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YONG LIU

On value distribution of difference polynomials

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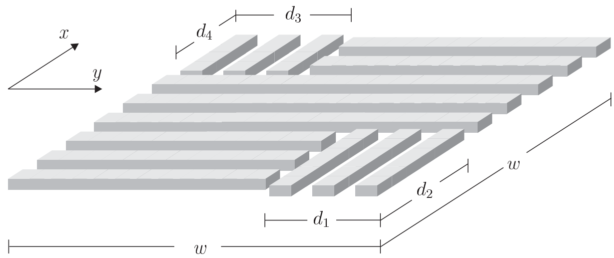

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ABSTRACT

We study the growth and value distribution of meromorphic solutions of difference equations arising from difference polynomials of various forms, mostly concentrating on the finite order case. For instance, we show that demanding the existence of a transcendental meromorphic solution of a large class of difference equations containing difference Painlevé I reduces the class either to a smaller one which still contains the difference Painlevé I, or to a class of first-order equations. In addition, for certain classes of equations, we prove the existence of rational solutions and give their forms. A number of results are obtained concerning the exponents of convergence of zeros of differences $g(z)$, $g_k(z)$, $\frac{g(z)}{f(z)}$, and $\frac{g_k(z)}{f^k(z)}$, where $g(z) = f(z+c_1) + f(z+c_2) + \cdots + f(z+c_k) - kf(z)$ and $g_k(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_k) - f^k(z)$. Finally, we investigate the zeros of the difference polynomial $F_n(z) = \sum_{j=1}^{k} a_j(z)f(z+c_j) - a(z)f^n(z)$.

AMS Mathematics Subject Classification: 39B32, 39A10, 30D35.

Keywords: Mathematical analysis; Nevanlinna theory; Difference equations; Polynomials; Value distribution theory; Functions, Meromorphic.

Preface

First of all, I am deeply in gratitude to my supervisors Professor Risto Korhonen and Professor emeritus Ilpo Laine for their valuable advice and constant encouragement.

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Joensuu, 2013 *Yong Liu*

LIST OF PUBLICATIONS

I Yong Liu and Hongxun Yi, *On zeros of differences of meromorphic functions,* Annales Polonici Mathematici, 100.2. (2011), 167-178.

II Yong Liu and Hongxun Yi, *On growth and zeros of differences of some meromorphic functions*, Annales Polonici Mathematici, 108.3. (2013), 305-318.

III Yong Liu, *Meromorphic solutions of certain difference equations of first order*, Aequationes Mathematicae, DOI 10.1007/s00010-013-0192-z.

IV Yong Liu, *On finite-order meromorphic solutions of a class containing a difference Painlevé I equation*, submitted, 25pp.

V Yong Liu and Hongxun Yi, *Properties of some difference polynomials*, Proceedings of the Japan Academy, Series A, 2. (2013), 29-33.

AUTHOR'S CONTRIBUTION

Paper **I** originates from private discussions between the first author and the second author. The first author made the main contribution to complete the paper.

The original version of Paper **II** is given by the first author. The second author checked the proofs of the paper.

The original version of Paper **V** is given by the first author. The second author checked the proofs of the paper.

Contents

1 Introduction

Nevanlinna's value distribution theory was originally formed as a deep generalization and quantification of Picard's theorem for meromorphic functions. Later on due to work of Wittich, Clunie, Mohon'ko, and many others, Nevanlinna theory has been applied as a powerful tool to consider complex oscillation and value distribution of meromorphic solutions of complex differential equations. In recent years these considerations have been extended to the field of complex difference equations.

The properties of meromorphic solutions of difference equations have been studied by using Nevanlinna theory by Yanagihara and others already in 1980's, but more systematic studied had to wait until the idea of Ablowitz, Halburd and Herbst in 2000. They suggested that the existence of sufficiently many finite-order meromorphic solutions of a difference equation is a good sign that the equation is of Painlevé type. This can be described as the first purely complex analytic candidate for the discrete Painlevé property. To verify their claim, new tools had to be developed to deal with various properties of meromorphic solutions of difference equations.

The lemma on the logarithmic derivative is an essential part of the proof of the second main theorem of Nevanlinna theory. It has also proved to be extremely useful in the analysis of value distribution of meromorphic solutions of differential equations [40]. Similarly, difference analogues of the lemma on the logarithmic derivatives have proved to be among the most important technical tools in the development of the Nevanlinna theoretical tools for difference equations. The first versions of this result were obtained by Halburd-Korhonen [23] and Chiang-Feng [14], independently. Combining a difference analogue of the lemma on the logarithmic derivative with methods from Nevanlinna theory, Chiang and Feng obtained a uniform lower bound for the order of meromorphic solutions of large classes of linear difference equations. Using the difference analogue of the lemma on the logarithmic derivative, Halburd and Korhonen [23] gave difference counterparts of the Clunie and Mohon'ko lemmas [16]. Laine and Yang [42] generalized the difference analogue of the Clunie lemma to a substantially larger class of difference equations.

Eremenko, Langley and Rossi have shown that the derivative functions of relatively slowly growing transcendental meromorphic functions have infinitely many zeros. Bergweiler and Langley gave analogous results for differences and divided differences of transcendental meromorphic functions. Hayman has studied a natural class of differential polynomials $f^n + af' - b = P(z, f)$ showing that if f is a transcendental meromorpic function, and $n \geq 5$, then $P(z, f)$ has infinitely many zeros. Hayman's result has prompted many generalizations, and related results over the years. The first difference analogue was given by Laine and Yang, who showed that an analogous difference polynomial has infinitely many zeros, provided that f is a transcendental meromorphic function of finite order.

In this thesis, relying on the methods introduced by Bergweiler and Langley, combined with some comparatively standard reasoning based on Wiman-Valiron theory, we get estimates about the zeros of differences $g(z) = f(z + c_1) + \cdots + f(z + c_k) - kf(z)$ and $g_k(z) = f(z + c_1) \cdots f(z + c_k) - f(z)^k$, for order < 1. Using methods from Nevanlinna theory, we continue to consider the zeros and fixed points of $g(z)$ and $g_k(z)$, for order ≥ 1. We also investigate the growth and value distribution of meromorphic solutions of a first order difference equation with small coefficients in the complex plane.

For the difference equation

$$f(z+1) + f(z-1) = R(z, f(z)), \tag{0.1}$$

Ablowitz, Halburd and Herbst [1] proved that if f has a non-rational meromorphic solution of finite order, then $\deg_w(R) \leq 2$. Under the supposition that (0.1) has an admissible meromorphic solution of finite order, Halburd and Korhonen [25] derived a class of difference equations containing

$$f(z+1)+f(z)+f(z-1)=\frac{\pi_1+\pi_2}{f(z)}+\kappa_1, \tag{0.2}$$

$$f(z+1)+f(z-1)=pf(z)+q, \tag{0.3}$$

where π_k, κ_1, p, q are distinct small functions related to f, κ_1 is an arbitrary periodic function of period 1, and π_k are arbitrary finite-order periodic functions with period k. We study a more general classes of difference equations

$$A_1(z)w(z+c_1)+A_2(z)w(z)+A_3(z)w(z+c_2)=\frac{A_4(z)}{w(z)}+A_5(z), \tag{0.4}$$

$$A_1(z)w(z+c_1)+A_2(z)w(z+c_2)=A_3(z)w(z)+A_4(z) \tag{0.5}$$

than (0.2) and (0.3). In fact, we investigate the growth and value distribution of finite order transcendental meromorphic solutions for difference equations (0.4) and (0.5). For instance, let $c_1+c_2=0$. If $w(z)$ is a transcendental meromorphic solution of (0.4) with meromorphic coefficients such that $T(r,A_j)=S(r,w)(j=1,2,\cdots,5)$, $A_1(z)A_2(z)A_4(z)\not\equiv 0, A_2(z+c_1)A_2(z)-A_1(z)A_3(z+c_1)\not\equiv 0$, then $\rho(w)=\infty$. This implies that if difference equation (0.4) has at least one admissible meromorphic solution of finite order, then either $A_1(z)A_2(z)A_4(z)\equiv 0$, or $A_2(z+c_1)A_2(z)-A_1(z)A_3(z+c_1)\equiv 0$. In addition, for difference equations (0.4) and (0.5), we prove the existence of rational solutions and give their forms.

Finally, for the difference polynomial $F_n(z)=\sum_{j=1}^{k}a_j(z)f(z+c_j)-a(z)f^n(z)$, we obtain an estimate of the number of $b-$ points, namely, $\lambda(F_n(z)-b)=\rho(f)$. The results we obtain improve those of Liu and Laine [43].

The rest of this survey is structured as follows: In section 2, we recall the basic notations of Nevanlinna theory, as well as some elementary results from complex differential equations. In section

3, we study the value distribution of differences of meromorphic functions. In section 4, we concentrate mainly on many recent results on meromorphic solutions of complex difference equations. In section 5, we summarize the contents of the Papers **I–V**.

2 General background

2.1 AN OUTLINE OF NEVANLINNA THEORY

As was mentioned in the introduction, Nevanlinna theory, or in particular, Nevanlinna's second main theorem, generalizes Picard's theorem in a very natural and powerful way. Nevanlinna theory is also a powerful tool for studying meromorphic solutions of differential and difference equations, and it is applied frequently for this purpose in this thesis as well. Therefore we go through the basics of Nevanlinna theory in this section.

For a meromorphic function f in $\mathbb{C}$, we first recall some definitions needed in this section.

The proximity function describes how close f is on average to a on the circle $\{|z| = r\}$.

Definition 2.1.1 $(Proximity\ \ function)[40, p.22]$

$$m(r, \frac{1}{f-a}) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \Big|\frac{1}{f(re^{i\phi}) - a}\Big| d\phi$$

suppose $f \not\equiv a \in \mathbb{C}$ *and*

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi,$$

with $\log^+ x := \max\{\log x, 0\} (x > 0)$.

We use the notation $D(z_0, r)$ to denote an open disc of radius r centered at $z_0 \in \mathbb{C}$. The counting function describes the number of times f takes the value a on the closed disc $\overline{D(0, r)}$.

Definition 2.1.2 $(Unintegrated\ \ counting\ \ function)[40, p.20]$
Let $f \not\equiv a \in \mathbb{C}$. *Let* $i(z, a, f)$ *denote the multiplicity of an* $a-$ *point of* f *at* z. *Then we define*

$$n(r, a, f) = n(r, \frac{1}{f-a}) = n(r, a) := \sum_{|z| \le r,\, f(z) = a} i(z, a, f),$$

i.e., $n(r,a,f)$ *counts the number of the roots of* $f(z) = a$ *on the closed disc* $\overline{D(0,r)}$*, each root according to its multiplicity. For the pole of* f*, similarly we define*

$$n(r,\infty,f) = n(r,f) = n(r,\infty) := \sum_{|z|\leq r,\ f(z)=\infty} i(z,\infty,f).$$

The integrated counting function is a sort of integrated logarithmic measure of the number of roots of $f(z) = a$. Counting function is continuous function of r.

Definition 2.1.3 $(Counting\ \ function)[40, p.20]$

$$N\Big(r, \frac{1}{f-a}\Big) := \int_0^r \frac{n(t,\frac{1}{f-a}) - n(0,\frac{1}{f-a})}{t}dt + n(0,\frac{1}{f-a})\log r,$$

suppose $f \not\equiv a \in \mathbb{C}$ *and*

$$N(r,f) := \int_0^r \frac{n(t,f) - n(0,f)}{t}dt + n(0,f)\log r.$$

The integrated counting function takes into account each $a-$ point only once.

Definition 2.1.4 $(The\ \ reduced\ \ counting\ \ function\ \ of\ \ f(z)\text{-}a)[66, p.31]$

$$\overline{N}\Big(r, \frac{1}{f-a}\Big) := \int_0^r \frac{\overline{n}(t,\frac{1}{f-a}) - \overline{n}(0,\frac{1}{f-a})}{t}dt + \overline{n}(0,\frac{1}{f-a})\log r,$$

where a *is a finite complex number,* $\overline{n}(r,\frac{1}{f-a})$ *denotes the number of zeros of* $f(z) - a$ *on the closed disc* $\overline{D(0,r)}$*, each zero being counted only once, and*

$$\overline{n}\Big(0, \frac{1}{f-a}\Big) = \begin{cases} 1, & if\ f(0)=a. \\ 0, & if\ f(0)\neq a. \end{cases}$$

Similarly, we can define $\overline{N}(r,f)$. An explicit expression of the characteristic function can be given in terms of the counting and proximity functions. We have

Definition 2.1.5 $(Characteristic\ \ function)[40, p.22]$

$$T(r,f) = m(r,f) + N(r,f).$$

The growth of the function $T(r,f)$ is a good measure of the complexity of the meromorphic function $f(z)$. The order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r} \quad \text{and} \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

The exponent of convergence $\lambda(f)$ of zeros of f is defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r,\frac{1}{f})}{\log r}.$$

The function characterizes the density of zeros of $f(z)$. For the characteristic functions of products and sums of a finite number of meromorphic functions, we introduce the following elementary inequalities, which come from the properties of the positive logarithmic function and of the Nevanlinna counting function.

