Value of H_0 in the Inhomogeneous Universe

Ido Ben-Dayan,[1] Ruth Durrer,[2] Giovanni Marozzi,[2] and Dominik J. Schwarz[3]

[1]*Deutsches Elektronen-Synchrotron DESY, Theory Group, D-22603 Hamburg, Germany*
[2]*Université de Genève, Département de Physique Théorique and CAP,*
24 quai Ernest-Ansermet, CH-1211 Genève 4, Switzerland
[3]*Fakultät für Physik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany*
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Local measurements of the Hubble expansion rate are affected by structures like galaxy clusters or voids. Here we present a fully relativistic treatment of this effect, studying how clustering modifies the mean distance- (modulus-)redshift relation and its dispersion in a standard cold dark matter universe with a cosmological constant. The best estimates of the local expansion rate stem from supernova observations at small redshifts ($0.01 < z < 0.1$). It is interesting to compare these local measurements with global fits to data from cosmic microwave background anisotropies. In particular, we argue that cosmic variance (i.e., the effects of the local structure) is of the same order of magnitude as the current observational errors and must be taken into account in local measurements of the Hubble expansion rate.

The Hubble constant H_0 determines the present expansion rate of the Universe. For most cosmological phenomena, a precise knowledge of H_0 is of utmost importance. In a perfectly homogeneous and isotropic world, H_0 is defined globally. But the Universe contains structures like galaxy clusters and voids. Thus, the local expansion rate measured by means of cepheids and supernovae at small redshifts does not necessarily agree with the expansion rate of an isotropic and homogeneous model that is used to describe the Universe at the largest scales.

Recent local measurements of the Hubble rate [1,2] are claimed to be accurate at the few percent level, e.g., Ref. [1] finds $H_0 = (73.8 \pm 2.4)$ km s^{-1} Mpc^{-1}. In the near future, observational techniques will improve further, such that the local value of H_0 will be determined at 1% accuracy [3], competitive with the current precision of indirect measurements of the global H_0 via the cosmic microwave background (CMB) anisotropies [4].

The observed distance modulus μ is related to the bolometric flux Φ and the luminosity distance d_L by ($\log \equiv \log_{10}$)

$$\mu = -2.5 \log[\Phi/\Phi_{10\ \mathrm{pc}}] = 5 \log[d_L/(10\ \mathrm{pc})]. \quad (1)$$

The relation between the intrinsic luminosity L, the bolometric flux Φ, and the luminosity distance d_L of a source is $\Phi = L/4\pi d_L^2$. In a flat cold dark matter universe with a cosmological constant (ΛCDM) with present matter density parameter Ω_m, the luminosity distance as a function of redshift z is given by

$$d_L(z) = \frac{1+z}{H_0/c} \int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + 1 - \Omega_m}}. \quad (2)$$

As long as we consider only small redshifts, $z \lesssim 0.1$, the dependence on cosmology is weak, $d_L(z) \simeq c[z + (1 - 3\Omega_m/4)z^2]/H_0$, and the result varies by about 0.2% when Ω_m varies within the 2σ error bars determined by Planck [4]. However, neglecting the model-dependent quadratic term induces an error of nearly 8% for $z \simeq 0.1$.

The observed Universe is inhomogeneous and anisotropic on small scales, and the local Hubble rate is expected to differ from its global value for two reasons. First, any supernova (SN) sample is finite (sample variance), and, second, we observe only one realization of a random configuration of the local structure (cosmic variance). Thus, even for arbitrarily precise measurements of fluxes and redshifts, the local H_0 differs from the global H_0. Sample variance is fully taken into account in the literature, but cosmic variance is usually not considered.

In the context of Newtonian cosmology, cosmic variance of the local H_0 has been estimated in Refs. [5–8]. The first attempts to estimate cosmic variance of the local Hubble rate in a relativistic approach can be found in Refs. [9,10] (see, also, Ref. [11]) based on the ensemble variance of the expansion rate averaged over a spatial volume. It has been shown that this approach agrees very well with the Newtonian one [9], and it predicts a cosmic variance which depends on the sampling volume on the subpercent to percent level. However, this approach still neglects the fact that observers probe the past light cone and not a spatial volume. Also, the measured quantity is not an expansion rate but a set of the bolometric fluxes and redshifts.

