Bull. Korean Math. Soc. **48** (2011), No. 6, pp. 1235–1243
http://dx.doi.org/10.4134/BKMS.2011.48.6.1235

VALUE SHARING RESULTS OF A MEROMORPHIC FUNCTION $f(z)$ AND $f(qz)$

Xiaoguang Qi, Kai Liu, and Lianzhong Yang

Abstract. In this paper, we investigate sharing value problems related to a meromorphic function $f(z)$ and $f(qz)$, where q is a non-zero constant. It is shown, for instance, that if $f(z)$ is zero-order and shares two valves CM and one value IM with $f(qz)$, then $f(z) = f(qz)$.

1. Introduction

In what follows, a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a value $a \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). We assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [5, 10].

As usual, by $S(r, f)$ we denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set of finite linear measure. In addition, denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r, a) = o(T(r, f))$, for $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. In particular, we denote by $S_1(r, f)$ any quality satisfying $S_1(r, f) = o(T(r, f))$ for all r on a set of logarithmic density 1.

The classical results due to Nevanlinna [9] in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems:

Theorem A. *If two meromorphic functions f and g share five distinct values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \cup \{\infty\}$ IM, then $f \equiv g$.*

Theorem B. *If two meromorphic functions f and g share four distinct values $a_1, a_2, a_3, a_4 \in \mathbb{C} \cup \{\infty\}$ CM, then $f \equiv g$ or $f \equiv T \circ g$, where T is a Möbius transformation.*

Received July 13, 2010; Revised October 5, 2010.
2010 *Mathematics Subject Classification.* 30D35, 39A05.
Key words and phrases. meromorphic functions, uniqueness, q-difference, sharing value.
This work was supported by the NNSF of China (No. 10671109) and the NSF of Shandong Province, China (No.ZR2010AM030).

It is well-known that 4 CM can not be improved to 4 IM, see [3]. Further, Gundersen [4, Theorem 1] has improved the assumption 4 CM to 2 CM+2 IM, while 1 CM+3 IM is still an open problem.

In recent papers [6], Heittokangas et al. started to consider the uniqueness of a finite order meromorphic function sharing values with its shift. They concluded that:

Theorem C. *Let f be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \mathcal{S}(f) \cup \{\infty\}$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Closely related to difference expressions are q-difference expressions, where the usual shift $f(z+c)$ of a meromorphic function will be replaced by the q-shift $f(qz)$, $q \in \mathbb{C} \setminus \{0\}$. The Nevanlinna theory of q-difference expressions and its applications to q-difference equations have recently been considered, see [1, 7]. In addition, some results about solutions of zero-order for complex q-difference equations, can be found in the introduction in [1].

A natural question is: what is the uniqueness result in the case when $f(z)$ shares values with $f(qz)$ for a zero-order meromorphic function $f(z)$. Corresponding to this question, we get the following result:

Theorem 1.1. *Let f be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ be three distinct values. If $f(z)$ and $f(qz)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(qz)$.*

Remark 1. Indeed, from the proof of Theorem 1.1, we know the assumption that share a_3 IM can be replaced by one of the following assumptions:

(1) if there exists a point z_0 such that $f(z_0) = f(qz_0) = a_3$; or
(2) if a_3 is a Picard exceptional value of f.
However, we give Theorem 1.1 just as a q-difference analogue of Theorem C.

If f is an entire function in Theorem 1.1, then the conclusion will be improved.

Theorem 1.2. *Let f be a zero-order entire function, $q \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. If $f(z)$ and $f(qz)$ share a_1 and a_2 IM, then $f(z) = f(qz)$.*

Remark 2. As a corollary of Theorem 1.1, we just know that $f(z) = f(qz)$ provided that $f(z)$ and $f(qz)$ share values under the condition that "1 CM + 1 IM".

In the following, we consider the value sharing problems relative to $F(z) = f^n$ and $F(qz)$, and we obtain the following results:

Theorem 1.3. *Let f be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share $a \in \mathbb{C} \setminus \{0\}$ and ∞ CM, then $f(z) = tf(qz)$ for a constant t that satisfies $t^n = 1$.*

Remark 3. Theorem 1.3 is not true, if $a = 0$. This can be seen by considering $f(z) = z$ and $f(\frac{1}{2}z) = \frac{1}{2}z$. Then $f(z)^n$ and $f(\frac{1}{2}z)^n$ share 0 and ∞ CM, however, $f(z) = 2f(\frac{1}{2}z)$, $2^n \neq 1$, where n is a positive integer.

