle for $\operatorname{Br}(F)$, which we now describe, since it is of some interest in its own right.

Take a prime number p and a field F_0 with $\mu_p \subseteq F_0$ with two independent valuations w_1 and w_2 such that $\operatorname{char}(\overline{F_{0,i}}) \neq p$. (The independence means that there is no valuation ring $T \not\subseteq F_0$ such that T contains the valuation ring of each w_i.) Let L_i be a field such that $F_0 \subseteq L_i \subseteq F_{0\,\mathrm{sep}}$ (the separable algebraic closure of F_0) such that L_i has a Henselian valuation v_i with $v_i|_{F_0} = w_i$ for $i = 1, 2$. Let $F = L_1 \cap L_2$, which has the induced valuations $v_1|_F$, $v_2|_F$. Let $G_p(F) = \mathcal{G}(\widetilde{F}_p/F)$ where $\widetilde{F}_p$ is the maximal p-extension of F, which is the compositum of all the finite Galois extensions of F of degree a power of p; so $G_p(F)$ is a pro-p-group. Let $\operatorname{Br}(F)(p)$ denote the p-primary part of $\operatorname{Br}(F)$. Since $\mu_{p^n} \subseteq \widetilde{F}_p$ for all n, it follows from the Merkurjev-Suslin Theorem that $\operatorname{Br}(\widetilde{F}_p)(p) = 0$, so $\operatorname{Br}(F)(p)$ is precisely the kernel of the map $\operatorname{Br}(F) \to \operatorname{Br}(\widetilde{F}_p)$. For $[A] \in \operatorname{Br}(F)(p)$, let p-$\operatorname{ind}(A)$ be the minimal degree over F of fields $K \supseteq F$ such that K splits A and $K \subseteq \widetilde{F}_p$ (so $\operatorname{ind}(A) \leq p\text{-}\operatorname{ind}(A)$). The following local-global principle was proved in [JW$_1$, Th. 4.3, Th. 4.11].

Theorem 5.10 *In the setup just defined,*

(a) *$G_p(F) \cong G_p(L_1) *_p G_p(L_2)$, where $*_p$ denotes the free product in the category of pro-p groups.*

(b) *The scalar extension maps $\operatorname{Br}(F) \to \operatorname{Br}(L_i)$ induce an isomorphism*

$$\operatorname{Br}(F)(p) \;\cong\; \operatorname{Br}(L_1)(p) \times \operatorname{Br}(L_2)(p).$$

(c) *For any central simple algebra A over F with $[A] \in \mathrm{Br}(F)(p)$, we have*

$$\exp(A) \;=\; \operatorname{lcm}\big(\exp(A \otimes_F L_1), \exp(A \otimes_F L_2)\big)$$

and

$$p\text{-ind}(A) \;=\; \operatorname{lcm}\big(\,p\text{-ind}(A \otimes_F L_1), p\text{-ind}(A \otimes_F L_2)\big).$$

Example 5.11 With F_0, L_i, v_i, and $F = L_1 \cap L_2$ as in Th. 5.10, assume $\mu_{p^3} \subseteq F_0$ and pick algebras B_1 and $B_2 \in \mathcal{D}(F_0)$, each of degree p^3 such that w_1 on F_0 extends to a valuation on B_1 so that B_1 is TTR with $\Gamma_{B_1}/\Gamma_{F_0,1} \cong (\mathbb{Z}/p\,\mathbb{Z})^6$ and $\Gamma_{B_1} \cap \Gamma_{L_1} = \Gamma_{F_0,1}$, while L_2 splits B_1; likewise, choose B_2 so that L_1 splits B_2 while w_2 extends to a valuation on B_2 so that B_2 is TTR and $\Gamma_{B_2}/\Gamma_{F_0,2} \cong (\mathbb{Z}/p^3\,\mathbb{Z})^2$ and $\Gamma_{B_2} \cap \Gamma_{L_2} = \Gamma_{F_0,2}$. (It is easy to build such B_1 and B_2 as a tensor product of symbol algebras, using the approximation theorem to get the desired conditions with respect to each valuation. See [JW$_1$, §5] for details.) Then, let D be the underlying division algebra of $(B_1 \otimes_{F_0} B_2) \otimes_{F_0} F$, so $D \in \mathcal{D}(F)$. We have $[D \otimes_F L_1] = [B_1 \otimes_{F_0} L_1]$ in $\mathrm{Br}(L_1)$, while $[D \otimes_F L_2] = [B_2 \otimes_F L_2]$ in $\mathrm{Br}(L_2)$. The valuation on B_i shows that $B_i \otimes_{F_0} L_i$ is a division algebra (see Prop. 2.8 above), and v_i on L_i extends to a valuation on $B_i \otimes_{F_0} L_i$ with $B_i \otimes_{F_0} L_i$ TTR and $\Gamma_{B_i \otimes_{F_0} L_i}/\Gamma_{L_i} \cong \Gamma_{B_i}/\Gamma_{F_0,i}$, so in particular, $\mathrm{ind}(B_i \otimes_{F_0} L_i) = p\text{-ind}(B_i \otimes_{F_0} L_i) = p^3$. Hence, $\mathrm{ind}(D \otimes_F L_i) = p\text{-ind}(D \otimes_F L_i) = p^3$, for $i = 1, 2$. It follows by Th. 5.10(c) that $\mathrm{ind}(D) = p\text{-ind}(D) = p^3$. Therefore, $D \otimes_F L_i \cong B_i \otimes_{F_0} L_i$, so the valuation on $B_i \otimes_{F_0} L_i$ restricts to a valuation on D such that D is TTR, with $\Gamma_{D,i}/\Gamma_{F,i} \cong \Gamma_{B_i \otimes_{F_0} L_i}/\Gamma_{L_i} \cong \Gamma_{B_i}/\Gamma_{F_0,i}$. So, the conditions of Prop. 5.9 are met, and D is not a crossed product. Note that this occurs over a field F for which $\mathrm{Br}(F)(p)$ is relatively well-understood, by Th. 5.10.

In the paper [JW$_1$], by further application of Th. 5.10 and similar results when we do not require $\mu_p \subseteq \overline{F}$, noncrossed products of index p^2 were constructed (when $p \neq \mathrm{char}(F)$), and also noncrossed products with index exceeding the exponent, and also decomposable noncrossed product algebras.

In [Br$_1$] E. Brussel found another way to play one valuation off against another to build noncrossed product algebras. He thereby obtained noncrossed products over $L((x))$ and $L(x)$ for any algebraic number field L. This generalizes to show (see [Br$_8$]) that there are noncrossed product division algebras over any field F finitely generated over its prime field P, provided that F has transcendence degree at least 1 over P if $\mathrm{char}(F) = 0$, and transcendence degree at least 2 over P if $\mathrm{char}(F) > 0$. Thus, we have a striking dichotomy between (i) division algebras over algebraic number fields or over algebraic function fields in one variable over a finite field—these division algebras are always cyclic crossed products; and (ii) division algebras over any larger fields finitely generated over their prime fields—these division algebras need not be crossed products.

The existence of noncrossed products over $L((x))$ for L an algebraic number field is somewhat more surprising than over $L(x)$. For, even though $L((x))$ is a much bigger field than $L(x)$ it is arithmetically much simpler, due to its Henselian valuation.

Here is a sketch of Brussel's approach. Let F_0 be an algebraic number field, and let $F = F_0((x))$. Let B be a suitably chosen division algebra over F_0 and let

L be a cyclic Galois extension of F_0, say with $\mathcal{G}(L/F_0) = \langle\sigma\rangle$. Then, let

$$E = (B \otimes_{F_0} F) \otimes_F (L((x))/F, \sigma, x), \tag{5.2}$$

and let D be the underlying division algebra of E. While E is always a crossed product (since we can find a maximal subfield of B Galois over F_0 and linearly disjoint to L over F_0), Brussel shows that under some conditions D will not be a crossed product. He does this by getting different and incompatible information about the Galois group G of a maximal subfield of D Galois over F, using properties of B and L with respect to different $\mathfrak{p}$-adic completions of F_0. Another, equivalent, way of looking at this is to observe that the discrete valuation rings of F_0 can be composed with the standard complete discrete valuation on $F_0((x))$ (see the discussion preceding Prop. 3.1) to yield rank 2 valuations on $F_0((x))$ with value group $\mathbb{Z}\times\mathbb{Z}$. One can arrange that D has different properties with respect to some of these valuations to find incompatible constraints on G.

This approach yields noncrossed product algebras over $F = F_0((x))$, for F_0 an algebraic number field. But with a little care we also get noncrossed products over $F_0(x)$. For, with the B and L as in (5.2), let $E_1 = (B \otimes_{F_0} F_0(x)) \otimes_{F_0(x)} (L(x)/F_0(x), \sigma, x)$, and let D_1 be the underlying division algebra of E_1. Since $E_1 \otimes_{F_0(x)} F \cong E$, whenever $\operatorname{ind}(E_1) = \operatorname{ind}(E)$, and this is often easily arranged, then $D_1 \otimes_{F_0(x)} F \cong D$. If D is a noncrossed product over F, then D_1 is necessarily a noncrossed product over $F_0(x)$.

We now give a specific example constructed by Brussel in [Br$_1$]. We work in the setup of (5.2). Note first that with respect to the standard complete discrete (so Henselian) valuation on $F = F_0((x))$, the E of (5.2) is the tensor product of an inertial division algebra $B \otimes_{F_0} F$ and a nicely semiramified division algebra $(L((x))/F, \sigma, x)$. So, by Th. 3.7, for the underlying division algebra of D of E we have $\Gamma_D = \Gamma_{(L((x))/F,\sigma,x)} = \frac{1}{[L\,:\,F]}\mathbb{Z}$, and $\overline{D} = B_L$, the underlying division algebra of $B \otimes_{F_0} L$. So, $[D:F] = [B_L : F_0][L : F_0]$ and $\exp(D) = \operatorname{lcm}\big(\exp(B), [L:F]\big)$.

Example 5.12 Fix a prime number p. For our algebraic number field F_0, let r and s be the maximal integers such that $\mu_{p^r} \subseteq F_0$ and $\mu_{p^s} \subseteq F_0(\mu_{p^{r+1}})$. If $r = 0$, set $\ell = s+1$, $m = 0$, and $n = s$; if $r > 0$, set $\ell = 2r+1$ and $m = n = r$. Pick four prime spots $\mathfrak{q}_0$, $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$ of F_0 not dividing 2 or p, such that $\mathfrak{q}_0$ and $\mathfrak{q}_3$ each split completely in $F_0(\mu_{p^{\ell-n}})$, $\mathfrak{q}_1$ splits completely in $F_0(\mu_{p^m})$ but not in $F_0(\mu_{p^{m+1}})$, and $\mathfrak{q}_2$ splits completely in $F_0(\mu_{p^n})$ but not in $F_0(\mu_{p^{n+1}})$. (The Cebotarev Density Theorem (see, e.g., [N, Ch. 5, Th. 6.4]) assures the existence of such $\mathfrak{q}_i$.) Then, let B be the F_0-division algebra of degree p^ℓ with local invariants

$$\operatorname{inv}(B_{\mathfrak{q}_0}) = \operatorname{inv}(B_{\mathfrak{q}_1}) = -\operatorname{inv}(B_{\mathfrak{q}_2}) = -\operatorname{inv}(B_{\mathfrak{q}_3}) = 1/p^\ell \text{ in } \mathbb{Q}/\mathbb{Z},$$

and $\operatorname{inv}(B_{\mathfrak{p}}) = 0$ for all other prime spots $\mathfrak{p}$ of F_0. (One knows that such a B exists because the sum of the local invariants is 0, cf. [CF, p. 188, Th. B].) Let L be a cyclic Galois field extension of F_0 of degree p^n such that L has local degree p^n at each of the $\mathfrak{q}_i$ and L/F_0 is unramified at $\mathfrak{q}_0$, $\mathfrak{q}_1$, and $\mathfrak{q}_3$, and is totally ramified at $\mathfrak{q}_2$. Such an L exists by the Grunwald-Wang Theorem [AT, Ch. 10, Th. 5]. Then, Brussel shows in [Br$_1$, Lemma 5] that there is no Galois extension K of F_0 of degree p^ℓ such that $K \supseteq L$ and K splits B. Then for the underlying division algebra D of the central simple F-algebra E of (5.2), Brussel shows that D has degree p^ℓ and deduces from the absence of K as just described that D is not a crossed product. Note that the degree of D is p^{2r+1} if $\mu_p \subseteq F_0$, and is p^{s+1} if $\mu_p \not\subseteq F_0$.

