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VALUE DISTRIBUTION OF THE PRODUCT OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVE

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Abstract

In the paper we discuss the value distribution of the product of a meromorphic function and its derivative and we improve a recent result of K. W. Yu.

1. Introduction and definitions

Let f be a transcendental meromorphic function defined in the open complex plane $\boldsymbol{C}$. Hayman [5] proved the following theorem.

Theorem A. *If n (≥ 3) is an integer then $\psi = f^n f'$ assumes all finite values, except possibly zero, infinitely many times.*

He further conjectured [7] that Theorem A remains valid even if $n = 1$ or 2. Mues [9] proved the result for $n = 2$ and the case $n = 1$ was proved by Bergweiler and Eremenko [1] and independently by Chen and Fang [3].

A natural question of investigating the value distribution of $ff' - a$, where $a = a(z)$ is a non-zero meromorphic function satisfying $T(r, a) = S(r, f)$, was raised and a number of researchers have worked on the problem.

We call a meromorphic function $a \equiv a(z)$ a small function of f if $T(r, a) = S(r, f)$.

Following two theorems can be derived from two inequalities proved by Zhang {[12], see also [11]}.

Theorem B. *If $\delta(\infty; f) > 7/9$ then $ff' - a$ has infinitely many zeros, where a $(\not\equiv 0, \infty)$ is a small function of f.*

Theorem C. *If $2\delta(0; f) + \delta(\infty; f) > 1$ then $ff' - a$ has infinitely many zeros, where a $(\not\equiv 0, \infty)$ is a small function of f.*

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However in Theorem C the condition $2\delta(0;f)+\delta(\infty;f)>1$ can easily be replaced by the weaker condition $2\Theta(0;f)+\Theta(\infty;f)>1$.

The following result of Bergweiler [2] is worth mentioning.

Theorem D. *If f is of finite order and a is a polynomial then $ff'-a$ has infinitely many zeros.*

In Theorem B and Theorem C we see that some conditions have to be imposed on f to achieve the desired result. On the other hand, though in Theorem D no restriction, except the order restriction, is imposed on f, the desired result is achieved only for polynomials in contrast to arbitrary small functions as the target.

Recently Yu [11] treated the general case but instead of a single small function he achieved the result for a small function and its negative as a pair of targets. His result can be stated as follows.

Theorem E. *If a $(\not\equiv 0,\infty)$ is a small function of f then at least one of $ff'-a$ and $ff'+a$ has infinitely many zeros.*

In the paper we prove a result on the value distribution of $(f)^{n_0}(f^{(k)})^{n_1}$, where n_0 (≥ 2), n_1, k are positive integers and as a consequence of this we improve Theorem E though most probably one should not expect any corresponding improvement of Theorem D because of the condition $n_0\geq 2$.

Throughout the paper we denote by f a transcendental meromorphic function defined in the open complex plane $\boldsymbol{C}$. We do not explain the standard notations and definitions of the value distribution theory as those are available in [6].

Definition [8]. Let m be a positive integer. We denote by $N(r,a;f\,|\,{\leq}m)$ $(N(r,a;f\,|\,{\geq}m))$ the counting function of those a-points of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.

In a like manner we define $N(r,a;f\,|\,{<}m)$ and $N(r,a;f\,|\,{>}m)$.

Also $\overline{N}(r,a;f\,|\,{\leq}m)$, $\overline{N}(r,a;f\,|\,{\geq}m)$, $\overline{N}(r,a;f\,|\,{<}m)$ and $\overline{N}(r,a;f\,|\,{>}m)$ are defined similarly where in counting the a-points of f we ignore the multiplicities.

Finally we agree to take $\overline{N}(r,a;f\,|\,{\leq}\infty)\equiv\overline{N}(r,a;f)$ and $N(r,a;f\,|\,{\leq}\infty)\equiv N(r,a;f)$.

2. Lemma

In this section we prove a lemma which is required in the sequel.

Lemma. *If $N(r,0;f^{(k)}\,|\,f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}\,|\,f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f\,|\,{<}k)+k\overline{N}(r,0;f\,|\,{\geq}k)+S(r,f).$$

Proof. By the first fundamental theorem and Milloux theorem {p. 55 [6]} we get

$$\begin{aligned} N(r,0;f^{(k)} \mid f \neq 0) &\leq N\left(r,0;\frac{f^{(k)}}{f}\right) \\ &\leq N\left(r,\frac{f^{(k)}}{f}\right) + m\left(r,\frac{f^{(k)}}{f}\right) + O(1) \\ &\leq k\overline{N}(r,\infty;f) + N(r,0;f \mid <k) + k\overline{N}(r,0;f \mid \geq k) + S(r,f). \end{aligned}$$

This proves the lemma. □

3. The main result

In this section we discuss the main result of the paper.

THEOREM. *Let* $\psi = (f)^{n_0}(f^{(k)})^{n_1}$, *where* n_0 (≥ 2), n_1 *and* k *are positive integers such that* $n_0(n_0-1)+(1+k)(n_0n_1-n_0-n_1)>0$. *Then*

$$\left[1-\frac{1+k}{n_0+k}-\frac{n_0(1+k)}{(n_0+k)\{n_0+(1+k)n_1\}}\right]T(r,\psi) \leq \overline{N}(r,a;\psi)+S(r,\psi)$$

for any small function a $(\not\equiv 0,\infty)$ *of* f.