(i) $T(r, \Sigma_{i=1}^{n} f_i) \leq \Sigma_{i=1}^{n} T(r, f_i) + O(1)$,

(ii) $T(r, \prod_{i=1}^{n} f_i) \leq \Sigma_{i=1}^{n} T(r, f_i)$,

where $f_i (i = 1,2,\cdots,n)$ are meromorphic functions and $r \geq 1$. See [66, P.8]

By applying the Jensen-Nevanlinna formula

$$T(r,f) = T(r,\frac{1}{f}) + O(1)$$

(see [66, P. 7]) to $f(z) - a$, we can easily get the first main theorem which gives a relation between characteristic functions $T\left(r, \frac{1}{f-a}\right)$ and $T(r,f)$. The first main theorem implies that $T\left(r, \frac{1}{f-a}\right)$ does not really depend on a, and the smaller $m\left(r, \frac{1}{f-a}\right)$ is, the larger $N\left(r, \frac{1}{f-a}\right)$ is.

Theorem 2.1.6 [First main theorem] *Let f be a meromorphic function, and let $a \in \mathbb{C}$. Then*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1).$$

The following theorem due to Valiron and Mohon'ko, see [48], is a more general version of the first main theorem. It is of essential importance, for instance, in the studies of the theory of complex differential and difference equations. For example, by the following theorem, Ablowitz, Halburd and Herst proved that If $\rho(f) < \infty$ and $y(z+1) + y(z-1) = R(z,y)$, then $\deg R(z) \leq 2$. A proof of this result can also be found, for instance, in [40, p. 29].

Theorem 2.1.7 [48] *Let f be a meromorphic function, and let $R(z,f)$ be an irreducible rational function in f of the form*

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j}, \tag{2.1.1}$$

where the coefficients are all small functions with respect to f. Then

$$T(r, R(z,f)) = \max\{p,q\} T(r,f) + S(r,f).$$

The lemma on the logarithmic derivative is one of the key results needed in the proof of the Nevanlinna's second fundamental theorem. The lemma is interesting on itself and is applied frequently. For the proof, see, e,g [29, p.36].

Lemma 2.1.8 [29] *Let f be a non-constant meromorphic function. If $\rho(f) < \infty$, then*

$$m\Big(r, \frac{f'}{f}\Big) = O(\log r), (r \to \infty),$$

while if $\rho(f) = \infty$, then

$$m\Big(r, \frac{f'}{f}\Big) = O\Big(\log r + \log T(r,f)\Big), (r \to \infty,\ r \notin E),$$

where E is a set of finite linear measure.

In what follows, $S(r,f)$ means any quantity of growth $o(T(r,f))$ as $r \to \infty$ outside of a set of finite linear measure. As usual, the lemma on the logarithmic derivative can be written in the form:

$$m\Big(r, \frac{f'}{f}\Big) = S(r,f).$$

The following lemma due to Gundensen is a sharp pointwise version of the lemma on the logarithmic derivative in the finite-order case. This result has been applied, for instance, in obtaining sharp growth estimates for entire solutions of linear differential equations with polynomial coefficients, and of arbitrary order.

Lemma 2.1.9 $[22, Corollary\ 1, Corollary\ 2]$ *Let $f(z)$ be a transcendental meromorphic function of finite-order σ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin H \cup [0, 1]$ and for all $k, j, 0 \leq j < k$, one has*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)} \tag{2.1.2}$$

Similarly, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that if $\psi \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that, for all z satisfying $\arg z = \psi$ and $|z| \geq R_0$, and for all $k, j, 0 \leq j < k$, we have (2.1.2) holds.

The following theorem is the deepest and the most important result of the value distribution theory. It implies a generalized version of the classical Picard's theorem.

Theorem 2.1.10 [Second main theorem] *Let f be a non-constant meromorphic function, let $q \geq 2$, and let $a_1, \ldots, a_q$ be distinct complex constants. Then*

$$m(r, f) + \sum_{k=1}^{q} m\Big(r, \frac{1}{f - a_k}\Big) \leq 2T(r, f) - N_1(r) + S(r, f),$$

where

$$N_1(r) := 2N(r, f) - N(r, f') + N\Big(r, \frac{1}{f'}\Big).$$

Combining Theorem 2.1.6 and Theorem 2.1.10, we can get a modified form of second main theorem. In loose terms, the following theorem implies that the counting function $N(r, a)$ must usually be much larger than the proximity function $m(r, a)$.

Theorem 2.1.11 *Let f be a non-constant meromorphic function, let $q \geq 2$, and let $a_1, \ldots, a_q$ be distinct complex constants. Then*

$$(q-1)T(r,f) \leq \overline{N}(r,f) + \sum_{k=1}^{q} \overline{N}\Big(r, \frac{1}{f-a_k}\Big) + S(r,f).$$

The second main theorem is also valid, if complex numbers are replaced by small functions, see Yamanoi [60, Corollary 1].

In addition to first and second main theorem, we recall the following two results which play an important role in the proofs of many results in Papers II-V. The first is the Hadamard's factorization theorem of meromorphic functions of finite order.

Theorem 2.1.12 [65, *Theorem* 2.7] *Let f be a meromorphic function of finite order $\rho(f)$. If*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots \quad (c_k \neq 0,\ k \in \mathbb{Z})$$

near $z = 0$, then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are the canonical products of $f(z)$ formed with the non-zero zeros and poles of $f(z)$, respectively, and $Q(z)$ is a polynomial of degree at most $\rho(f)$.

The following theorem is a generalization of Borel's Theorem on linear combinations of entire functions.

Lemma 2.1.13 $[65, pp.79-80]$ *Let $f_j(z)(j=1,2,\cdots,n)(n \geq 2)$ be meromorphic functions, $g_j(z)(j=1,2,\cdots,n)$ be entire functions, and let them satisfy the following conditions:*

(i) $f_1(z)e^{g_1(z)} + \cdots + f_k(z)e^{g_k(z)} \equiv 0$;

(ii) when $1 \leq j < k \leq n$, then $g_j(z) - g_k(z)$ is not a constant;

(iii) when $1 \leq j \leq n,\ \ 1 \leq h < k \leq n$, then

$$T(r,f_j) = o\{T(r,e^{g_h - g_k})\} \ \ (r \to \infty,\ \ r \notin E),$$

where $E \subset (1,\infty)$ is of finite logarithmic measure.

Then $f_j \equiv 0 (j = 1, \cdots, n)$.

2.2 WIMAN-VALIRON THEORY

Wiman-Valiron theory is another important tool for considering entire solutions of differential equations, for example. In this thesis, Wiman-Valiron theory plays an important role in the proof of Lemma 2.3 in Paper I. We will now introduce the basics of Wiman-Valiron theory, by following [40].

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the largest integer m such that $|a_m|r^m = \max_{n\geq 0} |a_n|r^n$ is called the central index and is denoted by $\nu(r, f)$. As we know, for a transcendental entire function f, $\nu(r, f)$ is increasing, piecewise constant, right-continuous and tends to infinity as $r \to \infty$. With the notation of the central index, the order of f can be expressed as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ \nu(r, f)}{\log r}.$$

and the hyper-order of f can be expressed as

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log \log \nu(r, f)}{\log r}.$$

See [40, p.55] or [37, p.36-37]. The following theory offers a useful method for order considerations of entire solutions of differential equations. The result can also found, for instance, in [40, p. 51].

Theorem 2.2.1 [37, *p*.187] *(Wiman-Valiron) Let f be a transcendental entire function, let $0 < \delta < \frac{1}{4}$ and z be such that*

$$|z| = r \quad \text{and} \quad |f(z)| > M(r, f)\nu(r, f)^{-\frac{1}{4}+\delta}.$$

Then

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu(r, f)}{z}\right)^n (1 + o(1))$$

holds for all $n \in \mathbb{N}$ and all $r \notin F \cup [0, 1]$, where $F \subset (1, \infty)$ is an exceptional set of finite logarithmic measure.

2.3 DIFFERENCE TOOLS FROM NEVANLINNA THEORY

A good number of tools from Nevanlinna theory to consider meromorphic solutions of difference equations have been developed over the recent years. The development of such tool is largely motivated by the groundbreaking 2000 paper due to Ablowitz, Halburd and Herbst on the extension of complex analytic Painlevé property to difference equations. However, there is still a lot to be done before one can talk about a comprehensive value distribution theory of difference equations. In this section we give a short survey of some of the most important results in this field from the point of view of this thesis.

A shift of $f(z)$ is defined as $f(z+c)$, while the differences $\Delta_c^n f(z)$ are defined in the standard way by

$$\Delta_c f(z) = f(z+c) - f(z), \quad \Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f), \quad n = 2, 3, \ldots.$$

The following theorem due to Halburd-Korhonen [23] is a difference analogue of the logarithmic derivative lemma.

Theorem 2.3.1 $[23, Corollary\ 2.2]$ *Let f be a non-constant meromorphic function of finite order, $c \in \mathbb{C}$, $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)}{r^\delta}\right) \tag{2.3.1}$$

for all r outside of an exceptional set F of finite logarithmic measure.

For a meromorphic function f of finite order, [25, Lemma 2.1] implies that $T(r+|c|, f) = (1+o(1))T(r, f)$ for all r outside of a set of finite logarithmic measure. Thus Theorem 2.3.1 can be expressed as follows:

Theorem 2.3.2 $[24, Theorem\ 2.1]$ *Let f be a non-constant meromorphic function of finite order, and let $c \in \mathbb{C}$, $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^\delta}\right), \tag{2.3.2}$$

for all r outside of an exceptional set with finite logarithmic measure.

Chiang and Feng [14] obtained a similar result as the above theorem, independently of [23], in a study concerning finite-order meromorphic solutions of linear difference equations.

Theorem 2.3.3 [14, *Corollary* 2.5] *Let f be a non-constant meromorphic function of finite order $\rho(f) = \rho$, and let $c \in \mathbb{C}$. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\rho-1+\varepsilon}). \tag{2.3.3}$$

Recently, Halburd, Korhonen and Tohge obtained the following theorem, which generalized the results above and extended the logarithmic difference lemma to meromorphic functions of hyper-order less than one:

Theorem 2.3.4 [26, *Theorem* 5.1] *Let f be a non-constant meromorphic function and $c \in \mathbb{C}$. If $\rho_2(f) = \rho_2 < 1$ and $\varepsilon > 0$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_2-\varepsilon}}\right) \tag{2.3.4}$$

for all r outside of a set of finite logarithmic measure.

The following theorem due to Chiang and Feng [14] gave a relation between $T(r,f)$ and $T(r,f(z+c))$, in the case when $f(z)$ is of finite order of growth.

Theorem 2.3.5 [14, *Theorem* 2.1] *Let f be a non-constant meromorphic function of finite order $\rho(f)$, and let c be a non-zero constant. Then, for each $\varepsilon > 0$, we have*

$$T(r,f(z+c)) = T(r,f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

The following lemma is an analogous estimate for the pole counting function of $f(z+c)$.

Lemma 2.3.6 [14, *Theorem* 2.2] *Let f be a meromorphic function with finite exponent of convergence of poles $\lambda(\frac{1}{f})$ and c is a non-zero complex constant. Then, for each $\varepsilon > 0$, we have*

$$N(r,f(z+c)) = N(r,f) + O(r^{\lambda(\frac{1}{f})-1+\varepsilon}) + O(\log r).$$

We next need the following notions: We consider difference products and difference polynomials. By a difference product, we mean a difference monomial, that is, an expression of type $\prod_{j=1}^{s} f(z+\gamma_j)^{\nu_j}$, where $\gamma_1, \cdots, \gamma_s$ are complex numbers and $\nu_1, \cdots, \nu_s$ are natural numbers. The degree of a difference monomial $\prod_{j=1}^{s} f(z+\gamma_j)^{\nu_j}$ is $\nu_1 + \nu_2 + \cdots + \nu_s$. A difference polynomial is a finite sum of difference products, that is, an expression of the form

$$U(z,f) = \sum_J \alpha_J(z)\Big(\prod_{j\in J} f(z+c_j)\Big),$$

where $c_j, j \in J$, is a set of distinct complex numbers. In what follows, we assume that the coefficients of difference polynomials are, unless otherwise stated, small functions as understood in the usual Nevanlinna theory; that is, their characteristic is of type $S(r,f)$. The total degree of $U(z,f)$ is the maxima of the degree of each single term in $U(z,f)$. See [42, p.557].

By these definitions it would perhaps be more natural to call "difference polynomial" by the name "shift polynomials." However, since the former terminology has been already widely adopted in the literature, we choose to stick with it. Also, by using the substitution $f(z+c_j) = f(z+c_j) - f(z) + f(z) = \Delta_{c_j} f + f$, we can always express "shift polynomials" as polynomials in differences of f.

As for difference counterparts of the Clunie lemma [16], see [23; Corollary 3.3]. The following lemma due to Laine and Yang is a significantly more general version than the original difference Clunie. It is applicable, for instance, to the difference Painlevé III equation, whereas [16] is not (see [39]). This fact played a key role in the classification of Ronkainen in [52], where he isolated the difference Painlevé III equations from a large class of difference equations using the criterion due to Ablowitz, Halburd and Herbst.

Theorem 2.3.7 [42, *Theorem* 2.3] *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f), P(z,f)$, and $Q(z,f)$ are difference polynomials such that the total degree $\deg U(z,f) = n$ *in* $f(z)$ *and its shifts, and* $\deg Q(z,f) \leq n$. *Moreover, we assume that* $U(z,f)$ *contains just one term of maximal total degree in* $f(z)$ *and its shifts. Then for each* $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

Possibly outside of an exceptional set of finite logarithmic measure.

The following lemma due to Halburd and Korhonen [23] is a difference analogue of a result due to A. Z. Mohon'ko and V. D. Mohon'ko [48] on differential equations. It enables the analysis of the value distribution of finite-order meromorphic solutions of difference equations for finite values.