Corollary 1.4. *Let f be a zero-order entire function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 3$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share 1 CM, then $f(z) = tf(qz)$ for a constant t that satisfies $t^n = 1$.*

Corollary 1.5. *Let f be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $n \geq 4$ be an integer, and let $F = f^n$. If $F(z)$ and $F(qz)$ share 0 and 1 CM, then $f(z) = tf(qz)$ for a constant t that satisfies $t^n = 1$.*

Remark 4. By simply calculations, we get $|q| = 1$ in above results. And some ideas of this paper are from [8].

2. Some lemmas

Lemma 2.1 ([1, Theorem 1.1]). *Let f be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.2 ([1, Theorem 2.1]). *Let f be a zero-order meromorphic function, let $q \in \mathbb{C} \setminus \{0, 1\}$, and let $a_1, \ldots, a_p \in \mathbb{C}$, $p \geq 2$, be distinct points. Then*

$$m(r, f) + \sum_{k=1}^{p} m\left(r, \frac{1}{f - a_k}\right) \leq 2T(r, f) - N_{pair}(r, f) + S_1(r, f),$$

where

$$N_{pair}(r, f) = 2N(r, f) - N(r, \Delta_q f) + N\left(r, \frac{1}{\Delta_q f}\right)$$

and $\Delta_q f = f(qz) - f(z)$.

Lemma 2.3 ([11, Theorem 1.1 and Theorem 1.3]). *Let f be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z)) \tag{2.1}$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z)) \tag{2.2}$$

on a set of lower logarithmic density 1.

Remark. From Remark 1 after Theorem 1.1 in [11], we know that $f(z)$ and $f(qz)$ are simultaneously of order zero.

Lemma 2.4 ([10, Theorem 2.17])**.** *Let f and g be meromorphic functions, and the order of f and g is less than 1. If f and g share 0 and ∞ CM, then $f \equiv kg$, where k is a non-zero constant.*

3. Proof of Theorem 1.1

If $q = 1$, then the conclusion holds. Now we consider the case that $q \neq 1$. Suppose first that a_1, a_2, $a_3 \in \mathbb{C}$. Denote

$$g(z) = \frac{f(z) - a_1}{f(z) - a_2} \frac{a_3 - a_2}{a_3 - a_1},$$

then

$$g(qz) = \frac{f(qz) - a_1}{f(qz) - a_2} \frac{a_3 - a_2}{a_3 - a_1}.$$

From the assumption of Theorem 1.1, we know $g(z)$ and $g(qz)$ share 0, ∞ CM.

Suppose first that 1 is not a Picard exceptional value of $g(z)$ and $g(qz)$. Then by Lemma 2.4, we get that $g(z) = kg(qz)$ for some constant $k \neq 0$. Take now z_0 such that $g(z_0) = 1$. Since $a_1 \neq a_2$, we deduce that $f(z_0) = a_3$. Since $f(z)$ and $f(qz)$ share a_3 IM, we have $g(qz_0) = 1$. Therefore, $k = 1$ and so $g(z) = g(qz)$, hence $f(z) = f(qz)$ as well.

Suppose next that 1 is a Picard exceptional value of $g(z)$ and $g(qz)$. Assume that $g(z) \neq g(qz)$, and from Lemma 2.2, we obtain

$$\begin{aligned} &m(r, g) + m\left(r, \frac{1}{g}\right) + m\left(r, \frac{1}{g-1}\right) \\ &\leq 2T(r, g) - 2N(r, g) + N(r, \Delta_q g) - N\left(r, \frac{1}{\Delta_q g}\right) + S_1(r, g), \end{aligned}$$

and so

$$\begin{aligned} T(r, g) \leq{}& N(r, g) + N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) + N(r, g(qz)) \\ &+ N(r, g) - 2N(r, g) - N\left(r, \frac{1}{\Delta_q g}\right) + S_1(r, g). \end{aligned} \tag{3.1}$$