Brussel has built some interesting variations on this type of example in [Br_1] and subsequent papers. For example, in [Br_1] he constructed a division algebra D over $F = F_0((x))$ (F_0 an algebraic number field) such that D is a crossed product, but every Galois group of a maximal subfield Galois over F is isomorphic to the same nonabelian group of order p^3. In [Br_2] he gives an example of a division algebra which not only is not a crossed product but cannot be embedded in any crossed product algebra with the same center. In [Br_3] he analyzes very thoroughly the decomposability and the different possible tensor product decompositions of every division algebra underlying the E as in (5.2). This work was based in part on his analysis in [Br_5] of the totally ramified subfields of tame division algebras over $F_0((x))$. The information he gets about the totally ramified subfields is remarkably complete, given that there is no easy classification of such fields. This is in marked contrast to the situation with unramified subfields of D, which correspond (up to isomorphism) to the subfields of $\overline{D}$ which are separable over $\overline{F}$.

The noncrossed product algebras constructed by Brussel over $F_0((x))$ were all inertially split with respect to the standard valuation on $F_0((x))$ (indeed, in the discrete rank 1 case everything is inertially split). Recently, T. Hanke in [Han] has proved a nice criterion for inertially split algebras over any Henselian field to be noncrossed products:

Theorem 5.13 *Let D be an inertially split division algebra over a Henselian valued field. Then D is a crossed product iff $\overline{D}$ has a maximal subfield which is Galois over $\overline{F}$ (not just over $Z(\overline{D})$).*

Recently, in [Br_7], Brussel has given a construction of noncrossed product algebras over $\widehat{\mathbb{Q}}_p(x)$, where $\widehat{\mathbb{Q}}_p$ is the p-adic completion of $\mathbb{Q}$. For this, he did not pull back from $\widehat{\mathbb{Q}}_p((x))$, because the residue field of its x-adic valuation, $\widehat{\mathbb{Q}}_p$, has only one useful valuation; hence, there is no apparent possibility of playing off one valuation against another over $\widehat{\mathbb{Q}}_p((x))$. (It is quite possible that every division algebra over $\widehat{\mathbb{Q}}_p((x))$ is a crossed product.) However, there is another significant discrete valuation on $\widehat{\mathbb{Q}}_p(x)$, the so-called Gaussian valuation: Let v_p be the usual complete discrete rank 1 valuation on $\widehat{\mathbb{Q}}_p$, with $\Gamma_{\widehat{\mathbb{Q}}_p} = \mathbb{Z}$ and $\overline{\widehat{\mathbb{Q}}_p} \cong \mathbb{Z}/p\,\mathbb{Z}$. For $f = \sum_{i=0}^{n} c_i x^{\ell} \in \widehat{\mathbb{Q}}_p[x] - \{0\}$, define $v(f) = \min\{\, v_p(c_i) \mid c_i \neq 0\}$. It is easy to check that v extends to a well-defined valuation v on $\widehat{\mathbb{Q}}_p(x)$ via $v(f/g) = v(f) - v(g)$, with value group $\mathbb{Z}$ and residue field $\mathbb{Z}/p\,\mathbb{Z}(x)$. This residue field is a global field, with lots of discrete valuations to work with, which can be composed with v to yield corresponding rank 2 valuations on $\widehat{\mathbb{Q}}_p(x)$. Brussel uses the residue valuations to construct noncrossed product algebras on the completion of $\widehat{\mathbb{Q}}_p(x)$ with respect to v, and arranges that his noncrossed products actually be defined over $\widehat{\mathbb{Q}}_p(x)$. There are significant technical obstacles in all of this, which we will not pursue here.

Very recently, there has been an interesting new construction of noncrossed product algebras by Reichstein and Youssin, the first which does not involve either generic methods or valuation theory. Instead, they use methods of invariant theory and algebraic group actions on varieties. See their paper [RY] for details.

6 $SK_1(D)$

If A is a central simple algebra over a field F, then for any $a, b \in A^*$, we have $\mathrm{Nrd}(aba^{-1}b^{-1}) = 1$ in F^* since the reduced norm is multiplicative and the target group is abelian. Thus, every product of commutators in A^* has reduced norm 1. It is natural to ask whether the converse is true. To formalize this, let $SL(A)$ be the special linear group of A

$$SL(A) = \{a \in A^* \mid \mathrm{Nrd}(a) = 1\}, \tag{6.1}$$

let $[A^*, A^*]$ denote the multiplicative commutator group of A, and let

$$SK_1(A) = SL(A)\big/[A^*, A^*]. \tag{6.2}$$

(This is the "special" subgroup of $K_1(A) = A^*/[A^*, A^*]$.) Our question then becomes whether $SK_1(A)$ is trivial.

The question of the triviality of $SK_1(A)$ was open for many years. It was called the "Artin-Tannaka problem" in the Russian literature. It was rather a surprise when Platonov gave in 1975 the first examples of central simple algebras with nontrivial SK_1 (see [P_1], [P_2], [P_4]). For, it seems that the general feeling at that time was that $SK_1(A)$ was probably always trivial. The results proved up to then had seemed to point in that direction. For, it was known that when we express $A \cong M_n(D)$ with D a division ring, then $SK_1(A) \cong SK_1(D)$ via the Dieudonné determinant (see [D_2, pp. 147–148, Cor. 1; p. 155, (3)]). Also, Wang had shown in 1950 in [Wa] that if $\mathrm{ind}(A)$ is square-free, then $SK_1(A)$ is trivial. He showed also in the same paper that if the ground field F is an algebraic number field, then $SK_1(A) = 1$ for every central simple algebra A over F. (The corresponding result for F a local field had been done earlier by Nakayama and Matsushima in [NM].) Also, in the late 1950's papers had been published purporting to prove the triviality of $SK_1(A)$ for every A over every field F.

A significant source of the interest in the question of the triviality of $SK_1(A)$ came from the theory of algebraic groups. Let $\mathfrak{G}$ be a quasisimple algebraic group over a field F, and let $G = \mathfrak{G}(F)$, its group of F-rational points. (Quasisimple means that every proper normal algebraic subgroup of $\mathfrak{G}$ is finite.) If $\mathfrak{G}$ is isotropic (i.e., G has a subgroup isomorphic to the additive group of F), let G^+ be the normal subgroup of G generated by transvections. Then the factor group of G/G^+ is called the *reduced Whitehead group* of $\mathfrak{G}$ over F, denoted $W(\mathfrak{G}, F)$. The Kneser-Tits conjecture [Ti_1, p. 315] was that $W(\mathfrak{G}, F) = 1$ for every simply connected quasi-simple algebraic group $\mathfrak{G}$ over every field F. However, take any $D \in \mathcal{D}(F)$ and let $\mathfrak{G} = SL_n(D)$ for $n \geq 2$, i.e., $\mathfrak{G}$ is the algebraic group with L-points $\mathfrak{G}(L) = SL(M_n(D \otimes_F L))$ for each field $L \supseteq F$. Then, $\mathfrak{G}$ is simply-connected and isotropic (as $n \geq 2$), and $G = \mathfrak{G}(F) = SL(M_n(D))$; also, one knows that $W(\mathfrak{G}, F) = SK_1(M_n(D)) \cong SK_1(D)$. Thus, examples of D with $SK_1(D)$ nontrivial also yield counterexamples to the Kneser-Tits conjecture. See Tits' Séminaire Bourbaki report [Ti_2] for more on $W(\mathfrak{G}, F)$ and $SK_1(D)$, including a discussion of the classes of algebraic groups for which the conjecture is known to be true.

Building on his work on the nontriviality of SK_1, Platonov was also able to give counterexamples settling other significant open questions about algebraic groups. He showed in [P_8], [P_9] that there exist division rings of arbitrary degree (> 1) over a suitable field F with discrete rank 1 valuation v and completion $\widehat{F}_v$, such that the closure of $SL(D)$ in $SL(D \otimes_F \widehat{F}_v)$ is not all of $SL(D \otimes_F \widehat{F}_v)$. This defeated the possibility suggested by Kneser of a weak approximation theorem for simply connected algebraic groups. Also, it had been an open question whether the underlying affine variety of a simply connected algebraic group is rational (i.e., its function field is purely transcendental over the ground field). Platonov had shown in [P_{10}] that if the variety of $SL_1(D)$ were rational, then there would be a finite upper bound n, such that every element of $SL(D)$ is a product of at most n commutators; hence, $SK_1(D)$ would have to be trivial. Thus, for every division algebra D with $SK_1(D)$ nontrivial, the variety of $SL(D)$ is not rational.

A further interesting characterization of $SK_1(D)$ was given by Voskresenskiĭ in [V_1] in terms of R-equivalence. If X is an irreducible variety over a field F, two points x, y of $X(F)$ (i.e., F-rational points of X) are said to be *strongly R-equivalent* if there exist a morphism $\alpha : \mathbb{P}^1_F \to X$ and F-rational points a, b of $\mathbb{P}^1_F$ such that $\alpha(a) = x$ and $\alpha(b) = y$. Then, x and y are said to be *R-equivalent* if there are $x_0, \ldots, x_n \in X(F)$ with $x_0 = x$, $x_n = y$ with each x_i and x_{i+1} strongly R-equivalent. Clearly, R-equivalence is an equivalence relation, and one writes $X(F)/R$ for the set of R-equivalence classes. When X is an algebraic group, $X(F)/R$ has the structure of a group. R-equivalence was introduced by Manin in [Ma], where he was studying cubic surfaces; this notion was first studied extensively for algebraic groups by Colliot-Thélène and Sansuc in [CTS]. Voskresenskiĭ showed in [V_1] (see also [V_2, p. 186, Cor. 1]) that if A is a central simple algebra over a field F, then (for the algebraic group $\mathfrak{G} = SL_1(A)$, for which $\mathfrak{G}(F) = SL(A)$),

$$SL(A)/R \cong SK_1(A).$$

Thus, it is all the more of interest not only to know when $SK_1(A)$ is nontrivial, but also to compute $SK_1(A)$ when possible.

Valuation theory was a key part of Platonov's original construction of division algebras D with nontrivial $SK_1(D)$, and it has been used in most of the work on this topic since then. We will here first give a very elementary example pointed out by Draxl in [D_2, p. 168, Th. 1] of a division algebra with $SK_1(D)$ nontrivial; then we will describe Platonov's construction; then we will sketch an approach of Ershov, which gives a way of computing $SK_1(D)$ for D a tame division algebra over a Henselian valued field.

Example 6.1 Let F_0 be a field containing a primitive 4-th root of unity ω (so $\mathrm{char}(F_0) \neq 2$), let $F = F_0((x_1))((x_2))((x_3))((x_4))$ with its standard Henselian valuation, and let D be the tensor product of quaternion algebras $\left(\frac{x_1, x_2}{F}\right) \otimes_F \left(\frac{x_3, x_4}{F}\right)$. Then, $\mathrm{Nrd}(\omega) = 1$. But $\omega \notin [D^*, D^*]$. Hence $SK_1(D)$ is nontrivial. For, as $\omega \in F$ and $\deg(D) = 4$, $\mathrm{Nrd}(a) = \omega^4 = 1$. By Ex. 4.4(ii) above D is a division ring with $\Gamma_D = (\frac{1}{2}\mathbb{Z})^4$, and D is TTR. Thus, we have the canonical pairing $\beta_D : \Lambda_D \times \Lambda_D \to \mu(\overline{F})$ of (4.1), where recall $\beta_D(v(a) + \Gamma_F, v(b) + \Gamma_F) = \overline{aba^{-1}b^{-1}}$ in $\overline{F}$. Since $\exp(\Lambda_D) = 2$, we have $\mathrm{im}(\beta_D) \subseteq \{\pm 1\}$. (This is all elementary to verify, and does not use the nondegeneracy of β_D.) It follows that for the canonical projection $\pi : V_D \to \overline{D} = \overline{F}$, we have $\pi^{-1}\{\pm 1\}$ contains every commutator. Hence, $[D^*, D^*] \in \pi^{-1}\{\pm 1\}$, while $\omega \notin \pi^{-1}\{\pm 1\}$.

In fact, $|SK_1(D)| = 2$ here. This is not so easy to see, but follows from Remark 6.2 below. It is quite remarkable that a question that had been open and of interest for so long should have such an elementary counterexample. Of course, in Ex. 6.1 we could have used $F_0(x_1, \ldots, x_4)$ instead of the iterated Laurent power series field for the ground field F. Also, the same elementary construction can be given, using two symbol algebras of degree n for any integer $n \geq 2$ over a field $F_0((x_1))\cdots((x_4))$ containing a primitive n^2 root of unity to obtain a D of degree n^2 with $SK_1(D)$ nontrivial.

Ex. 6.1 is an elementary example of a tensor product of two quaternion algebras with nontrivial SK_1. Merkurjev has actually shown that this nontriviality holds "in general." Specifically, he proved in [Me$_1$, Prop. 2] that if A is a central simple division algebra over any field F of characteristic not 2, and if $4 \mid \mathrm{ind}(A)$, then there is a field $L \supseteq F$ such that $SK_1(A \otimes_F L)$ is nontrivial.