Theorem 2.3.8 [23, 42, *Theorem* 2.4] *Let* $f(z)$ *be a transcendental meromorphic solution of finite order* ρ *of difference equation*

$$P(z,f) = 0,$$

where $P(z,f)$ *is a difference polynomial in* $f(z)$ *and its shift. If* $P(z,a) \not\equiv 0$ *for a slowly moving target function* a, *that is,* $T(r,a) = S(r,f)$, *then*

$$m\Big(r, \frac{1}{f-a}\Big) = O(r^{\rho-1+\varepsilon}) + S(r,f),$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 1. By observing the proofs of Theorem 2.3.7 and Theorem 2.3.8 one can see that the two terms in the error term $O(r^{\rho-1+\varepsilon}) + S(r,f)$ arises from two different sources. Namely, the term $O(r^{\rho-1+\varepsilon})$ comes from applying Theorem 2.3.3 to a difference quotient of f in various parts of the proof, while $S(r,f)$ arises from sums of characteristic functions of the coefficients of difference polynomials. Therefore, if one replaces Theorem 2.3.3 by Theorem 2.3.2 in the proofs of Theorem 2.3.7 and Theorem 2.3.8, the error term in both theorems becomes

$$o\Big(\frac{T(r,f)}{r^{\delta}}\Big) + S(r,f) = S(r,f),$$

where $\delta \in (0,1)$.

By using Theorem 2.3.4, Theorem 2.3.8 can be extended to meromorphic functions f such that $\rho_2(f) < 1$. This has been remarked in [26].

2.4 OTHER IMPORTANT LEMMAS

We recall some notation and a lemma from [25]. We use the notation ∞^k to denote a pole of w with multiplicity k. Similarly, 0^k and $a + 0^k$ denote a zero and a-point of w, respectively, with the multiplicity k.

The following lemma was originally introduced in [25, Lemma 3.1] in order to show explicitly that zeros and poles of small coefficient functions of difference equations cannot essentially interfere with the pole behavior of admissible meromorphic solutions. Let us consider the first-order equation $f(z+1) = a(z)f^2(z)$ as an example. Suppose $f(z)$ has a pole at z_0 of order k_0. Then $f(z)$ has (possibly) another pole at least of order $2k_0 - p_0$ at $z_0 + 1$, where p_0 is the order of (possible) zero of $a(z)$ at z_0. If $a(z_0) \neq 0$, we denote $p_0 = 0$. Now if $p_0 \geq 2k_0$, then in fact $f(z_0 + 1)$ is finite and the iteration process is stopped. Suppose that $f(z)$ has another pole at $z_1 \neq z_0$, of order k_1, say. Then $f(z)$ has again another pole at $z_1 + 1$, unless $a(z)$ has a zero of order $2k_1$ at least at z_1. Suppose that $a(z_1) \neq 0$. Then $f(z_1 + 1)$ is a pole of order $2k_1$ and $f(z_1 + 2)$ is a pole of order $4k_1 - p_1$, unless $p_1 \geq 4k_1$, where p_1 is the order of zero of $a(z)$ at $z_1 + 1$. If $p_1 = 0$, and there is no further interference from the zeros and poles of $a(z)$ to the iteration process, it follows that

$$f(z_1 + n) = \infty^{2^n k_1}, \tag{2.4.1}$$

for all $n \in \mathbb{N}$. This gives as a lower bound for the pole counting function of f, which immediately implies a lower bound 1 for the hyper-order of f. Whether or not we can always find such a sequence of poles depends on $a(z)$ and the nature of the solution $f(z)$. If, for instance, $a(z)$ is rational, and $f(z)$ transcendental, with infinitely many poles, pole sequence of the type (2.4.1) can be found (without Lemma 2.4.1). If $a(z)$ is non-rational, and satisfies $T(r, a) = S(r, f)$, then, in principle, $a(z)$ may have a zero if and only if $f(z)$ has a pole even if $N(r, f) = T(r, f) + S(r, f)$. In such a case, we cannot find a pole sequence, which is exactly of the type (2.4.1). However, with the help of Lemma 2.4.1 below, we can deduce that there is a pole sequence of $f(z)$ which behaves "almost" like (2.4.1), and in particular implies the same lower bound 1 for the hyper-order of f as above.

Lemma 2.4.1 [25, *Lemma* 3.1] *Let w be a meromorphic function having more than $S(r, w)$ poles, and let $a_s, s = 1, \cdots, n$, be small meromorphic functions with respect to w. Denote by m_j the maximum order of zeros and poles of the functions a_s at z_j. Then for any $\varepsilon > 0$, there are at most $S(r, w)$ points z_j such that*

$$w(z_j) = \infty^{k_j},$$

where $m_j \geq \varepsilon k_j$.

The following lemma is needed to get rid of an exceptional set of finite logarithmic measure. See [2] for the first part of the following lemma. See [21] for the second part of the following lemma.

Lemma 2.4.2 [21, *Lemma* 5] *Let $F(r)$ and $G(r)$ be monotone increasing functions such that $F(r) \leq G(r)$ outside of exceptional set E that is of finite logarithmic measure. Then for any $\alpha > 1$, there exists $r_0 > 1$ such that $F(r) \leq G(\alpha r)$ for all $r \geq r_0$.*

The following lemma finds a uniform upper bound for a finite order meromorphic function. It is a useful tool in dealing with the value distribution of finite-order meromorphic solutions of differential equations, both linear and nonlinear.

Lemma 2.4.3 [9, *Lemma* 1] *Let $f(z)$ be a meromorphic function with $\rho(f) = \eta < \infty$. Then for any given $\varepsilon > 0$, there is a set $E_1 \subset (1, +\infty)$ that has finite logarithmic measure, such that*

$$|f(z)| \leq \exp\{r^{\eta+\varepsilon}\},$$

holds for $|z| = r \notin [0, 1] \cup E_1$.

3 Zeros of differences of meromorphic functions

3.1 ASYMPTOTIC BEHAVIOR OF DIFFERENCES

One way to study the properties of differences of meromorphic functions is to first find an asymptotic relation between derivatives and differences, and they apply known tools for derivative functions of meromorphic functions. This approaches appears to work best for slowly growing functions.

Bergweiler and Langley [4] have shown that differences of meromorphic functions of order less than one behave asymptotically like their derivatives in the complex plane.

Theorem 3.1.1 [4, *Lemma* 3.5] *Let $f(z)$ be transcendental and meromorphic of order less than 1 in the plane. Let $h > 0$. Then there exists an $\varepsilon -$ set E' such that*

$$f(z+c) - f(z) = cf'(z)(1 + o(1)) \text{ as } z \to \infty \text{ in } \mathbb{C} \setminus E'.$$

uniformly in c for $|c| \leq h$.

Here, following Hayman [30, pp.75-76], we define an $\varepsilon-$set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an $\varepsilon-$set then the set of $r \geq 1$ for which the circle $S(0,r) = \{z \in \mathbb{C} : |z| = r\}$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Another useful result due to Bergweiler and Langley shows that for transcendental meromorphic functions of order less than one, both logarithmic derivative and difference quotient are small asymptotically, outside of an $\varepsilon -$ set.

Theorem 3.1.2 [4, *Lemma* 3.3] *Let f be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an $\varepsilon-$ set E such that*

$$\frac{f'(z+c)}{f(z+c)} = o(1), \quad f(z+c) = (1+o(1))f(z) \quad \text{as } z \to \infty \ \text{ in } \mathbb{C} \setminus E$$

uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large z not in E the function f has no zeros or poles in $|\varsigma - z| \leq h$.

The following theorem due to Bergweiler and Langley [4] gives an asymptotic identity on a circle of radius r involving a meromorphic function of lower order less than one, its derivative and its shift.

Theorem 3.1.3 [4, *Lemma* 3.6] *Let f be a function transcendental and meromorphic in the plane of the lower order $\mu(f) < \mu < 1$. Then, there exists arbitrarily large R with the following properties. First,*

$$T(32R, f') < R^{\mu}.$$

Second, there exists a set $J_R \subseteq [\frac{R}{2}, R]$ of linear measure $(1-o(1))\frac{R}{2}$ such that, for $r \in J_R$,

$$f(z+c) - f(z) = (1+o(1))f'(z) \quad \text{on } |z| = r.$$

Asymptotic relations between differences and derivatives can be found also in the higher order case. The following result of this type is also due to Bergweiler and Langley.

Theorem 3.1.4 [4, *Lemma* 4.2] *Let $n \in \mathbb{N}$. Let $f(z)$ be non-rational and meromorphic of order less than 1 in the plane. Then there exists an $\varepsilon -$ set E_n such that*

$$\Delta_c^n f(z) = (1+o(1))f^{(n)}(z) \text{ as } z \to \infty \text{ in } \mathbb{C} \backslash E_n.$$

3.2 RESULTS ON ZEROS OF DIFFERENCES OF MEROMORPHIC FUNCTIONS

We begin by introducing the zeros of derivative functions of meromorphic functions, and then move on to the zeros of differences of meromorphic functions. The following theorem considers the zeros of derivative functions of slowly growing meromorphic functions.

Theorem 3.2.1 [3, 17, 33] *Let f be transcendental and meromorphic in the plane with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

Then f' has infinitely many zeros.

Bergweiler and Langley [4] investigated the existence of zeros of $\triangle f$ and $\frac{\triangle f(z)}{f(z)}$, and obtained several results, which may be viewed as discrete analogues of the above theorem on the zeros of f'. It was observed by Bergweiler and Langley, that if f satisfies the hypotheses of Theorem 3.2.1, by Hurwitz's theorem, it follows that if z_0 is a zero of $f'(z)$ then $\triangle_c f(z) = f(z+c) - f(z)$ has a zero near z_0 for all sufficiently small $c \in \mathbb{C} \setminus \{0\}$. Hence it is natural to ask whether $\triangle_c f(z)$ must have infinitely many zeros or not. Bergweiler and Langley [4] answered this question, and obtained the following theorems.

Theorem 3.2.2 [4, *Theorem* 1.3] *There exists $\delta_0 \in (0, \frac{1}{2})$ with the following property. Let f be a transcendental entire function with order*

$$\sigma(f) < \frac{1}{2} + \delta_0 < 1$$

Then

$$G(z) = \frac{\triangle_c f(z)}{f(z)}$$

has infinitely many zeros.

Theorem 3.2.3 [4, *Theorem* 1.4] *Let $f(z)$ be a function transcendental and meromorphic of lower order $\mu(f) < 1$ in the plane. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of $f(z)$ satisfy $z_j - z_k = c$. Then $\triangle_c f(z)$ has infinitely many zeros.*

If f is a transcendental entire function of order < 1, then $\triangle_c f(z)$ is also a transcendental entire function of order < 1. By repetition of this argument, we know that so is each difference $\Delta^n f(z) (n \geq 1)$. Hence, $\Delta^n f(z)$ has infinitely many zeros. So it is natural to consider $G_n(z)$ instead of $\Delta^n f(z)$ in some cases. Using Theorem 3.1.4 and the standard Wiman-Valiron theory, Bergweiler and Langley obtained the following result.

Theorem 3.2.4 [4, *Theorem* 1.2] *Let $n \in \mathbb{N}, c \in \mathbb{C} \setminus \{0\}$ and f be a transcendental entire function of order $\sigma < \frac{1}{2}$. If*

$$G_n(z) = \frac{\triangle^n f(z)}{f(z)}$$

is transcendental, then $G_n(z)$ has infinitely many zeros.

The results above show that at least for relatively slow growing meromorphic functions, there are a large amount of zeros of differences and divided differences in the complex plane.

Chen and Shon [6] considered zeros and fixed points of differences and divided differences of entire functions with order of growth $\rho(f) = 1$ and obtained the following theorem.

Theorem 3.2.5 [6, *Theorem* 3] *Let $c \in \mathbb{C} \setminus \{0\}$ and let f be a transcendental entire function of order $\rho(f) = \rho = 1$, that has infinitely many zeros with the exponent of convergence of zeros $\lambda(f) = \lambda < 1$. Then $\triangle_c f(z)$ has infinitely many zeros and infinitely many fixed points.*

Recently, Chen and Shon [7] considered the following three problems about the zero distribution of differences of meromorphic functions:

(i) What conditions will guarantee that the difference $\triangle_c f(z)$ has infinitely many zeros for a meromorphic function $f(z)$?

(ii) What is the exponent of convergence of zeros of the difference $\triangle_c f(z)$ if it has infinitely many zeros?

(iii) What can we say about the zeros of

$$\triangle_c f(z) - l(z) \text{ and } \frac{\triangle_c f(z)}{f(z)} - l(z),$$

where $l(z)$ is a polynomial?

About question (i), the following theorem shows the condition that f satisfies $\lambda(\frac{1}{f}) < \lambda(f) < 1$ will guarantee that the difference $\triangle_c f(z)$ has infinitely many zeros without the hypothesis on c.

Theorem 3.2.6 [7, *Theorem* 1] *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a meromorphic function of order $\rho(f) = \rho \leq 1$. Suppose that f satisfies $\lambda(\frac{1}{f}) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then $\triangle_c f(z)$ has infinitely many zeros and satisfies*

$$\lambda(\triangle_c f) = \lambda(f).$$

Concerning question (ii), Theorem 3.2.6 also shows that if $f(z)$ has infinitely many zeros under certain circumstances, then

$$\lambda(\triangle_c f(z)) = \lambda(f).$$

About question (iii), the following two theorems show that

$$\triangle_c f(z) - l(z) \text{ and } \frac{\triangle_c f(z)}{f(z)} - l(z),$$

may have infinitely many zeros, respectively.