In this Letter, we present the first fully relativistic estimation of the effects of clustering on the local measurement of the Hubble parameter without making any special hypothesis about how the fluctuations can be modeled around us. Considering only the measured

quantities and the cosmological standard model with stochastic inhomogeneities, we study the effect of cosmic structures on the local determination of H_0, i.e., taking light propagation effects fully into account. Other relativistic approaches were recently proposed in Refs. [12,13]. In Ref. [12], a "Swiss cheese" model was used in modeling the local Universe; in Ref. [13], a "Hubble bubble" model was used and the perturbation of the expansion rate, which is not directly measurable, was considered.

We shall find that the mean value of the Hubble parameter is modified at the subpercent level, while the contribution from clustering to the error budget is larger, typically 2% to 3%, hence, as large as observational errors quoted in the literature [1]. As we shall see, the small modification of the mean of the Hubble parameter can be reduced by a factor of 3 by using the flux instead of the distance modulus. On the other hand, the cosmic variance induced by inhomogeneities on H_0 is independent of the observable used. Finally, we find that even for an infinite number of SN Ia within $0.01 < z < 0.1$ with identical redshift distribution compared to a finite sample considered, clustering induces a minimal error of about 2% for a local determination of H_0.

Following Refs. [14,15], we use cosmological perturbation theory up to second order with an almost scale-invariant initial power spectrum to determine the mean perturbation of the bolometric flux (and of the distance modulus) from a standard candle and its variance.

Let us first consider the fluctuation of the mean on a sphere at fixed observed redshift z. We denote the light-cone average [16] over a surface at fixed redshift by $\langle \cdots \rangle$ and a statistical average by $\overline{}$. Using the results of Refs. [17,18] (see, also, Ref. [19]), the fluctuation of the flux $\Phi \propto d_L^{-2}$, away from its background value in the Friedmann-Lemaître universe [denoted by $(d_L^{FL})^{-2}$] is given by

$$d_L^{-2} = (d_L^{FL})^{-2}[1 + \Phi_1/\Phi_0 + \Phi_2/\Phi_0], \qquad (3)$$

where we expand $\Phi = \Phi_0 + \Phi_1 + \Phi_2$ up to second order in perturbation theory. The ensemble average of $\langle \Phi_1/\Phi_0 \rangle$ vanishes at first order but not at second order and must be added to another second order contribution from Φ_2/Φ_0; we obtain (see, e.g., Ref. [20])

$$\overline{\langle d_L^{-2} \rangle}(z) = (d_L^{FL})^{-2}[1 + f_\Phi(z)], \qquad (4)$$

where for $z \ll 1$,

$$f_\Phi(z) \simeq -\left(\frac{1}{\mathcal{H}(z)\Delta\eta}\right)^2 \overline{\langle (\vec{v}_s \cdot \vec{n})^2 \rangle}. \qquad (5)$$

Here, $\vec{n}$ denotes the direction to a given SN and $\vec{v}_s$ its peculiar velocity, η is conformal time, $\Delta\eta = \eta_0 - \eta(z)$ is the difference between the present time and the time at redshift

z, and $\mathcal{H}$ is the conformal Hubble parameter. In Ref. [15], the full contribution is given in terms of 39 Fourier integrals over the dimensionless power spectrum of the Bardeen potential today, $\mathcal{P}_\psi(k) = (k^3/2\pi^2)|\Psi_k(\eta_0)|^2$ with different kernels. We have removed the observer velocity since observations are usually quoted in the CMB frame corresponding to $\vec{v}_0 = 0$. A nonvanishing observer velocity would nearly double the effect in Eq. (5). The dominant peculiar velocity contribution at low redshift gives

$$f_\Phi(z) \simeq -\left(\frac{1}{\mathcal{H}(z)\Delta\eta}\right)^2 \frac{\tau^2(z)}{3} \int_{H_0}^{k_{UV}} \frac{dk}{k} k^2 \mathcal{P}_\psi(k), \quad (6)$$

where

$$\tau(z) = \int_{\eta_{in}}^{\eta_s} d\eta \frac{a(\eta)}{a(\eta_s)} \frac{g(\eta)}{g(\eta_0)}.$$

$g(\eta)$ is the growth factor, and the source and the observer times are indicated with the suffixes s and 0.

