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Research Article

Value Sharing Results for q-Shifts Difference Polynomials

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We investigate the zero distribution of q-shift difference polynomials of meromorphic functions with zero order and obtain some results that extend previous results of K. Liu et al.

1. Introduction and Main Results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see, e.g., [1, 2]). Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of r with finite linear measure. Then the meromorphic function α is called a small function of $f(z)$, if $T(r, \alpha) = S(r, f)$. If $f(z) - \alpha$ and $g(z) - \alpha$ have same zeros, counting multiplicity (ignoring multiplicity), then we say that $f(z)$ and $g(z)$ share the small function α CM (IM). The logarithmic density of a set F_n is defined as follows:

$$\limsup_{r \to \infty} \frac{1}{\log r} \int_{[1,r] \cap F_n} \frac{1}{t} dt. \tag{1}$$

Currently, many articles have focused on value distribution in difference analogues of meromorphic functions (see, e.g., [3–11]). In particular, there has been an increasing interest in studying the uniqueness problems related to meromorphic functions and their shifts or their difference operators (see, e.g., [8, 12–15]). Our aim in this article is to investigate the uniqueness problems of q-difference polynomials.

Recently, Liu et al. [13] considered uniqueness of difference polynomials of meromorphic functions, corresponding to uniqueness theorems of meromorphic functions sharing values (see, e.g., [9, 16]). They got the following.

Theorem A. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that c is a nonzero complex constant and n is an integer. If $n \geq 14$ and $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.*

Theorem B. *Under the conditions of Theorem A, if $n \geq 26$ and $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.*

In this paper, we consider the case of q-shift difference polynomials and extend Theorem A as follows:

Theorem 1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f) = \rho(g) = 0$. Suppose that q and c are two nonzero complex constants and n is an integer. If $n \geq 14$ and $f^n(z)f(qz+c)$ and $g^n(z)g(qz+c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.*

It is natural to ask whether Theorem 1 holds if $f^n(z)f(qz+c)$ and $g^n(z)g(qz+c)$ share 1 IM. Corresponding to this question, we get the following result.

Theorem 2. *Under the conditions of Theorem 1, if $n \geq 26$ and $f^n(z)f(qz+c)$ and $g^n(z)g(qz+c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.*

Next, we consider the uniqueness of q-difference products of entire functions and obtain the following results.

Theorem 3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f) = \rho(g) = 0$, and let q and c be two nonzero complex constants, and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a nonzero polynomial, where $a_n (\neq 0)$, $a_{n-1}, \ldots, a_0$ are complex constants, and k denotes the number of the distinct zero of $P(z)$. If $n > 2k+1$ and $P(f(z))f(qz+c)$ and $P(g(z))g(qz+c)$ share 1 CM, then one of the following results holds:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and*

$$\lambda_j = \begin{cases} n+1, & a_j = 0, \\ j+1, & a_j \neq 0, \end{cases} \qquad j = 0, 1, \ldots, n; \tag{2}$$

(2) *$f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where*

$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c). \tag{3}$$

Remark 4. A similar result can be found in [15], but the method of this paper is more concise, and the condition of this paper is better.

2. Preliminary Lemmas

The following lemma is a q-difference analogue of the logarithmic derivative lemma.

Lemma 5 (see [14]). *Let $f(z)$ be a meromorphic function of zero order, and let c and q be two nonzero complex numbers. Then one has*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r, f) \tag{4}$$

on a set of logarithmic density 1.

Lemma 6 (see [7]). *If $T : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that*

$$\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = 0, \tag{5}$$

then the set

$$E := \{r : T(C_1 r) \geq C_2 T(r)\} \tag{6}$$

has logarithmic density 0 for all $C_1 > 1$ and $C_2 > 1$.

The following lemma is essential in our proof and is due to Heittokangas et al., see [12, Theorems 6 and 7].