Theorem 3.2.7 [7, *Theorem* 2] *Let c and $f(z)$ satisfy the conditions of Theorem 3.2.6. Suppose that $l(z)$ is a polynomial. Then $\triangle_c f(z) - l(z)$ has infinitely many zeros and satisfies $\lambda(\triangle_c f - l) = \rho(f)$.*

Theorem 3.2.8 [7, *Theorem* 3] *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a meromorphic function of order $\rho(f) = \rho < 1$ or of the form $f(z) = h(z)e^{az}$ where $a \neq 0$ is a constant, $h(z)$ being a transcendental meromorphic function with $\rho(h) < 1$. Suppose that $l(z)$ is a nonconstant polynomial. Then*

$$G_1(z) = \frac{\triangle_c f(z)}{f(z)} - l(z)$$

has infinitely many zeros.

Some improvements of Theorems 3.2.6– 3.2.8 can be found in Paper II.

4 Solutions of complex difference equations

4.1 EXISTENCE OF SOLUTIONS

We next introduce some results about the existence of meromorphic solutions of difference equations, by following [1][20][27][31][56]. We begin by discussing the case of differential equations, and then move on to difference equations. Concerning the case of first-order differential equations, Malmquist [47, P. 311] has shown a century ago that, the only equation of the form

$$y' = R(z, y),$$

where R is rational in both arguments, that can have transcendental meromorphic solutions, is the Riccati equation:

$$y' = a_0(z) + a_1(z)y + a_2(z)y^2.$$

In 1954, Wittich [59] obtained the result that if the coefficients $a_j(z)$ are rational functions, then all meromorphic solutions of the Riccati equation are of finite order.

In the second-order case Picard [50] raised the following problem in 1889: If $R(z, w, w')$ is rational in w and w' and analytic in z, what are the second order ordinary differential equations of the form

$$w'' = R(z, w, w') \tag{4.1.0}$$

with the property that the singularities other than poles of any solutions of (4.1.0) depend only on the equation and not on the constants of integration? (The problem can also be found, for instance, in [20, p.1]). About the question, Painlevé [49] and Gambier [19] showed that there exist fifty canonical equations of the form (4.1.0) such that the property posed by Picard is satisfied. This property is now known as the Painlevé property. It turned out that out of the list of 50 equations all except 6 were solvable or transformable into another equation in the list, or to a linear equation. The remaining 6 equations are called as Painlevé equations. These equations are as follows:

$$w'' = 6w^2 + z, \tag{P_I}$$

$$w'' = 2w^3 + zw + \alpha, \tag{P_{II}}$$

$$w'' = \frac{(w')^2}{w} - \frac{1}{z}w' + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}, \tag{P_{III}}$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{P_{IV}}$$

$$\begin{aligned} w'' &= \frac{3w-1}{2w(w-1)}(w')^2 - \frac{1}{z}w' + \frac{1}{z^2}(w-1)^2\left(\alpha w + \frac{\beta}{w}\right) \\ &+ \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \end{aligned} \tag{P_V}$$

$$\begin{aligned} w'' &= \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \\ &+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right), \end{aligned} \tag{P_{VI}}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex constants. In the above classification result, those equations in the list of 50 equations that are solvable explicitly possess meromorphic general solutions (expressible e.g. in terms of elliptic functions) and so they have the Painlevé property. Painlevé equations cannot in general be solved explicitly, but it has been shown [20, p.6, p.11, p.23] that all solutions of P_I, P_{II} and P_{IV} are meromorphic. Equations P_{III} and P_V can be transformed into equations that have only meromorphic solutions, but such transformation is not possible for P_{VI}. Nevertheless all solutions of P_{VI} are meromorphic outside of the fixed singularities [20, p.30]. Hence all equations in the original Painlevé-Gambier list of 50 equations indeed possess the Painlevé property. The orders of all transcendental solutions of equations P_I, P_{II} and P_{IV} are finite (being at most $\frac{5}{2}$, 3 and 4, respectively) [53, 54, 58].

Next let us consider difference equations. The following theorem due to Shimomura [55] shows the existence of entire solutions of a large class of first-order difference equations.

Theorem 4.1.1 [55] *For any non-constant polynomial $P(w)$, the difference equation*

$$w(z+1) = P(w) \tag{4.1.1}$$

has a non-trivial entire solution.

Yanagihara [62] considered the case where the right side of (4.1.1) is rational in w, and he proved the following theorem which extends Theorem 4.1.1.

Theorem 4.1.2 [62, *Theorem* 2.5] *For any non-constant rational function $R(w)$, the difference equation*

$$w(z+1) = R(w) \tag{4.1.2}$$

has a non-trivial meromorphic solution.

Using a similar method as in the proof of Theorem 4.1.2, Yanagihara obtained the following theorem which shows the existence of meromorphic solutions for a class of n^{th} order difference equations.

Theorem 4.1.3 [61] *For any rational function*

$$R(w) = \frac{a_p w^p + \cdots + a_0}{b_q w^q + \cdots + b_0},$$

where $a_p, \cdots, a_0 \in \mathbb{C}, b_q, \cdots, b_0 \in \mathbb{C}$ and $p \geq q+2$, the difference equation

$$\alpha_n w(z+n) + \cdots + \alpha_1 w(z+1) = R(w),\ \alpha_N, \cdots, \alpha_1 \in \mathbb{C},$$

has a non-trivial meromorphic solution.

Theorems 4.1.1-4.1.3 show that, for difference equations, the existence of meromorphic solutions appears to be much more common than in the case of differential equations. Therefore, if one is looking for analogues of the Painlevé property, and of Malmquist's theorem in the difference case, further restrictions than just the existence of meromorphic solutions is needed.

Applying methods from Nevanlinna theory into first-order difference equations, Yanagihara obtained a result which can be seen as a difference analogue of Malmquist's theorem.

Theorem 4.1.4 [62, *Theorem* 2] *If the first-order difference equation*

$$w(z+1) = R(z, w), \tag{4.1.3}$$

where $R(z, w)$ is rational in both arguments, admits a non-rational meromorphic solution of finite order, then $\deg w(R) = 1$.

Many scholars have extended Theorem 4.1.4 to higher order difference equations, see [1, 31, 41, 51, 64]. For instance, Ablowitz et al proved the following theorem.

Theorem 4.1.5 [1, *Theorem* 3, *Theorem* 5] *If the second-order difference equation*

$$w(z+1) \star w(z-1) = R(z, w), \tag{4.1.4}$$

admits a non-rational meromorphic solution of finite order, where $R(z, w)$ is rational in both of its arguments, and the operation $\star$ stands for either the addition or the multiplication, then $\deg w(R) \leq 2$.

Ablowitz, Halburd and Herbst also suggested that the existence of sufficiently many finite-order meromorphic solutions is a good difference analogue of the Painlevé property.

The class of equations (4.1.4) includes many equations regarded as Painlevé I-III, but also many other (non-Painlevé) equations.

Theorem 4.1.6 [51] *If the second order difference equation*

$$(w(z+1) + w(z))(w(z-1) + w(z)) = \frac{P(z, w)}{Q(z, w)}, \tag{4.1.5}$$

where $P(z, w)$ and $Q(z, w)$ are polynomials in w having rational coefficients and no common roots, admits a non-rational meromorphic solution of finite order, then $\deg_w(P) \leq 4$ *and* $\deg w(Q) \leq 2$.

This class of equations (4.1.5) with $\deg_w(P) \leq 4$ and $\deg w(Q) \leq 2$ includes some equations called as the difference Painlevé IV equation. Ramani et al [51] also obtained an analogue of Theorem 4.1.6, where they derived a class of equations including the difference Painlevé V equation .

The following theorem due to Halburd and Korhonen appears to indicate that "sufficiently many" in the conjecture of Ablowitz, Halburd and Herbst, could mean "one admissible meromorphic solution of finite order".

Theorem 4.1.7 [25, *Theorem* 1.1] *If equation*

$$w(z+1) + w(z-1) = R(z,w), \tag{4.1.6}$$

where $R(z,w)$ is rational and irreducible in w and meromorphic in z, has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation

$$w(z+1) = \frac{p(z+1)w(z) + q(z)}{w(z) + p(z)}, \tag{4.1.7}$$

where $p, q \in S(w)$, where $S(w)$ denotes the field of small functions with respect to w, or equation (4.1.6) can be transformed to one of the following equations:

$$w(z+1) + w(z) + w(z-1) = \frac{\pi_1 z + \pi_2}{w(z)} + \kappa_1 \tag{4.1.8}$$

$$w(z+1) - w(z) + w(z-1) = \frac{\pi_1 z + \pi_2}{w(z)} + (-1)^z \kappa_1 \tag{4.1.9}$$

$$w(z+1) + w(z-1) = \frac{\pi_1 z + \pi_3}{w(z)} + \pi_2 \tag{4.1.10}$$

$$w(z+1) + w(z-1) = \frac{\pi_1 z + \kappa_1}{w(z)} + \frac{\pi_2}{w^2(z)} \tag{4.1.11}$$

$$w(z+1) + w(z-1) = \frac{(\pi_1 z + \kappa_1)w(z) + \pi_2}{(-1)^{-z} - w^2(z)} \tag{4.1.12}$$

$$w(z+1) + w(z-1) = \frac{(\pi_1 z + \kappa_1)w(z) + \pi_2}{1 - w^2(z)} \tag{4.1.13}$$

$$w(z+1)w(z) + w(z)w(z-1) = p \tag{4.1.14}$$

$$w(z+1) + w(z-1) = pw(z) + q \tag{4.1.15}$$

where $\pi_k, \kappa_k \in S(w)$ are arbitrary finite-order periodic functions with period k.

Equations (4.1.8), (4.1.10) and (4.1.11) are known alternative forms of difference Painlevé I equation, equation (4.1.13) is a difference Painlevé II and (4.1.14) and (4.1.15) are linear difference equations. Therefore Theorem 4.1.7 says that the property suggested by Ablowitz, Halburd and Herbst is sufficient to single out difference Painlevé equations out of a large class of difference equations. Whether or not this property is necessary, is still an open question.

S. Shimomura [56] discussed meromorphic solutions mainly for (4.1.8) in the case where π_1, π_2 and k_1 are constants. He showed that (4.1.8) has an asymptotic solution in a certain domain containing the positive real axis, which may be continued meromorphically to the whole complex plane, and he constructed formal solutions of (4.1.8). Using a holomorphic function asymptotic to one of these formal solutions, he derived a nonlinear difference equation equivalent to (4.1.8). Moreover, analogous results for (4.1.11) and (4.1.13) are obtained by similar arguments in [56]. However, whether or not these meromorphic solutions are of finite order, remains open.

For higher order difference equations, Heittokangas. et al [31] proved the following theorem.

Theorem 4.1.8 $[31, Proposition\ 8]$ *Let $c_1, \cdots, c_n$ be a non-zero complex constants. If the difference equation*

$$\sum_{k=1}^{n} f(z+c_k) = \frac{a_0(z) + \sum_{i=1}^{p} a_i(z) f(z)^i}{b_0(z) + \sum_{i=1}^{q} b_i(z) f(z)^i}$$

with rational coefficients $a_i(z), b_i(z)$ admits a transcendental meromorphic solution of finite order, then $\max\{p, q\} \leq n$.

4.2 GROWTH OF SOLUTIONS

The following Lemma due to Chiang and Feng is a pointwise estimate for the difference quotient which is a counterpart of Gundersen's logarithmic derivative estimate [22].

Lemma 4.2.1 [14, *Theorem* 8.2] *Let $f(z)$ be a meromorphic function of finite order $\rho < \infty$, c be a non-zero complex number, and let $\varepsilon > 0$, then there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure, such that for all $|z| = r \notin E \cup [0, 1]$, we have*

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \left|\frac{f(z+c)}{f(z)}\right| \leq \exp\{r^{\rho-1+\varepsilon}\}.$$

Lemma 4.2.1 is a good tool to deal with the growth of solutions of higher order linear difference equations. We can find some applications in [14]. The following theorem on the growth of meromorphic difference equations is an example of such applications

Theorem 4.2.2 [14, *Theorem* 9.2] *Let $A_0(z), \cdots, A_n(z)$ be entire functions such that there exists an integer $l, 0 \leq l \leq n$, such that*

$$\rho(A_l) > \max_{0 \leq l \leq n,\, j \neq l} \{\rho(A_j)\} \tag{4.2.1}$$

If $f(z)$ is a meromorphic solution to

$$A_n(z)y(z+n) + \cdots + A_1(z)y(z+1) + A_0(z)y(z) = 0, \tag{4.2.2}$$

then we have $\rho(f) \geq \rho(A_l) + 1$.

Ishizaki and Yanagihara [36] considered the growth of transcendental entire solutions of difference equation

$$Q_n(z)\Delta^n f(z) + \cdots + Q_1(z)\Delta f(z) + Q_0(z)f(z) = 0, \tag{4.2.3}$$

where $Q_n(z), \cdots, Q_0(z)$ are polynomials, $\Delta f(z) = f(z+1) - f(z)$ and $\Delta^n f(z) = \Delta(\Delta^{n-1} f(z))$, and obtained the following theorem. The Newton polygon for (4.2.3) is defined as the convex hull of $\bigcup_{j=0}^{n} \{(x, y) : x \geq j, y \leq \deg a_{n-j}(z) - (n-j)\}$.

