MEMORANDUM
RM-5842-PR
JULY 1969

VALUES OF NON-ATOMIC GAMES, PART II: THE RANDOM ORDER APPROACH

Robert Aumann and Lloyd Shapley

PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is supported by the United States Air Force under Project RAND—Contract No. F44620-67-C-0045—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this study should not be interpreted as representing the official opinion or policy of the United States Air Force.

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

Published by The RAND Corporation

PREFACE

This is a continuation of RM—5468—PR: Values of non—atomic games, Part I: The axiomatic approach. Non—atomic games are models for competitive situations in which there are many participants, none of whom has any appreciable influence as an individual.

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SUMMARY

The value of an n-person game is a function that associates to each player a number that, intuitively speaking, represents an a priori opinion of what it is worth to him to play in the game. A non-atomic game is a special kind of infinite-person game in which no individual player has significance; such games have recently attracted considerable attention as models for mass phenomena in economics and game theory. This is the second in a series of papers in which the value concept is extended to certain classes of non-atomic games.

In this Memorandum the value concept for non-atomic games is developed not from axioms, as in the first part, but by measuring the expected marginal worths of the players after a sufficient amount of "shuffling" has randomized the order in which they enter into coalition. Some consideration is also given to a third, "asymptotic" approach, in which the game with a continuous infinity of players is treated as a limit of finite games.

CONTENTS

VALUES OF NON-ATOMIC GAMES
PART II: THE RANDOM ORDER APPROACH

11. INTRODUCTION TO PART II*

Shapley's original paper [S_1] contained two equivalent approaches to the value of games with finitely many players: one based on axioms, and one based on the idea of the expected marginal payoff to a player when the players are ordered at random. Part I was devoted to a generalization of the first of these—the axiomatic one—to games with a continuum of players. Most of Part II will be devoted to investigating analogues of the second of Shapley's approaches.

It turns out that a direct generalization of the random order approach is impossible; this will be shown in Sec. 12. In Secs. 13 through 17 we present an approach based on mixing transformations. In Sec. 13 we define the concept of "mixing value" and state the basic theorems; this section also contains a guide to Secs. 14 through 17.

In Sec. 18 we briefly discuss an asymptotic approach due to Kannai [K_2], in which a game with a continuum of players is treated as a limit of games with finitely many players. In Sec. 19 we discuss a remarkable common property of the axiomatic, mixing, and asymptotic approaches, namely the "diagonal property".

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*To enable easy reference to Part I, the sections of this part will be numbered 11, 12, and so on. Conventions introduced in Part I will be continued here, and familiarity with the definitions and results of Part I will be assumed throughout.

Both the mixing and the asymptotic approaches have in common that they give an expression for the value directly in terms of the set function v. Although the axiomatic approach also yields expressions for the value, these are not directly in terms of the set function v, but in terms of some representation of v—for example, as a sum of scalar measure functions or as a C^1 function of a vector measure.

Another similarity between the mixing and the asymptotic approaches, which is not shared by the axiomatic approach, is that both of the former work on individual games, whereas the latter works only on spaces of games.

12. ORDER AND RANDOM ORDER

The generalization to games with a continuum of players of Shapley's random order approach to value would, if it were possible, proceed as follows: first one defines (and possibly restricts) in an appropriate manner the notion of an order $\mathcal{R}$ on the player space. Then with each characteristic function v and each $\mathcal{R}$, one associates a measure $\varphi^{\mathcal{R}} v$ on the player space; intuitively $(\varphi^{\mathcal{R}} v)(S)$ is the marginal contribution of the coalition S to the payoff if the players "enter the scene" according to the order $\mathcal{R}$. Next, one defines the notion of measurability, and a probability measure ω, on the space Ω of orders; and finally, one defines the value φv to be the expectation of $\varphi^{\mathcal{R}} v$, i. e.,

$$(12.1) \qquad (\varphi v)(S) = \int_{\Omega} (\varphi^{\mathcal{R}} v)(S)\, d\,\omega(\mathcal{R}).$$

Let us see how far we can get in carrying out this program.

An <u>order</u> on the player space I is a relation $\mathcal{R}$ on I that is transitive, irreflexive, and complete.* For each order $\mathcal{R}$ and each $s \in I$, define the <u>initial segment</u> $I(s; \mathcal{R})$ by

$$I(s; \mathcal{R}) = \{t:\ s \mathcal{R} t\}.$$

The entire space I and the empty set $\emptyset$ will also be considered initial segments, and as such will be denoted $I(\infty; \mathcal{R})$ and $I(-\infty; \mathcal{R})$ respectively;

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*I. e., for all $s, t \in I$, one and only one of the three statements $s \mathcal{R} t$, $t \mathcal{R} s$, $s = t$ holds.

it will be understood that $\infty \mathcal{R}\, s\, \mathcal{R}(-\infty)$ for all $s \in I$, and we will denote* $\{-\infty\} \cup I \cup \{\infty\}$ by $\bar{I}$. The intuitive requirement on the measure $\varphi^{\mathcal{R}}v$ that was given in the previous paragraph is then mathematically expressed as follows:

(12.2) $$(\varphi^{\mathcal{R}}v)\,(I(s;\mathcal{R})) = v(I(s;\mathcal{R}))$$

for all $s \in \bar{I}$. For this equation to be meaningful, $I(s;\mathcal{R})$ must be measurable for all s; and for it uniquely to define the measure $\varphi^{\mathcal{R}}v$ on the player space $(I, \mathcal{C})$ it is necessary that all coalitions (members of $\mathcal{C}$) be in the σ–field $F(\mathcal{R})$ generated by all the initial segments $I(s, \mathcal{R})$. These two requirements may be summed up by

(12.3) $$\mathcal{C} = F(\mathcal{R});$$

orders $\mathcal{R}$ obeying (12.3) will be called measurable, and we will henceforth restrict the discussion to measurable orders.

We now digress to prove a number of lemmas that will be useful in the sequel. Given an order $\mathcal{R}$ on I, we will let "$s \overset{\mathcal{R}}{=} t$" denote "$s \mathcal{R} t$ or $s = t$." Then a subset R of I will be called $\mathcal{R}$–dense if for all s and t in I such that $s \mathcal{R} t$, there is a member r of R such that $s \overset{\mathcal{R}}{=} r \overset{\mathcal{R}}{=} t$.

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*Formally, we extend $\mathcal{R}$ to $\bar{I}$ by the condition $\infty \mathcal{R}\, s\, \mathcal{R}(-\infty)$ for all $s \in I$. This however is a notational device only; we are not adding anything to the underlying space, and all set functions, measures, etc. continue to be defined on subsets of I only. Note that if I has an "$\mathcal{R}$–first" element t, then $I(t;\mathcal{R}) = \emptyset = I(-\infty;\mathcal{R})$.

LEMMA 12.4. If $\mathcal{R}$ is a measurable order on I, then I has a denumerable $\mathcal{R}$–dense subset.

Proof. For each family $\mathcal{S}$ of subsets of I, let $F(\mathcal{S})$ denote the σ–field generated by $\mathcal{S}$, i.e., the smallest σ–field that includes $\mathcal{S}$. From (12.3) we know that every measurable subset S of I is in $F(\mathcal{R})$, which by definition is the σ–field generated by all the initial segments. Our first claim is that to generate a given measurable S, only a denumerable number of initial segments are needed; more precisely, for each $S \in \mathcal{C}$ there are initial segments $I_1, I_2, \ldots$ such that $S \in F(\{I_1, I_2, \ldots\})$. To show this, note first that it is trivially true when S is itself an initial segment. Next, if it is true for a given S then it is also true for its complement, and if it is true for a denumerable sequence of S_i then it is also true for their union. Hence it is true for every S in the smallest σ–field containing all the initial segments, and as we noted above, this is precisely $\mathcal{C}$. Since the player space is isomorphic to $([0, 1], \mathcal{B})$, it follows that $\mathcal{C}$ is countably generated; that is, there is a countable family $S^1, S^2, \ldots$ of measurable sets such that $\mathcal{C} = F(\{S^1, S^2, \ldots\})$. For each S^i there are initial segments $I^i_1, I^i_2, \ldots$ such that $S^i \in F(\{I^i_1, I^i_2, \ldots\})$. Then if we renumber the I^i_j so that they form a sequence $I_1, I_2, \ldots$, we obtain

$$\mathcal{C} = F(\{I_1, I_2, \ldots\}) . \tag{12.5}$$

Define $r_1, r_2, \ldots$ by $I_i = I(r_i; \mathcal{R})$ for all i, and let $R = \{r_1, r_2, \ldots\}$.

Suppose $s \mathcal{R} t$; we claim that there is an r_i such that $s \overset{\mathcal{R}}{=} r_i \mathcal{R} t$. Indeed, if not, then each of the initial segments I_i either contains both s and t or neither one. Then from (12.5) it follows that every measurable S either contains both s and t or neither one; this contradicts the fact that $(I, \mathcal{C})$ is isomorphic to $([0, 1], \mathcal{B})$.

Thus $\{r_1, r_2, \ldots\}$ is a denumerable $\mathcal{R}$–dense subset of I, and the proof of the lemma is complete.

The following corollary is of some interest, though it will not be used in the sequel.

<u>COROLLARY 12.6.</u> <u>If</u> $\mathcal{R}$ <u>is a measurable order on</u> I, <u>then</u> $\{(s, t)\colon\ s \mathcal{R} t\}$ <u>is a measurable subset of</u> $I \times I$.

<u>Proof</u>. Sets containing a single point are measurable; hence each set of the form $\{s\colon\ s \mathcal{R} t\}$ is also measurable, since $\{s\colon\ s \mathcal{R} t\} = I \setminus (I(t; \mathcal{R}) \cup \{t\})$. Now if $\{r_1, r_2, \ldots\}$ is an $\mathcal{R}$–dense subset, then

$$\begin{aligned}\{(s, t)\colon\ s \mathcal{R} t\} = {} & \cup_{i=1}^{\infty} \{s\colon\ s \mathcal{R} r_i\} \times \{t\colon\ r_i \mathcal{R} t\} \\ & \cup \cup_{i=1}^{\infty} \{r_i\} \times \{t\colon\ r_i \mathcal{R} t\} \\ & \cup \cup_{i=1}^{\infty} \{s\colon\ s \mathcal{R} r_i\} \times \{r_i\}\,.\end{aligned}$$

This completes the proof.

The converse of Corollary 12.6 is false. Take the unit square with its Borel subsets as the underlying space. If $\mathcal{R}$ is the lexicographic order, then $\mathcal{R}$ is not measurable, but $\{(s, t): s\mathcal{R}t\}$ is a measurable subset of $I \times I$.