Lemma 7. *Let $f(z)$ be a meromorphic function of finite order, and let $c \neq 0$ be fixed. Then*

$$\begin{gathered} \overline{N}(r, f(z+c)) \leq \overline{N}(r, f(z)) + S(r, f), \\ N\left(r, \frac{1}{f(z+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f), \\ N(r, f(z+c)) \leq N(r, f(z)) + S(r, f), \\ \overline{N}\left(r, \frac{1}{f(z+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f). \end{gathered} \tag{7}$$

Lemma 8. *Let $f(z)$ be a meromorphic function with $\rho(f) = 0$, and let c and q be two nonzero complex numbers. Then*

$$\begin{gathered} \overline{N}(r, f(qz+c)) \leq \overline{N}(r, f(z)) + S(r, f), \\ N\left(r, \frac{1}{f(qz+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f), \\ N(r, f(qz+c)) \leq N(r, f(z)) + S(r, f), \\ \overline{N}\left(r, \frac{1}{f(qz+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f). \end{gathered} \tag{8}$$

Proof of Lemma 8. We only prove the case $|q| \geq 1$. For the case $|q| \leq 1$, we can use the same method in the proof. By a simple geometric observation, we obtain

$$N\left(r, \frac{1}{f(qz+c)}\right) \leq N\left(|q|r, \frac{1}{f(z+(c/q))}\right). \tag{9}$$

Combining $\rho(f) = 0$ with Lemma 6, we obtain

$$N\left(|q|r, \frac{1}{f(z+(c/q))}\right) \leq N\left(r, \frac{1}{f(z+(c/q))}\right) + S(r, f) \tag{10}$$

on a set of logarithmic density 1. On the other hand, we have

$$N\left(r, \frac{1}{f(z+(c/q))}\right) \leq N\left(|qr|, \frac{1}{f(z+(c/q))}\right). \tag{11}$$

Therefore,

$$N\left(|q|r, \frac{1}{f(z+(c/q))}\right) = N\left(r, \frac{1}{f(z+(c/q))}\right) + S(r, f) \tag{12}$$

on a set of logarithmic density 1. From (9) and (12), we have

$$N\left(r, \frac{1}{f(qz+c)}\right) \leq N\left(r, \frac{1}{f(z+(c/q))}\right) + S(r, f). \tag{13}$$

By Lemma 7, we have

$$\begin{aligned} N\left(r, \frac{1}{f(qz+c)}\right) &\leq N\left(r, \frac{1}{f(z+(c/q))}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f(z)}\right) + S(r, f). \end{aligned} \tag{14}$$

Similarly, we have

$$\begin{gathered}\overline{N}\left(r, f\left(qz+c\right)\right) \leq \overline{N}\left(r, f\left(z\right)\right)+S\left(r, f\right), \\ N\left(r, f\left(qz+c\right)\right) \leq N\left(r, f\left(z\right)\right)+S\left(r, f\right), \\ \overline{N}\left(r, \frac{1}{f\left(qz+c\right)}\right) \leq \overline{N}\left(r, \frac{1}{f\left(z\right)}\right)+S\left(r, f\right).\end{gathered} \tag{15}$$

□

Lemma 9. *Let f be a nonconstant meromorphic function of zero order, and let c and q be two nonzero complex numbers. Then*

$$T\left(r, f\left(qz+c\right)\right) \leq T\left(r, f\left(z\right)\right)+S\left(r, f\right) \tag{16}$$

on a set of logarithmic density 1.

Proof of Lemma 9. By Lemmas 5 and 8, we have

$$\begin{aligned}T\left(r, f\left(qz+c\right)\right) &= m\left(r, f\left(qz+c\right)\right)+N\left(r, f\left(qz+c\right)\right) \\ &\leq m\left(r, \frac{f\left(qz+c\right)}{f\left(z\right)}\right)+m\left(r, f\left(z\right)\right) \\ &\quad +N\left(r, f\left(z\right)\right)+S\left(r, f\right) \\ &= T\left(r, f\left(z\right)\right)+S\left(r, f\right)\end{aligned} \tag{17}$$

on a set of logarithmic density 1. □

Lemma 10. *Let $f(z)$ be an entire function with $\rho(f)=0$, let q and c be two fixed nonzero complex constants, and let $P(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0$ be a nonzero polynomial, where $a_n(\neq 0), a_{n-1}, \ldots, a_0$ are complex constants. Then*

$$T\left(r, P\left(f\left(z\right)\right) f\left(qz+c\right)\right)=T\left(r, P\left(f\left(z\right)\right) f\left(z\right)\right)+S\left(r, f\right). \tag{18}$$