Theorem 4.2.3 [36, *Theorem* 1.1] *Let $f(z)$ be transcendental entire solutions of (4.2.3) and of order $\chi < \frac{1}{2}$. Then we have*

$$\log M(r, f) = Lr^{\chi}(1 + o(1)),$$

where a rational number χ is a slope of the Newton polygon for Eq. (4.2.3), and $L > 0$ is a constant. In particular, we have $\chi > 0$.

Comparing Theorem 4.2.2 with Theorem 4.2.3, we see that although Eq. (4.2.2) can be rewritten as (4.2.3), the condition (4.2.1) and $A_j(0 \leq j \leq k)$ being entire, the polynomial P_l is dominating coefficient, guarantees that all transcendental meromorphic solutions of (4.2.2) satisfy $\rho(f) \geq 1$.

The condition (4.2.4) below guarantees that all transcendental meromorphic solutions of (4.2.5) satisfy $\rho(f) \geq 1$ and $\rho(f) = \lambda(f)$.

Theorem 4.2.4 [8, *Theorem* 1] *Let $F(z), P_n(z), \cdots, P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$ and*

$$\deg(P_n + \cdots + P_0) = \max\{\deg P_j : j = 0, \cdots, n\} \geq 1. \tag{4.2.4}$$

Then every finite order transcendental meromorphic solution $f(z)$ of equation

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = F(z) \tag{4.2.5}$$

satisfies $\rho(f) \geq 1$ and $\rho(f) = \lambda(f)$.

In the previous theorem we considered equation (4.2.5) with $F \not\equiv 0$. For the case $F = 0$, we get the following theorem.

Theorem 4.2.5 [8, *Theorem* 2] *Let $P_n(z), \cdots, P_0(z)$ be polynomials such that $P_nP_0 \not\equiv 0$ and satisfy (4.2.4). Then every finite order transcendental meromorphic solution $f(z)$ of equation*

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = 0$$

satisfies $\rho(f) \geq 1$ and assumes every non-zero value $a \in \mathbb{C}$ infinitely often and $\rho(f) = \lambda(f-a)$.

5 Summary of Papers I-V

One of the main themes of this thesis is the investigation of zero distribution of difference polynomials of the form

$$F_n(z) = \sum_{j=1}^{k} a_j(z) f(z+c_j) - a(z) f^n(z). \qquad (5.0.0)$$

In paper I and II, we consider the case $n = 1, a_j(z) = 1, (j = 1, \cdots, k), a(z) = k$. For this case we can write $F_n(z)$ as $F_1(z) = g(z) = f(z+c_1) + \cdots + f(z+c_k) - kf(z)$. In paper I, using the methods introduced by Bergweiler and Langley, and Wiman-Valiron theory, we consider the zeros of $g(z)$ under the assumption $\rho(f) < 1$. In paper II, we consider the zeros of $g(z)$ under the assumption $\rho(f) \geq 1$, by using the standard methods of Nevanlinna theory, for example, Theorem 2.1.12 and Lemma 2.1.13 from this survey. In paper V, we consider the case $n \geq 2$.

Another theme of this thesis is the application of the methods described above to the study of value distribution and growth properties of meromorphic solutions of difference equations. For instance, we consider the class

$$A_1(z)w(z+c_1) + A_2(z)w(z) + A_3(z)w(z+c_2) = \frac{A_4(z)}{w(z)} + A_5(z)$$

in paper IV and

$$\begin{aligned} &a_0(z)f^2(z+c) + (b_2(z)f^2(z) + b_1(z)f(z) + b_0(z))f(z+c) \\ &= d_4(z)f^4(z) + d_3(z)f^3(z) + d_2(z)f^2(z) + d_1(z)f(z) + d_0(z) \end{aligned}$$

in paper III, showing that under certain assumptions on the coefficients, all meromorphic solutions of these equations are of infinite order.

5.1 SUMMARY OF PAPER I

Differences of the forms $f(z_j + c_1) + f(z_j + c_2), f(z_j + c_1)f(z_j + c_2)$ appear in a natural way as a part of many important difference equations including difference Painlevé I-III, [1, 14, 29, 39].

Thus, it is natural to ask the following questions.

Problem A. *What are the exponents of convergence of zeros of differences and divided differences?*

Problem B. *What can be said about zeros of differences* $g(z) = f(z + c_1) + f(z + c_2) + \cdots + f(z + c_k) - kf(z)$ *and* $g_k(z) = f(z + c_1)f(z + c_2)\cdots f(z + c_k) - f^k(z)$*?*

For $k = 2$, Chen and Shon [5, Theorem 1–Theorem 6] obtained some estimates for zeros of differences $g(z) = f(z + c_1) + f(z + c_2)$, $g_2(z) = f(z + c_1)f(z + c_2)$.

For the general case, we obtain the following results. The following theorem shows the conditions that $\rho(f) < 1$ and $c_1 + c_2 + \cdots + c_k \neq 0$ will guarantee that $g(z)$ has infinitely many zeros.

Theorem 5.1.1 *Let $f(z)$ be a transcendental entire function of order $\rho < 1$. Let $c_1, c_2, \cdots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + c_2 + \cdots + c_k \neq 0$. Then $g(z)$ has infinitely many zeros and satisfies $\lambda(g) = \rho(g) = \rho$.*

In particular, if f has at most finitely many zeros z_j satisfying $f(z_j + c_1) + f(z_j + c_2) + \cdots + f(z_j + c_k) = 0$, then $G(z) = \frac{g(z)}{f(z)}$ satisfies $\lambda(G) = \rho(G) = \rho$.

The exponents of convergence of zeros of differences $g_k(z)$ are estimated accurately in the following theorem.

Theorem 5.1.2 *Let f, c_j $(j = 1, 2, \cdots, k)$ satisfy the conditions of Theorem 5.1.1. Then $g_k(z)$ has infinitely many zeros and satisfies $\lambda(g_k) = \rho(g_k) = \rho$.*

In particular, if a set $H = \{z_j\}$ consists of all distinct zeros of $f(z)$,

satisfying any one of the following two conditions,
(i) at most finitely many zeros z_i, z_l *satisfy* $z_i - z_l = c_j$ $(j = 1, 2, \cdots, k)$*;*
(ii) $\liminf_{j\to\infty} |\frac{z_{j+1}}{z_j}| = l > 1$,

then $G_k(z) = \frac{g_k(z)}{f^k(z)}$ *has infinitely many zeros and satisfies* $\lambda(G_k) = \rho(G_k) = \rho$.

In Theorem 5.1.1, we consider the exponents of convergence of zeros of differences $g(z)$, when $f(z)$ is a transcendental entire function. A natural question arises: what is the exponent of convergence of zeros of the difference $g(z)$, if $f(z)$ is a transcendental meromorphic function? About this question, we give the following theorem.

Theorem 5.1.3 *Let* $f(z)$ *be a transcendental meromorphic function of order* $\rho < 1$. *Let* $c_1, c_2, \cdots, c_k \in \mathbb{C} \setminus \{0\}$ *be such that* $c_1 + c_2 + \cdots + c_k \neq 0$. *If* f *has at most finitely many poles* b_j, b_s *satisfying*

$$b_j - b_s = k_1 c_{l_1} + k_2 c_{l_2} \ \ (k_d = 0, \pm 1,\ d = 1, 2;\ l_1, l_2 \in (1, 2, \cdots, k)), l_1 \neq l_2$$

then $g(z)$ *has infinitely many zeros and satisfies* $\lambda(g) = \rho(g) = \rho$.

In particular, if f *has at most finitely many zeros* z_j *satisfying* $f(z_j + c_1) + f(z_j + c_2) + \cdots + f(z_j + c_k) = 0$, *then* $G(z) = \frac{g(z)}{f(z)}$ *has infinitely many zeros and satisfies* $\lambda(G) = \rho(G) = \rho$.

What can we say about the zeros of $g_k(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_k) - f^k(z)$, where $f(z)$ is a transcendental meromorphic function? Concerning this question, we give the following Theorem 5.1.4.

Theorem 5.1.4 *Let* f, c_j $(j = 1, 2, \cdots, k)$ *satisfy the conditions of Theorem 5.1.3. If* f *has at most finitely many poles* b_j *satisfying*

$$f(b_j + k_1 c_{l_1} + k_2 c_{l_2}) = 0, \infty \quad (k_d = 0, \pm 1,\ d = 1, 2;\ l_1, l_2 \in (1, 2, \cdots, k),$$

$l_1 \neq l_2$), *then* $g_k(z)$ *has infinitely many zeros and satisfies* $\lambda(g_k) = \rho(g_k) = \rho$.

In particular, if a set $H = \{z_j\}$ *consists of all distinct zeros of* $f(z)$, *satisfying any one of the following two conditions,*
(i) at most finitely many zeros z_i, z_l *satisfy* $z_i - z_l = c_j$ $(j = 1, 2, \cdots, k)$*;*

(ii) $\liminf_{j\to\infty} |\frac{z_{j+1}}{z_j}| = l > 1$,

then $G_k(z) = \frac{g_k(z)}{f^k(z)}$ *has infinitely many zeros and satisfies* $\lambda(G_k) = \rho(G_k) = \rho$.

5.2 SUMMARY OF PAPER II

The aim of the paper is to generalize Theorems 3.2.7–3.2.9. In [7], Chen and Shon consider the zeros of the differences $\triangle_c f(z)$ under the assumption $\rho(f) \leq 1$. We study the zeros of the sum $g_k(z) = \triangle_{c_1} f(z) + \triangle_{c_2} f(z) + \cdots + \triangle_{c_k} f(z)$ of differences under the assumption $\rho(f) < \infty$. In particular, we study the densities of zeros of $g_k(z) - l(z)$ and of

$$G_k(z) = \frac{f(z+c_1) + f(z+c_2) + \cdots + f(z+c_k) - kf(z)}{f(z)} - l(z). \tag{5.2.1}$$

For this purpose, we prove the following three theorems.

According to Theorem 3.2.7 due to Chen and Shon [7], if f is a meromorphic function of order $\rho(f) \leq 1$ such that $\lambda(\frac{1}{f}) < \lambda(f) < 1$, then the difference $\Delta_c f$ of f satisfies $\lambda(\Delta_c f) = \lambda(f)$. The following theorem is a generalization of this result for $g_k(z)$.

Theorem 5.2.1 *Let* $f(z)$ *be a finite order meromorphic function,* $\lambda(\frac{1}{f}) < \lambda(f) < 1$. *Let* $c_1, c_2, \cdots, c_k \in \mathbb{C} \setminus \{0\}$ *be such that* $c_1 + c_2 + \cdots + c_k \neq 0$, *and let* $g_k(z) \not\equiv 0$. *Then*
(i): If $\rho(f) = \rho < 1$, *we have* $\lambda(g_k) = \lambda(f)$.
(ii): If $1 \leq \rho(f) = \rho < \infty$, *we have* $\lambda(g_k) \geq \lambda(f)$.

Theorem 5.2.1 was proved in [45]. We reproduce clarified a version of the proof here, where we give more details.

Proof of Theorem 5.2.1. By assumption $\lambda(\frac{1}{f}) < \lambda(f) < 1$.
Proof of Claim (i):
We suppose that f satisfies $\rho(f) < 1$. By Lemma 2.3 in [44], we

have that $g_k(z)$ is transcendental. Let $f(z) = \frac{u(z)}{v(z)}$, where $u(z)$ and $v(z)$ are canonical products ($v(z)$ may be a polynomial) formed by zeros and poles of $f(z)$, respectively, and

$$\lambda(\frac{1}{f}) = \lambda(v) = \rho(v) < \lambda(f) = \lambda(u) = \rho(u).$$

By Theorem 3.1.1, there exists an ε-set E such that

$$g_k(z) = (c_1 + c_2 + \cdots + c_k)f'(z)(1 + o(1)) \text{ as } z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{5.2.2}$$

Set

$$H = \{|z| : z \in E \text{ or } g_k(z) = 0 \text{ or } f'(z) = 0\},$$

then H is of finite linear measure. By (5.2.2), for $|z| = r \notin H$, we obtain

$$\begin{aligned} & |g_k(z) - (c_1 + c_2 + \cdots + c_k)f'(z)| \\ = & |o(1)(c_1 + c_2 + \cdots + c_k)f'(z)| \\ < & |(c_1 + c_2 + \cdots + c_k)f'(z)|. \end{aligned} \tag{5.2.3}$$

Thus $g_k(z)$ and $-(c_1 + c_2 + \cdots + c_k)f'(z)$ satisfy the conditions of Rouché's theorem. Applying Rouché's theorem and (5.2.3), for $|z| = r \notin H$, we have

$$n(r, \frac{1}{g_k}) - n(r, g_k) = n(r, \frac{1}{f'}) - n(r, f'). \tag{5.2.4}$$

Since $f' = \frac{u'(z)v(z) - u(z)v'(z)}{v^2(z)}$, $\lambda(\frac{1}{f}) < \lambda(f) = \rho(f) < 1$, $\rho(f') = \rho(f)$, we have

$$\lambda(\frac{1}{f'}) = \lambda(\frac{1}{f}) < \lambda(f) = \rho(f) = \rho(f').$$

By $\lambda(\frac{1}{f'}) < \rho(f') < 1$, we obtain $\lambda(f') = \rho(f')$. From $\lambda(\frac{1}{f}) < \lambda(f) = \rho(f')$, we obtain

$$\lambda(\frac{1}{f}) < \lambda(f) = \lambda(f') = \rho(f').$$

Combining this and $g_k(z) = f(z+c_1) + f(z+c_2) + \cdots + f(z+c_k) - kf(z)$, we obtain

$$\lambda(\frac{1}{g_k}) \leq \lambda(\frac{1}{f}) < \lambda(f) = \lambda(f'). \tag{5.2.5}$$

Hence, with (5.2.4) and (5.2.5), we obtain

$$\lambda(g_k) = \lambda(f') = \lambda(f),$$

since $\lambda\left(\frac{1}{f'}\right) < \lambda(f')$. Thus (i) holds.