Platonov's original examples of division algebras D with $SK_1(D)$ nontrivial were over twice iterated power series fields. We indicate the key steps in computing $SK_1(D)$ for these examples without proof, following the description given in [Ti$_2$], where proofs can be found. Thorough treatments may also be found in [DK] or in [P$_4$].

We can analyze Platonov's examples by first considering the case of a ground field F with a Henselian discrete rank 1 valuation. Let $D \in \mathcal{D}(F)$, and assume that D is tame over F (see Prop. 3.9), which in this case means that $Z(\overline{D})$ is separable over $\overline{F}$ and $\mathrm{char}(\overline{F}) \nmid |\ker(\theta_D)|$. The comments after Th. 3.10 show that D is inertially split. So, Th. 3.4 shows that $Z(\overline{D})$ is cyclic Galois over $\overline{F}$, and $\Gamma_D/\Gamma_F \cong \mathcal{G}(Z(\overline{D})/\overline{F})$ via θ_D. The crucial result which allows one to express $SK_1(D)$ in terms of residue data is the following inclusion (the "congruence theorem" in Platonov's terminology):

$$(1 + M_D) \cap SL(D) \subseteq [D^*, D^*]. \tag{6.3}$$

(See [H$_2$] for a nice short proof of (6.3) (assuming $\mathrm{char}(\overline{F}) \nmid \deg(D)$) using Wedderburn's factorization theorem.) It follows immediately from (6.3) that (with bars denoting images in $\overline{D}$)

$$SK_1(D) \cong \overline{SL(D)}\big/\overline{[D^*, D^*]}. \tag{6.4}$$

Fix any generator π of the maximal ideal M_D of V_D, and let $\gamma : \overline{D}^* \to \overline{D}^*$ be the automorphism given by $\overline{d} \mapsto \overline{\pi d \pi^{-1}}$. For the terms in (6.4), one calculates that

$$\overline{SL(D)} = \{\overline{d} \in \overline{D} \mid N_{Z(\overline{D})/\overline{F}}(\mathrm{Nrd}(\overline{d})) = 1\}, \tag{6.5}$$

where $N_{Z(\overline{D})/\overline{F}}$ denotes the norm from $Z(\overline{D})$ to $\overline{F}$, and

$$\overline{[D^*, D^*]} = \{\gamma(\overline{d})\overline{d}^{-1} \bigm| \overline{d} \in \overline{D}^*\} \cdot [\overline{D}^*, \overline{D}^*]. \tag{6.6}$$

If we let $N = \mathrm{Nrd}(\overline{D}^*) \subseteq Z(\overline{D})^*$, $A = \{\overline{d} \in N \bigm| N_{Z(\overline{D})/\overline{F}}(\overline{d}) = 1\}$, and $B = \{\gamma(\overline{d})\overline{d}^{-1} \bigm| \overline{d} \in N\}$, then it follows that there is an exact sequence

$$SK_1(\overline{D}) \longrightarrow SK_1(D) \longrightarrow A/B \longrightarrow 1. \tag{6.7}$$

In his original construction Platonov considered a ground field $F = F_0((x))((y))$ and a pair of linearly disjoint cyclic Galois field extensions L_1 and L_2 of F_0, say with $\mathcal{G}(L_i/F_0) = \langle \sigma_i \rangle$, $i = 1, 2$, and computed $SK_1(E)$ for

$$E = (L_1 \cdot F/F, \sigma_1, x) \otimes_F (L_2 \cdot F/F, \sigma_2, y). \tag{6.8}$$

Since L_1 and L_2 are linearly disjoint over F_0, we know by Ex. 2.7 and Prop. 2.8 that E is a division algebra over F; clearly, $\deg(E) = [L_1 : F][L_2 : F]$. We are used to viewing F from the perspective of its standard rank 2 Henselian valuation, as in Ex. 3.6. But here instead, let v be the complete discrete rank 1 (so Henselian) y-adic valuation on F, so $V_F = F_0((x))[[y]]$ and $\overline{F} = F_0((x))$. For the extension of this valuation to the E of (6.8), we have E is inertially split with $\Gamma_E = \frac{1}{[L_2\,:\,F_0]}\mathbb{Z}$ and $Z(\overline{E}) = L_2((x))$, and

$$\overline{E} \;=\; (L((x))/L_2((x)), \sigma_1, x),$$

where $L = L_1 \cdot L_2$. Here, we can take a uniformizer π of V_D to be the standard generator of $(L_2{\cdot}F/F, \sigma_2, y)$ such that $\pi^{[L_2\,:\,F_0]} = y$ and $\pi c \pi^{-1} = \sigma_2(c)$ for $c \in L_2{\cdot}F$ (where σ_2 acts on an iterated Laurent series by acting on the coefficients in L_2). So, the map γ on $Z(\overline{E}) = L_2((x))$ is σ_2. Now, we have the complete discrete rank 1 x-adic valuation on $Z(\overline{E})$, which extends to a valuation on $\overline{E}$. If we let $\overline{\overline{E}}$ be the residue division algebra for this valuation on $\overline{E}$, then $\overline{\overline{E}} = L$. We now invoke the exact sequence (6.7) with $\overline{E}$ for D; since $\overline{\overline{E}}$ is a field, Hilbert's Th. 90 shows that $A = B$ in (6.7), so $SK_1(\overline{E}) = 1$. Another application of (6.7), now with E for D, yields for the E of (6.8)

$$SK_1(E) \;\cong\; A/B, \tag{6.9}$$

where $N = \operatorname{Nrd}(\overline{E}^*) \subseteq L_2((x))$, $A = \{a \in N \mid N_{L_2((x))/F_0((x))}(a) = 1\}$, and $B = \{\sigma_2(b)b^{-1} \mid b \in N\}$.

Platonov gives in [P$_4$] a number of ways of reformulating (6.9) to make the dependence of $SK_1(E)$ on the fields F_0, L_1, L_2 clearer. For this, note that for the A, B of (6.9), we have $A, B \subseteq U_{L_2((x))}$, the units with respect to the x-adic valuation on $L_2((x))$. Let $\overline{A}$, $\overline{B}$ be the images of A, B in the residue field $\overline{L_2((x))} = L_2$. Then, one finds that

$$\overline{A} \;=\; \{a \in \operatorname{im}(N_{L/L_2}) \mid N_{L_2/F_0}(a) = 1\} \quad \text{and} \quad \overline{B} \;=\; \{\sigma_2(b)b^{-1} \mid b \in \operatorname{im}(N_{L/L_2})\}.$$

(For $\overline{A}$ use that a valuation unit of $\overline{E}$ has the form cu with $c \in L$ and $u \in 1 + M_{\overline{E}}$; then $\operatorname{Nrd}(cu) = N_{L/L_2}(c)\operatorname{Nrd}(u)$, with $\operatorname{Nrd}(u) \in 1 + M_{L_2((x))}$. For $\overline{B}$, take ρ a standard generator of $\overline{E}$ such that $\rho^{[L\,:\,L_2]} = x$; then $\operatorname{Nrd}(\rho) = (-1)^{\varepsilon}x \in N$, where $\varepsilon = [L : L_2] - 1$. So, if we take any $b \in N$, then $b = d[(-1)^{\varepsilon}x]^i$ with $d \in U_{L_2((x))}$, so $d \in N$ and $\sigma_2(b)b^{-1} = \sigma_2(u)u^{-1}$.) Furthermore, $1 + M_{L_2((x))} \subseteq \operatorname{im}(N_{L((x))/L_2((x))}) \subseteq N$, as the valuation on $L_2((x))$ is Henselian (cf. [Er$_2$, Prop. 2] or [S, Ch. V, §2, Prop. 3(a)]). It follows easily from this that the natural map $A \to \overline{A}$ induces an isomorphism

$$A/B \;\cong\; \overline{A}/\overline{B}. \tag{6.10}$$

For a group G and a G-module C, let $\widehat{H}^{-1}(G, C)$ denote the (-1)-st Tate cohomology group of G with coefficients in C, i.e.,

$$\widehat{H}^{-1}(G, C) \;=\; \{c \in C \mid \sum_{g \in G} g \cdot c = 0\}\big/\langle\{g \cdot c - c \mid g \in G, c \in C\}\rangle.$$

In the situation at hand, with $L = L_1{\cdot}L_2$, and $\mathcal{G}(L/F_0) = \langle\sigma_1\rangle \times \langle\sigma_2\rangle$, this translates to

$$\widehat{H}^{-1}(\mathcal{G}(L/F_0), L^*) \;\cong\; \{c \in L^* \mid N_{L/F_0}(c) = 1\}\big/\langle\{\sigma_1(c)c^{-1}, \sigma_2(c)c^{-1} \mid c \in L^*\}\rangle. \tag{6.11}$$

Thus, for the $\overline{A}$ and $\overline{B}$ of (6.10) the norm map N_{L/L_2} induces an isomorphism

$$\widehat{H}^{-1}(\mathcal{G}(L/F_0), L^*) \;\cong\; \overline{A}/\overline{B}. \tag{6.12}$$

The group of "special projective conorms" $\mathcal{P}(L, L_2, F_0)$ in Platonov's terminology [P_4] is defined to be

$$\mathcal{P}(L, L_2, F_0) \;=\; \{a \in L_2^* \mid \sigma_2(a)a^{-1} \in N_{L/L_2}(L^*)\}\big/\big(F_0^* \cdot N_{L/L_2}(L^*)\big).$$

Observe that the group homomorphism $a \mapsto \sigma_2(a)a^{-1}$ induces an isomorphism

$$\mathcal{P}(L, L_2, F_0) \cong \overline{A}/\overline{B}. \tag{6.13}$$

Also, Serre pointed out that

$$\mathcal{P}(L, L_2, F_0) \cong \operatorname{Br}(L/F_0)\big/\big(\operatorname{Br}(L_1/F_0) + \operatorname{Br}(L_2/F_0)\big), \tag{6.14}$$

where $\operatorname{Br}(L/F_0)$ denotes the relative Brauer group $\ker(\operatorname{Br}(F_0) \to \operatorname{Br}(L))$. See [$P_4$, §4] or [Su, Lemma 1.13(2)] for a proof of (6.14). By combining (6.9)–(6.14), we have the following isomorphisms for the E of (6.8):

$$\begin{aligned} SK_1(E) \;&\cong\; A/B \;\cong\; \overline{A}/\overline{B} \;\cong\; \widehat{H}^{-1}(\mathcal{G}(L/F_0), L^*) \;\cong\; \mathcal{P}(L, L_2, F_0) \\ &\cong\; \operatorname{Br}(L/F_0)\big/\big(\operatorname{Br}(L_1/F_0) + \operatorname{Br}(L_2/F_0)\big). \end{aligned} \tag{6.15}$$

If, for example, F_0 is an algebraic number field or a local field, then one can use (6.15) together with standard results in class-field theory to compute $|SK_1(E)|$ for various choices of L_1 and L_2. Several cases of this type were treated in [P_4].

If $D \in \mathcal{D}(F)$ and $\operatorname{ind}(D) = p_1^{r_1} \cdots p_k^{r_k}$ with the p_i distinct primes and each $r_i > 0$, then Wang showed (see [Wa, p. 334] or [D_2, p. 164, Th. 4]) that the abelian group $SK_1(D)$ is torsion, with exponent dividing $\operatorname{ind}(D)/(p_1 \cdots p_k)$. By using (6.15) above, Gräbe has shown in [Gr] that every countably infinite torsion abelian group of finite exponent occurs as $SK_1(D)$ for some division algebra D. (Draxl had shown earlier in [D_1] that every finite abelian group occurs as $SK_1(D)$.)