Proof of Lemma 10. By $\rho(f)=0$ and Lemma 5, we obtain

$$\begin{aligned}T\left(r, P\left(f\left(z\right)\right) f\left(qz+c\right)\right) &= m\left(r, P\left(f\left(z\right)\right) f\left(qz+c\right)\right) \\ &\leq m\left(r, P\left(f\left(z\right)\right) f\left(z\right)\right) \\ &\quad +m\left(r, \frac{f\left(qz+c\right)}{f\left(z\right)}\right) \\ &= T\left(r, P\left(f\left(z\right)\right) f\left(z\right)\right)+S\left(r, f\right)\end{aligned} \tag{19}$$

on a set of logarithmic density 1. Using the similar method as above, we also get

$$T\left(r, P\left(f\left(z\right)\right) f\left(z\right)\right) \leq T\left(r, P\left(f\left(z\right)\right) f\left(qz+c\right)\right)+S\left(r, f\right) \tag{20}$$

on a set of logarithmic density 1.

Hence, we have $T(r, P(f(z))f(z)) = T(r, P(f(z))f(qz+c)) + S(r, f)$ on a set of logarithmic density 1. □

Lemma 11 (see [17]). *Let F and G be two nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

(i)

$$\begin{aligned}&\max\left\{T\left(r, F\right), T\left(r, G\right)\right\} \\ &\quad \leq N_2\left(r, \frac{1}{F}\right)+N_2\left(r, F\right)+N_2\left(r, \frac{1}{G}\right)+N_2\left(r, G\right) \\ &\quad\quad +S\left(r, F\right)+S\left(r, G\right),\end{aligned} \tag{21}$$

(ii) $F=G$,

(iii) $FG\equiv 1$,

where $N_2(r, 1/F)$ denotes the counting function of zero of F, such that simple zero are counted once and multiple zeros are counted twice.

In order to prove Theorem 2, we need the following lemma.

Lemma 12 (see [16]). *Let F and G be two nonconstant meromorphic functions, and let F and G share 1 IM. Let*

$$H=\frac{F''}{F'}-2\frac{F'}{F-1}-\frac{G''}{G'}+2\frac{G'}{G-1}. \tag{22}$$

If $H\not\equiv 0$, then

$$\begin{aligned}T\left(r, F\right)+T\left(r, G\right) &\leq 2\left(N_2\left(r, \frac{1}{F}\right)+N_2\left(r, F\right)\right. \\ &\quad \left.+N_2\left(r, \frac{1}{G}\right)+N_2\left(r, G\right)\right) \\ &\quad +3\left(\overline{N}\left(r, F\right)+\overline{N}\left(r, G\right)+\overline{N}\left(r, \frac{1}{F}\right)\right. \\ &\quad \left.+\overline{N}\left(r, \frac{1}{G}\right)\right)+S\left(r, F\right)+S\left(r, G\right).\end{aligned} \tag{23}$$

3. Proof of Theorem 1

Let $F(z)=f^n(z)f(qz+c)$ and $G(z)=g^n(z)g(qz+c)$. Thus, F and G share 1 CM. Combining the first main theorem with Lemma 9, we obtain

$$\begin{aligned}nT\left(r, f\left(z\right)\right) &\leq T\left(r, f^n\left(z\right) f\left(qz+c\right)\right)+T\left(r, f\left(z\right)\right) \\ &\quad +O\left(1\right).\end{aligned} \tag{24}$$

Hence, we obtain

$$\left(n-1\right) T\left(r, f\left(z\right)\right) \leq T\left(r, F\left(z\right)\right)+S\left(r, f\right). \tag{25}$$