Proof. First we note that {cf. [4, 10]}

$$T(r,f)+S(r,f) \leq CT(r,\psi)+S(r,\psi)$$

and

$$T(r,\psi) \leq \{n_0+(1+k)n_1\}T(r,f)+S(r,f),$$

where C is a constant.

So it is clear that if a $(\not\equiv 0,\infty)$ is a small function of f then a is also a small function of ψ and vice-versa. Hence by Nevanlinna's three small functions theorem {p. 47 [6]} we get

$$T(r,\psi) \leq \overline{N}(r,0;\psi)+\overline{N}(r,\infty;\psi)+\overline{N}(r,a;\psi)+S(r,\psi), \tag{1}$$

where $\overline{N}(r,a;\psi)=\overline{N}(r,0;\psi-a)$.

Now by the lemma we get

$$\begin{aligned} \overline{N}(r,0;\psi) &\leq \overline{N}(r,0;f)+N(r,0;f^{(k)} \mid f \neq 0) \\ &\leq \overline{N}(r,0;f)+k\overline{N}(r,\infty;f)+N(r,0;f \mid <k) \\ &\quad + k\overline{N}(r,0;f \mid \geq k)+S(r,f) \\ &\leq (1+k)\overline{N}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f). \end{aligned} \tag{2}$$

Again we see that

$$N(r,0;\psi) - \overline{N}(r,0;\psi) \geq \{(1+k)n_0 + n_1 - 1\}\overline{N}(r,0;f\,|\geq 1+k) + (n_0 - 1)\overline{N}(r,0;f\,|\leq k).$$

Hence from (2) we get

$$\begin{aligned}\overline{N}(r,0;\psi) \leq{}& (1+k)\overline{N}(r,0;f\,|\geq 1+k) + k\overline{N}(r,\infty;f)\\ &+ \frac{1+k}{n_0-1}[N(r,0;\psi) - \overline{N}(r,0;\psi)\\ &- \{(1+k)n_0 + n_1 - 1\}\overline{N}(r,0;f\,|\geq 1+k)] + S(r,f)\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{n_0+k}{n_0-1}\overline{N}(r,0;\psi) \leq{}& \frac{1+k}{n_0-1}N(r,0;\psi) + k\overline{N}(r,\infty;f)\\ &+ \left[1 + k - \frac{(1+k)\{(1+k)n_0 + n_1 - 1\}}{n_0 - 1}\right]\overline{N}(r,0;f\,|\geq 1+k)\\ &+ S(r,f)\\ \leq{}& \frac{1+k}{n_0-1}N(r,0;\psi) + k\overline{N}(r,\infty;f) + S(r,f)\end{aligned}$$

i.e.,

$$\overline{N}(r,0;\psi) \leq \frac{1+k}{n_0+k}N(r,0;\psi) + \frac{k(n_0-1)}{n_0+k}\overline{N}(r,\infty;f) + S(r,f). \tag{3}$$

If z_0 is a pole of f with multiplicity p then z_0 is a pole of ψ with multiplicity $n_0 p + (p+k)n_1 \geq n_0 + (1+k)n_1$. Hence

$$N(r,\infty;\psi) \geq \{n_0 + (1+k)n_1\}\overline{N}(r,\infty;\psi). \tag{4}$$

Since $\overline{N}(r,\infty;\psi) = \overline{N}(r,\infty;f)$ and $S(r,\psi) = S(r,f)$, from (1), (3) and (4) we get

$$\begin{aligned}T(r,\psi) \leq{}& \frac{1+k}{n_0+k}N(r,0;\psi) + \left\{1 + \frac{k(n_0-1)}{n_0+k}\right\}\overline{N}(r,\infty;f) + \overline{N}(r,a;\psi) + S(r,\psi)\\ \leq{}& \frac{1+k}{n_0+k}N(r,0;\psi) + \frac{n_0(1+k)}{(n_0+k)\{n_0 + (1+k)n_1\}}N(r,\infty;\psi) + \overline{N}(r,a;\psi)\\ &+ S(r,\psi)\end{aligned}$$

i.e.,

$$\left[1 - \frac{1+k}{n_0+k} - \frac{n_0(1+k)}{(n_0+k)\{n_0 + (1+k)n_1\}}\right]T(r,\psi) \leq \overline{N}(r,a;\psi) + S(r,\psi).$$

This proves the theorem. □

The following corollary improves Theorem E.

COROLLARY. *Let $F = ff^{(k)}$, where k is a positive integer. Then for any small function a $(\not\equiv 0, \infty)$ of f*

$$\Theta(a;F) + \Theta(-a;F) \le 2 - \frac{2}{(2+k)^2}.$$

Proof. Since a^2 is also a small function of f, we get from the theorem for $n_0 = n_1 = 2$

$$\left[1 - \frac{(1+k)(3+k)}{(2+k)^2}\right] T(r, F^2) \le \bar{N}(r, a^2; F^2) + S(r, F)$$

i.e.,

$$2\left[1 - \frac{(1+k)(3+k)}{(2+k)^2}\right] T(r, F) \le \bar{N}(r, a; F) + \bar{N}(r, -a; F) + S(r, F),$$

which shows that

$$\Theta(a;F) + \Theta(-a;F) \le \frac{2(1+k)(3+k)}{(2+k)^2} = 2 - \frac{2}{(2+k)^2}.$$

This proves the corollary. □

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