Proof of Claim (ii):

Since $1 \leq \rho(f) < \infty$ and $\lambda(\frac{1}{f}) < \lambda(f) < 1$, it follows from the Hadamard factorization theorem, that

$$f(z) = h(z)e^{P(z)} = \frac{u(z)}{v(z)}e^{P(z)},$$

where $P(z)$ is a nonconstant polynomial, $h(z)$ is a meromorphic function such that $h(z) = \frac{u(z)}{v(z)}$, $u(z)$ and $v(z)$ are canonical products ($v(z)$ may be a polynomial) formed by zeros and poles of $f(z)$, respectively, and

$$\lambda(\frac{1}{f}) = \lambda(v) = \rho(v) = \lambda(\frac{1}{h}) < \lambda(f) = \lambda(u) = \rho(u) = \lambda(h) = \rho(h)$$

< 1. Hence

$$\begin{aligned} g_k(z) &= f(z+c_1) + f(z+c_2) + \cdots + f(z+c_k) - kf(z) \\ &= h(z+c_1)e^{P(z)+R_1(z)} + h(z+c_2)e^{P(z)+R_2(z)} + \cdots \\ &\quad + h(z+c_k)e^{P(z)+R_k(z)} - kh(z)e^{P(z)} \\ &= \Big(h(z+c_1)e^{R_1(z)} + h(z+c_2)e^{R_2(z)} + \cdots \\ &\quad + h(z+c_k)e^{R_k(z)} - kh(z)\Big)e^{P(z)} \\ &= w(z)e^{P(z)}, \end{aligned}$$

where $R_j(z) = P(z+c_j) - P(z) (j = 1, 2, \cdots, k)$, and

$$w(z) = h(z+c_1)e^{R_1(z)} + h(z+c_2)e^{R_2(z)} + \cdots + h(z+c_k)e^{R_k(z)} - kh(z). \tag{5.2.6}$$

By this, we get $\lambda(\frac{1}{w}) \leq \lambda(\frac{1}{h}) = \lambda(\frac{1}{f}) < \lambda(f) < 1$. Since $g_k(z) \not\equiv 0$, we have $w(z) \not\equiv 0$.
Next, suppose, contrary to the assertion, that $\lambda(g_k) < \lambda(f) < 1$.

If $1 \leq \rho(w) < \infty$, then there exist a nonconstant polynomial $R_0(z)$ and a nonzero meromorphic function $Q(z)$ such that

$$w(z) = Q(z)e^{R_0(z)} = \frac{u_1(z)}{v_1(z)}e^{R_0(z)}, \tag{5.2.7}$$

where $Q(z) = \frac{u_1(z)}{v_1(z)}$ with $u_1(z)$ and $v_1(z)$ being the canonical products formed by zeros and poles of $w(z)$, respectively, and

$$\lambda(\frac{1}{Q}) = \lambda(v_1) = \rho(v_1) = \lambda(\frac{1}{w}) \leq \lambda(\frac{1}{f}) < 1$$
$$\lambda(u_1) = \rho(u_1) = \lambda(Q) = \lambda(w) = \lambda(g_k) < 1$$

So, we obtain that $\rho(Q) = \max\{\lambda(Q),\ \lambda(\frac{1}{Q})\} < 1$. Without loss of generality, we assume that $c_i \neq c_j$, for arbitrary $i, j \in \{1, 2, \cdots, k\}, i \neq j$. Let $c_{k+1} = 0, h(z) = h(z)e^{R_{k+1}(z)}$, where $R_{k+1}(z) = 0$. We next consider the following two cases.

Case (1.1): Suppose there exist i_0, j_0 $(i_0, j_0 = 0, 1, 2, \cdots, k+1)$ such that $R_{j_0}(z) - R_{i_0}(z) = A$ is a constant. We need to consider two subcases ($R_{j_0}(z) - R_0(z)$ is not a constant for all j_0 $(j_0 = 1, 2, \cdots, k+1)$ or $R_{j_0}(z) - R_0(z)$ is a constant for some j_0 $(j = 1, 2, \cdots, k+1)$) separately.

Subcase (1.1.1): Suppose that $R_{j_1}(z) - R_0(z)$ is not a constant, for all j_1 $(j_1 = 1, 2, \cdots, k+1)$. We are assuming that there exist $1 \leq i_1, j_1 \leq k+1$ such that $R_{j_1}(z) - R_{i_1}(z) = A$ is a constant. Then we have that $P(z + c_{j_1}) - P(z + c_{i_1}) = A$. Since $P(z)$ is a polynomial, it must have the form $P(z) = az + d$ and $a \neq 0$. Hence we have that $R_{j_1} = ac_{j_1}$ is constant for $j_1 = 1, 2, \cdots, k+1$. From

$$w(z) = h(z + c_1)e^{R_1(z)} + h(z + c_2)e^{R_2(z)} + \cdots + h(z + c_k)e^{R_k(z)} - kh(z),$$

we get $\rho(w) < 1$. This is a contradiction.

Subcase (1.1.2): Suppose there exists a j_2 $(j_2 = 1, 2, \cdots, k+1)$ such that $R_{j_2}(z) - R_0(z) = A$ is a constant. If there also exists i_2 $(i_2 = 1, \cdots, j_2 - 1, j_2 + 1, \cdots, k+1)$ such that $R_{i_2}(z) - R_0(z) = B$

is a constant, then we have $R_{j_2}(z) - R_{i_2}(z) = A - B$. As in Subcase (1.1.1), we have R_{j_2} is a constant for $j_2 = 1, 2, \cdots, k+1$. Therefore, then R_0 is a constant, a contradiction. If now $R_i(z) - R_0(z)$ is not constant for any $i \neq j_2$, then

$$\begin{aligned} & h(z+c_1)e^{R_1(z)} + h(z+c_2)e^{R_2(z)} + \cdots + (e^A h(z+c_{j_2}) - Q(z))e^{R_0(z)} \\ & + h(z+c_k)e^{R_k(z)} - kh(z) = 0. \end{aligned} \tag{5.2.8}$$

For any $\alpha, \beta \in \{0, 1, \cdots, j_2 - 1, j_2 + 1, \cdot, k+1\}, \alpha \neq \beta$, since $\deg(R_\alpha(z) - R_\beta(z)) \geq 1, e^{R_\alpha(z) - R_\beta(z)}$ is of regular growth (see, e.g., [29, p. 7]), $\rho(h(z+c_i)) < 1$ and $\rho(e^A h(z+c_{j_2}) - Q(z)) < 1$, we conclude that

$$T(r, h(z+c_i)) = o\{T(r, e^{R_\alpha(z) - R_\beta(z)})\}$$

$$T(r, e^A h(z+c_j) - Q(z)) = o\{T(r, e^{R_\alpha(z) - R_\beta(z)})\}$$

Thus, from Lemma 2.1.13 and (5.2.8), we have $h(z) \equiv 0$, a contradiction.

Case (1.2): Suppose then that $R_j(z) - R_i(z)$ is not a constant for all $i,\ j (i, j = 0, 1, \cdots, k+1, i \neq j)$. By Lemma 2.1.13, $h(z+c_j) \equiv 0$ $(j = 1, \cdots, k)$, $h(z) \equiv 0$, a contradiction.

Therefore, $\rho(w) < 1$. We break the rest of the proof into three cases.

Case (2.1): Suppose there exists exactly one j $(j \in \{1, 2, \cdots, k\})$ such that $R_j(z)$ is a nonconstant polynomial. From (5.2.6), we get $\rho(w) \geq 1$, a contradiction.

Case (2.2): Suppose there exists at least two i_0, j_0 $(i_0, j_0 = 1, 2, \cdots, k)$, with $1 \leq i_0 < j_0 \leq k$, such that $R_{i_0}(z), R_{j_0}(z)$ are nonconstant polynomials. Without loss of generality, we suppose $R_1(z), R_2(z), \cdots, R_m(z) (m \geq 2)$ are nonconstant polynomials, where $R_{m+1}, \cdots, R_k$ are constants. We now rewrite $w(z)$ as follows

$$\begin{aligned} w(z) &= h(z+c_1)e^{R_1(z)} + h(z+c_2)e^{R_2(z)} + \cdots + h(z+c_m)e^{R_m(z)} \\ & + h(z+c_{m+1})e^{R_{m+1}} + \cdots + h(z+c_k)e^{R_k} - kh(z). \end{aligned}$$

If there exist $1 \leq i, j \leq m$ such that $R_i - R_j$ is a constant, we may apply Subcase (1.1.1) to deduce that $R_i(z)$ is a constant for $i = 1, 2, \cdots, m$, a contradiction. Then for arbitrary $i,\ j\ (i, j \in$

$\{1, \cdots, m\}, i \neq j)$ we have that $R_i - R_j$ is not a constant. By Lemma 2.1.13, we have $h(z + c_j) \equiv 0$, a contradiction.

Case (2.3): Suppose R_j is constant for all $j = 1, \cdots, k$. Using the method of Subcase (1.1.1), we see that $P(z) = az + b, a \neq 0$. Substituting $P(z) = az + b$ into $w(z)$, we have

$$w(z) = h(z + c_1)e^{ac_1} + h(z + c_2)e^{ac_2} + \cdots + h(z + c_k)e^{ac_k} - kh(z).$$

By Theorem 3.1.2, there exists an ε-set E such that

$$h(z + c) = h(z)(1 + o(1)), \tag{5.2.9}$$

as $z \to \infty$ in $\mathbb{C} \setminus E$. By (5.2.9), we obtain that

$$\begin{aligned} & w(z) = (e^{ac_1} + e^{ac_2} + \cdots + e^{ac_k})h(z)(1 + o(1)) - kh(z) \\ = & \quad (e^{ac_1} + e^{ac_2} + \cdots + e^{ac_k} - k)h(z)(1 + o(1)). \end{aligned} \tag{5.2.10}$$

By (5.2.10) and $w(z) \neq 0$, we have $e^{ac_1} + e^{ac_2} + \cdots + e^{ac_k} \neq k$. Since $h(z)$ is transcendental, we know that $w(z)$ is transcendental. Set

$$H = \{|z| : z \in E \text{ or } w(z) = 0 \text{ or } h(z) = 0\},$$

then H is of finite linear measure. By (5.2.10), for $|z| = r \notin H \cup [0, 1]$, we obtain that

$$\begin{aligned} & |w(z) - (e^{ac_1} + e^{ac_2} + \cdots + e^{ac_k} - k)h(z)| \\ = & \quad |(e^{ac_1} + e^{ac_2} + \cdots + e^{ac_k} - k)o(1)| \\ < & \quad |(e^{ac_1} + e^{ac_2} + \cdots + e^{ac_k} - k)h(z)|. \end{aligned} \tag{5.2.11}$$

Applying Rouché's theorem and (5.2.11), comparing (5.2.11) and (5.2.3), $w(z)$ and $g_k(z)$, $(e^{ac_1} + e^{ac_2} + \cdots + e^{ac_k} - k)h(z)$ and $(c_1 + c_2 + \cdots + c_k)f'(z)$, and using a similar method as in the proof of (i), we obtain

$$\lambda(w) = \lambda(h) = \lambda(u) = \lambda(f),$$

a contradiction. Hence $\lambda(g_k) = \lambda(w) \geq \lambda(f)$. Theorem 5.2.1 is thus proved.

According to Theorem 3.2.8 due to Chen and Shon [7], if f satisfies the conditions of Theorem 3.2.7, and $l(z)$ is a polynomial, then $\lambda(\Delta_c f - l) = \lambda(f)$. The following theorem is a generalization of this result for $g_k(z,l) := g_k(z) - l(z)$.

Theorem 5.2.2 *Let f, c_j $(j = 1, 2, \cdots, k), g_k(z)$ satisfy the conditions of Theorem 5.2.1. Suppose that $l(z)$ is a nonconstant polynomial, and let $g_k(z,l) := g_k(z) - l(z)$. Then*
(i): If $\rho(f) < 1$, we have $\lambda(g_k(z,l)) = \rho(f)$.
(ii): If $1 \leq \rho(f) < \infty$, we have $\lambda(g_k(z,l)) \geq 1$.

According to Theorem 3.2.9 due to Chen and Shon [7], if f is a meromorphic function of order $\rho(f) = \rho < 1$ or of the form $f(z) = h(z)e^{az}$, where $a \neq 0$ is a constant, $h(z)$ is a transcendental meromophic function such that $\rho(h) < 1$, then $G_1(z) = \frac{f(z+c)-f(z)}{f(z)} - l(z)$ has infinitely many zeros. The following theorem is a generalization of this result for $G_k(z)$.