Since 1 is a Picard exceptional value of $g(z)$, by combining (2.2) and (3.1), it follows that

$$T(r, g) \leq N(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{\Delta_q g}\right) + S_1(r, g). \tag{3.2}$$

Since $g(z)$ and $g(qz)$ share 0, ∞ CM, we get

$$N(r, g) + N\left(r, \frac{1}{g}\right) \leq N\left(r, \frac{1}{\Delta_q g}\right). \tag{3.3}$$

From (3.2) and (3.3), we conclude that

$$T(r, g) = S_1(r, g),$$

which is impossible. Hence, we conclude that $f(z) = f(qz)$.

It remains to consider the case that one of $a_j (j = 1,\ 2,\ 3)$ is infinite. Without loss of generality, we suppose that $a_1 = \infty$, while $a_2, a_3 \in \mathbb{C}$. Take $d \in \mathbb{C} \setminus \{a_2, a_3\}$ and denote $h(z) = \frac{1}{f(z)-d}$, $b_2 = \frac{1}{a_2-d}$ and $b_3 = \frac{1}{a_3-d}$. Then $b_2, b_3 \in \mathbb{C} \setminus \{0\}$ are two distinct values. Hence $h(z)$ and $h(qz)$ share 0, b_2 CM and b_3 IM. By the above argument, we get $h(z) = h(qz)$, and therefore $f(z) = f(qz)$.

4. Proof of Theorem 1.2

From the fact that a non-constant meromorphic function of zero-order can have at most one Picard exceptional value (see, e.g., [2, p. 114]), we obtain that $N(r, \frac{1}{f-a_1}) \neq 0$ and $N(r, \frac{1}{f-a_2}) \neq 0$. Let

$$F(z) = \frac{f(z) - a_1}{a_2 - a_1} \quad \text{and} \quad F(qz) = \frac{f(qz) - a_1}{a_2 - a_1}. \tag{4.1}$$

Then $F(z)$ and $F(qz)$ share 0 and 1 IM. Clearly, neither 0 nor 1 is a Picard exceptional value of $F(z)$. From Lemma 2.3, we obtain that

$$T(r, F(qz)) = T(r, F(z)) + S_1(r, F). \tag{4.2}$$

Denote

$$V(z) = \frac{F'(z)(F(qz) - F(z))}{F(z)(F(z) - 1)}. \tag{4.3}$$

Lemma 2.1 and the lemma on logarithmic derivative yield that $m(r, V) = S_1(r, F)$. From (4.3), we know the poles of $V(z)$ are at the zeros and 1-points of $F(z)$. Since $F(z)$ and $F(z + c)$ share 0 and 1, we get $N(r, V) = S(r, F)$. Therefore, $T(r, V) = S_1(r, F)$.

Case 1. If $V \neq 0$, then $F(z) \neq F(qz)$. From (4.3) and Lemma 2.1, we have

$$\begin{aligned}
&\overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}\left(r, \frac{1}{F(z) - 1}\right) \\
&= N\left(r, \frac{F'(z)}{F(z)(F(z) - 1)}\right) + S(r, F) \\
&= N\left(r, \frac{V}{F(qz) - F(z)}\right) + S(r, F) \\
&\leq T(r, F(qz) - F(z)) + S_1(r, F) = m(r, F(qz) - F(z)) + S_1(r, F) \\
&\leq m\left(r, \frac{F(qz) - F(z)}{F(z)}\right) + m(r, F(z)) + S_1(r, F) \\
&\leq T(r, F) + S_1(r, F).
\end{aligned}$$

According to second main theorem and above inequality, we get

$$T(r, F) = \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - 1}\right) + S_1(r, F). \tag{4.4}$$

Now we define

$$U(z)=\frac{F'(qz)(F(qz)-F(z))}{F(qz)(F(qz)-1)}. \tag{4.5}$$

Using the same argument as above, we know that $T(r,U)=S_1(r,F(qz))=S_1(r,F(z))$.

