IJMMS 28:2 (2001) 83–91
PII. S0161171201011036
http://ijmms.hindawi.com

VALUE DISTRIBUTION OF CERTAIN DIFFERENTIAL POLYNOMIALS

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(Received 5 October 2000 and in revised form 1 April 2001)

Abstract. We prove a result on the value distribution of differential polynomials which improves some earlier results.

2000 Mathematics Subject Classification. 30D35.

1. Introduction and definitions. Let f be a transcendental meromorphic function in the open complex plane. The problem of possible Picard values of derivatives of f reduces to the problem of whether certain polynomials in a meromorphic function and its derivatives necessarily have zeros. We do not explain the standard definitions and notations of value distribution theory as those are available in [6].

Definition 1.1. A meromorphic function "a" is said to be a small function of f if $T(r,a) = S(r,f)$.

Definition 1.2 (see [1, 4, 10]). Let $n_{0j}, n_{1j}, \dots, n_{kj}$ be nonnegative integers. The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $P[f] = \sum_{i=1}^{l} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\gamma_P = \max\{\gamma_{M_j} : 1 \le j \le l\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \le j \le l\}$, where $T(r,b_j) = S(r,f)$ for $j = 1,2,\dots,l$.

The numbers $\underline{\gamma}_P = \min\{\gamma_{M_j} : 1 \le j \le l\}$ and k (the highest order of the derivative of f in $P[f]$) are called, respectively, the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\gamma_P = \underline{\gamma}_P$.

Also $P[F]$ is called a quasi differential polynomial generated by f if, instead of assuming $T(r,b_j) = S(r,f)$, we just assume that $m(r,b_j) = S(r,f)$ for the coefficients $b_j (j = 1,2,\dots,l)$.

Definition 1.3. Let m be a positive integer. We denote by $N(r,a;f| \le m)$ ($N(r,a;f| \ge m)$) the counting function of those a-points of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.

In a similar manner, we define $N(r,a;f| < m)$ and $N(r,a;f| > m)$.

Also $\overline{N}(r,a;f| \le m)$, $\overline{N}(r,a;f| \ge m)$, $\overline{N}(r,a;f| < m)$, and $\overline{N}(r,a;f| > m)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Finally, we agree to take $\overline{N}(r,a;f| \le \infty) \equiv \overline{N}(r,a;f)$ and $N(r,a;f| \le \infty) \equiv N(r,a;f)$.

Definition 1.4. For two meromorphic functions f, g and positive integer m, we denote by $N(r,a;f|g = b, > m)$ the counting function of those a-points of f, counted with proper multiplicities, which are the b-points of g with multiplicities greater than m.

Definition 1.5 (see [2]). Let m be a positive integer. We denote by $N_m(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity μ is counted μ times if $\mu \le m$ and m times if $\mu > m$.

As the standard convention, we mean by $N(r,f)$ and $\overline{N}(r,f)$ the counting functions $N(r,\infty;f)$ and $\overline{N}(r,\infty;f)$, respectively.

Hayman [5] proved the following theorems.

Theorem 1.6. *If f is a transcendental meromorphic function and n (≥ 5) is a positive integer, then $\psi = f' - af^n$ assumes all finite values infinitely often.*

Theorem 1.7. *If f is a transcendental meromorphic function and n (≥ 3) is a positive integer, then $\psi = f'f^n$ assumes all finite values, except possibly zero, infinitely often.*

When f is transcendental, entire conclusions of Theorems 1.6 and 1.7 hold, respectively for $n \ge 3$ (cf. [5]) and $n \ge 1$ (cf. [3]).

To study the value distribution of differential polynomials Yang [7] proved the following results.