In [Er_1] and [Er_2] Ershov gave a calculation of $SL_1(D)$ for an arbitrary tame $D \in \mathcal{D}(F)$ for any Henselian valued field F. We describe his result here. For such D, let

$$\ell \;=\; \sqrt{|\ker(\theta_D)|}, \tag{6.16}$$

where θ_D is the map of (2.13) above. (So, ℓ is an integer by Prop. 3.9(iii) and Prop. 2.6, and $\ell = 1$ iff D is inertially split, cf. Th. 3.4.) As usual, let U_D denote the group of valuation units of D, and M_D the maximal ideal of the valuation ring V_D. By using the fact that for any tamely ramified field extension K of F, we have $N_{K/F}(1 + M_K) = 1 + M_F$ ([Er_2, p. 65, Prop. 2]), Ershov verified that the congruence theorem, (6.3) above, holds in this setting with the same proof as in the discretely valued case; more precisely, in [Er_2, Prop. 3] he shows,

$$(1 + M_D) \cap SL(D) \;\subseteq\; [U_D, D^*] \;\; (= \langle\{udu^{-1}d^{-1} \mid u \in U_D,\ d \in D^*\}\rangle)\,. \tag{6.17}$$

For $d \in V_D$, we have $\overline{\operatorname{Nrd}(d)} = (N_{Z(\overline{D})/\overline{F}}(\operatorname{Nrd}(\overline{d})))^{\ell}$ (see [Er_2, p. 65, Cor. 2]) for the ℓ of (6.16). (The exponent ℓ did not appear in the discrete case considered earlier, since there every tame D is inertially split, so $\ell = 1$.) Ershov deduces [Er_2, p. 66, Lemma 1] that

$$\overline{SL(D)} = \{c \in \overline{D} \mid N_{Z(\overline{D})/\overline{F}}(\operatorname{Nrd}(c))^{\ell} = 1\}\,.$$

Let

$$\begin{aligned} C &= \{u \in SL(D) \mid N_{Z(\overline{D})/\overline{F}}(\mathrm{Nrd}(\overline{u})) = 1\}\big/[U_D, D^*] \\ &\cong \{c \in \overline{D} \mid N_{Z(\overline{D})/\overline{F}}(\mathrm{Nrd}(c)) = 1\}\big/\overline{[U_D, D^*]}\,. \end{aligned}$$

(The isomorphism follows from (6.17).) Then, Ershov proves in [Er_2, (4)] that the following diagram with the obvious maps has exact rows and columns:

$$\begin{array}{ccccccccccc} & & & & & & 1 & & & & \\ & & & & & & \downarrow & & & & \\ & & 1 & \longrightarrow & \mathcal{K} \longrightarrow SK_1(\overline{D}) & \longrightarrow & C & \longrightarrow & \widehat{H}^{-1}(\mathcal{G}(Z(\overline{D})/\overline{F}), \mathrm{Nrd}(\overline{D}^*)) & \longrightarrow & 1 \\ & & & & & & \downarrow & & & & \\ 1 & \longrightarrow & [D^*, D^*]\big/[U_D, D^*] & \longrightarrow & & & SL(D)\big/[U_D, D^*] & \longrightarrow & SK_1(D) & \longrightarrow & 1 \\ & & & & & & \downarrow & & & & \\ & & & & & & \mu_\ell \cap \overline{F}^* & & & & \\ & & & & & & \downarrow & & & & \\ & & & & & & 1 & & & & \end{array} \tag{6.18}$$

where $\mathcal{K} = \big(\overline{[U_D, D^*]} \cap SL(\overline{D})\big)\big/[\overline{D}^*, \overline{D}^*]$ and $\mu_\ell \cap \overline{F}^*$ denotes the group of those ℓ-th roots of unity which lie in $\overline{F}$. He shows further [Er_2, p. 69, 5.] that

The group $[D^*, D^*]\big/[U_D, D^*]$ is generated by commutators $aba^{-1}b^{-1}$ as a and b range over representatives of generators of the group Γ_D/Γ_F. (6.19)

Observe that for the E of (6.8) considered by Platonov, the formula $SK_1(E) \cong \widehat{H}^{-1}(\mathcal{G}(L/F_0), L^*)$ in (6.15) falls out quickly from this, since then $\ell = 1$, $[E^*, E^*] = [U_E, E^*]$ by (6.19) as Γ_E/Γ_F is cyclic, $\overline{D} = \overline{\overline{E}} = L$, a field, and $SK_1(\overline{D}) = 1$.

Remark 6.2 Consider the case where F is a field with Henselian valuation and $D \in \mathcal{D}(F)$ is TTR. Then, the ℓ in (6.16) equals $\deg(D)$, and the C in (6.18) is trivial. By (6.19) it follows that the image of $[D^*, D^*]\big/[U_D, D^*]$ in $\mu_\ell \cap \overline{F}^*$ in (6.18) coincides with the image of the canonical pairing of (4.1) above. Since the canonical pairing is nondegenerate, by Prop. 4.2, it follows from (6.18) that

$$SK_1(D) \text{ is a cyclic group with } |SK_1(D)| = |\mu_{\deg(D)} \cap \overline{F}^*|\big/\exp(\Gamma_D/\Gamma_F)\,. \tag{6.20}$$

In particular, we can see that for the D of Ex. 6.1 above, we have $|SK_1(D)| = 2$, but if we did not have $\mu_4 \subseteq F_0$ there, then $SK_1(D)$ would be trivial.

For another perspective on $SK_1(D)$, the reader may wish to consult Suslin's interesting paper, [Su]. In particular Suslin conjectures [Su, Conj. 1.16]: If $D \in \mathcal{D}(F)$ for any field F, and $\mathrm{char}(F) \nmid n$ where $n = \deg(D)$, then there is a canonical homomorphism

$$f : SK_1(D) \to H^4(F, \mu_n^{\otimes 3})\big/\big([D] \cup H^2(F, \mu_n^{\otimes 2})\big)$$

(where $[D]$ is the image of D in $H^2(F, \mu_n)$) such that Platonov's description of $SK_1(E)$ for the E of (6.8) over $F_0((x))((y))$ would be obtainable by composing f with successive residue maps associated with discrete valuations. Suslin does not prove this conjecture but gets something very close (off by a factor of 2), assuming that $\mu_{n^3} \subseteq F_0$, by using some sophisticated K-theoretic techniques.

The simply connected algebraic groups of inner type with Dynkin diagram of type A_n over a field F are all of the form $SL_1(A)$ where A is a central simple algebra of degree $n+1$ over F. (That is $SL_1(A)$ is the algebraic group with K-points $SL_1(A)(K) = SL(A \otimes_F K)$ for any field $K \supseteq F$.) The simply connected outer type groups of type A_n are the special unitary groups $SU(B, \sigma)$, where B is a central simple algebra of degree $n+1$ over separable quadratic extension L of F, and σ is an L/F involution of the second kind on B, i.e., an involution on B such that $\sigma|_L$ is the nontrivial F-automorphism of L. Then, (the F-points of) $SU(B, \sigma)$ equals $\{b \in B^* \mid \mathrm{Nrd}(b) = 1 \text{ and } b\sigma(b) = 1\}$. The associated reduced Whitehead group for $SU(B, \sigma)$ is $W(SU(B, \sigma)) = B^*/\Sigma_\sigma(B)$, where $\Sigma_\sigma(B) = \langle \{b \in B^* \mid \sigma(b) = b\} \rangle$. There is a theory for these groups $W(SU(B, \sigma))$ analogous to that for $SK_1(B)$, developed principally by Yanchevskiĭ and Platonov. Again, the reduced Whitehead group can be nontrivial, but has been computed for certain cases, such as for the E of (6.8) over $F_0((x))((y))$. See, e.g., [Y_4], [Y_5], [PY_3].

7 Other constructions

We give in this section some further examples of division algebras with interesting properties which have been obtained using valuation theory.

(a) indecomposable division algebras.

A division algebra $D \in \mathcal{D}(F)$ is said to be *decomposable* if $D \cong D_1 \otimes_F D_2$ where each $\deg(D_i) > 1$. It is obviously desirable to know whether a given D is decomposable, since if so D can be studied in terms of the smaller division algebras D_i. Of course, one always has the primary decomposition of D (also called the "Sylow decomposition" of D): If $\deg(D) = p_1^{r_1} \dots p_k^{r_k}$ where the p_i are distinct primes, then $D \cong D_1 \otimes_F \cdots \otimes_F D_k$ where $\deg(D_i) = p_i^{r_i}$, and each D_i is uniquely determined up to isomorphism (though typically there are many different copies of each D_i in D). Thus, the study of decomposability is immediately reduced to the case where $\deg(D)$ is a prime power. Valuation theory has been used in many of the constructions of indecomposable division algebras.

The first serious investigation of decomposability in the prime power case seems to have been given by D. Saltman in [Sa_3]. Suppose $\deg(D) = p^n$, where p is prime. It is immediate that if $\exp(D) = \deg(D)$, then D is indecomposable. So, the interesting division algebras for decomposability questions are those of degree p^n and exponent p^m where $m < n$. Saltman gave in [Sa_3] the first example of such a division algebra which is indecomposable. His approach is analogous to Amitsur's noncrossed product constructions. Namely, he first constructed a division algebra $R(p^n, p^m)$ which is generic among division algebras of degree p^n and exponent p^m in that there is a specialization property for them analogous to the one for $UD(F; n)$ given in (5.1) above. Using this, he showed that if there is a nontrivial decomposition $R(p^n, p^m) = A_1 \otimes A_2$ then every division algebra D of degree p^n and exponent dividing p^m over any field $L \supseteq F$ admits a decomposition $D \cong E_1 \otimes_L E_2$ with $\deg(E_i) = \deg(A_i)$ for $i = 1, 2$. Thus, to show that $R(p^n, p^m)$ is indecomposable, it suffices to exhibit division algebras D_1, D_2 satisfying the index and exponent conditions, which may be decomposable, but do not admit tensor factors of the same degree. We have already seen such algebras in §4:

Example 7.1 (a) Let D_1 be a TTR division algebra over a Henselian valued field F, with $\Lambda_{D_1} = \Gamma_{D_1}/\Gamma_F \cong (\mathbb{Z}/p^3\,\mathbb{Z})^2 \times (\mathbb{Z}/p\,\mathbb{Z})^2$; so $\deg(D_1) = p^4$ and $\exp(D_1) = \exp(\Lambda_{D_1}) = p^3$. Then if $D_1 \cong A \otimes_F B$ we have $\Lambda_{D_1} \cong \Lambda_A \times \Lambda_B$, by Th. 4.5(c), (d) (which shows $\Lambda_A \cap \Lambda_B = (0)$) and Prop. 2.8. Since A and B must be TTR, each must have a nondegenerate canonical pairing (see Prop. 4.2) so the invariant factors for Λ_A and Λ_B occur in pairs. Thus, the only possibility is $\Lambda_A \cong (\mathbb{Z}/p^3\,\mathbb{Z})^2$ and $\Lambda_B \cong (\mathbb{Z}/p\,\mathbb{Z})^2$, so $\deg(A) = p^3$ and $\deg(B) = p$, or vice versa.

(b) Likewise, let D_2 be a TTR division algebra over F with $\Lambda_{D_2} \cong (\mathbb{Z}/p^2\,\mathbb{Z})^4$. Then, if $D_2 \cong S \otimes_F T$ nontrivially, then we must have $\Lambda_S \cong \Lambda_T \cong (\mathbb{Z}/p^2\,\mathbb{Z})^2$, so $\deg(S) = \deg(T) = p^2$.

We can build such D_i as in Ex. 7.1, for example, with $F = F_0((x_1))\cdots((x_4))$, where F_0 is a field with $\mu_{p^3} \subseteq F_0$ (see Ex. 4.4(ii)). It follows by Saltman's specialization result that for any field F with $\mathrm{char}(F) \neq p$, the division ring $R(p^4, p^3)$ over F is indecomposable.

Another way to produce indecomposable division algebras is to build a division algebra $D \in \mathcal{D}(F)$ with two different valuations v_1 and v_2 which produce incompatible constraints on the size of a tensor factor of D. For example, if D is TTR with respect to each v_i and has Γ_D/Γ_F like D_1 in Ex. 7.1(a) relative to v_1 and like D_2 for v_2, then the same reasons as given there show that D must be indecomposable. Such a D can be constructed where F is the intersection of two Henselian valued fields by the methods used to build noncrossed products in [JW$_1$], cf. Ex. 5.11 above.

Another constraint on the decomposability of D was pointed out by Saltman in [Sa$_3$, Lemma 3.2]: For $n \in \mathbb{N}$, let $D^{\otimes n}$ denote the underlying division algebra of $\bigotimes_{i=1}^{n} D$. An old result of Albert [A$_5$, p. 76, Lemma 7] says that if a prime number $p \mid \mathrm{ind}(D)$, then $p\,\mathrm{ind}(D^{\otimes p}) \mid \mathrm{ind}(D)$. Saltman noted that if $\mathrm{ind}(D)$ is a power of p and $\mathrm{ind}(D^{\otimes p}) = \mathrm{ind}(D)/p$, then D is indecomposable. For, if $D = A \otimes_F B$ is a nontrivial decomposition, then

$$\mathrm{ind}(D^{\otimes p}) \;=\; \mathrm{ind}(A^{\otimes p} \otimes B^{\otimes p}) \;\Big|\; \mathrm{ind}(A^{\otimes p}) \cdot \mathrm{ind}(B^{\otimes p}) \;\Big|\; \mathrm{ind}(D)/p^2,$$

contrary to the assumption on $\mathrm{ind}(D^{\otimes p})$. This observation can be combined with the index reduction formula for function fields of Brauer-Severi varieties to obtain further indecomposable division algebras, as described in Ex. 7.2 (no valuation theory is used here).