Returning to the matter at hand, we remark that although the measurability condition (12.3) is a necessary condition for Eq. (12.2) to define a unique measure $\varphi^{\mathcal{R}}v$, it is not sufficient. For a simple example, let the player space be $([0, 1], \mathcal{B})$, and let the order $\mathcal{R}$ be the usual order $>$ (which is clearly measurable); in this case $I(s; \mathcal{R}) = [0, s)$. Let $v = f \circ \lambda$, where

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \geqq \frac{1}{2} \end{cases} .$$

Then there is no measure $\varphi^{\mathcal{R}}v$ satisfying (12.2).

It might be thought that what makes this example possible is the discontinuity in the function f; but this is not the case. Indeed, suppose again that the player space is $([0, 1], \mathcal{B})$, and let the order be defined as follows: for two points s and t that are either both in the Cantor set, or both outside of the Cantor set, $s\mathcal{R}t$ iff $s > t$; and if t is in the Cantor set whereas s is not, then $s\mathcal{R}t$ (i.e., the entire Cantor set comes "before" its complement). Let $v = f \circ \lambda$, where f is the Cantor function. Then there is no measure $\varphi^{\mathcal{R}}v$ satisfying (12.2).

If, however, f is <u>absolutely</u> continuous, then a counterexample of this kind is impossible. Indeed, we have

PROPOSITION 12.7. *If* $v \in AC$, *then for each measurable order* $\mathcal{R}$, *there is a unique measure* $\varphi^{\mathcal{R}} v$ *satisfying* (12.2). *For this measure, we have*

$$\|\varphi^{\mathcal{R}} v\| \leqq \|v\|.$$

Finally, if μ *is a nonatomic probability measure such that* $v \ll \mu$, *then* $\varphi^{\mathcal{R}} v \ll \mu$, *uniformly** *for all measurable* $\mathcal{R}$; *in particular,* $\varphi^{\mathcal{R}} v$ *is nonatomic.*

Remark. $\varphi^{\mathcal{R}} v \ll \mu$ uniformly for all $\mathcal{R}$ means that the δ appearing in the definition of absolute continuity (see Sec. 5) depends on ϵ only and not on $\mathcal{R}$. If $|\varphi^{\mathcal{R}} v|$ is the "total variation" of $\varphi^{\mathcal{R}} v$ [H_1, p. 122], this may be restated as follows: For each $\epsilon > 0$ there is a $\delta > 0$ —depending on ϵ and v only, and not on $\mathcal{R}$—such that $\mu(S) \leqq \delta$ implies $|\varphi^{\mathcal{R}} v|(S) \leqq \epsilon$. Of course, this condition of uniform absolute continuity implies the ordinary absolute continuity, i.e., it implies that $\varphi^{\mathcal{R}} v \ll \mu$ in the ordinary sense, for each $\mathcal{R}$ separately.

Proof. Let $H(\mathcal{R})$ be the field (not σ-field) generated by the initial segments $I(s; \mathcal{R})$; if we define a *half-open* $\mathcal{R}$*-interval* (or simply an $\mathcal{R}$-interval for short) to be the set difference between two initial segments, then the members of $H(\mathcal{R})$ are precisely the finite unions of $\mathcal{R}$-intervals. Clearly there is a unique finitely additive measure ν on $H(\mathcal{R})$ such that

*See the remark below.

$$\nu(I(s; \mathcal{R})) = v(I(s; \mathcal{R}))$$

for all s.

Let $U \in H(\mathcal{R})$. Then we may write

$$U = \cup_{i=1}^{k} [s_i, t_i)_{\mathcal{R}},$$

where the right side contains a self–explanatory notation for $\mathcal{R}$–intervals, and

$$t_k \mathcal{R} s_k \mathcal{R} \ldots \mathcal{R} t_1 \mathcal{R} s_1.$$

Let

$$S_i = I(s_i; \mathcal{R}),$$

$$T_i = I(t_i; \mathcal{R})),$$

let Ω be the chain

$$\emptyset \subset T_1 \subset S_1 \subset \ldots \subset T_k \subset S_k \subset I,$$

and let Λ be the subchain whose links are the $\{T_i, S_i\}$; we will say that Ω and Λ are <u>associated</u> with U. Then

(12.8)
$$\begin{aligned} \|v\|_{\Lambda} &= \sum_{i=0}^{k} |v(S_i) - v(T_i)| \\ &= \sum_{i=0}^{k} |\nu(S_i \setminus T_i)| \\ &\geqq \left|\sum_{i=0}^{k} \nu(S_i \setminus T_i)\right| \\ &= |\nu(U)|. \end{aligned}$$

Furthermore, for any nonnegative measure μ we have

$$\|\mu\|_{\Lambda} = \sum_{i=0}^{k} (\mu(S_i) - \mu(T_i)) = \mu(U).$$

Now by a well—known theorem,[*] in order to prove that ν can be extended to a completely additive measure $\varphi^{\mathcal{R}}v$ on $F(\mathcal{R}) = \mathcal{C}$, it is sufficient to prove that it is bounded, and that it is completely additive on $H(\mathcal{R})$; that is, if $U_1, U_2, \ldots$ is an infinite sequence of pairwise disjoint sets in $H(\mathcal{R})$ whose union U is also in $H(\mathcal{R})$, we must show that

$$\nu(U) = \sum_{i=1}^{\infty} \nu(U_i). \tag{12.9}$$

To prove that ν is bounded, let $U \in H(\mathcal{R})$, and let Λ be the subchain associated with U; then

$$|\nu(U)| \leqq \|v\|_{\Lambda} \leqq \|v\|.$$

To prove (12.9), let us for the moment fix k, and let $W_k = U \setminus \cup_{i=1}^{k} U_i$. Then $W_k \in H(\mathcal{R})$, so there is a subchain Λ associated with W_k; for this Λ we have

$$\|v\|_{\Lambda} \geqq |\nu(W_k)| .$$

Now let μ be a probability measure such that $v \ll \mu$. Let $\epsilon > 0$ be given, and choose $\delta > 0$ to correspond to ϵ in accordance with the definition of $v \ll \mu$ (Sec. 5). Let k be such that $\|\mu\|_{\Lambda} = \mu(W_k)$ $= \mu(U \setminus \cup_{i=1}^{k} U_i) < \delta$; the existence of such a k follows from the complete additivity of μ and $U = \cup_{i=1}^{\infty} U_i$. Then by (12.8),

$$|\nu(U) - \sum_{i=1}^{k} \nu(U_i)| = |\nu(W_k)| \leqq \|v\|_{\Lambda} < \epsilon ;$$

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[*] See for example [Dun—S, III. 5. 9, p. 136].

or in other words,

$$\nu(U) - \epsilon \leqq \sum_{i=1}^{k} \nu(U_i) \leqq \nu(U) + \epsilon \ .$$

Letting $k \to \infty$, we obtain

$$\nu(U) - \epsilon \leqq \liminf_{k\to\infty} \sum_{i=1}^{k} \nu(U_i) \leqq \limsup_{k\to\infty} \sum_{i=1}^{k} \nu(U_i) \leqq \nu(U) + \epsilon \ .$$

Here the middle two terms are independent of ϵ; so if we let $\epsilon \to 0$ we find that the limit exists and equals $\nu(U)$. But since by definition the limit is $\Sigma_{i=1}^{\infty} \nu(U_i)$, it follows that

$$\nu(U) = \sum_{i=1}^{\infty} \nu(U_i) \ .$$

This completes the proof of the first part of the proposition. Note that we may now rewrite (12.8) as

$$\| v \|_{\Lambda} \geqq |(\varphi^{\mathcal{R}} v)(U)| ,$$

where Λ is the subchain associated with a coalition U in $H(\mathcal{R})$.

To prove $\| \varphi^{\mathcal{R}} v \| \leqq \| v \|$, let $I = S^+ \cup S^-$ be a Hahn decomposition of I w.r.t. $\varphi^{\mathcal{R}} v$ [H_1, p. 121]. By a standard approximation theorem,* every measurable S can be approximated by members of $H(\mathcal{R})$ w.r.t. $\varphi^{\mathcal{R}} v$; that is, there is a $U^+ \in H(\mathcal{R})$ such that

$$|(\varphi^{\mathcal{R}} v)(S^+) - (\varphi^{\mathcal{R}} v)(U^+)| < \epsilon \ ;$$

setting $U^- = I \backslash U^+$, we deduce easily that

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*One uses [H_1, p. 56, Theorem D] on the "total variation" $|\varphi^{\mathcal{R}} v|$.

$$|(\varphi^{\mathcal{R}}v)(S^-) - (\varphi^{\mathcal{R}}v)(U^-)| < \epsilon \ .$$

Now the chain Ω associated with U^+ is also associated with U^-; denote by Λ_+ and Λ_- the subchains associated with U^+ and U^- respectively. Then

$$\|v\| \geqq \|v\|_\Omega = \|v\|_{\Lambda_+} + \|v\|_{\Lambda_-} \geqq |(\varphi^{\mathcal{R}}v)(U^+)| + |(\varphi^{\mathcal{R}}v)(U^-)|$$

$$\geqq |(\varphi^{\mathcal{R}}v)(S^+)| - \epsilon + |(\varphi^{\mathcal{R}}v)(S^-)| - \epsilon = \|\varphi^{\mathcal{R}}v\| - 2\epsilon \ .$$

Letting $\epsilon \to 0$, we obtain $\|v\| \geqq \|\varphi^{\mathcal{R}}v\|$.

Finally, to prove the uniform absolute continuity, let $\epsilon > 0$ be given, and let $\delta > 0$ correspond to $\epsilon/2$ in accordance with the definition of $v \ll \mu$. Let $\mathcal{R}$ be a measurable order, let $U \in H(\mathcal{R})$, and let the subchain Λ be associated with U; then

$$\mu(U) = \|\mu\|_\Lambda \leqq \delta \Rightarrow |(\varphi^{\mathcal{R}}v)(U)| = \|v\|_\Lambda \leqq \epsilon/2 \ .$$

Now by a standard approximation theorem,* every measurable S can be approximated by members of $H(\mathcal{R})$, simultaneously w.r.t. μ and w.r.t. $\varphi^{\mathcal{R}}v$; hence

$$\mu(S) < \delta \Rightarrow |(\varphi^{\mathcal{R}}v)(S)| \leqq \epsilon/2,$$

for every measurable S. If, as before, $I = S^+ \cup S^-$ is a Hahn decomposition, then it follows that

$$\mu(S) < \delta \Rightarrow \mu(S \cap S^+) < \delta \Rightarrow |(\varphi^{\mathcal{R}}v)(S \cap S^+)| \leqq \frac{\epsilon}{2} ,$$

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*One uses [H_1, p. 56, Theorem D] on the measure $\mu + |\varphi^{\mathcal{R}}v|$.

and similarly

$$\mu(S) < \delta \Rightarrow |(\varphi^{\mathcal{R}} v)(S \cap S^-)| \leqq \frac{\epsilon}{2} .$$

Hence

$$\mu(S) \leqq \frac{\delta}{2} < \delta \Rightarrow |\varphi^{\mathcal{R}} v|(S) = |(\varphi^{\mathcal{R}} v)(S \cap S^+)| + |(\varphi^{\mathcal{R}} v)(S \cap S^-)|$$

$$\leqq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon ,$$

and Proposition 12. 7 is proved.