Theorem 1.8. *Let f be a transcendental meromorphic function with $N(r,f) = S(r,f)$, and let $\psi = f^n + P[f]$, where n (≥ 2) is an integer and $P[f]$ is a differential polynomial generated by f with $\gamma_P \le n-2$. Then $\delta(a;\psi) < 1$ for $a \ne 0,\infty$.*

Theorem 1.9. *Let f be a transcendental meromorphic function with $N(r,f) = S(r,f)$, and let $\psi = f^n P[f]$, where n (≥ 2) is an integer and $P[f]$ is a differential polynomial generated by f. Then $\delta(a;\psi) < 1$ for $a \ne 0,\infty$.*

Improving all the above results, Yi [9] proved the following theorem.

Theorem 1.10. *Let f be a transcendental meromorphic function and $Q_1[f]$, $Q_2[f]$ be two differential polynomials generated by f such that $Q_1[f] \not\equiv 0$, $Q_2[f] \not\equiv 0$, and $P[f] = \sum_{j=0}^{n} a_j f^j$ $(a_n \not\equiv 0)$, where $a_1, a_2, \ldots, a_n$ are small functions of f. If $F = P[f]Q_1[f] + Q_2[f]$, then*

$$(n-\gamma_{Q_2})T(r,f) \le \overline{N}(r,0;F) + \overline{N}(r,0;P[f]) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) + S(r,f). \quad (1.1)$$

In Theorem 1.10 we see that the influence of $Q_1[f]$ on the value distribution of F is ignored. In this paper, we show that Theorem 1.10 can further be improved if the influence of $Q_1[f]$ is taken into consideration. Throughout, we ignore zeros and poles of any small function of f because the corresponding counting function is absorbed in $S(r,f)$.

2. Lemmas.

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1 (see [4]). *Let f be a nonconstant meromorphic function and $Q^*[f]$, $Q[f]$ denote differential polynomials generated by f with arbitrary meromorphic coefficients*

$q_1^*, q_2^*, \dots, q_s^*$ and $q_1, q_2, \dots, q_t$, respectively. Further let $P[f] = \sum_{j=0}^n a_j f^j$ ($a_n \not\equiv 0$) and $\gamma_Q \le n$. If $P[f]Q^*[f] = Q[f]$, then

$$m(r, Q^*[f]) \le \sum_{j=1}^{s} m(r, q_j^*) + \sum_{j=1}^{t} m(r, q_j) + S(r,f). \tag{2.1}$$

Lemma 2.2. *Let $Q[f] = \sum_{j=1}^{l} b_j M_j[f]$ be a differential polynomial generated by f of order and lower degree k and $\underline{\gamma}_Q$, respectively. If z_0 is a zero of f with multiplicity μ $(> k)$ and z_0 is not a pole of any of the coefficients b_j $(j = 1, 2, \dots, l)$, then z_0 is a zero of $Q[f]$ with multiplicity at least $(\mu - k)\underline{\gamma}_Q$.*

Proof. Clearly z_0 is a zero of $M_j[f]$ with multiplicity

$$\begin{aligned} &\mu n_{0j} + (\mu-1)n_{1j} + \cdots + (\mu-k)n_{kj} \\ &\quad = \mu\gamma_{M_j} - (\Gamma_{M_j} - \gamma_{M_j}) = (\mu-k)\gamma_{M_j} + (k+1)\gamma_{M_j} - \Gamma_{M_j} \\ &\quad \ge (\mu-k)\gamma_{M_j} \ge (\mu-k)\underline{\gamma}_Q. \end{aligned} \tag{2.2}$$

Since z_0 is assumed not to be a pole of the coefficients b_j $(j = 1, 2, \dots, l)$ we see that z_0 is a zero of $Q[f]$ with multiplicity at least $(\mu - k)\underline{\gamma}_Q$. This proves the lemma. □

Lemma 2.3 (see [1]). *The following inequality holds:*

$$N(r, P[f]) \le \gamma_P N(r,f) + (\Gamma_P - \gamma_P)\overline{N}(r,f) + S(r,f). \tag{2.3}$$