Example 7.2 Let p be a prime number, and let $E \in \mathcal{D}(F)$ with $\mathrm{ind}(E) = \exp(E) = p^n$ for $n \geq 3$. For any integer $i, 2 \leq i \leq n-1$, let K be the function field of the Brauer-Severi variety of $E^{\otimes p^i}$. So, K is a generic splitting field of $E^{\otimes p^i}$. Let $D = E \otimes_F K$. Then,

$$\mathrm{ind}(D) = p^n,\ \exp(D) = p^i,\ \text{and}\ \mathrm{ind}(D^{\otimes p}) = p^{n-1}. \tag{7.1}$$

Hence, D is indecomposable by the remarks above. To verify the assertions in (7.1), first note that $\mathrm{ind}(E^{\otimes p}) = p^{n-1}$ since $p^{n-1} = \exp(E^{\otimes p}) \mid \mathrm{ind}(E^{\otimes p}) \mid \mathrm{ind}(E)/p$. Likewise, $\mathrm{ind}(E^{\otimes p^j}) = \exp(E^{\otimes p^j}) = p^{n-j}$ for $0 \leq j \leq n$. Then, one can invoke the index reduction formula of Schofield and van den Bergh [SB, Th. 1.6] which says that for any central simple algebra A over F,

$$\mathrm{ind}(A \otimes_F K) = \gcd_{1 \leq k \leq \exp(E^{\otimes p^i})} \big(\mathrm{ind}(A \otimes_F E^{\otimes p^i k})\big). \tag{7.2}$$

(That is, the index reduction of A on passing to K is no more than what is forced by the fact that K splits $E^{\otimes p^i}$.) It follows that $\mathrm{ind}(D^{\otimes p^j}) = \mathrm{ind}(E^{\otimes p^j}) = p^{n-j}$ for

$0 \le j < i$, while K splits $E^{\otimes p^i}$, yielding 7.1. For more on index reduction formulas on passage to certain kinds of function field extensions, see [MPW_1], [MPW_2], [Me_2], and the references given there.

The preceding examples yield indecomposable division algebras of degree p^n and exponent p^i for $2 \le i \le n-1$, but do not yield indecomposables of exponent p. The first examples of indecomposable division algebras D of prime exponent were constructed by Amitsur, Rowen, and Tignol in [ART] for $p = 2$ (with $\deg(D) = 8$) and by Tignol in [T_6] for p odd (with $\deg(D) = p^2$). In light of Albert's result in [A_3, Th. 6] that a division algebra of degree 4 and exponent 2 is a tensor product of two quaternion algebras, the value of $\deg(D)$ in the [ART] paper was as small as possible. The constructions in these papers are also easier to understand using valuation theory, even though valuations did not appear explicitly in them. The key point is given in the following proposition; for this, recall from (3.14) above the group $\mathrm{Dec}(K/F)$ for an abelian Galois extension K of a field F.

Proposition 7.3 *Let p be a prime number and let F be a Henselian valued field with $\mu_p \subseteq \overline{F}$. Take any p-Kummer extension L of $\overline{F}$, and any division algebra $B \in \mathrm{Br}(L/\overline{F})$ with $\exp(B) = p$. Let $I \in \mathcal{D}(F)$ be the inertial lift of B to F, let N be any nicely semiramified division algebra over F with $\overline{N} \cong L$, and let D be the underlying division algebra of $I \otimes_F N$. Then $\overline{D} = L$, $\mathrm{ind}(D) = [L : F]$, and $\exp(D) = p$. Furthermore, D is a tensor product of p-symbol algebras iff $[B] \in \mathrm{Dec}(L/\overline{F})$.*

The information on $\overline{D}$, $\mathrm{ind}(D)$, and $\exp(D)$ in Prop. 7.3 follows from Th. 3.7 above. The last assertion of the proposition follows by [JW_2, Prop. 4.8, Th. 5.15(c)]. Note that the required N will exist whenever $|\Gamma_F/p\Gamma_F| \ge [L : \overline{F}]$. Thus, we can assure that such an N is available if we take, say $F = \overline{F}((x_1))\cdots((x_k))$ with k sufficiently large. Prop. 7.3 shows that in order to find an indecomposable division algebra D of degree p^2 for p odd, it suffices to find a p-Kummer extension L of a field $\overline{F}$ with $[L : \overline{F}] = p^2$ and $B \in \mathrm{Br}(L/\overline{F}) - \mathrm{Dec}(L/\overline{F})$ with $\exp(B) = p$. This, which is still by no means easy, was done by Tignol in [T_6]. Likewise, in [ART] a 2-Kummer field L of $\overline{F}$ was found with $[L : \overline{F}] = 8$ and a $B \in \mathrm{Br}(L/\overline{F}) - \mathrm{Dec}(L/\overline{F})$ with $\exp(B) = 2$. The corresponding D given by Prop. 7.3 is then indecomposable since otherwise Albert's result on division algebras of degree 4 and exponent 2 would show that D is a tensor product of quaternion algebras, which is ruled out by Prop. 7.3.

In [J_2] B. Jacob used an elaboration of this technique to construct indecomposable division algebras of index p^n and exponent p for any prime p and any $n \ge 2$ ($n \ge 3$ if $p = 2$). Specifically, he constructed an example of a p-Kummer extension L of a field $\overline{F}$ with $\mu_p \subseteq \overline{F}$ and $[L : \overline{F}] = p^n$, and a division algebra B over $\overline{F}$ such that $\exp(B) = p$ and $[B] \in \mathrm{Br}(L/\overline{F})$, but $[B] \notin \mathrm{Dec}(L/\overline{F}) + \mathrm{Br}(L_1/\overline{F}) + \mathrm{Br}(L_2/\overline{F})$ for any proper subfields L_1, L_2 of L such that $L_1 \cap L_2 = \overline{F}$. For the I, N, and D built as in Prop. 7.3 from B and L over a Henselian valued field F with residue field the given $\overline{F}$, it is shown in [J_2, Th. 3.3] that D is indecomposable of exponent p and index p^n.

The examples of indecomposables of prime exponent given in [ART], [T_6], and [J_2] were all over fields of characteristic 0, because they required $\overline{F}$ to have characteristic 0 since they used arithmetic properties of a valuation on $\overline{F}$ with residue field of characteristic p in order to obtain the desired $B \notin \mathrm{Dec}(L/\overline{F})$. Another approach to obtaining indecomposable division algebras of prime exponent, not using valuation theory, was given subsequently by N. Karpenko. Let E be a division algebra over a field F (of any characteristic) with $\mathrm{ind}(E) = \exp(E) = p^n$ where p is prime and $n \geq 2$ ($n \geq 3$ if $p = 2$). Let K be the function field of the Brauer-Severi variety of $E^{\otimes p}$. Then, $\exp(E \otimes_F K) = p$, and the Schofield-van den Bergh index reduction formula given in (7.2) above (where now $i = 1$) shows that $\mathrm{ind}(E \otimes_F K) = p^n$, i.e., $E \otimes_F K$ is a division algebra. By using Chow group calculations, Karpenko shows in [Ka$_1$, Th. 3.1] that $E \otimes_F K$ is indecomposable.

(b) common subfields.

If $D_1, D_2 \in \mathcal{D}(F)$, their tensor product $D_1 \otimes_F D_2$ is often not a division algebra. When possible, one would like to find what it is that relates D_1 and D_2 and "causes" the zero divisors in $D_1 \otimes_F D_2$. The most apparent circumstance preventing $D_1 \otimes_F D_2$ from being a division algebra is if each D_i contains a copy of a field $L \supsetneqq F$. For, then $D_1 \otimes_F D_2$ contains $L \otimes_F L$, which always has zero divisors, as $1 < [L : F] < \infty$. The question naturally arose long ago whether all appearances of zero divisors in $D_1 \otimes_F D_2$ could be accounted for in this way:

If $D_1 \otimes_F D_2$ is not a division ring, must D_1 and D_2 contain a common subfield $L \supsetneqq F$? (7.3)

This did not seem very likely. But there was the positive result of Albert in 1931 in [A$_1$, Th. 3] saying that the answer is yes if D_1 and D_2 are both quaternion algebras, i.e., $\deg(D_1) = \deg(D_2) = 2$. Risman gave a generalization of Albert's result in [Ri$_2$] by showing that if D_2 is a quaternion algebra and $D_1 \otimes_F D_2$ is not a division ring, then either D_1 and D_2 contain a common field $L \supsetneqq F$ or D_1 contains a splitting field K of D_2 with $[K : F] > 2$ (or both). Risman also gave an example where the second case occurs but not the first. The difficulty in obtaining an example lies in the fact that the D_i can contain so many subfields that they are difficult to classify or even to describe completely. The first example providing a negative answer to question (7.3) for division algebras D_i of the same degree was given in [TW] in 1987. Here is that example:

Example 7.4 Let $n > 1$ be an odd integer. Let F_0 be a field with $\mu_n \subseteq F_0$ and let $\omega \in \mu_n^*(F_0)$. Let t, x, y be independent indeterminates, let $F = F_0(t)((x))((y))$, and let $D_1 = A_\omega(x, y; F)$ and $D_2 = A_\omega(x(t-1)/y, xt; F)$. Then, with respect to the standard Henselian valuation on F with $\overline{F} = F_0(t)$ and $\Gamma_F = \mathbb{Z} \times \mathbb{Z}$, we have D_1 and D_2 are each tame totally ramified division algebras (see Ex. 4.4(i)). One can check that $D_1 \otimes_F D_2 \cong M_n(A_\omega(xt^s, ty^2/(t-1); F))$, where $s = (n+1)/2$. However, the subfields L of each D_i are determined up to isomorphism by Γ_L, by Th. 4.5(c), so it is easy to enumerate them and to check that there is no $L \supsetneqq F$ shared by D_1 and D_2.

At the time this example appeared in [TW, Prop. 5.1], D. Saltman pointed out that for these D_i, even though $D_1 \otimes_F D_2$ is not a division algebra, $D_1^{op} \otimes_F D_2$ is a division algebra, where D_1^{op} is the opposite algebra of D_1. Thus, since D_1 and D_1^{op} have the same subfields, they can have none in common with D_2. This led to suggestions by Saltman and Rowen that a more reasonable question than (7.3) might be:

If $\mathrm{ind}(D_1 \otimes_F D_2^{\otimes k}) < \mathrm{ind}(D_1) \cdot \mathrm{ind}(D_2)$ for all $k, 1 \leq k < \exp(D_2)$, must D_1 and D_2 have a common subfield $L \supsetneqq F$? (7.4)

Mammone gave a negative answer to this question in [Mam, §3] with the following example:

Example 7.5 For any $n > 1$, let F_0 be a field with $N_n \le F_0$, and let $\omega \in \mu_n^*(F_0)$. Let x, y, z, t_1, t_2 be independent indeterminates, and let $F = F_0(x, y, z)((t_1))((t_2))$ with its standard Henselian valuation with $\overline{F} = F_0(x, y, z)$ and $\Gamma_F = \mathbb{Z} \times \mathbb{Z}$. Let

$$D_1 \,=\, A_\omega(x(1-y), t_1; F) \otimes_F A_\omega(x(1-z), t_2; F) \quad \text{and} \quad D_2 \,=\, A_\omega(x, yz; F)\,,$$

which are division algebras over F, by Ex. 3.6 and Ex. 2.7. Then, D_2 is unramified over F, while D_1 is semiramified with $\overline{D_1} = \overline{F}\big(\sqrt[n]{x(1-y)},\ \sqrt[n]{x(1-z)}\big)$. Mammone pointed out that in $\mathrm{Br}(F)$,

$$D_1 \otimes_F D_2^{\otimes k} \,\sim\, A_\omega(x(1-y), t_1 y^k; F) \otimes_F A_\omega(x(1-z), t_2 z^k; F)\,,$$

which has index at most n^2. However, a nontrivial common subfield of the D_i would have to be unramified over F. Mammone ruled this out by showing that if L is any minimal proper field extension of $\overline{F}$ in $\overline{D_1}$, then $L \otimes_{\overline{F}} \overline{D_2}$ is a division ring (so $L \otimes_F D_2$ is a division ring by Prop. 2.8 above).

Question (7.4) was particularly tantalizing in the case where D_1 and D_2 each have prime index. A negative answer for this case was given in [JW$_3$]. The key to obtaining a counterexample was provided by the following proposition (see [JW$_3$, Th. 1]). Here, we write $N_{F_0}(a)$ for the image of the norm map $N\colon F_0(\sqrt[p]{a}) \to F_0$.