Using the similar method as above, we have

$$\left(n-1\right) T\left(r, g\left(z\right)\right) \leq T\left(r, G\left(z\right)\right)+S\left(r, g\right). \tag{26}$$

From Lemma 9, we have

$$T\left(r, F\right) \leq \left(n+1\right) T\left(r, f\right)+S\left(r, f\right), \tag{27}$$

$$T\left(r, G\right) \leq \left(n+1\right) T\left(r, g\right)+S\left(r, g\right). \tag{28}$$

By the second main theorem, Lemma 9, and (28), we obtain

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(\frac{1}{F-1}\right) + S(r,F) \\ &\le \overline{N}(r,f) + \overline{N}(r,f(qz+c)) + \overline{N}\left(r,\frac{1}{f}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{f(qz+c)}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) + S(r,f) \\ &\le 4T(r,f) + T(r,G) + S(r,f) \\ &\le 4T(r,f) + (n+1)T(r,g) + S(r,g) + S(r,f). \end{aligned} \tag{29}$$

Hence, (25) and (29) imply that

$$(n-5)T(r,f) \le (n+1)T(r,g) + S(r,f) + S(r,g). \tag{30}$$

Similarly, we have

$$(n-5)T(r,g) \le (n+1)T(r,f) + S(r,f) + S(r,g). \tag{31}$$

Equations (30) and (31) imply that $S(r,f) = S(r,g)$. Together the definition of F with Lemma 9, we have

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &\le 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f(qz+c)}\right) + S(r,f) \\ &\le 3T(r,f) + S(r,f). \end{aligned} \tag{32}$$

Similarly,

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) &\le 3T(r,g) + S(r,f), \\ N_2(r,F) &\le 3T(r,f) + S(r,f), \\ N_2(r,G) &\le 3T(r,g) + S(r,f). \end{aligned} \tag{33}$$

Thus, together (21) with (32)-(33), we obtain

$$\begin{aligned} T(r,F) + T(r,G) &\le 2N_2\left(r,\frac{1}{F}\right) + 2N_2(r,F) + 2N_2\left(r,\frac{1}{G}\right) \\ &\quad + 2N_2(r,G) + S(r,f) + S(r,g) \\ &\le 12(T(r,f) + T(r,g)) + S(r,f) \\ &\quad + S(r,g). \end{aligned} \tag{34}$$

Then, by (25), (26), and (34), we obtain

$$(n-13)[T(r,f) + T(r,g)] \le S(r,f) + S(r,g), \tag{35}$$

which is a contradiction since $n \ge 14$. By Lemma 11, we have $F \equiv G$ or $FG \equiv 1$. If $F \equiv G$, that is, $f^n(z)f(qz+c) = g^n(z)g(qz+c)$. Set $H(z) = f(z)/g(z)$. Suppose that $H(z)$ is not a constant. Then we obtain

$$H^n(z)H(qz+c) = 1. \tag{36}$$

Lemma 9 and (36) imply that

$$\begin{aligned} nT(r,H(z)) = T(r,H^n(z)) &= T\left(r,\frac{1}{H(qz+c)}\right) \\ &\le T(r,H(z)) + S(r,H). \end{aligned} \tag{37}$$

Hence, $H(z)$ must be a nonzero constant, since $n \ge 14$. Set $H(z) = t$. By (36), we know $t^{n+1} = 1$. Thus, $f(z) = tg(z)$, where $t^{n+1} = 1$.

If $FG = 1$, that is,

$$f^n(z)f(qz+c)g^n(z)g(qz+c) = 1. \tag{38}$$

Let $L(z) = f(z)g(z)$. Using the similar method as above, we also obtain that $L(z)$ must be a nonzero constant. Thus, we have $fg = t$, where $t^{n+1} = 1$.