Lemma 2.4 (see [7]). *Let $P[f] = \sum_{i=0}^n a_i f^i$, where $a_n (\not\equiv 0), a_{n-1}, \dots, a_1, a_0$ are small functions of f. Then $m(r, P[f]) = nm(r,f) + S(r,f)$.*

Lemma 2.5 (see [4]). *If $Q[f]$ is a differential polynomial generated by f with arbitrary meromorphic coefficients q_j $(1 \le j \le n)$, then*

$$m(r, Q[f]) \le \gamma_Q m(r,f) + \sum_{j=1}^{n} m(r, q_j) + S(r,f). \tag{2.4}$$

Lemma 2.6 (see [8]). *If $P[f]$ is as in Lemma 2.4, then $T(r, P[f]) = nT(r,f) + S(r,f)$.*

3. The main result. In this section, we present the main result of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function in the open complex plane, and $Q_1[f]$ $(\not\equiv 0)$, $Q_2[f]$ $(\not\equiv 0)$ be two differential polynomials generated by f such that k and $\underline{\gamma}_{Q_1}$ be the order and lower degree of $Q_1[f]$, respectively and $P[f] = \sum_{i=0}^n a_i f^i$, where $a_n (\not\equiv 0)$, $a_{n-1}, \dots, a_0$ are small functions of f. If*

$$F = P[f]Q_1[f] + Q_2[f], \tag{3.1}$$

then

$$\begin{aligned} (n - \gamma_{Q_2})T(r,f) \le{} & \overline{N}(r,0;F) + \overline{N}(r,0;P[f]) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) \\ & - \gamma\{N(r,0;f) - N_{k+1}(r,0;f)\} + S(r,f), \end{aligned} \tag{3.2}$$

where $\gamma = \underline{\gamma}_{Q_1}$ if $n \ge \gamma_{Q_2}$ and $\gamma = 0$ if $n < \gamma_{Q_2}$.

Proof. If $n < \gamma_{Q_2}$, the theorem is obvious. So we suppose that $n \geq \gamma_{Q_2}$. Differentiating (3.1) we get

$$F' = P'[f]Q_1[f] + P[f]Q_1'[f] + Q_2'[f], \tag{3.3}$$

where $P'[f] = (d/dz)P[f]$ and $Q_i'[f] = (d/dz)Q_i[f]$ for $i = 1,2$.

Multiplying (3.1) by (F'/F), and substituting in (3.3) we get

$$P[f]Q^*[f] = Q[f], \tag{3.4}$$

where

$$Q^*[f] = \left(\frac{F'}{F} - \frac{P'[f]}{P[f]}\right)Q_1[f] - Q_1'[f], \tag{3.5}$$

$$Q[f] = Q_2'[f] - \left(\frac{F'}{F}\right)Q_2[f]. \tag{3.6}$$

First we suppose that $Q^*[f] \not\equiv 0$. By Lemma 2.1, it follows from (3.4) that $m(r, Q^*[f]) = S(r,f)$ because $\gamma_Q = \gamma_{Q_2} \leq n$.

Since $P[f] = Q[f]/Q^*[f]$, we get by Lemma 2.5 and the first fundamental theorem

$$\begin{aligned} m(r,P[f]) &\leq m(r,Q[f]) + m(r,0;Q^*[f]) \\ &\leq \gamma_{Q_2} m(r,f) + m(r,Q^*[f]) + N(r,Q^*[f]) - N(r,0;Q^*[f]) + S(r,f) \\ &= \gamma_{Q_2} m(r,f) + N(r,Q^*[f]) - N(r,0;Q^*[f]) + S(r,f). \end{aligned} \tag{3.7}$$

So by Lemma 2.4

$$(n - \gamma_{Q_2})m(r,f) \leq N(r,Q^*[f]) - N(r,0;Q^*[f]) + S(r,f). \tag{3.8}$$