Proposition 7.6 *Let p be a prime number, and let F_0 be a field with $\mu_p \subseteq F_0$, and let $\omega \in \mu_p^*(F_0)$. Let $a, c, d \in F_0^*$ with $a, c \notin F_0^p$ and $F_0(\sqrt[p]{a}) \ne F_0(\sqrt[p]{c})$. Let $F = F_0((x))$, and let $D_1 = A_\omega(a, x; F)$ and $D_2 = A_\omega(c, dx; F)$, which are each division algebras of degree p over F. Then,*

(a) *For $1 \le i \le p-1$, $D_1^{\otimes i} \otimes_F D_2$ is not a division algebra iff $A_\omega\big(c, d; F_i(\sqrt[n]{a^i c})\big)$ is not a division algebra;*
(b) *D_1 and D_2 have no common subfield $L \supsetneq F$ iff $d \notin N_{F_0}(a) \cdot N_{F_0}(c)$.*

As in so many constructions using valuation theory, Prop. 7.6 reduces question (7.4) for D_1 and D_2 to a more stringent question concerning the residue algebras: Can we find a field F_0 with elements a, c, d such that $A_\omega(c, d; F_0(\sqrt[p]{a^i c}))$ is split for $1 \le i \le p-1$ but $d \notin N_{F_0}(a) \cdot N_{F_0}(c)$? In [JW$_3$] an example was given for each odd prime p of a field F_0 with a, c, d satisfying these conditions. The argument used a valuation in F_0 such that the residue field has characteristic p.

Further light has been shed on common subfield questions in an interesting recent paper [Ka$_2$] by Karpenko, which uses Chow group calculations and nontrivial results in algebraic geometric intersection theory. Among other things, Karpenko shows: Let A be division algebra of prime degree p over a field F, and let $B_1, \ldots, B_{p-1}$ be division algebras of degrees $p_1^{n_1}, \ldots, p_{p-1}^{n_{p-1}}$; let $n = n_1 + \cdots + n_{p-1}$. If no $A \otimes_F B_i$ is a division algebra, then there is a field extension E of F with $[E : F] \le p^n$ such that E splits each of $A, B_1, \ldots, B_{p-1}$. Notice that when $p = 2$ this recovers Risman's generalization of Albert's result.

8 Orderings on finite dimensional division algebras

Valuation theory has been a key tool in the study of orderings on fields, see, e.g., [L$_1$], [Pr]. Some of this has been generalized to orderings on division algebras infinite dimensional over their centers, as in the work of M. Marshall and his associates in [LMZ], [MZ_1], and [MZ_2]. However, our focus here is on finite dimensional division algebras. For these, one might think that the consideration of orderings was ended by Albert's theorem in [A_4] (or see [L_2, Th. 18.10, Cor. 18.11]), which says that there is no ordering on a noncommutative division ring finite dimensional over its center—that is, no total ordering satisfying the usual conditions that the sum and the product of positive elements is positive. Nonetheless, there has been some work in the finite dimensional case. Albert's result has been sidestepped by considering division algebras D with an involution σ, and defining orderings so that a sign is attached to every element $d \in D$ with $\sigma(d) = d$, but not to all elements of D. Different possible definitions of such orderings have been given, and valuation theory has been helpful in understanding them. The reader can find an extensive account of this topic in the paper [Cr_7] by Craven in these proceedings. Therefore, we will not pursue it here.

9 Total valuation rings

In the preceding sections, we have considered the invariant valuation rings associated with valuations on finite dimensional division algebras. We now turn to more general types of valuation rings, for which there is no associated valuation, at least in the familiar sense. In this section we will consider total valuation rings. Recall that a subring V of a division ring D is called a *total valuation ring* of D if for every $d \in D^*$, we have $d \in V$ or $d^{-1} \in V$. The terminology seems to be due to P. M. Cohn [C_1]. It follows that the left ideals of V are linearly ordered by inclusion, since this holds for the principal ideals; likewise for the right ideals. But, the left ideals are not the same as the right ideals, unless V is actually an invariant valuation ring. Clearly, if E is any sub-division ring of D, then $V \cap E$ is a total valuation ring of E. While every invariant valuation ring is also a total valuation ring, the following example demonstrates that there do exist non-invariant total valuation rings on finite dimensional division algebras. The first such example was constructed by Gräter in [G_1].

Example 9.1 Let F be a field, and let $D \in \mathcal{D}(F)$ such that D has an invariant valuation ring W_D, with residue ring $\overline{D}$, and $\pi_D : W_D \to \overline{D}$ the canonical projection. Set $W_F = W_D \cap F$, a valuation ring of F, say with residue field $\overline{F}$, and let $\pi_F : W_F \to \overline{F}$ be the canonical map. Suppose $\overline{D}$ is a field; then $\overline{D}$ is normal over $\overline{F}$ by Prop. 2.5. Suppose that $\overline{D}$ strictly contains $\overline{F}$, and that there is a valuation ring Y of $\overline{F}$, such that Y has k different extensions to $\overline{D}$, call them $Z_1, \ldots, Z_k$, with $k > 1$. Let $V_i = \pi_D^{-1}(Z_i)$, for $1 \le i \le k$, and let $V_F = \pi_F^{-1}(Y)$. The V_i are "composites" of the Z_i and W_D, analogous to the commutative case described before Prop. 3.1 above. An easy calculation shows that each V_i is a total valuation ring of D with $V_i \cap F = V_F$. Since each Z_i extends Y, there is $\sigma_i \in \mathcal{G}(\overline{D}/\overline{F})$ with $\sigma_i(Z_1) = Z_i$. Then for the surjective (see Prop. 2.5) map $\theta_D : \Gamma_D/\Gamma_F \to \mathcal{G}(\overline{D}/\overline{F})$ of (2.13) associated with its invariant valuation ring W_D, if we take any $a_i \in D^*$ with $\theta_D(v(a_i) + \Gamma_F) = \sigma_i$, then $a_i V_1 a_i^{-1} = V_i$. Consequently, $V_1, \ldots, V_k$ are all conjugate in D, and, as $k > 1$, the V_i are not invariant valuation rings. It is known (see Th. 10.3(b) below) that since W_D is an invariant valuation ring, it is the only total valuation ring of D contracting to W_F in F. From this it follows easily that $V_1, \ldots, V_k$ are the only total valuation rings of D contracting to V_F in F.

Let us give a more specific example illustrating Ex. 9.1. Let n be any positive integer, with $n > 1$. Let F_0 be a field containing a primitive n-th root of unity ω. Let x and y be indeterminates, and let $F = F_0(x)((y))$. Let $D = A_\omega(1{+}x, y; F)$. By Ex. 2.7 above, the complete discrete rank 1 y-adic valuation of F (with valuation ring $W_F = F_0(x)[[y]]$ and residue field $\overline{F} = F_0(x)$) extends to a valuation on the division ring D, with valuation ring say W_D, and residue ring the field $\overline{D} = F_0(\sqrt[n]{1+x})$. Because $1 + x$ is a 1-unit with respect to the x-adic valuation on $F_0(x)$ (whose valuation ring is $Y = F_0[x]_{(x)}$), this x-adic valuation has n different extensions to $\overline{D}$, one for each of the maximal ideals of the ring $Y[\sqrt[n]{1+x}]$, which is the integral closure of Y in $\overline{D}$. So, Ex. 9.1 yields n different but conjugate total noninvariant valuation rings of D, each contracting to the rank 2 valuation ring $V_F = F_0[x]_{(x)} + y(F_0(x)[[y]])$ of F.

Theorem 9.2 *Let F be a field, and let V be a valuation ring of F; let $D \in \mathcal{D}(F)$. Then, there is a total valuation ring W of D with $W \cap F = V$ iff the set $T = \{d \in D \mid d$ is integral over $V\}$ is a ring. When this occurs, there are only finitely many different total valuation rings, say $W_1, \dots, W_k$, of D with $W_i \cap F = V$. Furthermore, $W_1 \cap \dots \cap W_k = T$ and k is the matrix size of $D \otimes_F F^h$, where F^h is the Henselization of F with respect to the valuation of V. Hence, $k \mid \deg(D)$.*

The matrix size of $D \otimes_F F^h$ means the integer k such that $D \otimes_F F^h \cong M_k(E)$, where E is a division ring. See [W_4, Th. G, Th. A] or [MMU_2, Th. 8.11, Th. 8.12, Th. 8.14] for proofs of Th. 9.2. Most of Th. 9.2, except for the formula for k as a matrix size, were proved originally in [BG_1], where it was proved that $k \le \deg(D)$. For the division algebra E over F^h such that $D \otimes_F F^h \cong M_k(E)$ the Henselian valuation ring on F^h extends to an invariant valuation ring V_E on E, and the residue division algebra $\overline{E}$ of V_E coincides with the residue division ring of each W_i, and Γ_E is isomorphic to a suitable notion of value group for W_i. This will be discussed in the more general context of Dubrovin valuation rings in the next section (see Th. 10.4).

Regrettably, there does not seem to be any good effective criterion for when a valuation ring of a field F extends to a total valuation ring of a given $D \in \mathcal{D}(F)$.

The example in Ex. 9.1 is rather typical of noninvariant total valuation rings. Specifically, it is known that such rings W can arise only for $D \in \mathcal{D}(F)$ when for $V = W \cap F$, there is a localization of V at a nonmaximal prime ideal P such that V_P extends to an invariant valuation ring Y on D and the valuation ring V/P of the residue field $\overline{V_P}$ has more than one extension to the center of the residue division algebra $\overline{Y}$. See [W_4, Th. D, Th. E] for more precise information. In particular, if V is a rank 1 valuation ring of F, then any total valuation ring W of $D \in \mathcal{D}(F)$ with $W \cap F = V$ is actually invariant. This was proved earlier by Cohn in [C_1, Th. 3].

10 Dubrovin valuation rings

One of the major difficulties in working with invariant or total valuation rings is their relative scarcity. Given $D \in \mathcal{D}(F)$ and a valuation ring V of F there is often no invariant or even total valuation ring W of D with $W \cap F = V$. A further challenge arises because invariant and total valuation rings are defined only in division algebras, and not for general central simple algebras. Even when we start out with division algebras, very often we need to work with central simple algebras with zero divisors, which arise, e.g., from scalar extensions or tensor products of division algebras.

A very nice way of overcoming these difficulties was found by N. Dubrovin in [Du_2], [Du_3]. He introduced a general concept of a valuation ring based on the notion of a place in the category of simple Artinian rings. (His definition is close to that of Manis in [Man] for valuation rings in commutative rings with zero divisors.) There is in general no valuation associated with Dubrovin's rings (but see Th. 10.5, Th. 10.6 below), but he showed that they enjoy many properties similar to those of valuation rings on fields (see Th. 10.1, Th. 10.2 below). These rings occur much more frequently than invariant or total valuation rings, yet still enjoy a certain amount of uniqueness (see Th. 10.3). Since Dubrovin's pioneering work, a substantial theory of these rings has been developed, and connections have been established with invariant and total valuation rings. In some cases, theorems about invariant valuation rings have been proved where essential and seemingly unavoidable use has been made of Dubrovin valuation rings.

We can here only indicate some of the results that have been proved about Dubrovin valuation rings. The proofs are in many cases quite difficult. Fortunately, there is now a book by Marubayashi, Miyamoto, and Ueda [MMU_2] which brings together many of the major results about Dubrovin valuation rings, with complete proofs. In a number of cases, the book gives later published proofs of results which are considerably simplified and clearer than the original proofs, though still by no means easy.

We begin with the definition: Let S be a simple Artinian ring (we do not assume that $\dim_{Z(S)} S < \infty$). A subring B of S, with Jacobson radical $J(B)$, is called a *Dubrovin valuation ring* of S if

(i) $B/J(B)$ is a simple Artinian ring;

(ii) For every $s \in S - B$ there exist $r, r' \in B$ such that $rs, sr' \in B - J(B)$. (10.1)

(Think of B as the ring of a place λ from the simple Artinian ring S to the simple Artinian ring $B/J(B)$, i.e., $\lambda : S \to B/J(B) \cup \infty$, where $\lambda(b) = b + J(B)$ for $b \in B$ and $\lambda(s) = \infty$ for $s \in S - B$.) It is easy to check that a Dubrovin valuation ring B of S is a total valuation ring iff $B/J(B)$ is a division ring. Hence, if S is a (commutative) field, then the Dubrovin valuation rings of S are exactly the usual valuation rings of S. Dubrovin proved in [Du_2], [Du_3] the further characterizations of Dubrovin valuation rings given in Th. 10.1, and the significant properties given in Th. 10.2.