In what follows, we denote $S_{f\sim g(m,n)}(a)$ for the set of those points $z\in\mathbb{C}$ such that z is an a-point of f with multiplicity m and an a-point of g with multiplicity n. Let $N_{(m,n)}(r,\frac{1}{f-a})$ and $\overline{N}_{(m,n)}(r,\frac{1}{f-a})$ denote the counting function and reduced counting function of f with respect to the set $S_{f\sim g(m,n)}(a)$, respectively.

For any point $z_0\in S_{F(z)\sim F(qz)(m,n)}(0)$, we have $mn\neq 0$, since 0 is not a Picard exceptional value of $F(z)$ as we discuss above. From (4.3), (4.5) and the Taylor expansion of $F(z)$ and $F(qz)$ at z_0, by calculating carefully, we get that

$$-V(z_0)=m\left(\frac{F'(qz_0)}{n}-\frac{F'(z_0)}{m}\right) \tag{4.6}$$

and

$$-U(z_0)=n\left(\frac{F'(qz_0)}{n}-\frac{F'(z_0)}{m}\right). \tag{4.7}$$

From (4.6) and (4.7), we know $nV(z_0)=mU(z_0)$.

If $nV=mU$, then we deduce that

$$n\left(\frac{F'(z)}{F(z)-1}-\frac{F'(z)}{F(z)}\right)=m\left(\frac{F'(qz)}{F(qz)-1}-\frac{F'(qz)}{F(qz)}\right),$$

which implies that

$$\left(\frac{F-1}{F}\right)^n=b\left(\frac{F(qz)-1}{F(qz)}\right)^m,$$

where b is a non-zero constant. If $m\neq n$, then we get from above equality and (4.2) that

$$nT(r,F(z))=mT(r,F(qz))+S_1(r,F)=mT(r,F(z))+S_1(r,F),$$

which is a contradiction. If $m=n$, then we get

$$\left(\frac{F'(z)}{F(z)-1}-\frac{F'(z)}{F(z)}\right)=\left(\frac{F'(qz)}{F(qz)-1}-\frac{F'(qz)}{F(qz)}\right).$$

Hence

$$\frac{F(z)-1}{F(z)}=d\frac{F(qz)-1}{F(qz)}, \tag{4.8}$$

where d is a non-zero constant. If $d=1$, then we obtain $F(z)=F(qz)$, which contradicts the assumption of Case 1. It remains to consider the case that

$d \neq 1$. It follows from (4.8) that

$$\frac{d-1}{d}\frac{F(z)+\frac{1}{d-1}}{F(z)} = \frac{1}{F(qz)}.$$

Since $N(r, F(z)) = N(r, F(qz)) = 0$, we get $N(r, \frac{1}{F(z)-\frac{1}{1-d}}) = 0$. Clearly, $\frac{1}{1-d} \neq 0$ and $\frac{1}{1-d} \neq 1$, then apply the second main theorem, resulting in

$$2T(r,F) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) + S(r,F),$$

which contradicts (4.4).

Hence $nV \neq mU$. By the above argument, we know any point $z_0 \in S_{F(z)\sim F(qz)(m,n)}(0)$ satisfies that $nV(z_0) = mU(z_0)$. Therefore,

$$\overline{N}_{(m,n)}\left(r, \frac{1}{F}\right) \leq N\left(r, \frac{1}{nU-mV}\right) = S_1(r,F).$$

Using the same reason, we get

$$\overline{N}_{(m,n)}\left(r, \frac{1}{F-1}\right) \leq N\left(r, \frac{1}{nU-mV}\right) = S_1(r,F).$$

It follows that

$$\overline{N}_{(m,n)}\left(r, \frac{1}{F}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{F-1}\right) = S_1(r,F). \tag{4.9}$$