From (3.5) we see that possible poles of $Q^*[f]$ occur at the poles of f and zeros of F and $P[f]$. Also we note that the zeros of F and $P[f]$ are at most simple poles of $Q^*[f]$. Let z_0 be a pole of f with multiplicity μ. Then z_0 is a pole of $Q[f]$ with multiplicity not exceeding $(\mu-1)\gamma_{Q_2} + \Gamma_{Q_2} + 1 = \mu\gamma_{Q_2} + \Gamma_{Q_2} - \gamma_{Q_2} + 1$ and z_0 is a pole of $P[f]$ with multiplicity $n\mu$. Hence, from (3.4) it follows that z_0 is a pole of $Q^*[f]$ with multiplicity not exceeding $\mu\gamma_{Q_2} + \Gamma_{Q_2} - \gamma_{Q_2} + 1 - n\mu = \Gamma_{Q_2} - \gamma_{Q_2} + 1 - (n - \gamma_{Q_2})\mu$. Therefore

$$\begin{aligned} N(r,Q^*[f]) \leq{}& \overline{N}(r,0;F) + \overline{N}(r,0;P[f]) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) \\ &- (n - \gamma_{Q_2})N(r,f) + S(r,f). \end{aligned} \tag{3.9}$$

Now we note that the order of the differential polynomial $Q_1'[f]$ is $k+1$. Let z_0 be a zero of f with multiplicity $\mu > k+1$. Let $\underline{\gamma}_{Q_1} \geq 1$. Then by Lemma 2.2, we see that z_0 is a zero of $Q_1[f]$ with multiplicity at least $(\mu-1)\underline{\gamma}_{Q_1}$. Also z_0 may be a pole of $(F'/F) - P'[f]/P[f]$ with multiplicity not exceeding 1. So z_0 is a zero of $((F'/F) - P'[f]/P[f])Q_1[f]$ with multiplicity at least $(\mu-k)\underline{\gamma}_{Q_1} - 1$.

Since the lower degree of $Q_1'[f]$ is $\underline{\gamma}_{Q_1}$, it follows from Lemma 2.2 that z_0 is a zero of $Q_1'[f]$ with multiplicity at least $(\mu-k-1)\underline{\gamma}_{Q_1}$.

Therefore z_0 is a zero of $Q^*[f]$ with multiplicity at least $(\mu-k-1)\underline{\gamma}_{Q_1}$. Hence

$$\begin{aligned} N(r,0;Q^*[f]) & \\ &\geq N(r,0;Q^*[f]|f=0,>k+1) \\ &\geq \underline{\gamma}_{Q_1}N(r,0;f|>k+1)-\underline{\gamma}_{Q_1}(k+1)\overline{N}(r,0;f>k+1)+S(r,f) \\ &= \underline{\gamma}_{Q_1}N(r,0;f)-\underline{\gamma}_{Q_1}\{N(r,0;f|\leq k+1)+(k+1)\overline{N}(r,0;f|>k+1)\}+S(r,f). \end{aligned} \tag{3.10}$$

So

$$N(r,0;Q^*[f]) \geq \underline{\gamma}_{Q_1}\{N(r,0;f)-N_{k+1}(r,0;f)\}+S(r,f). \tag{3.11}$$

If $\underline{\gamma}_{Q_1}=0$, inequality (3.11) obviously holds. Now from (3.8), (3.9), and (3.11) we get

$$\begin{aligned}(n-\gamma_{Q_2})T(r,f) \leq \overline{N}(r,0;F)+\overline{N}(r,0;P[f])+(\Gamma_{Q_2}-\gamma_{Q_2}+1)\overline{N}(r,f) \\ -\underline{\gamma}_{Q_1}\{N(r,0;f)-N_{k+1}(r,0;f)\}+S(r,f).\end{aligned} \tag{3.12}$$