The program outlined at the beginning of this section has now been carried out up to the point at which it is necessary to impose a probability measure on the space Ω, which we now take to be the space of all measurable orders. It is true that in order to define the measures $\varphi^{\mathcal{R}} v$, we had to limit the domain of set functions rather drastically; nevertheless, at this point there might still be hope that the program would work out for $v \in AC$ (and in particular for $v \in pNA$).

Obviously, there are many ways to impose a probability measure on Ω; for a trivial example, one can concentrate all the probability on one order $\mathcal{R}$. This is unsatisfactory because of its arbitrariness; why one order rather than another order? The situation as given consists of the underlying space $(I, \mathcal{C})$, and a symmetric subspace Q of AC on which we wish to define a value. Singling out a specific order $\mathcal{R}$ on which to concentrate all of the probability adds a new, nonintrinsic element to the situation, which cannot be derived from what is originally given.

Let I = (I, $\mathcal{C}$) and I' = (I', $\mathcal{C}$') be two underlying spaces, Ω and Ω' the spaces of all measurable orders on I and I' respectively, Θ an isomorphism from I onto I'. Then to each order $\mathcal{R}$ on I, there corresponds an order $\Theta\mathcal{R}$ on I', defined by

$$\Theta s \;\Theta\mathcal{R}\; \Theta t \iff s\,\mathcal{R}\,t.$$

Note that $\Theta\mathcal{R}$ is measurable iff $\mathcal{R}$ is; thus Θ may be considered a one-one mapping from Ω onto Ω', in addition to being an isomorphism from I onto I'. Furthermore, if on the space Ω there is defined a σ-field of measurable subsets and a probability measure ω, then we may define a corresponding σ-field and a corresponding probability measure ω' on Ω', as follows: the measurable subsets of Ω' are precisely those of the form $\Theta\Gamma$, where Γ is measurable in Ω; and their probability $\omega'(\Theta\Gamma)$ is given by

$$\omega'(\Theta\Gamma) = \omega(\Gamma) .$$

Now set I' = I, whence $\Omega' = \Omega$. If Q is the symmetric space on which we wish to define the value, then because of its symmetry we get $Q' = \Theta Q = Q$ as well. Thus Θ transforms the given situation back into itself; it does not change anything, it is a symmetry of the situation. Therefore it should not change ω either, i.e., we should have $\omega' = \omega$. This means that for each automorphism Θ of the underlying space and each $\Gamma \subset \Omega$, we should have

(12.10) $\Theta\Gamma$ is measurable in Ω iff Γ is measurable, and

(12.11) $\omega\Theta = \omega$.

In the preceding discussion we have assumed that the probability distribution on Ω should depend on Q, I, and $\mathcal{C}$ only, but not on any specific set function v. It would, indeed, be curious if the definition of "random order" changed from game to game. This, however, is not a crucial point, our conclusions remain unchanged if we fix v and, in (12.10) and (12.11), restrict Θ to be a symmetry of v (i.e., demand $\Theta_* v = v$).

One more requirement must be made on the probability space Ω in order for it to enable us to carry out our program, namely

(12.12) $(\omega^{\mathcal{R}} v)(S)$ is measurable as a function of $\mathcal{R}$, for each fixed measurable $S \subset I$.

This condition follows at once from (12.1).

We are now ready to state the impossibility theorem that is the object of this section.

PROPOSITION 12.13. *There is a v in pNA such that it is impossible to define measurability and probability on the space Ω of all measurable orders on the underlying space I, in such a way that conditions (12.10), (12.11), and (12.12) are satisfied.* *

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*In either interpretation on the range of Θ.

It is convenient to procede the proof of this proposition by a number of lemmas. A subset J of I will be called an <u>initial set</u> (to be distinguished from an initial segment!) if

$$s \in J,\ s \mathcal{R} s' \Rightarrow s' \in J.$$

<u>LEMMA 12. 14.</u> <u>Let</u> $v \in AC$ <u>be monotonic</u>, $\mathcal{R}$ <u>a measurable order</u>, J <u>an initial set of</u> I. <u>Then</u> J <u>is measurable, and, setting</u> $\bar{J} = J \cup \{-\infty\}$, <u>we have</u>

$$(\varphi^{\mathcal{R}} v)(J) = v(J) = \sup_{s \in \bar{J}} v(I(s; \mathcal{R})) = \inf_{t \in \bar{I}/\bar{J}} v(I(t; \mathcal{R})).$$

<u>Proof.</u> Let R be a denumerable $\mathcal{R}$–dense subset of I (Lemma 12.4), let $\bar{R} = \{-\infty\} \cup R \cup \{\infty\}$, and let

$$J_1 = \cap_{q \in \bar{R} \setminus \bar{J}} I(q; \mathcal{R}),$$

$$J_0 = \cup_{q \in \bar{R} \cap \bar{J}} I(q; \mathcal{R}).$$

Then $J_1 \supset J \supset J_0$, and because R is $\mathcal{R}$–dense and each $q \in \bar{R}$ must be either in $\bar{R} \cap \bar{J}$ or in $\bar{R} \setminus \bar{J}$, it follows that $J_1 \setminus J_0$ can contain at most two points. Since the intersection and union defining J_1 and J_0 respectively are denumerable, it follows that J_1 and J_0, and therefore also J, are measurable. Furthermore, since the $I(q; \mathcal{R})$ are linearly ordered under inclusion, each finite intersection of the $I(q; \mathcal{R})$ is equal to one of them; hence

$$J_1 = \cap_i I(q_1^i; \mathcal{R}) ,$$

where $\{q_1^i\}$ is a sequence of points in $\bar{R} \setminus \bar{J}$ such that

$$q_1^1 \underset{=}{\mathcal{R}} q_1^2 \underset{=}{\mathcal{R}} \ldots,$$

i. e.,

$$I(q_1^1; \mathcal{R}) \supset I(q_1^2; \mathcal{R}) \supset \ldots .$$

Similarly

$$J_0 = \cup_i I(q_0^i; \mathcal{R}),$$

where $\{q_0^i\}$ is a sequence of points in $\bar{R} \cap \bar{J}$ such that

$$\ldots \underset{=}{\mathcal{R}} q_0^2 \underset{=}{\mathcal{R}} q_0^1 ,$$

i. e.,

$$I(q_0^1; \mathcal{R}) \subset I(q_0^2; \mathcal{R}) \subset \ldots .$$

Now since $\varphi^{\mathcal{R}} v$ is totally finite, we have

$$(\varphi^{\mathcal{R}} v)(J_1) = \lim_{i\to\infty} (\varphi^{\mathcal{R}} v)(I(q_1^i; \mathcal{R}))$$

$$(\varphi^{\mathcal{R}} v)(J_0) = \lim_{i\to\infty} (\varphi^{\mathcal{R}} v)(I(q_0^i; \mathcal{R})) .$$

But since $J_1 \setminus J_0$ consists of at most two points, and $\varphi^{\mathcal{R}} v$ is non–atomic (Proposition 12.7), it follows that $(\varphi^{\mathcal{R}} v)(J_1) = (\varphi^{\mathcal{R}} v)(J_0)$. Hence

$$v(I(q_1^i; \mathcal{R})) - v(I(q_0^i; \mathcal{R})) = (\varphi^{\mathcal{R}} v)(I(q_1^i; \mathcal{R})) - (\varphi^{\mathcal{R}} v)(I(q_0^i; \mathcal{R})) \to 0$$

as $i \to \infty$. But clearly, for each i we have

$$v(I(q_1^i; \mathcal{R})) \geqq \inf_{t \in \bar{I} \setminus \bar{J}} v(I(t; \mathcal{R})) \geqq \sup_{s \in \bar{J}} v(I(s; \mathcal{R})) \geqq v(I(q_0^i; \mathcal{R}));$$

hence if we let $i \to \infty$ we deduce that

$$\inf_{t \in \bar{I} \setminus \bar{J}} v(I(t; \mathcal{R})) = \sup_{s \in \bar{J}} v(I(s; \mathcal{R})) .$$

Now note that because v is monotonic, $(\varphi^{\mathcal{R}} v)(S)$ is nonnegative whenever S is a half–open $\mathcal{R}$–interval, and so by (12. 2) for all S. Since for $t \in \bar{I} \setminus \bar{J}$ and $s \in \bar{J}$ we have

$$I(t; \mathcal{R}) \supset J \supset I(s; \mathcal{R}),$$

it follows that

$$v(I(t; \mathcal{R})) = (\varphi^{\mathcal{R}} v)(I(t; \mathcal{R})) \geqq (\varphi^{\mathcal{R}} v)(J) \geqq (\varphi^{\mathcal{R}} v)(I(s; \mathcal{R})) = v(I(s; \mathcal{R}));$$

since v is monotonic, it also follows that

$$v(I(t; \mathcal{R})) \geqq v(J) \geqq v(I(s; \mathcal{R})).$$

Taking the inf w. r. t. t and the sup w. r. t. s and using the fact that they are equal, we deduce the conclusion of the lemma.

Now if $\mathcal{R}$ is a measurable order and μ a non–atomic probability measure on the underlying space, and if $\alpha \in [0, 1]$, let $J(\alpha; \mu, \mathcal{R})$ denote the intersection of all initial segments of μ–measure $> \alpha$.

LEMMA 12. 15. $J(\alpha; \mu, \mathcal{R})$ is measurable, and $\mu(J(\alpha; \mu, \mathcal{R})) = \alpha$.

Proof. The case $\alpha = 1$ is trivial, so we may exclude it without loss of generality. Clearly $J(\alpha; \mu, \mathcal{R})$ is an initial set; therefore by the previous lemma it is measurable. Set $v = \mu$. If $t \notin \bar{J}$, then there is an initial segment $I(t'; \mathcal{R})$ of μ–measure $> \alpha$ not containing t; hence $I(t; \mathcal{R}) > I(t'; \mathcal{R})$, so $\mu(I(t; \mathcal{R})) \geqq \mu(I(t'; \mathcal{R})) > \alpha$. Hence the inf appearing in the previous lemma is $\geqq \alpha$. If $s \in \bar{J}$, then we must have $\mu(I(s; \mathcal{R})) \leqq \alpha$; for if $\mu(I(s; \mathcal{R})) > \alpha$, then by definition $J < I(s; \mathcal{R})$, so $s \in I(s; \mathcal{R})$, a contradiction. Hence the sup appearing in the previous lemma is $\leqq \alpha$. Since the inf and the sup are equal, they both $= \alpha$, and the result follows from the previous lemma.