The brightness of supernovae is typically expressed in terms of the distance modulus μ. Because of the nonlinear function relating μ and Φ, one obtains different second order contributions,

$$\overline{\langle \mu \rangle} - \mu^{FL} = -\frac{2.5}{\ln(10)}\left[f_\Phi - \frac{1}{2}\overline{\langle (\Phi_1/\Phi_0)^2 \rangle}\right], \qquad (7)$$

where at $z \ll 1$, we also find

$$\overline{\langle (\Phi_1/\Phi_0)^2 \rangle} \simeq -4f_\Phi. \qquad (8)$$

The approximate equalities in Eqs. (5) and (8) are valid for $z \ll 1$, where the first order squared contribution of the peculiar velocity terms dominates over the other second order contributions. For $z \sim 0.3$ and larger, additional contributions notably due to lensing become relevant, see Refs. [14,15].

For measurements of the Hubble parameter, low redshift SNe are used in order to minimize the dependence of the result on cosmological parameters. As a consequence, Eqs. (5) and (8) are good approximations for the aim of this Letter.

Hereafter, we use the cosmological parameters from Planck [4], the linear transfer function given in Ref. [21] taking baryons into account, and $k_{UV} = 0.1\ h\,\mathrm{Mpc}^{-1}$, see Ref. [15] for details. Increasing the cutoff does not change our result due to two effects: the kernel $k^2\mathcal{P}_\psi(k)$ of the peculiar velocity contribution decreases at large k, and small-scale fluctuations are incoherent (see below) and their contribution to the variance decays like $1/N$, where N is the number of supernovae.

As an illustration for the effects of cosmic structure on the observed flux from a SN, we plot in Fig. 1 the average $\overline{\langle d_L^{-2} \rangle}(z)$ and its variance (as defined in Ref. [20]) using Eqs. (4)–(6) and (8). Figure 1 clearly shows how at low
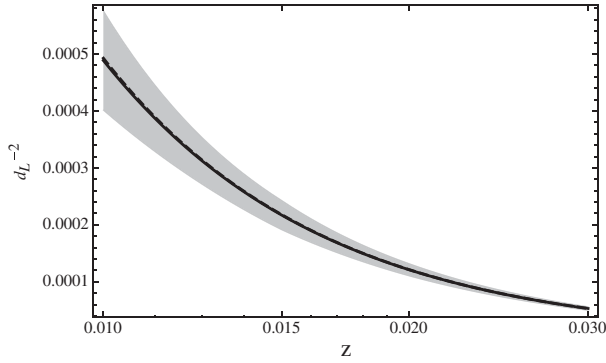
FIG. 1. The average $\overline{\langle d_L^{-2} \rangle}(z)$ of Eq. (5) in units of Mpc^{-2} (thick solid curve), its dispersion (shaded region), and the homogeneous value (dashed curve) are computed within a range $z = 0.01$ and $z = 0.03$. We have used a best-fit cosmology from Planck [4] and a UV cutoff of $k_{\rm UV} = 0.1 \, h \, {\rm Mpc}^{-1}$.

redshift the dispersion of the flux is much more important than the shift of the average (see, also, Refs. [14,15]).

Comparing Eqs. (4) and (7), we see that the flux averaged over a sphere at constant redshift experiences a different effect than the distance modulus averaged over the same sphere.