From Lemma 2.3, (4.4) and (4.9), we obtain that

$$\begin{aligned} T(r,F) &= \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F-1}) + S_1(r,F) \\ &= \sum_{m,n}(\overline{N}_{(m,n)}(r, \frac{1}{F}) + \overline{N}_{(m,n)}(r, \frac{1}{F-1})) + S_1(r,F) \\ &= \sum_{m+n\geq 5}(\overline{N}_{(m,n)}(r, \frac{1}{F}) + \overline{N}_{(m,n)}(r, \frac{1}{F-1})) + S_1(r,F) \\ &\leq \frac{1}{5}\sum_{m+n\geq 5}(N_{(m,n)}(r, \frac{1}{F}) + N_{(m,n)}(r, \frac{1}{F-1}) \\ &\quad + N_{(m,n)}(r, \frac{1}{F(qz)}) + N_{(m,n)}(r, \frac{1}{F(qz)-1})) + S_1(r,F) \\ &\leq \frac{2}{5}T(r,F) + \frac{2}{5}T(r,F(qz)) + S_1(r,F) \\ &= \frac{4}{5}T(r,F) + S_1(r,F), \end{aligned}$$

which is a contradiction.

Case 2. If $V = 0$, then $F(z) = F(qz)$. Clearly, $f(z) = f(qz)$. This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Let $G(z) = \frac{F(z)}{a}$, then we know $G(z)$ and $G(qz)$ share 1 and ∞ CM, and since the order of f is zero, it follows that

$$\frac{G(qz)-1}{G(z)-1} = \tau,$$

where τ is a non-zero constant. Rewriting the above equation, gives

$$G(z) + \frac{1}{\tau} - 1 = \frac{G(qz)}{\tau}. \tag{5.1}$$

Assume that $\tau \neq 1$. Noting (2.2) and (5.1), the second main theorem yields

$$\begin{aligned} nT(r, f(z)) = T(r, G(z)) &\leq \overline{N}(r, G(z)) + \overline{N}\left(r, \frac{1}{G(z)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G(z)-1+\frac{1}{\tau}}\right) + S(r, f) \\ &\leq \overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + S(r, f) \\ &\leq N(r, f(z)) + 2N\left(r, \frac{1}{f(z)}\right) + S_1(r, f) \\ &\leq 3T(r, f(z)) + S_1(r, f), \end{aligned} \tag{5.2}$$

which contradicts the assumption that $n \geq 4$. Hence, we get $\tau = 1$, which implies that $G(z) = G(qz)$, that is, $f^n(z) = f^n(qz)$. So we have $f(z) = tf(qz)$ for a constant t with $t^n = 1$.

References

[1] D. C. Barnett, R. G. Halburd, R. J. Korhonen, and W. Morgan, *Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), no. 3, 457–474.

[2] A. A. Goldberg and I. V. Ostrovskii, *Value Distribution of Meromorphic Functions*, Transl. Math. Monogr., vol. **236**, American Mathematical Society, Providence, RI, 2008, translated from the 1970 Russian original by Mikhail Ostrovskii, with an appendix by Alexandre Eremenko and James K. Langley.

[3] G. G. Gundersen, *Meromorphic functions that share three or four values*, J. London Math. Soc. (2) **20** (1979), no. 3, 457–466.

[4] ______, *Meromorphic functions that share four values*, Trans. Amer. Math. Soc. **277** (1983), no. 2, 545–567.

[5] W. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[6] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and J. L. Zhang, *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*, J. Math. Anal. Appl. **355** (2009), no. 1, 352–363.

[7] I. Laine and C. C. Yang, *Clunie theorem for difference and q-difference polynomials*, J. London. Math. Soc. (2) **76** (2007), no. 3, 556–566.

[8] P. Li and C. C. Yang, *When an entire function and its linear differential polynomial share two values*, Illinois J. Math. **44** (2000), no. 2, 349–362.

[9] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthiers-Villars, Paris, 1929.

[10] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, 2003.

[11] J. L. Zhang and R. Korhonen, *On the Nevanlinna characteristic of $f(qz)$ and its applications*, J. Math. Anal. Appl. **369** (2010), no. 2, 537–544.

Xiaoguang Qi
School of Mathematics
University of Jinan
Jinan, Shandong, 250022, P. R. China
E-mail address: xiaogqi@gmail.com, xiaogqi@mail.sdu.edu.cn

Kai Liu
Department of Mathematics
Nanchang University
Nanchang, Jiangxi, 330031, P. R. China
E-mail address: liukai418@126.com

Lianzhong Yang
School of Mathematics
Shandong University
Jinan, Shandong, 250100, P. R. China
E-mail address: lzyang@sdu.edu.cn