LEMMA 12.16. *If Θ is any automorphism of the underlying space, then*

$$\Theta J(\alpha; \mu, \mathcal{R}) = J(\alpha; \Theta_*^{-1}\mu, \Theta\mathcal{R}).$$

Proof. This is a straightforward consequence of $\Theta I(s; \mathcal{R}) = I(\Theta s; \Theta\mathcal{R})$; the details are omitted.

Proof of Proposition 12.13. W.l.o.g. let $(I, \mathcal{C}) = ([0, 1], \mathcal{B})$. Define $v = f \circ \lambda$, where

$$f(x) = \begin{cases} x, & \text{when } x \leqq \frac{1}{2} \\ \frac{1}{2}, & \text{when } x \geqq \frac{1}{2}. \end{cases}$$

For each measurable order $\mathcal{R}$, set $J = J(\mathcal{R}) = J(1/2; \lambda, \mathcal{R})$. Then for each $S \subset I$, we claim that

(12.17) $$\varphi^{\mathcal{R}}(v)(S) = \lambda(J \cap S).$$

To prove this, first let S be an initial segment. If $\lambda(S) > 1/2$, then $S \supset J$, so

$$\varphi^{\mathcal{R}}(v)(S) = v(S) = \tfrac{1}{2} = \lambda(J) = \lambda(J \cap S).$$

If $\lambda(S) \leqq \frac{1}{2}$, let $S = I(s; \mathcal{R})$. Then for all s' such that $\lambda(I(s'; \mathcal{R})) > 1/2$, we have $s' \mathcal{R} s$, i.e., $s \in I(s'; \mathcal{R})$. Hence $s \in J$, and so $S \subset J$. Hence

$$\varphi^{\mathcal{R}}(v)(S) = v(S) = \lambda(S) = \lambda(J \cap S).$$

Hence (12.17) is proved whenever S is an initial segment, and so, by (12.3) and the fact that both sides are measures, for all measurable S.

We now apply a trick due to Furstenberg (cf. [A_3]). Let Θ be a Lebesgue-measure-preserving automorphism of the underlying space, which is moreover (strongly) mixing.* For each fixed measurable $S \subset I$, define a sequence of random variables $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots$ on the probability space Ω by

$$\underset{\sim}{x}_n = \underset{\sim}{x}_n^{\mathcal{R}} = \lambda(J(\Theta^n \mathcal{R}) \cap S);$$

from (12.10), (12.12), and (12.17) it follows** that the $\underset{\sim}{x}_n$ are

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*Readers not familiar with this term may refer to the definition, which is given in the next section.

**Note that Θ and hence all the Θ^n are not only automorphism of the underlying space but also symmetries of v.

indeed random variables, i.e., measurable in $\mathcal{R}$. From (12.11) it follows that all the $\underset{\sim}{x}_n$ have the same distribution. On the other hand, since Θ^n is measure-preserving, Lemma 12.16 yields $J(\Theta^n \mathcal{R}) = \Theta^n J(\mathcal{R})$. Therefore, since Θ is mixing, it follows that

$$\underset{\sim}{x}_n = \lambda(\Theta^n J(\mathcal{R}) \cap S) \rightarrow \lambda(J(\mathcal{R}))\lambda(S) = \tfrac{1}{2}\lambda(S).$$

So $\underset{\sim}{x}_n$ tends to a constant (independent of $\mathcal{R}$); but since all the $\underset{\sim}{x}_n$ have the same distribution, it follows that for each n,

$$\underset{\sim}{x}_n = \tfrac{1}{2}\lambda(S)$$

with probability 1. In particular, setting n = 0, we deduce

(12.18) $$\lambda(J(\mathcal{R}) \cap S) = \tfrac{1}{2}\lambda(S)$$

with probability 1. In particular, this is true whenever S is a finite union of intervals with rational end points; therefore, since there are only denumerably many S of this kind, (12.18) holds simultaneously for <u>all</u> such S with probability 1. But from this, a simple approximation argument leads to the conclusion that with probability 1, (12.18) holds simultaneously for <u>all</u> measurable S. If we then take an $\mathcal{R}$ for which (12.18) does in fact hold for all S, and set $S = J(\mathcal{R})$, we obtain

$$\tfrac{1}{2} = \lambda(J(\mathcal{R})) = \lambda(J(\mathcal{R}) \cap S) = \tfrac{1}{2}\lambda(S) = \tfrac{1}{2}\lambda(J(\mathcal{R})) = \tfrac{1}{4},$$

an absurdity. This completes the proof of Proposition 12.13, and shows that the random order approach to values cannot be used for even the simplest non-atomic games.

Rather than paralleling a proof of the theorem of [A_3], it is also possible to use that theorem directly in order to prove the above impossibility result; but little if any space would be saved in the process.

Basically, the impossibility theorem we have just proved hinges on the demand that the probability measure ω on Ω be invariant under all automorphisms of the underlying space, or at least under those automorphisms that are also symmetries of the game v. Such a demand is unavoidable in the context of this paper. In a different context, though—if more "structure" were imposed on the game—it might be avoided; or if not avoided, at least rendered harmless. We could, for example, view the underlying space as having a topological as well as a measurable structure. That, of course, would change our whole outlook. We would restrict Θ to be a homeomorphism of the topological space as well as an automorphism of the measurable space; this restriction would, however, be applied not only to (12.10), but also to the axiomatic definition of value in Part I, and in particular to the symmetry condition (2.2). The outcome would be a value that is not the analog of the ordinary value for finite games as treated in [S_1], but rather of variants in which the symmetry axiom is relaxed in one way or another. We have not followed this direction any further (cf. [O]).

13. MIXING

The impossibility theorem in the previous section may be intuitively understood as follows: Suppose that there are a large finite number n of players, rather than a continuum. Let S be a coalition of these players which is "not too small;" to fix ideas, let $|S| = [n/4]$ (the greatest integer in n/4). For each ordering $\mathcal{R}$ of the players let $J(\mathcal{R})$ be the coalition consisting of the first $[n/2]$ players. If we assign probability 1/n! to each of the n! possible orderings $\mathcal{R}$ of the player set, then it may be seen that $|J(\mathcal{R}) \cap S|/n$ will be close to $1/2 \cdot 1/4$ with high probability; more precisely, for each $\epsilon > 0$,

$$\text{Prob}\left(\frac{|J(\mathcal{R}) \cap S|}{n} - \tfrac{1}{4} \cdot \tfrac{1}{2} > \epsilon \right) \to 0$$

as $n \to \infty$. Thus if we shuffle a deck of cards and then cut it in half, the proportion of hearts in the top half will with high probability be close to 1/4.

More generally, if we replace 1/4 by any other fixed number θ such that $0 < \theta \leqq 1$, i.e., if we take $|S| = [\theta n]$, then $|J(\mathcal{R}) \cap S|/n$ will be close to $\theta/2$ with high probability.

If we now "pass to the limit," i.e., use a continuum of players rather than a finite number, then we might expect to replace the phrase "close to" by "equal to," and "high probability" by "probability 1." An appropriate analogue for the relative number of players in a coalition T (i.e., the quantity $|T|/n$) might be the measure $\mu(T)$.

Of course it is not quite clear which measure μ is the correct one to use, and presumably this depends on just how one "passes to the limit;" but this is a technical point that need not concern us for the present, since the whole discussion is only meant to be suggestive. The upshot is that we would expect, for every S, that

$$\mu(J(\mathcal{R}) \cap S) = \tfrac{1}{2}\mu(S)$$

with probability 1; and this leads to a contradiction as in the end of the previous section (from (12.18) to the end of the proof).

We saw in the previous section that matters are not really as simple as all this. Nevertheless, it will be useful to keep in mind the principle behind the impossibility theorem: If it were possible to define the notion of random order, then the first "half" of the players would with probability 1 be evenly mixed among all the players, i.e., the first half would intersect each coalition S in the same proportion that it intersects the whole player space; and such evenly mixed sets do not exist (cf. the discussion following (12.18)).

This idea of "even mixing" reminds us of the notion of mixing transformation from Ergodic Theory. Let μ be a probability measure on the underlying space $(I, \mathcal{C})$. Recall that a <u>mixing transformation</u> of the measure space $(I, \mathcal{C}, \mu)$ is a μ-measure preserving automorphism Θ of $(I, \mathcal{C})$ such that for all $S, T \in \mathcal{C}$, we have

(13.1) $$\mu(S \cap \Theta^n T) \to \mu(S)\mu(T)$$

as $n \to \infty$. Roughly speaking, the sequence $(\Theta, \Theta^2, \ldots)$ accomplishes "in the long run" the mixing job that we would have liked to (but cannot) accomplish "in one fell swoop" with a single automorphism. We will now see how such a sequence can be used as an approach to the value concept.

First let us generalize slightly the notion of mixing transformation. All that really interests us is the property (13.1) of the sequence $\{\Theta, \Theta^2, \ldots\}$; the fact that it is a sequence of powers is irrelevant. W.r.t. a fixed non-atomic probability measure μ on $(I, \mathcal{C})$, we therefore define a μ–mixing sequence to be a sequence $\{\Theta_1, \Theta_2, \ldots\}$ of μ–measure preserving automorphisms of $(I, \mathcal{C})$; such that for all $S, T \in \mathcal{C}$, we have

$$\mu(S \cap \Theta_n T) \to \mu(S)\mu(T).$$

Now let $\mathcal{R}$ be a fixed measurable order (cf. Sec. 12) on $(I, \mathcal{C})$. Then for each automorphism Θ of $(I, \mathcal{C})$, $\Theta\mathcal{R}$ is also a measurable order. We shall be interested in measures of the form $\varphi^{\Theta\mathcal{R}}v$ (cf. (12.2)); since this notation will occasionally be cumbersome, we will sometimes write $\varphi(v; \Theta\mathcal{R})$ rather than $\varphi^{\Theta\mathcal{R}}v$.

DEFINITION 13.2. Let $v \in AC$. A set function* φv is said to be the mixing value of v if for all non–atomic probability measures μ such that $v \ll \mu$, for

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*We will see below that the mixing value, if it exists, must necessarily be a measure.

all μ-mixing sequences $\{\Theta_1, \Theta_2, \ldots\}$, for all measurable orders $\mathcal{R}$, and for all coalitions S, we have

(13.3) $$\varphi(v; \Theta_n \mathcal{R})(S) \to (\varphi v)(S) \text{ as } n \to \infty .$$

The mixing value, if it exists, is clearly unique. The set of all members of AC, that have mixing values will be denoted MIX.