On the other hand, the induced theoretical dispersion on the bare value of H_0, which is entirely due to squared first order perturbations, is independent of the observable considered to infer H_0. To determine the dispersion of H_0 from a sample of SNe, we consider that at small redshift $H_0^2 \simeq c^2 z^2 / d_L^2$. H_0 inferred from the observation of a single SN at redshift $z \ll 1$ is then expected to deviate from the true H_0 by approximately [20]

$$(\Delta H_0)^2 = \frac{H_0^2}{4} \overline{\langle (\Phi_1/\Phi_0)^2 \rangle}. \qquad (9)$$

Of course, in practice, observers do not have at their disposal many SNe at the same redshift, so the average over a sphere cannot be performed. Hence, we now go beyond this simplifying assumption of previous works.

Let us estimate the (ensemble) variance of the locally measured Hubble parameter H_0 from the covariance matrix of the fluxes, given an arbitrarily distributed sample of N observed SNe at positions $(z_i, \vec{n}_i)$, which reads

$$\left(\frac{\Delta H_0}{H_0} \right)^2 = \frac{1}{4N^2} \sum_{ij} \overline{\frac{\Phi_1(z_i, \vec{n}_i)}{\Phi_0(z_i)} \frac{\Phi_1(z_j, \vec{n}_j)}{\Phi_0(z_j)}}$$

$$= \frac{1}{N^2} \sum_{ij} \frac{V_{ij}}{\mathcal{H}(z_i)\Delta\eta_i \mathcal{H}(z_j)\Delta\eta_j}, \qquad (10)$$

with

$$V_{ij} = \tau(z_i)\tau(z_j) \int_{H_0}^{k_{\rm UV}} \frac{dk}{k} k^2 \mathcal{P}_\psi(k) I(k\Delta\eta_j, k\Delta\eta_i, (\vec{n}_i \cdot \vec{n}_j)), \qquad (11)$$

and

$$I(x, y, \nu) = \frac{1}{4\pi} \int d\Omega_{\hat{k}} e^{ix(\hat{k}\cdot\vec{n}_i)} e^{-iy(\hat{k}\cdot\vec{n}_j)} (\hat{k}\cdot\vec{n}_j)(\hat{k}\cdot\vec{n}_i)$$

$$= \frac{xy(1-\nu^2)}{R^2} j_2(R) + \frac{\nu}{3}[j_0(R) - 2j_2(R)], \qquad (12)$$

where $\nu = (\vec{n}_i \cdot \vec{n}_j)$ and $R = \sqrt{x^2 + y^2 - 2\nu xy} = kd$. Here, d is the comoving distance between the SNe at $(z_i, \vec{n}_i)$ and $(z_j, \vec{n}_j)$, j_ℓ denotes the spherical Bessel function of order ℓ, and $\hat{k}$ is the unit vector in direction $\vec{k}$. To arrive at Eq., (12), we have introduced the Fourier representation of $\Phi_1(z_i, \vec{n}_i) = 2/(\mathcal{H}(z_i)\Delta\eta_i)\vec{v}_s(\vec{k}) \cdot \vec{n}_i$ and used some well-known identities. Note that with $I(x, x, 1) = 1/3$ and Eqs. (6) and (8), the autocorrelation term reproduces Eq. (9).

If the fluxes are perfectly coherent for all SNe so that $\overline{\Phi_1(z_i, \vec{n}_i)\Phi_1(z_j, \vec{n}_j)} = 4\sigma^2 \Phi_0(z_j)\Phi_0(z_i)$ for all correlations, we obtain $(\Delta H_0/H_0)^2 = \sigma^2$, while in the incoherent case $\overline{\Phi_1(z_i, \vec{n}_i)\Phi_1(z_j, \vec{n}_j)} = \delta_{ij} 4\sigma^2 \Phi_0(z_j)\Phi_0(z_i)$, we obtain $(\Delta H_0/H_0)^2 = \sigma^2/N$. The reality lies somewhere in between, wavelengths with $kd < 1$ being rather coherent while those with $kd > 1$ are rather incoherent.

In order to estimate the effect of the cosmic (co)variance for a realistic sample of SNe, we consider the following setup. We calculate $\Delta H_0/H_0$ from Eqs. (10)–(12) considering the redshifts of a sample of 155 SNe selected to lie in the range $0.01 \leq z \leq 0.1$ from the CfA3 and OLD samples [22,23]. The redshift distribution of the sample is shown in Fig. 2. We do not use their actual positions on the sky (see below). We then also study the limiting case of infinitely many SNe.