Theorem 5.2.3 *Let f be a transcendental meromorphic function of order of growth $\rho(f) = \rho < 1$ or of the form $f(z) = h(z)e^{az}$ where $a \neq 0$ is a constant, and $h(z)$ is a transcendental meromorphic function with $\rho(h) < 1$. Let $c_1, c_2, \cdots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + c_2 + \cdots + c_k \neq 0$. Suppose that $l(z)$ is a nonconstant polynomial. Then $G_k(z)$ has infinitely many zeros.*

5.3 SUMMARY OF PAPER III

According to the classical Malmquist's theorem [47](see also, e.g. [40, p.192]), the only equation of the form

$$f' = R(z, f),$$

where $R(z, f)$ is rational in both arguments, that can have non-rational meromorphic solutions, is the Riccati equation. Let us consider a more general case of the following first-order algebraic differential equation

$$C(z, f)(f')^2 + B(z, f)f' + A(z, f) = 0, \tag{5.3.1}$$

where $C(z, f) \not\equiv 0, B(z, f)$, and $A(z, f)$ are polynomials in z and f. Here we refer the reader to Ishizaki's work [35], for the cases where the coefficients of the powers of f in $A(z, f), B(z, f)$ and $C(z, f)$ are transcendental functions. Steinmetz [56] showed that if (5.3.1) has a transcendental meromorphic solution, then the equation (5.3.1) can be reduced to the form

$$\begin{aligned} & a_0(z)f'^2 + (b_2(z)f^2 + b_1(z)f + b_0(z))f' \\ & = d_4(z)f^4(z) + d_3(z)f^3 + d_2(z)f^2 + d_1(z)f + d_0(z), \end{aligned} \tag{5.3.2}$$

where $a_0(z), b_i(z)(i = 0, 1, 2)$ and $d_j(z)(j = 0, ..., 4)$ are polynomials.

A natural discretization of equation (5.3.2) can be obtained by replacing f' in equation (5.3.2) with $f(z + c)$. We now consider the growth of meromorphic solutions to the following difference equation (5.3.3):

Theorem 5.3.1 *Let $c \in \mathbb{C} \setminus \{0\}$. If $f(z)$ is a transcendental meromorphic solution of*

$$\begin{aligned} & a_0(z)f^2(z + c) + (b_2(z)f^2(z) + b_1(z)f(z) + b_0(z))f(z + c) \\ & = d_4(z)f^4(z) + d_3(z)f^3(z) + d_2(z)f^2(z) + d_1(z)f(z) + d_0(z), \end{aligned} \tag{5.3.3}$$

with meromorphic coefficients satisfying $T(r, a_0) = S(r, f), T(r, b_i) = S(r, f)(i = 0, 1, 2)$, $T(r, d_j) = S(r, f)(j = 0, \cdots, 4)$ and $d_4(z) \not\equiv 0$, then $\rho(f) = \infty$.

In the previous theorem we considered equation (5.3.3) with $d_4(z) \not\equiv 0$. The following four theorems are about the case $d_4(z) \equiv 0$:
If $d_4(z) \equiv 0$, then (5.3.3) becomes

$$\begin{aligned} & a_0(z)f^2(z + c) + (b_2(z)f^2(z) + b_1(z)f(z) + b_0(z))f(z + c) \\ & = d_3(z)f^3(z) + d_2(z)f^2(z) + d_1(z)f(z) + d_0(z). \end{aligned} \tag{5.3.4}$$

The following theorem gives a relation between $\rho(f)$ and $\max\{\lambda(f), \lambda(\frac{1}{f})\}$, when $a_0(z)$, $b_i(z)(i = 0, 1, 2)$, and $d_j(z), (j = 0, 1, 2, 3)$ are polynomials such that $\deg(b_2 - d_3) = \max\{\deg b_2, \deg d_3\} \geq 1$, and $b_2(z)d_3(z) \not\equiv 0$.

Theorem 5.3.2 *Let $a_0(z)$, $b_i(z)(i = 0,1,2)$, and $d_j(z),(j = 0,1,2,3)$ be polynomials such that $\deg(b_2 - d_3) = \max\{\deg b_2, \deg d_3\} \geq 1$, and $b_2(z)d_3(z) \not\equiv 0$. Let $c \in \mathbb{C} \setminus \{0\}$. If $f(z)$ is a finite order transcendental meromorphic solution of (5.3.4), then*

$$1 \leq \rho(f) \leq \max\{\lambda(f), \lambda(\frac{1}{f})\} + 1. \tag{5.3.5}$$

The following theorem shows that if (5.3.4) has at least one admissible meromorphic solution of finite order, then either $d_3(z) = 0$ or $b_2(z) \not\equiv 0$.

Theorem 5.3.3 *Let $c \in \mathbb{C} \setminus \{0\}$, and let $a_0(z), b_i(z)(i = 0,1,2), d_j(z),(j = 0,1,2,3)$ be meromorphic functions of finite order. If $f(z)$ is a transcendental meromorphic solution of (5.3.4), then*
(i) If $\rho(a_0) < \rho(f)$, $\rho(b_i) < \rho(f)(i = 0,1,2)$, $\rho(d_j) < \rho(f)(j = 0,1,2,3)$, $\frac{f(z)}{f(z+c)} \not\equiv \frac{b_2(z)}{d_3(z)}$ and $\rho(f) = \rho < \infty$, then

$$\rho(f) = \max\{\lambda(f), \lambda(\frac{1}{f})\}.$$

(ii) If $T(r,a_0) = S(r,f)$, $T(r,b_i) = S(r,f)(i = 0,1)$, $T(r,d_j) = S(r,f)(j = 0,1,2,3)$, $d_3(z) \not\equiv 0$ and $b_2(z) \equiv 0$, then $\rho(f) = \infty$.

The proof of Theorem 5.3.3 (ii) has been omitted in [45], since it is similar to that of [45, Theorem 1.1]. We take the opportunity to present the details here.

Proof of Theorem 5.3.3 (ii): Let f be a meromorphic solution of (5.3.4). Suppose, contrary to the assertion, that $\rho(f) = \rho < \infty$. We write (5.3.4) in the form

$$d_3(z)f^3(z) = Q(z,f), \tag{5.3.6}$$

where

$$\begin{aligned} Q(z,f) \quad = \quad & a_0(z)f^2(z+c) + (b_1(z)f(z) + b_0(z))f(z+c) \\ & -d_2(z)f^2(z) - d_1(z)f(z) - d_0(z). \end{aligned}$$

Since the total degree of $Q(z,f)$ as a polynomial in $f(z)$ and its shifts, $\deg Q(z,f) \leq 2$, by Remark 1 and (5.3.6), we have

$$m(r,f) = S(r,f).$$

Hence, f has more than $S(r,f)$ poles counting multiplicities. In fact, $N(r,f) = T(r,f) + S(r,f)$. We use z_j to denote points in the pole sequence, and $k_j \geq 1$ to denote the order of the pole of $f(z)$ at $z = z_j$. Let m_j denote the maximum order of zeros and poles of the coefficients $a_0(z), b_i(z)(i = 0,1), d_j(z)(j = 0,1,2,3)$ at z_j. If z_j is not a pole or a zero of a_0, b_i, d_j, then $m_j = 0$ and so $\varepsilon k_j > m_j = 0$, for an arbitrarily small $\varepsilon \geq 0$. If z_j is a pole or a zero of a_0, b_i, d_j, then Lemma 2.4.1 implies that there are more than $S(r,f)$ points z_j such that k_j satisfies $\varepsilon k_j > m_j$. For $\varepsilon < \frac{1}{4}$, by (5.3.6), we either have that $z_j + 2c$ is a pole of f of multiplicity $k_{2,j} \geq (\frac{3}{2} - \varepsilon)k_{1,j}$, or $z_j + 2c$ is a zero of $d_3(z)$ with multiplicity greater than $\varepsilon k_{1,j}$. For the former case, By (5.3.6), we have that $z_j + 3c$ is a pole of f of multiplicity $k_{3,j}$, where

$$k_{3,j} \geq (\frac{3}{2} - \varepsilon)k_{2,j} \geq (\frac{3}{2} - \varepsilon)^2 k_{1,j}.$$

This implies that we either have that $z_j + 4c$ is a pole of f of multiplicity $k_{4,j} \geq (\frac{3}{2} - \varepsilon)k_{3,j}$, or $z_j + 4c$ is a zero of $d_3(z)$ with multiplicity greater than $\varepsilon k_{1,j}$. And so on. Not all sequences of iterates of this type can have a zero of $d_3(z)$ with the multiplicity greater than $\varepsilon k_{1,j}$ in them. Otherwise $d_3(z)$ has more than $S(r,w)$ zeros, counting multiplicities, and we get $d_3(z) \equiv 0$. This is a contradiction. Hence, there exist at least one infinite sequence $z_n = z_0 + nc$ ($n \in \mathbb{N}$) of poles of f, the multiplicity of which is k_n, such that $k_n \geq (\frac{3}{2} - \varepsilon)^{n-1}k_1 \geq (\frac{3}{2} - \varepsilon)^{n-1}$, and the multiplicity of $d_3(z_n) = 0$ is less than εk_0 for all $n \in \mathbb{N}$.
By a simple geometric observation, we have

$$z_n \in D(z_1, (n-1)|c|) \subset D(0, |z_1| + (n-1)|c|) = D(0, r_n).$$

As $n \to \infty$, we have $r_n \leq 2(n-1)|c|$, for n large enough. Therefore,

$$n(r_n, f) \geq (2-\varepsilon)^{n-1} > \left(\frac{5}{4}\right)^{n-1}.$$

Hence,

$$N(2r_n, f) \geq (\log 2)\left(\frac{5}{4}\right)^{n-1} \geq (\log 2)\left(\frac{5}{4}\right)^{\frac{r_n}{3|c|}}.$$

So, we get $\lambda(f) = \infty$, contradicting our hypothesis that $\rho(f) < \infty$. Hence $\rho(f) = \infty$.

We next investigate the exponent of convergence of zeros and poles of meromorphic solutions of the difference equation (5.3.4):

Theorem 5.3.4 *Let $c \in \mathbb{C} \setminus \{0\}$. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (5.3.4), where $a_0(z), b_i(z)(i = 0, 1, 2), d_j(z), (j = 0, 1, 2, 3)$ are finite order meromorphic functions such that $T(r, a_0) = S(r, f), T(r, b_i) = S(r, f)(i = 0, 1, 2), T(r, d_j) = S(r, f)(j = 0, 1, 2, 3)$.*
(i) If $d_0(z) \not\equiv 0$, then $\lambda(f) = \rho(f)$.
(ii) If $a_0(z) \equiv 0, b_i(z) \equiv 0(i = 0, 1)$ and there exist $i(i = 0, 1, 2)$ such that $d_i(z) \not\equiv 0$, then $\lambda(\frac{1}{f}) = \rho(f)$.

The following theorem shows that $f(z)$ has no finite Borel exceptional values under certain assumptions.

Theorem 5.3.5 *Let $a_0(z), b_i(z)(i = 0, 1, 2), d_j(z), (j = 0, 1, 2, 3)$ be finite order meromorphic functions such that $T(r, a_0) = S(r, f), T(r, b_i) = S(r, f)(i = 0, 1, 2)$, $T(r, d_j) = S(r, f)(j = 0, \cdots, 4)$, $d_0(z) \not\equiv 0$ and $(b_2(z) - d_3(z))z^3 + (a_0(z) + b_1(z) - d_2(z))z^2 + (b_0(z) - d_1(z))z - d_0(z) \not\equiv 0$. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (5.3.4). Then $f(z)$ has no finite Borel exceptional values.*

5.4 SUMMARY OF PAPER IV

In 2007, Halburd and Korhonen [24] proved that if equation (4.1.6) has an admissible meromorphic solution w of finite order, then either w satisfies a difference Riccati equation (4.1.7) or equation (4.1.6) can be transformed to one of the equations which are listed as (4.1.8)-(4.1.15).

Equations (4.1.8), (4.1.10) and (4.1.11) are known as alternative forms of difference Painlevé I equation, equation (4.1.13) is a difference Painlevé II and (4.1.14) and (4.1.15) are linear difference equations. Chen and Shon [9, 10] considered some value distribution problems of finite order meromorphic solutions of equations (4.1.7), (4.1.10), (4.1.11) and (4.1.13).

The main aims of paper IV are the following two points.

(1) We consider the order of growth, zeros and poles of meromorphic solutions for the following nonlinear differences equations (5.4.2) and (5.4.4), where equation (5.4.2) is the more general form of equation (4.1.8) and (4.1.9), and (5.4.4) is the more general form of equation (4.1.15).

(2) We consider properties of rational solutions of (5.4.3) and (5.4.5), where equation (5.4.3) is the more general form of equation (4.1.8) and (4.1.9), and (5.4.5) is the more general form of equation (4.1.15).

From the proof of the classification result by Halburd and Korhonen [25], it follows that if one chooses as a starting point, instead of (4.1.6), the equation

$$w(z+1)+w(z)+w(z-1)=\frac{A(z)}{w(z)}+B(z), \tag{5.4.1}$$

where $A(z)$ and $B(z)$ are rational, then (5.4.1) is reduced exactly to (4.1.8). This is due to the structure of the proof in [25, Theorem 1.1]. It is composed of many subcases that correspond to various subclasses of (4.1.6). Equation (5.4.1) corresponds essentially to one of these subclasses. The following theorem shows that demanding the existence of a transcendental meromorphic solution of a large class of difference equations containing (5.4.1) reduces this bigger class either to a smaller one which still contains (5.4.1) (and thus the difference Painlevé I), or to a class of first-order equations.