4. Proof of Theorem 2

Let $F(z) = f^n(z)f(qz+c)$ and $G(z) = g^n(z)g(qz+c)$, and let H be defined in Lemma 12. Using the similar proof as the proof of Theorem 1, we prove that (25)–(33) hold. By Lemma 9, we obtain

$$\begin{aligned} \overline{N}(r,F(z)) &\le \overline{N}(r,f(z)) + \overline{N}(r,f(qz+c)) + S(r,f) \\ &\le 2T(r,f) + S(r,f). \end{aligned} \tag{39}$$

Similarly, we obtain

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F(z)}\right) &\le 2T(r,f) + S(r,f), \\ \overline{N}(r,G(z)) &\le 2T(r,g) + S(r,g), \\ \overline{N}\left(r,\frac{1}{G(z)}\right) &\le 2T(r,g) + S(r,g). \end{aligned} \tag{40}$$

Together Lemma 12 with (32), (33), (39), and (40), we have

$$\begin{aligned} T(r,F(z)) + T(r,G(z)) &\le 24(T(r,f) + T(r,g)) + S(r,f) \\ &\quad + S(r,g). \end{aligned} \tag{41}$$

By (25), (26) and (41) yield that

$$\begin{aligned} (n-1)(T(r,f(z)) + T(r,g(z))) &\le 24(T(r,f) + T(r,g)) \\ &\quad + S(r,f) + S(r,g), \end{aligned} \tag{42}$$

which is impossible, since $n \ge 26$. Hence, we have $H \equiv 0$.

By integrating (22) twice, we have

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \tag{43}$$

which yields that $T(r,F) = T(r,G) + O(1)$. From (25)–(28), we obtain

$$(n-1)\,T(r,f) \le (n+1)\,T(r,g) + S(r,f) + S(r,g), \tag{44}$$

$$(n-1)\,T(r,g) \le (n+1)\,T(r,f) + S(r,f) + S(r,g). \tag{45}$$

Next, we will prove that $F = G$ or $FG = 1$.

Case 1 ($b \ne 0, -1$). If $a - b - 1 \ne 0$, by (43), we obtain

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{G - (a-b-1)/(b+1)}\right). \tag{46}$$

Together the Nevanlinna second main theorem with Lemma 9, (28), and (44), we obtain

$$\begin{aligned}
&(n-1)\,T(r,g)\\
&\quad\le T(r,G) + S(r,g)\\
&\quad\le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G-(a-b-1)/(b+1)}\right)\\
&\qquad + S(r,G) + S(r,g)\\
&\quad\le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,g) + S(r,f)\\
&\quad\le \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g(qz+c)}\right) + \overline{N}(r,g)\\
&\qquad + \overline{N}(r,g(qz+c)) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f(qz+c)}\right)\\
&\qquad + S(r,g)\\
&\quad\le 4T(r,g) + 2T(r,f) + S(r,g)\\
&\quad\le \left(4 + 2\frac{n+1}{n-1}\right)T(r,g) + S(r,g),
\end{aligned} \tag{47}$$

which yields that $n^2 - 8n + 3 \le 0$, which is impossible, since $n \ge 26$. Hence, we obtain $a - b - 1 = 0$, so

$$F(z) = \frac{(b+1)\,G(z)}{bG(z)+1}. \tag{48}$$

Using the similar method as above, we obtain

$$\begin{aligned}
&(n-1)\,T(r,g)\\
&\quad\le T(r,G) + S(r,g)\\
&\quad\le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G+1/b}\right) + S(r,G)\\
&\quad\le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}(r,F) + S(r,G)\\
&\quad\le \left(4 + 2\frac{n+1}{n-1}\right)T(r,g) + S(r,g),
\end{aligned} \tag{49}$$

which is impossible.

Case 2. If $b = -1$ and $a = -1$, then $FG = 1$ follows trivially. Therefore, we may consider the case $b = -1$ and $a \ne -1$. By (43), we have

$$F = \frac{a}{a+1-G}. \tag{50}$$

Similarly, we get a contradiction.

Case 3. If $b = 0$, $a = 1$ and then $F = G$ follows trivially. Therefore, we may consider the case $b = 0$ and $a \ne 1$. By (43), we obtain

$$F = \frac{G+a-1}{a}. \tag{51}$$

Similarly, we get a contradiction.