Let us review the idea behind this definition: We saw before that a random order (if there were such a thing) would be "thoroughly mixing" with probability 1. So rather than averaging the $\varphi^{\mathcal{R}} v$ over all $\mathcal{R}$ as we tried to do in the previous section, we could let $\mathcal{R}$ be a single "thoroughly mixing" order, and define the value to be $\varphi^{\mathcal{R}} v$ for this $\mathcal{R}$. This would especially convincing if $\varphi^{\mathcal{R}} v$ were the same for all "thoroughly mixing" $\mathcal{R}$. Unfortunately, as we have seen, there are no "thoroughly mixing" orders. But there are orders that are "approximately thoroughly mixing," namely the orders $\Theta_n \mathcal{R}$, where $\{\Theta_1, \Theta_2, \ldots\}$ is a μ-mixing sequence and $\mathcal{R}$ is arbitrary. So the measures $\varphi(v; \Theta_n \mathcal{R})$ may be considered "approximate values"; and if their limit exists and is independent of the various choices that must be made, we may feel justified in calling it a value.

The need for the condition $v \ll \mu$ may be illustrated as follows: if $(I, \mathcal{C}) = ([0,1], \mathcal{B})$, $v = \lambda^2$, and μ is defined by $\mu(S) = \lambda([0, \frac{1}{2}] \cap S)$, then a sequence whose behavior on $[\frac{1}{2}, 1]$ is perfectly arbitrary may be μ-mixing, and clearly this cannot always produce convergence in

(13.3). In Sec. 14 we will state an alternative condition on μ, similar in spirit to the condition $v \ll \mu$, and prove that it leads to the same definition of mixing value.

The main theorem of this part is that the mixing value exists for each member of pNA and coincides with the unique value on pNA as determined in Part I; this will be proved in Sec. 16. On the way to this theorem, we will prove that MIX is a closed symmetric linear subspace of BV, and that the operator φ that associates to each v its mixing value φv is a value on MIX (Sec. 15). Our proof will be independent of the proof of Theorems A and B given in Part I, and indeed it will enable us to give an alternative proof of the existence of a value on pNA and of formula (3.1)(Sec. 16). Also, we will show that the set function of Example 5.8 is not in MIX; combined with the fact that pNA $\subset$ MIX, this provides an independent proof that this set function is not in pNA, and also shows that MIX does not comprise all of AC (Sec. 17).

We remark that the lengthiness of the development of the mixing value is in part due to the fact that we have no underlying measure to work with. Thus, if we had restricted ourselves to set functions v that are absolutely continuous w.r.t. a fixed measure $\mu \in NA^+$, it would have simplified matters considerably.

14. AN ALTERNATIVE CHARACTERIZATION OF THE MIXING VALUE

PROPOSITION 14.1. *Let* $v \in AC$. *A necessary and sufficient condition that the set function* φv *be the mixing value of* v *is that there exist a measure* μ_v *in* NA^+ *such that for all* $\mu \gg \mu_v$,

(14.2) *for all* μ-*mixing sequences* $\{\Theta_1, \Theta_2, \ldots\}$, *all measurable orders* $\mathcal{R}$, *and all coalitions* S, *we have* $\varphi(v; \Theta_n \mathcal{R})(S) \to (\varphi v)(S)$ *as* $n \to \infty$.

If we think of a measure μ as being "more sensitive" than a measure ν whenever $\nu \ll \mu$ but not $\mu \ll \nu$, then Proposition 14.1 says that φv is the mixing value of v is and only if (14.2) holds for all "sufficiently sensitive" μ.

The proof of Proposition 14.1 is the object of this section; it will require a number of lemmas and auxiliary propositions, including some of independent interest.

It might occur to the reader to ask whether Proposition 14.1 could be used to extend the definition of mixing value to set functions v that are not in AC. The answer is no, since only for $v \in AC$ are we assured that the $\varphi(v; \Theta_i \mathcal{R})$ are defined (cf. Proposition 12.7).

LEMMA 14.3. *For every probability measure* μ *in* NA, *there exists a* μ-*mixing sequence.*

Proof. W.l.o.g. let $(I, \mathcal{C}) = ([0,1], \mathcal{B})$. Let Θ be a Lebesgue-measure-preserving mixing transformation, e.g., the bilateral 1-shift [H_2]. If $\mu = \lambda$ then $\{\Theta, \Theta^2, \ldots\}$ is a μ-mixing sequence. For general μ, let Φ be an automorphism such that $\Phi_* \mu = \lambda$ (Lemma 6.2); then $\{\Phi\Theta\Phi^{-1}, \Phi\Theta^2\Phi^{-1}, \ldots\}$ is a μ-mixing sequence.

LEMMA 14.4. *Let $\mathcal{R}$ be a measurable order on the player space $(I, \mathcal{C})$. Then there is a Borel subset D of the real line, such that if $\mathcal{D}$ is the family of Borel subsets of D, then $(I, \mathcal{C})$ is isomorphic to $(D, \mathcal{D})$ under an isomorphism Θ that preserves order, i.e., such that*

(14.5) $$s \mathcal{R} t \Leftrightarrow \Theta s > \Theta t.$$

Proof. According to Lemma 12.4 and Lemma II of* [D_3], there is a real-valued function—which we shall call Θ—on I, such that (14.5) holds. Define

$$D = \Theta(I).$$

Clearly Θ is a one-one transformation from I onto D. Let $\mathcal{E}$ be the σ-field of subsets of D consisting of all intersections of D with Borel sets. Now for an arbitrary real α, the set

$$J = \Theta^{-1}(D \cap (-\infty, \alpha))$$

has the property that

*There is an error in one of the proofs in [D_3]; this error is corrected in [D_4].

$$s \in J,\ s \mathcal{R} s' \Rightarrow s' \in J.$$

Hence by Lemma 12.14, J is measurable. The sets $D \cap (-\infty, \alpha)$ generate $\mathcal{E}$, and each of their inverse images is measurable; hence Θ is a measurable transformation from $(I, \mathcal{C})$ onto $(D, \mathcal{E})$. Then it follows from Theorem 3.2 of [M] that Θ is an isomorphism from $(I, \mathcal{C})$ onto $(D, \mathcal{E})$, and D is a Borel subset of the real line. Hence if $\mathcal{D}$ is the family of Borel subsets of D, then $\mathcal{D} = \mathcal{E}$, and the lemma is proved.

LEMMA 14.6. *On* I, *let* $\mathcal{R}$ *be a measurable order, and* ξ *a non-atomic probability measure. Let* λ *be Lebesgue measure on* $[0, 1]$. *Then there is an isomorphism* Ξ *from* $(I, \mathcal{C})$ *onto* $([0, 1], \mathcal{B})$ *such that*

$$\xi(S) = \lambda(\Xi(S))$$

for all $S \subset I$, *and such that there is a* $U \subset I$ *of* ξ*–measure* 0 *such that*

$$s \mathcal{R} t \Rightarrow \Xi s > \Xi t$$

whenever s *and* t *are in* $I \backslash U$.

Remark. The lemma says that the "ordered measure spaces" $(I, \mathcal{C}, \xi, \mathcal{R})$ and $([0,1], \mathcal{B}, \lambda, >)$ are "almost isomorphic", in the sense that there is a one–one mapping from one onto the other that preserves measurability and measure, and that "almost" preserves the order. Of course, it follows that any two "ordered measure spaces" are "almost isomorphic" in this sense, provided that the underlying measurable spaces are isomorphic to $([0,1], \mathcal{B})$, that the measures are non–atomic probability measures, and that the orders are measurable.

Proof. By the previous lemma, we may assume w.l.o.g. that I is a Borel subset of the real line, that $\mathcal{C}$ is the family of Borel subsets of I, and that the order is the natural order > ("greater than"). All measurable spaces appearing in this proof will consist of a Borel subset X of the real line E^1, together with the family of Borel subsets of X; thus identifying such a space by X only can cause no confusion.

Since $I \subset E^1$, we may extend ξ from I to a measure ξ' on all of E^1 by defining

$$\xi'(S) = \xi(S \cap I).$$

Let us define a mapping Ψ of E^1 into $[0,1]$ by

$$\Psi(s) = \xi'((-\infty, s)). \tag{14.7}$$

Since ξ is a non–atomic probability measure, so is ξ'; hence Ψ is a monotonic non–decreasing continuous function. Hence if $s \in E^1$, then from the continuity and monotonicity of Ψ, and the fact that the continuous image of a connected set is connected, if follows that $\Psi((-\infty, s))$ is an interval with end–points 0 and $\Psi(s)$. Hence from (14.7) we obtain

(14.8) $$\lambda(\Psi((-\infty,\ s))) = \Psi(s) = \xi'((-\infty,\ s)).$$

Next, let W be the set of points where Ψ fails to be univalent, i.e.,

$$W = \{s \in E^1 :\ \exists\, t \neq s \text{ with } \Psi(t) = \Psi(s)\}\ .$$

Because of the monotonicity and continuity of Ψ, W is a disjoint union of nondegenerate closed intervals, each of which are carried into a single point under Ψ, and each of which are of ξ—measure 0. Since they are nondegenerate and disjoint, there are at most denumerably many of them; hence $\Psi(W)$ is denumerable, and

(14.9) $$\xi'(W) = 0,$$

(14.10) $$\lambda(\Psi(W)) = 0.$$

Let Ψ^* be the restriction of Ψ to $E^1 \setminus W$, and let ξ^* be the restriction of ξ' to the Borel subsets of $E^1 \setminus W$. Then Ψ^* is an isomorphism from the measurable space $E^1 \setminus W$ to the measurable space $\Psi(E^1) \setminus \Psi(W)$. Furthermore for all $s \in E^1$ we have

$$\Psi^*((-\infty,\ s) \setminus W) = \Psi((-\infty,\ s) \setminus W) = \Psi((-\infty,\ s)) \setminus \Psi(W).$$

Therefore from (14.10) it follows that

$$\lambda(\Psi^*((-\infty,\ s) \setminus W)) = \lambda(\Psi((-\infty,\ s))),$$

and hence from (14.9) and (14.8) we obtain

$$(14.11)\qquad \xi^*((-\infty, s)\setminus W) = \xi'((-\infty, s)\setminus W)$$
$$= \xi'((-\infty, s)) = \lambda(\Psi((-\infty, s))) = \lambda(\Psi^*((-\infty, s)\setminus W)).$$

Now since Ψ^* is an isomorphism, $\lambda\Psi^*$ is a measure on $E^1\setminus W$. Formula (14.11) says that this measure coincides with ξ^* for all sets of the form $(-\infty, s)\setminus W$, which generate all the measurable subsets of $E^1\setminus W$; hence

$$(14.12)\qquad \xi^*(S) = \lambda(\Psi^*(S))$$

for all $S\subset E^1\setminus W$.