Next we suppose that $Q^*[f]\equiv 0$. Then from (3.4) it follows that $Q[f]\equiv 0$, and so using (3.1) we get $P[f]Q_1[f]=cQ_2[f]$, where c is a nonzero constant. Then in a similar line of calculation for inequalities (3.8), (3.9), and (3.11) we get

$$\begin{aligned}(n-\gamma_{Q_2})m(r,f) &\leq N(r,Q_1[f])-N(r,0;Q_1[f])+S(r,f), \\ N(r,Q_1[f]) &\leq (\Gamma_{Q_2}-\gamma_{Q_2}+1)\overline{N}(r,f)-(n-\gamma_{Q_2})N(r,f)+S(r,f), \\ N(r,0;Q_1[f]) &\geq \underline{\gamma}_{Q_1}\{N(r,0;f)-N_{k+1}(r,0;f)\}+S(r,f).\end{aligned} \tag{3.13}$$

Now from (3.13) we get

$$\begin{aligned}(n-\gamma_{Q_2})T(r,f) \leq \overline{N}(r,0;F)+\overline{N}(r,0;P[f])+(\Gamma_{Q_2}-\gamma_{Q_2}+1)\overline{N}(r,f) \\ -\underline{\gamma}_{Q_1}\{N(r,0;f)-N_{k+1}(r,0;f)\}+S(r,f).\end{aligned} \tag{3.14}$$

This proves the theorem. □

Remark 3.2. The following example shows that Theorem 3.1 is sharp.

Example 3.3. Let $f=e^z-2$, $P[f]=f+2$, $Q_1[f]=f$, and $Q_2[f]=1$. Then $F=P[f]Q_1[f]+Q_2[f]=(e^z-1)^2$ and $k=0$, $\underline{\gamma}_{Q_1}=1$, $\gamma_{Q_2}=0$, $n=1$. Also we see that

$$\begin{aligned}(n-\gamma_{Q_2})T(r,f) = \overline{N}(r,0;F)+\overline{N}(r,0;P[f])+(\Gamma_{Q_2}-\gamma_{Q_2}+1)\overline{N}(r,f) \\ -\underline{\gamma}_{Q_1}\{N(r,0;f)-N_{k+1}(r,0;f)\}+S(r,f).\end{aligned} \tag{3.15}$$

4. Applications. As applications of Theorem 1.10, Yi [9] proved the following theorems which improve Theorems 1.8 and 1.9.

Theorem 4.1. *Let f be a transcendental meromorphic function and $Q_1[f]$ $(\not\equiv 0)$, $Q_2[f]$ $(\not\equiv 0)$ be two differential polynomials generated by f. Let $F = f^n Q_1[f] + Q_2[f]$ and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f)}{T(r,f)} < n - \gamma_{Q_2}. \tag{4.1}$$

Then $\Theta(a;F) < 1$ for any small function a $(\not\equiv \infty, Q_2[f])$ of f.

Theorem 4.2. *Let $F = f^n Q[f]$, where $Q[f]$ is a differential polynomial generated by f and $Q[f] \not\equiv 0$. If*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f) + \overline{N}(r,f)}{T(r,f)} < n, \tag{4.2}$$

then $\Theta(a;F) < 1$, where a $(\not\equiv 0,\infty)$ is a small function of f.

Considering the following examples, Yi [9] claimed that Theorems 4.1 and 4.2 are sharp.

Example 4.3. Let $f = (e^{4z}+1)/(e^{4z}-1)$, $Q_1[f] = 1$, $Q_2[f] = f' - 1$, and $F = f^4 Q_1[f] + Q_2[f]$. Then $n = 4$, $\gamma_{Q_2} = 1$, $\Gamma_{Q_2} = 2$, and

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f)}{T(r,f)} = n - \gamma_{Q_2}. \tag{4.3}$$

Also we see that $\Theta(0;F) = 1$.

Example 4.4. Let $f = (e^z-1)/(e^z+1)$, $Q_1[f] = 1$, $F = f^n Q_1[f]$, where $n = 2$. It is easy to verify that

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f) + \overline{N}(r,f)}{T(r,f)} = n \tag{4.4}$$

and $\Theta(1;F) = 1$.