For the redshift distribution of the 155 SNe of this sample, Eq. (10) yields a dispersion induced by inhomogeneities between 2.2% and 3.3% for different angular distributions for the SNe. From this range, we infer
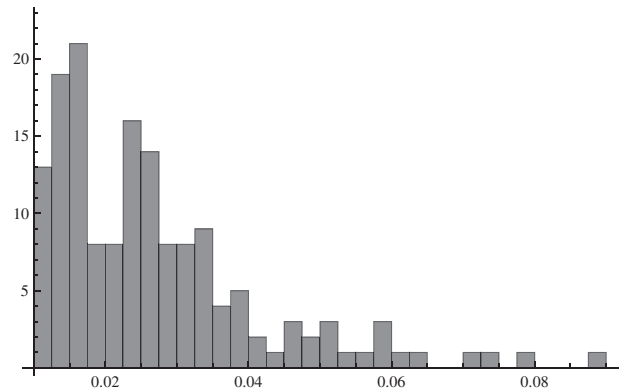

FIG. 2. The redshift distribution of the 155 SNe of the CfA3 + OLD sample [22,23] with redshift within 0.01 and 0.1 considered here.

$$\Delta H_0 = (1.6\text{–}2.4)\ \text{km s}^{-1}\,\text{Mpc}^{-1}, \tag{13}$$

with H_0 as given in Ref. [1]. We have kept ν constant to different values, and we have chosen a random distribution of directions over one hemisphere. The different choices give rise to the range quoted above. The smallest error corresponds to a random distribution of directions over one hemisphere, while the largest one corresponds to the case where all SNe are inside a narrow cone ($\nu \simeq 1$). The dispersion due to the actual angular distribution of real SN samples is left for future studies.

Let us also estimate the effect of inhomogeneities on the measured value of H_0 itself for this sample. In Ref. [1], a partial reconstruction of the peculiar velocity field has been applied, which, however, comes from the density field in the neighborhood of the SNe and, therefore, contributes only an incoherent part which we neglect. Considering a perfectly homogeneous universe, a measured Hubble parameter $\hat{H}_0$ is deduced from the measurement of $\mu(z \ll 1) \simeq 5\log(cz/\hat{H}_0) + C$, with C a constant, see Eqs. (1) and (2). However, this is not the true underlying H_0, since it ignores the local large-scale structure and, therefore, gives a biased value. The true underlying Hubble parameter is derived only by applying the appropriate correction due to this structure. Comparing Eq. (7) with the above expression, we have

$$H_0 \simeq \hat{H}_0 \left(1 - \frac{3}{2} f_\Phi\right). \tag{14}$$

We now consider the 155 SNe of the sample used here and generate the mean value of the corrected H_0 starting from a value of $\hat{H}_0$ and for the given redshift distribution. The final result is about 0.3% higher than $\hat{H}_0$ (choosing a larger cutoff affects only this result slightly). A similar global shift has already been included in the analysis of Ref. [1] as a consequence of the partial reconstruction of the peculiar velocity field [24]. Let us underline that the correction to H_0 would be 3 times smaller if we would consider the backreaction on the flux instead of the one on the distance modulus. In this case, Eq. (14) should be replaced by $H_0 \simeq \hat{H}_0(1 - \frac{1}{2} f_\Phi)$.

Considering the quoted observational error of 2.4 km/s/Mpc [1] and the additional variance (13), we obtain

$$H_0 = [73.8 \pm 2.4 \pm (1.6\text{–}2.4)]\ \text{km s}^{-1}\,\text{Mpc}^{-1}. \tag{15}$$

The tension with the Planck measurement [4], for which a value $(H_0)_{\text{CMB}} = 67.3 \pm 1.2$ km s^{-1} Mpc^{-1} is reported, is reduced when taking this additional variance into account. In particular, adding the above errors in quadrature, we obtain a deviation of 2.2σ to 1.9σ from $(H_0)_{\text{CMB}}$, while the difference is 2.7σ when using the error quoted in Ref. [1]. This analysis is insensitive to fluctuations on smaller scales

due to the incoherence of such contributions. Further modeling of these scales, e.g., Ref. [12] (see, also, Ref. [25]), might increase the uncertainty. However, effects from nearby small-scale structures are at least partly included in the analysis of Ref. [1].