From (14.9) it follows that $\xi^*(I\setminus W) = 1$; hence $I\setminus W$ is nondenumerable, and so contains a nondenumerable Borel set U' of ξ–measure 0. Let $V' = \Psi^*(U')$; by (14.12), V' is of Lebesgue measure 0. Now from (14.12) and (14.9) it follows that

$$\lambda(\Psi^*(I\setminus W)) = \xi^*(I\setminus W) = \xi'(I\setminus W) = \xi'(I) = \xi(I) = 1;$$

hence $[0, 1]\setminus\Psi^*(I\setminus W)$ is of Lebesgue measure 0. Let

$$U = U'\cup(I\cap W),$$
$$V = V'\cup([0, 1]\setminus\Psi^*(I\setminus W));$$

then U is a nondenumerable Borel subset of I whose ξ–measure is 0 (by (14.9)), V is a nondenumerable Borel subset of $[0, 1]$ whose λ–measure is 0, and Ψ^* is an isomorphism from $I\setminus U$ onto $[0, 1]\setminus V$. Let Θ be an arbitrary isomorphism from the measurable space U onto the measurable space V. Define a one–one mapping Ξ from I onto $[0, 1]$ by

$$\Xi(s) = \begin{cases} \Theta(s) & \text{when } s \in U \\ \Psi^*(s) & \text{when } s \in I \setminus U. \end{cases}$$

Then the demands of the lemma are satisfied, and so our proof is complete.

COROLLARY 14. 13. *Let $\mathcal{R}$ be a measurable order on* I, *and let* $\xi \in NA^+$. *Let* S *and* T *be measurable subsets of* I *of equal* ξ*–measure. Then there is an automorphism* Ξ *of* I *such that*

$$\Xi(S) = T$$

and such that there is a $U \subset I$ *of* ξ*–measure* 0 *such that*

$$s \mathcal{R} t \Leftrightarrow \Xi s \ \mathcal{R}\ \Xi t$$

whenever s *and* t *are in* $S \setminus U$, *and whenever* s *and* t *are in* $(I \setminus S) \setminus U$.

Proof. Assume first that

$$0 < \xi(S) = \xi(T) < \xi(I).$$

Consider the following two "ordered measure spaces": First, the space formed from S, its measurable subsets, the order $\mathcal{R}$ restricted to S, and the probability measure $\xi(\cdot)/\xi(S)$; and second, the space formed in a similar way from T. Because of Lemma 14. 6 and the remark following its statement, these two "ordered measure spaces" are "almost isomorphic"; let Ξ_1 be the appropriate "almost isomorphism." Similarly, the "ordered measure spaces" formed from $I\setminus S$ and $I\setminus T$ in a similar way are "almost isomorphic"; let Ξ_2 be the appropriate "almost isomorphism" If we combine Ξ_1 and Ξ_2, we obtain an automorphism of I that satisfies the required conditions.

If $0 = \xi(S) = \xi(T)$, then we substitute an arbitrary (nonorder-preserving) isomorphism of S onto T for Ξ_1; similarly if $\xi(S) = \xi(T) = \xi(I)$, we substitute an arbitrary (nonorder-preserving) isomorphism of $I\setminus S$ onto $I\setminus T$ for Ξ_2. This completes the proof of Corollary 14. 13.

LEMMA 14. 14. *Let* $u \ll \mu$, *where* $\mu \in NA^+$, *let* $T \subset I$ *be of* μ*-measure* 0, *and let* $\mathcal{R}$ *and* $\mathcal{R}'$ *be measurable orders such that*

$$s\,\mathcal{R}\,t \Leftrightarrow s\,\mathcal{R}'\,t$$

whenever both s *and* t *are in* $I\setminus T$. *Then*

$$\varphi^{\mathcal{R}}u = \varphi^{\mathcal{R}'}u\ .$$

Proof. For $s \notin T$, we have

$$\begin{aligned} I(s;\mathcal{R}) &= \{t:\ s\,\mathcal{R}\,t\} = \{t \notin T:\ s\,\mathcal{R}\,t\} \cup \{t \in T:\ s\,\mathcal{R}\,t\} \\ &= \{t \notin T:\ s\,\mathcal{R}'\,t\} \cup \{t \in T:\ s\,\mathcal{R}\,t\} \\ &= (\{t:\ s\,\mathcal{R}'\,t\} \setminus \{t \in T:\ s\,\mathcal{R}'\,t\}) \cup \{t \in T:\ s\,\mathcal{R}\,t\} \\ &\underset{\mu}{\cong} I(s;\mathcal{R}'), \end{aligned}$$

where $\underset{\mu}{\cong}$ means "equal except possibly for a set of μ–measure 0."
Since both $u \ll \mu$ and $\varphi^{\mathcal{R}'} u \ll \mu$ (Proposition 12.7), it follows that for $s \in I \setminus T$,

$$(\varphi^{\mathcal{R}} u)(I(s; \mathcal{R})) = u(I(s; \mathcal{R})) = u(I(s; \mathcal{R}'))$$
$$= (\varphi^{\mathcal{R}'} u)(I(s; \mathcal{R}')) = (\varphi^{\mathcal{R}'} u)(I(s; \mathcal{R})).$$

This is also true when $s = \pm\infty$, so that we have

$$(14.15) \qquad (\varphi^{\mathcal{R}} u)(I(s; \mathcal{R})) = (\varphi^{\mathcal{R}'} u)(I(s; \mathcal{R}))$$

whenever $s \in \bar{I} \setminus T$; we wish to establish (14.15) for all s. To this end, let $t \in T$, and apply Proposition 12.14 with

$$J = \cap \{ I(s; \mathcal{R}) : \; s \in \bar{I} \setminus T \text{ and } s \mathcal{R} t \}$$

and $v = \mu$. We deduce that for each $\epsilon > 0$ there are s_1 and s_2 such that $s_1 \in J$, $s_2 \in \bar{I} \setminus \bar{J}$ (whence in particular $s_1 \mathcal{R} s_2$), and

$$\mu(I(s_2; \mathcal{R}) \setminus I(s_1; \mathcal{R})) < \epsilon .$$

Since $s_2 \in \bar{I} \setminus J$, it follows that there is an $s \in \bar{I} \setminus T$ such that $s \mathcal{R} t$ and $s_2 \notin I(s; \mathcal{R})$, i.e., $s_2 \overset{\mathcal{R}}{=} s$. But from $s_1 \in J$ it follows that $s \mathcal{R} s_1$; hence

$$(14.16) \qquad \mu(I(s; \mathcal{R}) \setminus I(s_1; \mathcal{R})) < \epsilon .$$

Now clearly $t \in J$; hence if $t \overset{\mathcal{R}}{=} s_1$, it follows that

$$(14.17) \qquad \mu(I(s; \mathcal{R}) \setminus I(t; \mathcal{R})) < \epsilon \;;$$

if $s_1 \mathcal{R} t$, then from $s_1 \in J$ it follows that $I(s_1; \mathcal{R}) \setminus I(t; \mathcal{R}) \subset T$, hence $\mu(I(s_1; \mathcal{R}) \setminus I(t; \mathcal{R})) = 0$, and this together with (14.16) again yields (14.17). Hence for each $\epsilon > 0$ there is an $s \in \bar{I} \setminus T$ such that $s \mathcal{R} t$ and (14.17) holds; this, together with $\varphi^{\mathcal{R}} u \ll \mu$, $\varphi^{\mathcal{R}'} \mu \ll \mu$, and the truth of (14.15) for all $s \in \bar{I} \setminus T$ yields

$$(\varphi^{\mathcal{R}} u)(I(t; \mathcal{R})) = (\varphi^{\mathcal{R}'} u)(I(t; \mathcal{R})).$$

Hence (14.15) holds for all s. But the $I(s; \mathcal{R})$ generate all the measurable sets, and $\varphi^{\mathcal{R}}$ and $\varphi^{\mathcal{R}'}$ are both measures. Hence $(\varphi^{\mathcal{R}} u)(S) = (\varphi^{\mathcal{R}'} u)(S)$ for all S, as was to be proved.

<u>LEMMA 14.18</u>. <u>Let</u> $u \in AC$, <u>let</u> $\mu \in NA^+$ <u>be a probability measure such that</u> $u \ll \mu$, <u>and let</u> $\{\Theta_1, \Theta_2, \ldots\}$ <u>be a μ–mixing sequence</u>. <u>Let</u> $\mathcal{R}$ <u>be a measurable order, and let</u> $\tau \in NA^+$. <u>Then there is a probability measure</u> $\xi \in NA^+$ <u>with</u> $\mu + \tau \ll \xi$, <u>and a ξ–mixing sequence</u> $\{\Psi_1, \Psi_2, \ldots\}$, <u>such that</u>

$$\varphi(u; \Psi_k^{-1} \mathcal{R}) = \varphi(u; \Theta_k \mathcal{R})$$

<u>for all</u> k.

<u>Proof</u>. Let $\tau = \tau^{ac} + \tau^{\perp}$ be the Lebesgue decomposition of τ w.r.t. μ, i.e., τ^{ac} and $\tau^{\perp}$ are nonnegative measures such that $\tau^{ac} \ll \mu$ and $\tau^{\perp} \perp \mu$; then τ^{ac} and $\tau^{\perp}$ are necessarily non–atomic.

If $\tau^{\perp}$ vanishes identically, then $\tau \ll \mu$, so we may take $\xi = \mu$ and $\Psi_i = \Theta_i^{-1}$, and the proof is complete when we note that $\{\Theta_i^{-1}\}$ is a mixing sequence if and only if $\{\Theta_i\}$ is. Suppose, therefore, that $\tau^{\perp}$ does not vanish identically; w.l.o.g., assume that $\tau^{\perp}(I) = 1$. Define

$$\xi = \tfrac{1}{2}(\mu + \tau^{\perp}).$$

Let $W \subset I$ be such that $\mu(W) = 0$ and $\tau^{\perp}(W) = 1$.

The idea of the proof is to restrict $\{\Theta_i^{-1}\}$ to $I \backslash W$, to define an arbitrary $\tau^{\perp}$–mixing sequence $\{\Phi_i\}$ on W, and then to combine these into a ξ–mixing sequence $\{\Psi_k\}$ on all of I by "mixing them" together. The choice of the sequence $\{\Phi_i\}$ will not affect anything, because we are only interested in the effect of $\{\Psi_k\}$ on u, and because $u \ll \mu$, the set function u is "essentially defined" on $I \backslash W$ only. We now go on to the technical details of the proof.