The following examples suggest that some improvements of Theorems 4.1 and 4.2 are possible.

Example 4.5. Let $f = ((e^z-1)/(e^z+1))^2$, $Q_1[f] = f$, $Q_2[f] = 1$, and $F = f Q_1[f] + Q_2[f]$. Then $n = 1$, $\underline{\gamma}_{Q_1} = 1$, $\gamma_{Q_2} = 0$, $\Gamma_{Q_2} = 0$, and the order of the differential polynomial $Q_1[f]$ is zero. Clearly

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f)}{T(r,f)} = n - \gamma_{Q_2}. \tag{4.5}$$

Also we see that $\Theta(1;F) = \Theta(\infty;F) = 3/4$, $\Theta(2;F) = 1/2$ and so, by Nevanlinna's three small functions theorem (cf. [6, page 47]), $\Theta(a;F) \le 2 - 3/4 - 1/2 = 3/4$ for any small function a $(\not\equiv 1,2,\infty)$. However, we note that

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f\mid\le 1) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f)}{T(r,f)} = \frac{1}{2} < n - \gamma_{Q_2}. \tag{4.6}$$

Example 4.6. Let $f = ((e^z-1)/(e^z+1))^2$, $Q[f] = f$, and $F = fQ[f]$. Then $n = 1$, $\underline{\gamma}_Q = 1$, and the order of the differential polynomial $Q[f]$ is zero. Clearly

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f)+\overline{N}(r,f)}{T(r,f)} = n \tag{4.7}$$

and $\Theta(a;F) < 1$ for any small function a of f. We note that

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f\mid \le 1)+\overline{N}(r,f)}{T(r,f)} = \frac{1}{2} < n. \tag{4.8}$$

The following two theorems improve Theorems 4.1 and 4.2.

Theorem 4.7. *Let f be a transcendental meromorphic function and $Q_1[f]$, $Q_2[f]$ be two differential polynomials generated by f which are not identically zero. Let $F = f^nQ_1[f]+Q_2[f]$. If*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f\mid \le \chi_{Q_1}) + (\Gamma_{Q_2}-\gamma_{Q_2}+1)\overline{N}(r,f)}{T(r,f)} < n-\gamma_{Q_2}, \tag{4.9}$$

then $\Theta(a;F) < 1$ for any small function a $(\not\equiv \infty, Q_2[f])$ of f, where

$$\chi_{Q_1} = \begin{cases} 1+k & \text{if } \underline{\gamma}_{Q_1} \ge 1, \\ \infty & \text{if } \underline{\gamma}_{Q_1} = 0, \end{cases} \tag{4.10}$$

and k is the order of the differential polynomial $Q_1[f]$.

Theorem 4.8. *Let f be a transcendental meromorphic function and $Q[f]$ $(\not\equiv 0)$ be a differential polynomial generated by f. If $F = f^nQ[f]$ and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f\mid \le \chi_Q)+\overline{N}(r,f)}{T(r,f)} < n, \tag{4.11}$$

then $\Theta(a;F) < 1$ for every small function a $(\not\equiv 0,\infty)$ of f, where

$$\chi_Q = \begin{cases} 1+k & \text{if } \underline{\gamma}_Q \ge 1, \\ \infty & \text{if } \underline{\gamma}_Q = 0, \end{cases} \tag{4.12}$$

and k is the order of the differential polynomial $Q[f]$.

Remark 4.9. Theorem 4.7 improves Theorems 1.8 and 4.1, and Theorem 4.8 improves Theorems 1.9 and 4.2.

Remark 4.10. The following examples show that Theorems 4.7 and 4.8 are sharp.