Before concluding, we want to determine the ultimate error for an arbitrarily large sample when the SNe are distributed isotropically over directions. In this case, we can integrate $I(x, y, \nu)$ over all directions. With

$$\frac{1}{2}\int_{-1}^{1} d\nu I(x, y, \nu) = j_1(x) j_1(y),$$

we obtain for a normalized redshift distribution $s(z)$,

$$\left(\frac{\Delta H_0}{H_0}\right)^2 = \int \frac{dk}{k} k^2 \mathcal{P}_\psi(k) \left(\int dz \tau(z) s(z) \frac{j_1[k\Delta\eta(z)]}{\mathcal{H}(z)\Delta\eta(z)}\right)^2, \tag{16}$$

with $\int dz s(z) = 1$. Approximating the redshift distribution of our sample using an interpolating function of the histogram in Fig. 2, integrating from $z = 0.01$ to 0.1, we obtain a dispersion of about 1.8%, which corresponds to an error of

$$\Delta H_0 = 1.3\ \text{km s}^{-1}\,\text{Mpc}^{-1}. \tag{17}$$

This is the minimal dispersion of a SN sample with a redshift space distribution given by the one in Fig. 2. It is not much smaller than the value obtained for the real sample. Interestingly, this result is close to the ones obtained in Refs. [8,9,13], some of them with a very different analysis.

The errors from the nearby SNe with small $\Delta\eta(z)$ give the largest contribution. Therefore, the dispersion can be reduced by considering higher redshift SNe for which, however, the model dependence becomes more relevant. If we consider higher redshifts (close to or larger than 0.3), we have to take into account also the other contributions to the perturbation of the luminosity distance, see Refs. [17–19] for the full expression. As it is well known (see, for example, Refs. [14,15]), at redshift $z > 0.3$, the lensing term begins to dominate.

In Ref. [26], the peculiar velocity field has been reconstructed using the IRAS PSCz catalog [27]. As already mentioned above, this is subtracted in the analysis of Ref. [1]. It is clear that this procedure also modifies the expected mean and its variance in our method, but a detailed analysis of this is beyond the scope of this work. As the (minimal) cosmic variance Eq. (17) receives mainly contributions from scales larger than those considered in the reconstruction, we expect that it still has to be taken into account, in addition to the reconstructed peculiar velocities.

To conclude, in this Letter we estimate the impact of stochastic inhomogeneities on the local value of the Hubble

parameter and on its error budget for a given sample of standard candles. Equations (10)–(12) and (16) are the main results of this Letter, namely, a general formula for the cosmic variance contribution to ΔH_0 from a sample of SNe with $z \lesssim 0.2$, where the Doppler term dominates, and its limit for an arbitrarily large number of SNe isotropically distributed over directions. This general formula can be easily implemented and does not require an N-body simulation for each set of cosmological parameters. The required input is solely the linear power spectrum and the distribution of the observed SNe in position and redshift space. In particular, we find that for samples presently under consideration, this error is not negligible but of the same order as the experimental error, i.e., between 2.2% and 3.3%. We have also considered different samples (e.g., 95 SNe from Ref. [22]) in the range $0.01 < z < 0.1$ and find similar results. This cosmic variance is a fundamental barrier on the precision of a local measurement of H_0. It has to be added to the observational uncertainties, and it reduces the tension with the CMB measurement of H_0 [4].

Finally, even when the number of SNe is arbitrarily large, an irreducible error remains due to cosmic variance of the local Universe. We estimate this error and find it to be about 1.8% for SNe with redshift $0.01 < z < 0.1$ and a distribution given by the one in Fig. 2. This error can only be reduced by considering SNe with higher redshifts, but if too high redshifts are included, the result becomes strongly dependent on other cosmological parameters like Ω_m and curvature.

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