Construct a $\tau^{\perp}$–mixing sequence $\{\Phi_1, \Phi_2, \ldots\}$ such that for all i,

$$\Phi_i(W) = W$$

and Φ_i is the identity on $I \backslash W$; the possibility of constructing such a sequence follows from Lemma 14.3 applied to the underlying space W. Next, recall again that since $\{\Theta_i\}$ is μ–mixing, so is $\{\Theta_i^{-1}\}$. Hence it is possible to construct a μ–mixing sequence $\{\Theta'_1, \Theta'_2, \ldots\}$ such that for all i,

$$\Theta_i^!(I \setminus W) = I \setminus W,$$

such that $\Theta_i^!$ is the identity on W, and such that $\Theta_i^!$ differs from Θ_i^{-1} on a set of μ–measure 0 only; indeed, we may define

$$\Theta_i^!(t) = \begin{cases} t & \text{when } t \in \bigcup_{n=-\infty}^{\infty} \Theta_i^n(W) \\ \Theta_i^{-1}(t) & \text{otherwise.} \end{cases}$$

For i = 0, 1, 2, ..., let Ξ_i be a ξ–measure–preserving automorphism of the player space such that

$$\Xi_i(W) = \bigcup_{j=1}^{2^i} \{J(j/2^i; \xi, \mathcal{R}) \setminus J((j-\tfrac{1}{2})/2^i; \xi, \mathcal{R})\},$$

and such that there is a set U of ξ–measure 0 such that

$$(14.19) \qquad s \mathcal{R} t \Leftrightarrow \Xi_i s \mathcal{R} \Xi_i t$$

whenever both s and t are in $W \setminus U$ and whenever both are in $(I \setminus W) \setminus U$; such an automorphism exists because of Corollary 14.13. Clearly U may be chosen independent of i; moreover, it may be chosen so that $\Xi_i U = U$ for all i (otherwise replace U by $\bigcup_\alpha \Pi_\alpha U$, where Π_α ranges over all automorphisms formed by composing a finite string of Ξ_i's and their inverses).

For k = 1, 2, ... define automorphisms Λ_k by

$$\Lambda_k(s) = \begin{cases} \Phi_k(s) & \text{when } s \in W \\ \Theta_k^!(s) & \text{when } s \in I \setminus W; \end{cases}$$

for k = 0, set $\Lambda_0(s) = s$. Note that each of the Λ_k is ξ–measure-preserving, and define a sequence $\{\Psi_k\}$ of automorphisms by

$$\Psi_k = \Xi_k \Lambda_k .$$

Then we claim:

(14.20) $\{\Psi_k\}$ is a ξ–mixing sequence.

To prove this, we must show that

(14.21) $\xi(S \cap \Psi_k T) \to \xi(S)\, \xi(T)$

for all S, $T \subset I$. We first show this when S is of the form $J(h/2^i;\, \xi,\, \Lambda_i \mathcal{R})$. To this end, fix h and i, and consider the underlying spaces

$$I_1 = (I \setminus W) \setminus U$$

and

$$I_2 = W \setminus U.$$

On them, define orders $\mathcal{R}_1$ and $\mathcal{R}_2$, by restricting $\mathcal{R}$ to I_1 and I_2, respectively. Also, we will consider the underlying spaces $\Xi_k I_1$ and $\Xi_k I_2$, with the orders $\Xi_k \mathcal{R}_1$ and $\Xi_k \mathcal{R}_2$, respectively.

Now for $k \geqq i$, we have

$$\begin{aligned} S \setminus U &= J(h/2^i;\, \xi,\, \mathcal{R}) \setminus U \\ &= \cup_{j=1}^{h2^{k-i}} \{J((j - \tfrac{1}{2})/2^k;\, \xi,\, \mathcal{R}) \setminus J((j-1)/2^k;\, \xi,\, \mathcal{R})\} \setminus U \\ &\quad \cup \cup_{j=1}^{h2^{k-1}} \{J(j/2^k;\, \xi,\, \mathcal{R}) \setminus J((j - \tfrac{1}{2})/2^k;\, \xi,\, \mathcal{R})\} \setminus U \\ &= H_1^k \cup H_2^k , \end{aligned}$$

where, by (14.19),

$$H_1^k = S \cap \Xi_k I_1 \quad \text{is an initial set in } \Xi_k I_1,$$

and

$$H_2^k = S \cap \Xi_k I_2 \quad \text{is an initial set in } \Xi_k I_2.$$

Hence (again using (14.19)), $\Xi_k^{-1} H_1^k$ and $\Xi_k^{-1} H_2^k$ are initial sets in I_1 and I_2, respectively. On the other hand we have

$$\xi(\Xi_k^{-1} H_1^k) = \xi(H_1^k) = h/2^{i+1}$$

and

$$\xi(\Xi_k^{-1} H_2^k) = \xi(H_2^k) = h/2^{i+1} .$$

But since, of any two initial sets in the same underlying space, one must contain the other, it follows that initial sets of the same ξ–measure can differ by a set of ξ–measure 0 only. Hence

$$\Xi_k^{-1} H_1^k \cong_\xi \Xi_i^{-1} H_1^i$$

and

$$\Xi_k^{-1} H_2^k \cong_\xi \Xi_i^{-1} H_2^i$$

where $\cong_\xi$ means "equal except for a set of ξ–measure 0." Hence

$$\Xi_k^{-1} S \cong_\xi \Xi_k^{-1}(S \setminus U) = \Xi_k^{-1} H_1^k \cup \Xi_k^{-1} H_2^k$$

$$\cong_\xi \Xi_i^{-1}(H_1^i \cup H_2^i).$$

To prove (14.21) for the special S under consideration, let

$T = T_1 \cup T_2$, where $T_1 \subset I \setminus W$ and $T_2 \subset W$. Then since $\{\Theta'_1, \Theta'_2, \dots\}$ is a μ–mixing sequence and $\Xi_i^{-1} H_2^i \subset I_2 \subset W$, we have, as $k \to \infty$,

$$\xi(S \cap \Psi_k T_1) = \xi \Xi_k^{-1}(S \cap \Psi_k T_1)$$

$$= \xi(\Xi_k^{-1} S \cap \Theta'_k T_1)$$

$$= \tfrac{1}{2}\mu(\Xi_i^{-1}(H_1^i \cup H_2^i) \cap \Theta'_k T_1)$$

$$= \tfrac{1}{2}\mu(\Xi_i^{-1} H_1^i \cap \Theta'_k T_1)$$

$$\to \tfrac{1}{2}\mu(\Xi_i^{-1} H_1^i)\, \mu(T_1)$$

$$= \xi(\Xi_i^{-1} H_1^i)\mu(T_1) = \frac{h}{2^{i+1}}\, \mu(T_1) = \frac{h}{2^i}\, \tfrac{1}{2}\mu(T_1).$$

Similarly, using the fact that $\{\Phi_k\}$ is a $\tau^{\perp}$–mixing sequence, we obtain

$$\xi(S \cap \Psi_k T_2) \to \frac{h}{2^i}\, \tfrac{1}{2}\, \tau^{\perp}(T_2).$$

Hence

$$\xi(S \cap \Psi_k T) = \xi(S \cap \Psi_k T_1) + \xi(S \cap \xi_k T_2)$$

$$\to \frac{h}{2^i}(\tfrac{1}{2}\mu(T_1) + \tfrac{1}{2}\tau^{\perp}(T_2)) = \frac{h}{2^i}(\tfrac{1}{2}\mu(T) + \tfrac{1}{2}\tau^{\perp}(T))$$

$$= \frac{h}{2^i}\, \xi(T) = \xi(J(h/2^i;\, \xi,\, \mathcal{R}))\, \xi(T) = \xi(S)\, \xi(T),$$

as was to be proved.

Now we have proved (14.21) for certain initial sets S of measure $h/2^i$, and hence an easy approximation argument yields it for certain initial sets S of arbitrary measure. Hence it follows for all initial sets S, in particular, therefore, for all initial segments S, and so, by standard approximation arguments, for all S. This completes the proof of (14.21), and hence also of (14.20). From the latter it follows at once that

(14.22) $\{\Psi_k^{-1}\}$ is a ξ–mixing sequence.

For a fixed k, let

$$V_k = (\Theta_k')^{-1} U \cup \bigcup_{n=-\infty}^{\infty} \Theta_k^n(W).$$

Then

(14.23) $$\mu(V_k) = 0.$$

Furthermore, when s, $t \in I \setminus V_k$, it follows that $\Theta_k'(s)$ and $\Theta_k'(t)$ are not in $\Theta_k'(V_k)$, hence not in $W \cup U$, hence in $(I\setminus W)\setminus U$. Hence

(14.24) $$\Theta_k' s \mathrel{\mathcal{R}} \Theta_k' t \Leftrightarrow \Xi_k \Theta_k' s \mathrel{\mathcal{R}} \Xi_k \Theta_k' t$$

$$\Leftrightarrow \Theta_k' s \; (\Xi_k^{-1} \mathcal{R}) \; \Theta_k' t,$$

where the second equivalence follows from the definition of $\Xi_k^{-1} \mathcal{R}$. Also, when s, $t \in I\setminus V_k$, we have

$$\Theta_k' s = \Theta_k^{-1} s \text{ and } \Theta_k' t = \Theta_k^{-1} t.$$

Hence for s, $t \in I\setminus V_k$ we deduce from (14.24) that

$$
\begin{aligned}
s\ (\Psi_k^{-1}\mathcal{R})\ t &\Leftrightarrow s\ (\Theta_k^{\prime -1}\Xi_k^{-1}\mathcal{R})\ t \\
&\Leftrightarrow \Theta_k^{\prime} s\ (\Xi_k^{-1}\mathcal{R})\ \Theta_k^{\prime} t \\
&\Leftrightarrow \Theta_k^{\prime} s\ \mathcal{R}\ \Theta_k^{\prime} t \\
&\Leftrightarrow \Theta_k^{-1} s\ \mathcal{R}\ \Theta_k^{-1} t \\
&\Leftrightarrow s\ (\Theta_k \mathcal{R})\ t.
\end{aligned}
$$

Applying Lemma 14.14 with $T = V_k$ and recalling (14.23), we deduce that

$$\varphi(u;\ \Psi_k^{-1}\ \mathcal{R}) = \varphi(u,\ \Theta_k \mathcal{R}),$$

and the proof of Lemma 14.18 is complete.

Proof of Proposition 14.1. The necessity is immediate; indeed, we can choose μ_v to be any ν such that $v \ll \nu$ (such a ν exists because $v \in AC$). To prove the sufficiency, let $\mu \in NA^+$ be such that $v \ll \mu$, and let $\{\Theta_1, \Theta_2, \ldots\}$ be a μ–mixing sequence. Now apply Lemma 14.18 with $u = v$ and $\tau = \mu_v$. Then $\mu_v \ll \xi$, and so by (14.2),

$$\varphi(v;\ \Theta_n\ \mathcal{R})(S) = \varphi(v;\ \Psi_n^{-1}\mathcal{R})(S) \to (\varphi v)(S)$$

as $n \to \infty$, for all $\mathcal{R}$ and S. This completes the proof.