5. Proof of Theorem 3

Since $P(f(z))f(qz+c)$ and $P(g(z))g(qz+c)$ share 1 CM, we obtain

$$\frac{P(f(z))\,f(qz+c) - 1}{P(g(z))\,g(qz+c) - 1} = e^{l(z)}, \tag{52}$$

where $l(z)$ is an entire function. by $\rho(f) = 0$ and $\rho(g) = 0$, we have $e^{l(z)} \equiv \eta$ as a constant. We can rewrite (52) as follows:

$$\eta P(g(z))\,g(qz+c) = P(f(z))\,f(qz+c) - 1 + \eta. \tag{53}$$

If $\eta \ne 1$, by the first main theory, the second main theory, and Lemma 9, we have

$$\begin{aligned}
&T(r, P(f(z)))\,f(qz+c)\\
&\quad\le \overline{N}\left(r,\frac{1}{P(f(z))\,f(qz+c)}\right)\\
&\qquad + \overline{N}\left(r,\frac{1}{P(f(z))\,f(qz+c)-1+\eta}\right) + S(r,f)\\
&\quad= \overline{N}\left(r,\frac{1}{P(f(z))\,f(qz+c)}\right)\\
&\qquad + \overline{N}\left(r,\frac{1}{P(g(z))\,g(qz+c)}\right) + S(r,f)\\
&\quad\le (k+1)\,T(r,f(z)) + (k+1)\,T(r,g(z)) + S(r,f)\\
&\qquad + S(r,g).
\end{aligned} \tag{54}$$

By Lemma 10 and (54), we have

$$\begin{aligned}
(n+1)\,T(r,f(z)) &= T(r,P(f(z))\,f(z))\\
&= T(r,P(f(z)))\,f(qz+c) + S(r,f)\\
&\le (k+1)\,T(r,f(z)) + (k+1)\,T(r,g(z))\\
&\quad + S(r,f) + S(r,g).
\end{aligned} \tag{55}$$

Hence, we have

$$(n-k)\,T(r,f(z)) \leq (k+1)\,T(r,g(z)) + S(r,f) + S(r,g). \tag{56}$$

Similarly, we have

$$(n-k)\,T(r,g(z)) \leq (k+1)\,T(r,f(z)) + S(r,f) + S(r,g). \tag{57}$$

Equations (56) and (57) imply that

$$(n-2k-1)\left[T(r,f(z)) + T(r,g(z))\right] \leq S(r,f) + S(r,g), \tag{58}$$

which is impossible, since $n > 2k+1$. Hence, we have $\eta = 1$. We can rewrite (52) as follows:

$$P(f(z))\,f(qz+c) = P(g(z))\,g(qz+c). \tag{59}$$

Set $h(z) = f(z)/g(z)$. We break the rest of the proof into two cases.

Case 1. Suppose that $h(z)$ is a constant. Then by substituting $f = gh$ into (59), we obtain

$$g(qz+c)\left[a_n g^n\left(h^{n+1}-1\right) + a_{n-1} g^{n-1}\left(h^n-1\right) + \cdots + a_0(h-1)\right] \equiv 0, \tag{60}$$

where $a_n(\neq 0), a_{n-1}, \ldots, a_0$ are complex constants. By the fact that g is transcendental entire function, we have $g(qz+c) \not\equiv 0$. Hence, we obtain

$$a_n g^n\left(h^{n+1}-1\right) + a_{n-1} g^{n-1}\left(h^n-1\right) + \cdots + a_0(h-1) \equiv 0. \tag{61}$$

Equation (61) implies that $h^{n+1} = 1$ and $h^{i+1} = 1$ when $a_i \neq 0$ for $i = 0, 1, \ldots, n-1$. Therefore, $h^d = 1$, where d is defined as the assumption of Theorem 3.

Case 2. Suppose that h is not a constant, then we know by (59) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1,w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$.

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