Theorem 10.1 *Let S be a simple Artinian ring, and let B be a subring of S. Then, the following conditions are equivalent:*

(i) *B is a Dubrovin valuation ring of S.*

(ii) *$B/J(B)$ is simple Artinian, every finitely generated right ideal of B is principal, and B is a right order in S.*

(iii) *$B/J(B)$ is simple Artinian, every finitely generated right ideal of B is projective, and B is a right order in S.*

(iv) *$B/J(B)$ is simple Artinian and B is a right n-chain ring of S, where n is the matrix size of $B/J(B)$, i.e., for every $s_1, \ldots, s_k \in S$ with $k \geq n$, the B-module $s_1B + \cdots + s_kB$ is generated by some n of the s_i.*

Of course, (i) is equivalent also to the "left" versions of (ii), (iii), and (iv). That B is a right order in S means that the units B^* of B form a right Ore set in B, and the right ring of quotients $B_{B^*} \cong S$. See [MMU_2, Th. 5.11] for a proof of Th. 10.1.

Theorem 10.2 *Let B be a Dubrovin valuation ring of a simple Artinian ring S. Then,*

(a) *(two-sided ideals) The two-sided ideals B are linearly ordered by inclusion, as are the B-B-bimodules lying in S. For any two such bimodules I, J, we have $I \cdot J = J \cdot I$.*

(b) *(Morita invariance) If $S \cong M_n(T)$, then $B \cong M_n(A)$, where A is a Dubrovin valuation ring of T. Furthermore, $M_m(B)$ is a Dubrovin valuation ring of $M_m(S)$ for every positive integer m. Also, if e is any nonzero idempotent of S, then eBe is a Dubrovin valuation ring of eSe.*

(c) *(composition of places) If C is a subring of B with $J(B) \subseteq C$, then C is a Dubrovin valuation ring of S iff $C/J(B)$ is a Dubrovin valuation ring of $B/J(B)$.*

(d) *(overrings) If A is any overring of B, i.e., $B \subseteq A \subseteq S$, then A is a Dubrovin valuation ring of S, and $J(A)$ is a prime ideal B with $J(A) \subseteq J(B)$. Also, A is the right (and left) localization of B with respect to the Ore set of elements of B regular $\bmod J(A)$, and $B/J(A)$ is a Dubrovin valuation ring of $A/J(A)$.*

(e) *(relation with center) Let $F = Z(S)$, a field, and suppose $\dim_F(S) < \infty$. Let $V = B \cap F$. Then, $V = Z(B)$ and V is a valuation ring of F with maximal ideal $J(B) \cap F$. For any prime ideal Q of B, let $P = Q \cap V$. Then, P is a prime ideal of V; the central localization B_P of B (i.e., localization with respect to the set $V - P$) is an overring of B (so is a Dubrovin valuation ring); and $J(B_P) = Q$. There are one-to-one correspondences: $\{$prime ideals of $B\} \leftrightarrow \{$prime ideals of $V\} \leftrightarrow \{$overrings of $B\}$.*

See [MMU_2, Prop. 6.4, Prop. 5.14, Prop. 6.16, Th. 6.6, Th. 7.8] for proofs of the assertions in Th. 10.2. Among the examples of Dubrovin valuation rings are the following: Every invariant valuation ring and every total valuation ring is clearly a Dubrovin valuation ring. Every Azumaya algebra over a commutative valuation ring is a Dubrovin valuation ring (see [MMU_2, Prop. 7.13]). Also, if V is a discrete rank 1 valuation ring with quotient field F and S is a central simple F-algebra, then a subring B of S is a Dubrovin valuation ring of S with $B \cap F = V$ iff B is a maximal order of V in S (cf. [W_4, Ex. 1.15]). Thus, it is unsurprising that Dubrovin valuation rings have been used in studies of orders over central valuation rings in central simple algebras. In addition to the examples of Dubrovin valuation rings just mentioned, many more Dubrovin valuation rings can be constructed by "composing" valuation rings using Th. 10.2(c).

A significant feature of the Dubrovin valuation rings of a central simple algebra A is that they are determined up to isomorphism by their centers, which can be any valuation ring of (i.e., with quotient field) $Z(A)$:

Theorem 10.3 *Let A be a central simple algebra over field F, and let V be any valuation ring of F. Then,*

(a) *There is a Dubrovin valuation ring B of A with $B \cap F = V$.*

(b) *If B' is another Dubrovin valuation ring of A and $B' \cap F = V = B \cap F$, then there is $a \in A^*$ with $B' = aBa^{-1}$.*

The existence theorem, Th. 10.3(a), was proved by Dubrovin in [Du_3]; see also [MMU_2, Th. 9.4], which gives the improved proof from [BG_2, Th. 3.8]. The conjugacy theorem, Th. 10.3(b), was proved by Brungs and Gräter in [BG_2, Th. 5.4] for V of finite Krull dimension, and proved in general in [W_4, Th. A]; see [MMU_2, Th. 9.8] for the improved proof from [G_5, Th. 3.3]. Th. 10.3 suggests that Dubrovin valuation rings should be usable in analyzing the arithmetic properties of central simple algebras, much as valuation rings are used for fields.

There are close connections between Dubrovin valuation rings of central simple algebras and invariant valuation rings, which one can see via passage to the Henselization. This was first shown in [W_4]. To describe this, we observe first that there are structures associated to a Dubrovin valuation ring analogous to ones we have seen for invariant valuation rings. Let B be a Dubrovin valuation ring of a central simple algebra A over a field F, and let $V = B \cap F = Z(B)$. We write now M_V, $\overline{V} = V/M_V$ and Γ_V for the objects associated with V, instead of the notation M_F, $\overline{F}$, and Γ_F used earlier. Now, our ring B has a canonically associated residue ring

$$\overline{B} \;=\; B/J(B),$$

a simple Artinian ring. Let

$$t_B \;=\; \text{the matrix size of } \overline{B}, \tag{10.2}$$

i.e., $\overline{B} \cong M_{t_B}(E)$ for some division ring E. Since there is no valuation attached to B, it might be a little surprising that there is still an associated value group, as defined in [W_4]: Let

$$st(B) \;=\; \{a \in A^* \mid aBa^{-1} = B\}, \tag{10.3}$$

the stabilizer of B for the group action of A^* by conjugation on the set of Dubrovin valuation rings of A; $st(B)$ is also the normalizer of B^* as a subgroup of A^*. Then set

$$\Gamma_B \;=\; st(B)/B^*,$$

which we call the *value group* of B. Note that Γ_B classifies the two-sided fractional ideals I of B of the form $I = aB = Ba$ for some $a \in A^*$, and the operation in Γ_B corresponds to multiplication of these ideals. It follows from Th. 10.2(a) that Γ_B is an abelian group, which is totally ordered (by $aB^* \le a'B^*$ just when $aB \supseteq a'B$); so, we write the group operation in Γ_B additively. Note that there is also a well-defined group homomorphism

$$\theta_B : \Gamma_B/\Gamma_V \longrightarrow \mathcal{G}(Z(\overline{B})/\overline{V})$$

given by, for $a \in st(B)$ and $c \in B$ with $\overline{c} \in Z(\overline{B})$, $\theta_B(a+\Gamma_V)(\overline{c}) = \overline{aca^{-1}}$; this is analogous to the θ_D of (2.13) above. Now, let F^h be the Henselization of F with respect to the valuation ring $V = B \cap F$, and let V^h be the valuation ring of F^h which is the Henselization of V. Let

$$n_B \;=\; \text{matrix size of } A \otimes_F F^h, \tag{10.4}$$

so $A \otimes_F F^h \cong M_{n_B}(D^h)$ where $D^h \in \mathcal{D}(F^h)$. Define the *defect* of B by

$$\delta(B) \;=\; [A:F]\big/\big([\overline{B}:\overline{V}]|\Gamma_B:\Gamma_V|(n_B/t_B)^2\big). \tag{10.5}$$

Because V^h is Henselian, we know from Cor. 2.2 that there is a unique invariant valuation ring R of D^h with $R \cap F^h = V^h$. We write M_R, $\overline{R}$, Γ_R, θ_R for the objects associated with R that were earlier denoted M_{D^h}, $\overline{D^h}$, Γ_{D^h}, θ_{D^h}. Of course, R has a defect $\delta(R)$ defined by (10.5) with R replacing B, etc.; but since $n_R = t_R = 1$, note that $\delta(R)$ coincides with the defect $\delta(D^h)$ defined in (2.10) above. We have the following basic connections between the Dubrovin valuation ring B of A and the invariant valuation ring R of D^h.

Theorem 10.4 *For B and R as above,*

(a) $\overline{B} \cong M_{t_B}(\overline{R})$.
(b) $\Gamma_B \cong \Gamma_R$, *as ordered groups.*
(c) n_B/t_B *is a positive integer.*
(d) *There is a commutative diagram*

$$\begin{array}{ccc} \Gamma_B/\Gamma_V & \xrightarrow{\cong} & \Gamma_R/\Gamma_{V^h} \\ {\scriptstyle\theta_B}\big\downarrow & & \big\downarrow{\scriptstyle\theta_R} \\ \mathcal{G}(Z(\overline{B})/\overline{V}) & \xrightarrow{\cong} & \mathcal{G}(Z(\overline{R})/\overline{V}^h). \end{array}$$

(e) $\delta(B) = \delta(R) = \rho^c$, *where c is a nonnegative integer and* $\rho = \mathrm{char}(\overline{V})$ *if* $\mathrm{char}(\overline{V}) \neq 0$, *and* $\rho = 1$ *if* $\mathrm{char}(\overline{V}) = 0$.

Th. 10.4 was proved in [W_4, Th. B, Th. D, Th. C]. An immediate consequence is that the assertions about $Z(\overline{R})$ and the roots of unity in $\overline{V}$ given in Prop. 2.5 and Prop. 2.6 apply correspondingly for the Dubrovin valuation ring B. Part (e) of Th. 10.4 justifies calling $\delta(B)$ the defect of B. See Th. 10.11 below for a significant interpretation of the integer n_B/t_B.

While there is no valuation corresponding to a noninvariant Dubrovin valuation ring, Morandi found a somewhat less stringent type of function associated to certain Dubrovin valuation rings. For this, let A be a central simple algebra over a field F, let Γ be a totally ordered abelian group, and let $w : A - \{0\} \to \Gamma$ be a function satisfying, for all $a, b \in A - \{0\}$,

(i) $w(a + b) \geq \min(w(a), w(b))$, whenever $b \neq -a$;
(ii) $w(ab) \geq w(a) + w(b)$, whenever $ab \neq 0$;
(iii) $w(-1) = 0$;
(iv) $\mathrm{im}(w) = w(st(w))$, where $st(w) = \{a \in A^* \mid w(a^{-1}) = -w(a)\}$; (10.6)
(v) let $B_w = \{a \in A - \{0\} \mid w(a) \geq 0\} \cup \{0\}$, a subring of A, and $J_w = \{a \in A - \{0\} \mid w(a) > 0\} \cup \{0\}$, an ideal of B_w; we require that B_w/J_w be a simple Artinian ring.

A function w satisfying conditions (i)–(v) is called a *value function* of A.

Theorem 10.5 *If w is a value function on a central simple algebra A as in* (10.6) *above, then B_w is a Dubrovin valuation ring of A which is integral over its center. Furthermore, $J_w = J(B_w)$, and $st(B_w) = st(w)$, and* $\Gamma_B = \mathrm{im}(w)$.

Th. 10.5 was proved by Morandi in [M_2]; see also [MMU_2, Th. 23.3]. This theorem has proved very useful in building Dubrovin valuation rings; see Ex. 10.7 below for one example, and papers [M_2], [MW_1] for others. Th. 10.5 also provides some hint that Dubrovin valuation rings integral over their centers are rather special. In fact, we have the following characterizations of such rings:

Theorem 10.6 *Let B be a Dubrovin valuation ring of a central simple algebra A, let $Z(A) = F$, and let $V = B \cap F = Z(B)$. Let F^h (resp. V^h) be the*

Henselization of F (resp. V) with respect to V. Then, the following conditions are equivalent:

(i) *B is integral over V.*
(ii) *There is a value function $w : A - \{0\} \to \Gamma$ as in* (10.6) *above such that $B = B_w$.*
(iii) *Every principal two-sided ideal of B is principal as a left ideal and as a right ideal.*
(iv) *Every two-sided ideal of B is generated by elements of $st(B)$.*
(v) *For every ring T with $B \subseteq T \subseteq A$ (so T is Dubrovin by Th. 10.2(d)), for $W = T \cap F = Z(T)$, the valuation ring $V/J(W)$ of $\overline{W}$ has a unique extension to a valuation ring of $Z(\overline{T})$.*
(vi) *$n_B = t_B$.*
(vii) *$B \otimes_V V^h$ is a Dubrovin valuation ring of $A \otimes_F F^h$.*
(viii) *There is a Dubrovin valuation ring B^h of $A \otimes_F F^h$ with $B^h \cap A = B$.*

This theorem was proved in [W_4, Th. F]; see also [MMU_2, Th. 12.3, Cor. 23.4]. Note in particular that whenever a commutative valuation ring V has rank 1, then for any Dubrovin valuation ring B of a central simple algebra such that $Z(B) = V$, we have B is integral over V; for, condition (v) of Th. 10.6 holds trivially. Also, if V is a Henselian commutative valuation ring of arbitrary rank, then again for any Dubrovin valuation ring B of a central simple algebra such that $V = Z(B)$, we have B is integral over V.