Theorem 5.4.1 *Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If $w(z)$ is a transcendental mero-*

morphic solution of

$$A_1(z)w(z+c_1) + A_2(z)w(z) + A_3(z)w(z+c_2) = \frac{A_4(z)}{w(z)} + A_5(z), \tag{5.4.2}$$

with meromorphic coefficients such that $T(r, A_j) = S(r, w)(j = 1, 2, \cdots, 5)$, $A_1(z)A_2(z)A_4(z) \not\equiv 0, A_2(z+c_1)A_2(z) - A_1(z)A_3(z+c_1) \not\equiv 0$ *and* $c_1 + c_2 = 0$, *then we have* $\rho(w) = \infty$.

Suppose that $c_1 + c_2 = 0$. Then Theorem 5.4.1 implies that if (5.4.2) has at least one admissible meromorphic solution of finite order, then either $A_1(z)A_2(z)A_4(z) \equiv 0$, or $A_2(z+c_1)A_2(z) - A_1(z)A_3(z+c_1) \equiv 0$. If $A_1(z) \equiv 0$, then (5.4.2) is reduced to

$$A_3(z)w(z+c_2) = \frac{A_4(z) - A_2(z)w^2(z)}{w(z)} + A_5(z),$$

which is a difference equation of order one. By applying Theorem 4.1.7, it follows that $A_2(z) \equiv 0$, and so (5.4.2) is in fact a difference Riccati equation in this case. If $A_4(z) \equiv 0$, then (5.4.2) becomes a linear difference equation. In the case $A_2(z) \equiv 0$ and $A_5(z) \equiv 0$, (5.4.2) is a first order linear difference equation with respect to $W(z) = w(z)w(z+c_1)$, but if $A_2(z) \equiv 0$ and $A_5(z) \not\equiv 0$, then we obtain a class of equations which contains (4.1.10) instead of (4.1.8). Finally, if $A_2(z+c_1)A_2(z) - A_1(z)A_3(z+c_1) = 0$, then (5.4.1) becomes a special case of (5.4.2) with $A_1(z) = A_2(z) = A_3(z) = 1$.

In the previous theorem we considered equation (5.4.2) with $c_1 + c_2 = 0$. For the case $c_1 + c_2 \neq 0$, we obtain the following theorem which shows that the condition $A_j(z)(j = 1, 2, \cdots, 5)$ is a polynomial, guarantees that all transcendental meromorphic solutions of (5.4.2) satisfy $\rho(w) \geq 1$.

Theorem 5.4.2 *Let* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. *If* $w(z)$ *is a transcendental meromorphic solution of (1.12), then*
(i): If $A_j(z)(j = 1, 2, \cdots, 5)$ *is a polynomial and* $c_1 + c_2 \neq 0$, *then we have* $\rho(w) = \rho \geq 1$.
(ii): If $A_5(z) \equiv 0, A_4(z) \not\equiv 0$ *and* $A_j(z)(j = 1, 2, 3, 4)$ *is a polynomial, then we have* $\rho(w) = \rho \geq 1$.

The previous two theorems were about non-rational solutions of a generalized form of (4.1.8) and (4.1.9). We next consider the existence and forms of rational solutions of (5.4.3), and obtain the following theorem:

Theorem 5.4.3 *Let $k \in \mathbb{C} \setminus \{0\}$ and $R(z) = \frac{A(z)}{B(z)}$ be an irreducible rational function, where $A(z)$ and $B(z)$ are polynomials with $\deg A(z) = a$ and $\deg B(z) = b$.*
(i): Suppose that $a \geq b$ and $a - b$ is zero or an even number. If equation

$$w(z+1) + w(z) + w(z-1) = \frac{R(z)}{w(z)} + k \tag{5.4.3}$$

has an irreducible rational solution $w(z) = \frac{m(z)}{n(z)}$, where $m(z)$ and $n(z)$ are polynomials with $\deg m(z) = m$ and $\deg n(z) = n$, then

$$m - n = \frac{a-b}{2}.$$

(ii): Suppose that $a < b$. If equation (5.4.3) has an irreducible rational solution $w(z) = \frac{m(z)}{n(z)}$, then $w(z)$ satisfies one of the following two cases
(1): $w(z) = \frac{m(z)}{n(z)} = \delta + \frac{S(z)}{H(z)}$, where $\delta = \frac{k}{3}$, $S(z)$ and $H(z)$ are polynomials with $\deg S(z) = s$ and $\deg H(z) = h$, and $s - h = a - b$.
(2): $m - n = a - b$.

The following two theorems are concerned with rational and non-rational solutions of a generalized form of (4.1.15). In addition to order considerations, a result (see Theorem 5.4.4 (ii)) indicates that solutions having a finite Borel exceptional value seem to appear in special situations only.

Theorem 5.4.4 *Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Let $w(z)$ be a finite order transcendental meromorphic solution of*

$$A_1(z)w(z+c_1) + A_2(z)w(z+c_2) = A_3(z)w(z) + A_4(z), \tag{5.4.4}$$

where $T(r, A_j) = S(r, w) (j = 1, 2, 3, 4)$. Then we have
(i): If $A_4(z) \not\equiv 0$ and $\frac{A_1(z)+A_2(z)-A_3(z)}{A_4(z)}$ is not a constant, then $w(z)$ has no finite Borel exceptional values.
(ii): If $A_4(z) \equiv 0, \rho(w) > 1$ and $\rho(A_j) < \rho(w) - 1 (j = 1, 2, 3)$, then $\rho(w) \leq \max\{\lambda(w), \lambda(\frac{1}{w})\} + 1$.

The aim of the following theorem is to show the existence and the forms of rational solutions of (5.4.5).

Theorem 5.4.5 *Let $q \in \mathbb{C} \setminus \{0\}$ and $R(z) = \frac{A(z)}{B(z)}$ be an irreducible rational function, where $A(z)$ and $B(z)$ are polynomials with $\deg A(z) = a$ and $\deg B(z) = b$. Let $s = a - b$.*
(i): Suppose that $q = 2$ and $s = -2$. Then equation

$$w(z+1) + w(z-1) = R(z) + qw(z) \tag{5.4.5}$$

has no rational solutions.
(ii): Suppose that $q = 2$ and $s = -1$. Then (5.4.5) has no rational solutions.
(iii): Suppose that $q = 2$ and $s \geq 0$. If (5.4.5) has an irreducible rational solution $y(z) = \frac{m(z)}{n(z)}$, where $m(z)$ and $n(z)$ are polynomials with $\deg m(z) = m$ and $\deg n(z) = n$, then

$$m - n = s + 2.$$

(iv): Suppose that $q = 2$ and $s \leq -3$, and suppose that $y(z)$ is defined as in (iii). Then

$$m - n = s + 2 \quad or \quad m - n = 1 \quad or \quad m = n.$$

(v): Suppose that $q \neq 2$, and suppose that $y(z)$ is defined as in (iii). Then

$$m - n = s.$$

5.5 SUMMARY OF PAPER V

Hayman [26] proved two classical theorems which can be combined as follows:

Theorem 5.5.1 [26, *Theorem* 8]. *Let $f(z)$ be a transcendental meromorphic function and $a \neq 0, b$ be finite complex constants. Then $f^n(z) + af'(z) - b$ has infinitely many zeros for $n \geq 5$. If $f(z)$ is transcendental entire, this holds for $n \geq 3$, resp. $n \geq 2$, if $b = 0$.*

Recently, a number of papers (see, e.g., [4, 12-15, 22-24, 34, 39, 40]) have focused on complex difference equations and difference analogues of Nevanlinna's theory.

Liu and Laine [43] established partial difference counterparts of Theorem 5.5.1, and obtained the following:

Theorem 5.5.2 [43, *Theorem* 1.2]. *Let $f(z)$ be a transcendental entire function of finite order, not of period c, and let $s(z)$ be a nonzero function, small compared to f. Then $f^n(z) + f(z+c) - f(z) - s(z)$ has infinitely many zeros, provided $n \geq 3$, resp. $n \geq 2$, if $s = 0$.*

In 2010, Liu [43] extended the above result by considering meromorphic functions. In 2011, Chen [12] gave an estimate of the number of $b-$ points, namely, $\lambda(f(z+c) - f(z) - af^n(z) - b) = \rho(f)$, hence he also generalized Theorem 5.5.2.

In paper V, we consider the zeros of the difference polynomial

$$F_n(z) = \sum_{j=1}^{k} a_j(z) f(z + c_j) - a(z) f^n(z),$$

and obtain the following results which generalize Theorem 5.5.2. In Theorems 5.5.3 and 5.5.5, we consider the case when the coefficients of $F_n(z)$ are constants. The following theorem shows $F_n(z)(n \geq 3)$ have infinitely many zeros under the condition that $\sum_{j=1}^{k} a_j(z) f(z + c_j) \not\equiv b$.

Theorem 5.5.3 *Let $f(z)$ be a transcendental entire function of finite order $\rho(f)$, let $b, a, c_j, a_j (j = 1, 2, \cdots, k)$ be complex constants. Set $F_n(z) = \sum_{j=1}^{k} a_j f(z + c_j) - af^n(z)$, where $n \geq 3$ is an integer. Then $F_n(z)$ have infinitely many zeros and $\lambda(F_n(z) - b) = \rho(f)$ provided that $\sum_{j=1}^{k} a_j(z) f(z + c_j) \not\equiv b$.*

In the previous theorem, we consider difference polynomial $F_n(z)$ with $n \geq 3$. The following theorem is about the case $n = 2$:

Theorem 5.5.4 *Suppose that $f(z)$ is a finite order transcendental entire function with a Borel exceptional value d. Let $b(z), a(z)(\not\equiv 0), a_j(z)(j = 1, 2, \cdots, k)$ be polynomials, and let $c_j(j = 1, 2, \cdots, k)$ be complex constants. If either $d = 0$ and $\sum_{j=1}^{k} a_j(z) f(z + c_j) \not\equiv 0$, or, $d \neq 0$ and $\sum_{j=1}^{k} da_j(z) - d^2 a(z) - b(z) \not\equiv 0$, then $F_2(z) - b(z) = \sum_{j=1}^{k} a_j(z) f(z + c_j) - a(z) f^2(z) - b(z)$ has infinitely many zeros and $\lambda(F_2(z) - b(z)) = \rho(f)$.*

The following example shows the condition that $f(z)$ has a Borel exceptional value cannot be omitted in Theorem 5.5.4.

Example 5.5.5 *For $f(z) = \exp\{z\} + z$, $a(z) = 4$, $c_1 = 3\pi i$, $c_2 = \pi i$, $c_3 = 0$, $c_4 = 5\pi i$, $c_5 = 7\pi i$, $a_1(z) = z$, $a_2(z) = -3z$, $a_3(z) = 6z$, $a_4(z) = -1$, $a_5(z) = 1$, $a_6(z) = \cdots = a_k(z) = 0$, $b(z) = 2\pi i$, we have $F_2(z) - b(z) = \sum_{j=1}^{k} a_j(z) f(z + c_j) - a(z) f^2(z) - b(z) = -4\exp\{2z\}$. Here $f(z)$ has no Borel exceptional values, but $F_2(z) - b(z)$ has no zeros.*

The following example shows that the zero distribution of $f(z)F_2(z)$ is different from that of $F_2(z)$.

Example 5.5.6 *For $f(z) = \exp\{z\} + 1, a = 2, c_1 = \ln 2, c_2 = \ln 4, c_3 = \ln 3, a_1 = 3, a_2 = 1, a_3 = -2, a_4 = \cdots = a_k = 0$, we have $F_2(z) = \sum_{j=1}^{k} a_j f(z + c_j) - af^2(z) = -2\exp\{2z\}$. Here $F_2(z)$ has no zero, but $f(z)F_2(z) = -2\exp\{2z\}(\exp\{z\} + 1)$ has infinitely many zeros.*

Next, we consider the question of what can we say about $f(z)F_2(z)$ when $f(z)$ has infinitely many multi-order zeros. For this question, we obtain the following Theorem 5.5.7:

Theorem 5.5.7 *Let $f(z)$ be a finite order transcendental entire function, and let $b, a, a_j, c_j (j = 1, 2, \cdots, k)$ be complex constants. If $f(z)$ has infinitely many multi-order zeros, then $H(z) = f(z)(\sum_{i=1}^{k} a_j f(z + c_j) - af^2(z)) - b$ has infinitely many zeros.*

The following example shows that Theorem 5.5.3 may fail for entire functions of infinite order.

Example 5.5.8 *For* $f(z) = \exp\{\exp\{z\}\}$, $c_1 = \ln 3$, $c_2 = \ln 3$, $c_3 = 0$, $a = 2$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = \cdots = a_k = 0$, *we obtain*

$$F_3(z) = \sum_{j=1}^{3} a_j f(z + c_j) - af^3(z) = 2\exp\{\exp\{z\}\}.$$

Here $F_3(z) \not\equiv 0$.

This following example shows that Theorem 5.5.3 may be fail for $n = 2$ and that the condition $n \geq 3$ in Theorem 5.5.3 is the best possible.

Example 5.5.9 *For* $f(z) = \exp\{z\} + 2$, $a = 2$, $c_1 = \ln 3$, $c_2 = \ln 9$, $c_3 = \ln 4$, $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, $a_4 = \cdots = a_k = 0$, *we have* $F_2(z) = \sum_{j=1}^{k} a_j f(z + c_j) - af^2(z) = -2\exp\{2z\} - 6$. *Here* $F_2(z) \not\equiv -6$.

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This thesis mainly considers the growth and value distribution of meromorphic solutions of difference equations arising from difference polynomials of various forms, mostly concentrating on the finite order case.

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