Example 4.11. Let $f = e^z-1$, $Q_1[f] = f'-f$, $Q_2[f] = 2f'$, and $F = f^2Q_1[f]+Q_2[f]$. Then $n = 2$, $k = 1$, $\Gamma_{Q_2} = 2$, $\gamma_{Q_2} = 1$, and

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f\mid \le 2) + (\Gamma_{Q_2}-\gamma_{Q_2}+1)\overline{N}(r,f)}{T(r,f)} = n-\gamma_{Q_2}. \tag{4.13}$$

Also we see that $\Theta(1;F) = 1$.

Example 4.12. Let $f = e^z + 1$, $Q[f] = f - f'$, and $F = fQ[f]$. Then $\underline{\gamma}_Q = 1$, $k = 1$, $n = 1$, and

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f\mid \leq 2) + \overline{N}(r,f)}{T(r,f)} = n. \tag{4.14}$$

Also we see that $\Theta(1;F) = 1$.

As other applications of Theorem 3.1, we obtain the following results which improve Theorems 1.6 and 1.7.

Theorem 4.13. *Let f be a transcendental meromorphic function, and $F = f' - af^n$, where a $(\not\equiv 0)$ is a small function of f. If n (≥ 5) is an integer, then $\Theta(b;F) \leq 4/n$ for any small function b $(\not\equiv \infty)$ of f.*

Theorem 4.14. *Let f be a transcendental meromorphic function. If $F = f^n f'$ and n (≥ 3) is an integer, then $\Theta(a;F) \leq 4/(n+2)$ for any small function a $(\not\equiv 0,\infty)$ of f.*

We prove Theorems 4.8 and 4.14 only.

Proof of Theorem 4.8. First we treat the case $\underline{\gamma}_Q \geq 1$. Then by Theorem 3.1 we get

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,a;F) + \overline{N}(r,0;P[f]) + \overline{N}(r,f) \\ &\qquad - \underline{\gamma}_Q\{N(r,0;f) - N_{k+1}(r,0;f)\} + S(r,f) \\ &\leq \overline{N}(r,a;F) + \overline{N}(r,0;f) - N(r,0;f) \\ &\qquad + N_{k+1}(r,0;f) + \overline{N}(r,f) + S(r,f), \end{aligned} \tag{4.15}$$

that is,

$$nT(r,f) \leq \overline{N}(r,a;F) + \overline{N}(r,0;f\mid \leq k+1) + \overline{N}(r,f) + S(r,f). \tag{4.16}$$

Now we treat the case $\underline{\gamma}_Q = 0$. Then from Theorem 3.1 we get

$$nT(r,f) \leq \overline{N}(r,a;F) + \overline{N}(r,0;f) + \overline{N}(r,f) + S(r,f). \tag{4.17}$$

Combining (4.16) and (4.17), we obtain

$$nT(r,f) \leq \overline{N}(r,a;F) + \overline{N}(r,0;f\mid \leq \chi_Q) + \overline{N}(r,f) + S(r,f) \tag{4.18}$$

from which the theorem follows. □

Proof of Theorem 4.14. Proceeding in the line of the proof of Theorem 4.8 we get

$$nT(r,f) \leq \overline{N}(r,a;f) + \overline{N}(r,0;f\mid \leq k+1) + \overline{N}(r,f) + S(r,f), \tag{4.19}$$

that is,

$$(n-2)T(r,f) \leq \overline{N}(r,a;F) + S(r,f). \tag{4.20}$$

Now by Lemmas 2.3 and 2.5 we see that

$$T(r,F) \leq (n+2)T(r,f)+S(r,f). \tag{4.21}$$

If possible let $\Theta(a;F) > 4/(n+2)$. Then there exits an ε (>0) such that for all large values of r

$$\overline{N}(r,a;F) < \left(\frac{n-2}{n+2}-\varepsilon\right)T(r,F). \tag{4.22}$$

From (4.20), (4.21), and (4.22) we get

$$\varepsilon(n+2)T(r,f) \leq S(r,f), \tag{4.23}$$

which is a contradiction. This proves the theorem. □

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