Example 10.7 Let V be a valuation ring of a field F, and let B_i be a Dubrovin valuation ring of a central simple F-algebra A_i such that $B_i \cap F = V$ for $i = 1, 2$. Suppose $Z(\overline{B_1})$ is separable (hence abelian Galois) over $\overline{V}$ and $\delta(B_1) = 1$. Let $\mathcal{L} = Z(\overline{B_1}) \cap Z(\overline{B_2})$. Let ρ_i be the composition $\Gamma_{B_i} \to \Gamma_{B_i}/\Gamma_V \xrightarrow{\theta_{B_i}} \mathcal{G}(Z(\overline{B_i})/\overline{V}) \to \mathcal{G}(\mathcal{L}/\overline{V})$, where the right map is given by restriction. Suppose $\Gamma_{B_1} \cap \Gamma_{B_2} = \Gamma_V$. If B is a Dubrovin valuation ring of $A_1 \otimes_F A_2$ with $B \cap F = V$, then $Z(\overline{B}) \cong Z(\overline{B_1}) \otimes_{\mathcal{L}} Z(\overline{B_2})$, $\overline{B}$ is Brauer equivalent to $\overline{B_1} \otimes_{\mathcal{L}} \overline{B_2}$, and $\Gamma_B = \{\gamma_1 + \gamma_2 \mid \rho_1(\gamma_1) = \rho_2(\gamma_2)\}$. This example is taken from [MW_1, Cor. 3.12], where it is verified by reducing to the case where V is Henselian via Th. 10.4 above; then the B_i are integral over V and explicit constructions using value functions can be carried out to obtain a Dubrovin valuation ring of $A_1 \otimes_F A_2$. Note that the separability and defect hypotheses here are satisfied whenever V is Henselian and S_1 is a division algebra tame over F with respect to V.

When working with several valuation rings over a given field, the Approximation Theorem is a basic tool, see [R_1] or [R_2, p. 136, Th. 3] for the general version. Morandi proved the corresponding approximation theorem for Dubrovin valuation rings—see Th. 10.9 below—in [M_3]. At the same time, independently, Gräter studied in [G_5] what he called the Intersection Property for finite families of Dubrovin valuation rings, and it became apparent that the condition he found was equivalent to the one Morandi had identified as needed for the approximation theorem to hold. Gräter used the Intersection Property to illuminate properties of semilocal Bézout orders (see [G_7]), as well as clarifying integral extensions of commutative valuation rings in central simple algebras, and also providing a significant new interpretation of the integer n_B/t_B associated to a Dubrovin valuation ring B.

To describe these results, we need some more terminology. Recall first that two valuation rings V_1, V_2 of a field K are said to be *independent* if there is no valuation ring W of K with each $V_i \subseteq W$ and $W \subsetneqq K$; that is, the ring generated by V_1 and V_2 is all of K. Now, let B_i and B_j be Dubrovin valuation rings of a central simple algebra A. Let $B_{ij} = B_i \cdot B_j$, the subring of A generated by B_i and B_j. This B_{ij} is a Dubrovin valuation ring of A by Th. 10.2(d) (possibly $B_{ij} = A$), with $J(B_{ij}) \subseteq B_i \cap B_j$. If we set $\widetilde{B_i} = B_i/J(B_{ij})$ and $\widetilde{B_j} = B_j/J(B_{ij})$, then by Th. 10.2(c), $\widetilde{B_i}$ and $\widetilde{B_j}$ are each Dubrovin valuation rings of the central simple algebra $\overline{B_{ij}} = B_{ij}/J(B_{ij})$. Let $\mathcal{B}(B_i)$ denote the set of rings T such that $B_i \subseteq T \subseteq A$.

Theorem 10.8 *Let $B_1, \ldots, B_k$ be pairwise incomparable Dubrovin valuation rings of a central simple algebra A over a field F. Let $R = B_1 \cap B_2 \cap \cdots \cap B_k$. Then the following conditions are equivalent.*

- (i) *There is a well-defined order-reversing bijection $\mathcal{B}(B_1) \cup \cdots \cup \mathcal{B}(B_k) \to \{\text{prime ideals of } R\}$ given by $T \mapsto J(T) \cap R$.*
- (ii) *For every ring $T \in \mathcal{B}(B_1) \cup \cdots \cup \mathcal{B}(B_k)$, $J(T) \cap R$ is a prime ideal of R.*
- (iii) *For each distinct i and j, if we let $B_{ij} = B_i \cdot B_j$ and set $\widetilde{B_i} = B_i/J(B_{ij})$ and $\widetilde{B_j} = B_j/J(B_{ij})$ as above, then $Z(\widetilde{B_i})$ and $Z(\widetilde{B_j})$ are independent valuation rings of $Z(\overline{B_{ij}})$.*

Following the terminology of Gräter in [G_5], we say that $B_1, \ldots, B_k$ have the *Intersection Property* if they satisfy the equivalent conditions of Th. 10.8. Of the three conditions in Th. 10.8, (i)⇒(ii) is clear, (ii)⇒(i) was proved by Y. Zhao in [Z], and (i)⇔(iii) was proved by Gräter in [G_5, Cor. 6.2, Prop. 6.3, Cor. 6.7]; see also [MMU_2, Cor. 16.9].

Theorem 10.9 (Approximation Theorem) *Let $B_1, \ldots, B_k$ be pairwise incomparable Dubrovin valuation rings of a central simple algebra A such that $B_1, \ldots, B_k$ have the Intersection Property. Let $B_{ij} = B_i \cdot B_j$. Let I_i be a right ideal of B_i, for $1 \le i \le k$, such that $I_i B_{ij} = I_j B_{ij}$ for all distinct i, j. Take any $a_1, \ldots, a_k \in A$ such that $a_i - a_j \in I_i B_{ij}$ for all distinct i, j. Then, there is an $x \in A$, such that $x - a_i \in I_i$ for all i.*

The Approximation Theorem was proved by Morandi in [M_3, Th. 2.3], working from condition (iii) in Th. 10.8; see also [MMU_2, Th. 15.2]. Morandi also showed in [M_3, Prop. 2.1] that the Approximation Theorem fails to hold whenever $B_1, \ldots, B_k$ do not have the Intersection Property. The approximation theorem for total valuation rings in a division ring had been proved earlier by Gräter in [G_3, Satz 4.1].

The next three theorems indicate the ring theoretic significance of the Intersection Property.

Theorem 10.10 *Let A be a central simple algebra over a field F, and let $V_1, \ldots, V_n$ be pairwise incomparable valuation rings of F. Let $T = V_1 \cap \cdots \cap V_n$.*

- (a) *For some k, $1 \le k \le n$, let $B_1, \ldots, B_\ell$ be Dubrovin valuation rings of A satisfying the Intersection Property, such that $B_1 \cap \cdots \cap B_\ell \cap F = V_1 \cap \cdots \cap V_k$. Then, there exist Dubrovin valuation rings $B_{\ell+1}, \ldots, B_m$ of A such that $B_1, \ldots, B_m$ satisfies the Intersection Property, and for $R = B_1 \cap \cdots \cap B_m$, we have $R \cap F = T$ and R is integral over T.*
- (b) *Assume the $B_1, \ldots, B_m$ of part (a) are pairwise incomparable. If $B_1', \ldots, B_q'$ is another set of pairwise incomparable Dubrovin valuation rings of A satisfying the Intersection Property such that for $R' = B_1' \cap \cdots \cap B_q'$ we have*

$R' \cap F = T$ and R' is integral over T, then $q = m$ and there is $a \in A^*$ such that $R' = aRa^{-1}$.

Th. 10.10 was proved by Gräter in [G_5, Th. 6.11, Th. 6.12], see also [MMU_2, Th. 16.14, Th. 16.15]. The theorem shows in particular that if we have a finite family of Dubrovin valuation rings of A which satisfy the Intersection Property, then we can iteratively add to the family while preserving the Intersection Property and the intersection with $F = Z(A)$, until a point is reached where the intersection of the Dubrovin valuation rings is integral over the intersection with F. When this point is reached (which will occur after at most finitely many nontrivial steps), one cannot add to the collection of Dubrovin valuation rings nontrivially without losing the Intersection Property or shrinking the intersection with the center; also, the intersection of Dubrovin valuation rings obtained at this point is unique up to isomorphism.

In studying subrings of a central simple A, the set of all elements integral over a subring T of $Z(A)$ is usually not a subring of A. Thus, one instead studies orders of A integral over T as an approximation to an integral closure of T in A. Th. 10.10 suggests that when T is a finite intersection of valuation rings of $Z(A)$, then the ring R described there, with R integral over T, $R \cap Z(A) = T$, and R a finite intersection of Dubrovin valuation rings having the Intersection Property, could be considered the most natural candidate for an "integral closure of T in A." This R is usually not unique, but, as Th. 10.10(b) shows, it is unique up to isomorphism.

Consider now a single valuation ring V of the center F of a central simple algebra A. Th. 10.10 shows that there is a unique associated positive integer k such that there are k Dubrovin valuation rings $B_1, \dots, B_k$ of A satisfying the Intersection Property with each $B_i \cap F = V$, but there are no $k+1$ such B_i. Moreover, whenever we have k such B_i, their intersection is integral over V and is uniquely determined up to isomorphism. Gräter defined this k to be the *extension number* of V to A.

Theorem 10.11 *Let B be a Dubrovin valuation ring of a central simple algebra A over a field F, and let k be the extension number to A of the valuation ring $B \cap F$. Then $k = n_B/t_B$.*

Gräter proved Th. 10.11 in [G_5, Prop. 7.4]; see also [MMU_2, Prop. 19.2]. At the same time Gräter gave a new proof of the Ostrowski-type defect theorem for Dubrovin valuation rings, Th. 10.4(e) above, using the extension number in place of n_B/t_B.

There are also nice characterizations of the rings which arise as intersections of Dubrovin valuation rings with the Intersection Property:

Theorem 10.12 *Let R be an order in a central simple algebra A. The following conditions are equivalent:*

(i) *$R = B_1 \cap \cdots \cap B_n$, where $B_1, \dots, B_n$ are Dubrovin valuation rings with the Intersection Property.*
(ii) *R is semilocal (i.e., has just finitely many maximal two-sided ideals) and every finitely generated right (resp. left) ideal of R is principal.*
(iii) *R is semilocal and for every prime ideal P of R, the set $\mathcal{C}(P) = \{r \in R \mid r + P$ is regular in $R/P\}$ is a regular right and left Ore set of R, and the right (= left) localization $R_{\mathcal{C}(P)}$ is a Dubrovin valuation ring.*

The implications (i)⇔(ii) and (i)⇒(iii) in Th. 10.12 were proved by Gräter in [G_7, Cor. 3.5, Th. 3.6, Th. 2.6], while (iii)⇔(ii) was proved in [MUZM]; see also

[MMU_2, Th. 17.3]. Also, Morandi proved in [M_3, Lemma 3.2, Th. 3.4] a version of (i)⇒(ii), where instead of (i) he used the equivalent condition (iii) of Th. 10.8.

The theorems stated in this section give a number of the fundamental results concerning Dubrovin valuation rings and suggest the richness in the theory of these rings. They have been applied in a number of contexts, which we mention only briefly. As one might expect, Dubrovin valuation rings have been used in the study of orders in central simple algebras over central valuation rings, including maximal orders, semihereditary orders, extremal orders, Bézout orders, and Prüfer orders, see [HM], [HMW], [K_1], [K_2], [K_3], [MMU_2], [MMUZ], [MU_1], [MUZM], [M_4], [M_5]. Dubrovin valuation rings were also needed in showing in [HW_2] that the tame part of the Brauer group of a field F with Henselian valuation ring V is isomorphic to the graded Brauer group of the associated graded ring of V (this ring is actually a graded field). For more on the connections between valued and graded algebras, see also [B_1], [B_2], [B_3], [HW_1]. Dubrovin valuation rings had also been applied earlier in [W_5] in proving an algorithm for computing the residue algebra and the value group for the valuation ring (or Dubrovin valuation ring) of a tensor product of symbol algebras over a field with valuation. In the last two mentioned applications, Dubrovin valuation rings are not involved in the statement of the results, but seem essential for their proofs.

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