15. BASIC PROPERTIES OF THE SET "MIX" AND OF THE MIXING VALUE

In this section we shall prove, among other things, that MIX is a closed linear subspace of BV and that the mixing value is a value on MIX (in the sense of Sec. 2).

LEMMA 15.1. *Let $\mathcal{R}$ be a measurable order. Then $\varphi^{\mathcal{R}}$ is a continuous linear operator on* AC.

Proof. The linearity follows easily from (12.2) and the continuity from Proposition 12.7.

PROPOSITION 15.2. *Let* $v \in$ MIX, *with mixing value* φv. *Then* φv *is a non-atomic measure, and for all non-atomic probability measures* μ *such* $v \ll \mu$, *all* μ*-mixing sequences* $\{\Theta_1, \Theta_2, \ldots\}$ *and all measurable orders* $\mathcal{R}$, *we have*

$$\varphi(v; \Theta_n \mathcal{R}) \to \varphi v$$

in the weak topology on the space of all measures on the underlying space $(I, \mathcal{C})$.

Proof. Recall that "measure" means "completely additive totally finite signed measure." We first prove that the set $K_v = \{\varphi^{\mathcal{R}} v: \mathcal{R}$ is a measurable order$\}$ is weakly sequentially compact. The condition for weak sequential compactness of a set K of measures is that K be bounded in norm and that the countable additivity of the measures in K is uniform over all of K [Dun–S, Theorem IV. 9. 1, p. 305]. The boundedness condition is satisfied by K_v because of Lemma 15. 1. The uniform countable additivity means that for any decreasing sequence of measurable sets $\{S_n\}$ with $\cap_n S_n = \emptyset$, we have $(\varphi^{\mathcal{R}} v)(S_n) \to 0$ uniformly for all $\mathcal{R}$ [Dun–S, p. 160]. Now let $v \ll \mu$; then $\mu(S_n) \to 0$, and the uniform countable additivity follows from the uniformity of $\varphi^{\mathcal{R}} \ll v$ (Proposition 12. 7). Hence K_v is indeed weakly sequentially compact.

Suppose now that ν is a weak limit point of $\varphi(v; \Theta_n \mathcal{R})$, i. e., that there is a subsequence of $\{\varphi(v; \Theta_n \mathcal{R})\}$ that converges weakly to ν; the existence of such a weak limit point follows from the weak sequential compactness that we have just proved. Since for each measurable S, the linear functional that associates with each measure μ the value $\mu(S)$ is continuous, it follows that

$$\varphi(v; \Theta_n \mathcal{R})(S) \to \nu(S)$$

when $\varphi(v; \Theta_n \mathcal{R})$ is restricted to the above subsequence. But we know that

$$\varphi(v; \Theta_n \mathcal{R})(S) \to (\varphi v)(S),$$

since φv is a mixing value; hence $\varphi v = \nu$. This proves firstly that φv is a completely additive measure, and secondly that every limit point of $\varphi(v; \Theta_n \mathcal{R})$ is φv. Hence $\varphi(v; \Theta_n \mathcal{R})$ must converge weakly to

φv; for if there were a weak neighborhood of φv outside of which there lie infinitely many $\varphi(v; \Theta_n \mathcal{R})$, then this infinite sequence would have a weak limit point, which would have to be different from φv, contradicting what we just proved. The proof of Proposition 15.2 is complete.

COROLLARY 15.3. $\|\varphi v\| \leqq \|v\|$.

Proof. A convex set is closed if and only if it is weakly closed [Dun-S, V. 3.13, p. 422]. Since the ball with center 0 and radius $\|v\|$ in the space of completely additive measures is closed and convex, it must be weakly closed. By Propositions 12.7 and 15.2, φv is the weak limit of members of this ball, and the corollary follows.

PROPOSITION 15.4. MIX is a closed linear subspace of BV.

Proof. That $v \in$ MIX $\Rightarrow \alpha v \in$ MIX for all scalar α is immediate. Suppose that $v, w \in$ MIX, and that μ_v and μ_w correspond to v and w respectively in accordance with Proposition 14.1. Then $(\mu_v + \mu_w)/2$ corresponds to $v + w$. This proves that MIX is a subspace.

To prove that it is closed, let $v_i \to u$, where $v_i \in$ MIX. Let $\mu \in NA^+$ be a probability measure such that $u \ll \mu$, and let $\{\Theta_1, \Theta_2, \ldots\}$ be a μ-mixing sequence. Let $\mathcal{R}$ be a measurable order.

Choose probability measures ν_i in NA^+ such that $v_i \ll \nu_i$. Let $\tau = \Sigma_{i=1}^{\infty} \nu_i/2^i$, and choose ξ and $\{\Psi_i\}$ in accordance with Lemma 14.18.

Let φv_i be the mixing value of v_i. From $v_i \to u$ and Corollary 15.3 it follows that φv_i is a Cauchy sequence (in the variation norm); denote its limit by φu. Since

$$v_i \ll \nu_i \ll \tau \ll \xi,$$

it follows that for all $S \subset I$,

$$\varphi(v_i; \Psi_k^{-1}\mathcal{R})(S) \to (\varphi v_i)(S)$$

as $k \to \infty$. We wish to show that

$$\varphi(u; \Psi_k^{-1}\mathcal{R})(S) \to (\varphi u)(S) \tag{15.5}$$

for all $S \subset I$. To this end, let $S \subset I$ and $\epsilon > 0$ be given. Then there is a number $j = j(\epsilon)$ such that

$$\| v_j - u \| \leqq \frac{\epsilon}{3} \text{ and } \| \varphi v_j - \varphi u \| \leqq \frac{\epsilon}{3}.$$

For this j we may find a number $k_0 = k_0(\epsilon)$ such that for $k \geqq k_0$,

$$|\varphi(v_j; \Psi_k^{-1}\mathcal{R})(S) - (\varphi v_j)(S)| \leqq \epsilon/3.$$

For $k \geqq k_0$ it then follows from Proposition 12.7 that

$$|\varphi(u;\Psi_k^{-1}\mathcal{R})(S) - (\varphi u)(S)| \leqq |\varphi(u;\Psi_k^{-1}\mathcal{R})(S) - \varphi(v_j;\Psi_k^{-1}\mathcal{R})(S)|$$

$$+ |\varphi(v_j;\Psi_k^{-1}\mathcal{R})(S) - (\varphi v_j)(S)| + |(\varphi v_j)(S) - (\varphi u)(S)|$$

$$\leqq \|\varphi(u;\Psi_k^{-1}\mathcal{R}) - \varphi(v_j;\Psi_k^{-1}\mathcal{R})\| + \frac{\epsilon}{3} + \|\varphi v_j - \varphi u\|$$

$$\leqq \|u - v_j\| + \frac{\epsilon}{3} + \|\varphi v_j - \varphi u\|$$

$$\leqq \frac{3\epsilon}{3} = \epsilon;$$

this proves (15. 5). But then from Lemma 14. 18 it follows that

$$\varphi(u;\Theta_k\mathcal{R})(S) = \varphi(u;\Psi_k^{-1}\mathcal{R})(S) \to (\varphi u)(S)$$

for all S. But since μ was arbitrary chosen with $u \ll \mu$, and $\{\Theta_k\}$ was an arbitrarily chosen μ–mixing sequence, it follows that $u \in$ MIX; hence MIX is closed. This completes the proof of Proposition 15. 4.

PROPOSITION 15. 6. MIX is symmetric, and the operator φ that associates with each v in MIX its mixing value φv is a value on MIX (in the sense of Sec. 2).

Proof. The linearity of φ follows easily from (13. 2) and the linearity of $\varphi^{\mathcal{R}}$ (Lemma 15. 1). The continuity of φ follows from Corollary 15. 3. The normalization condition (2. 3) follows at once from (13. 2) and (12. 2). The positivity also follows from (13. 2) and (12. 2); alternatively, it follows from Corollary 15. 3 and Proposition 4. 6.

The proof that MIX is symmetric and that φ satisfies the symmetry condition (2.2) is straightforward: Let $v \in$ MIX, let Ψ be an automorphism of the underlying space, let μ be a probability measure in NA such that $\Psi_* v \ll \mu$, let $\{\Theta_n\}$ be a μ-mixing sequence, and let $\mathcal{R}$ be a measurable order. Then $v \ll \Psi_*^{-1}\mu$; $\{\Psi\Theta_n\Psi^{-1}\}$ is a $\Psi_*^{-1}\mu$-mixing sequence; and $\Psi\mathcal{R}$ is a measurable order. Hence for all measurable S,

$$\varphi(v;\ \Psi\Theta_n\Psi^{-1}\Psi\mathcal{R})(S) \to (\varphi v)(S).$$

Now for any s,

$$\begin{aligned}\varphi(\Psi_* v;\ \Theta_n\mathcal{R})(I(s;\ \Theta_n\mathcal{R})) &= (\Psi_* v)(I(s;\ \Theta_n\mathcal{R}))\\ &= v(\Psi(I(s;\ \Theta_n\mathcal{R})))\\ &= v(I(\Psi s;\ \Psi\Theta_n\mathcal{R}))\\ &= \varphi(v;\ \Psi\Theta_n\mathcal{R})(I(\Psi s;\ \Psi\Theta_n\mathcal{R}))\\ &= \varphi(v;\ \Psi\Theta_n\mathcal{R})(\Psi(I(s;\ \Theta_n\mathcal{R}))).\end{aligned}$$

Thus for a fixed n,

$$\varphi(\Psi_* v;\ \Theta_n\mathcal{R})(S) = \varphi(v;\ \Psi\Theta_n\mathcal{R})(\Psi S)$$

for all S of the form $I(s;\ \Theta_n\mathcal{R})$; and so, by (12.3), for all S. Hence

$$\begin{aligned}\varphi(\Psi_* v;\ \Theta_n\mathcal{R})(S) &= \varphi(v;\ \Psi\Theta_n\Psi^{-1}\Psi\mathcal{R})(\Psi S)\\ &\to (\varphi v)(\Psi S)\end{aligned}$$

as $n \to \infty$. This shows that $\Psi_* v \in$ MIX, so MIX is symmetric. But then by the definition of the mixing value for $\Psi_* v$, we have

$$(\varphi\Psi_* v)(S) = \lim_{n\to\infty} \varphi(\Psi_* v; \Theta_n \mathcal{P})(S)$$

$$= (\varphi v)(\Psi S) = (\Psi_* \varphi v)(S).$$

This shows that $\varphi\Psi_* = \Psi_* \varphi$, establishing the symmetry condition (2.2). The proof of the proposition is complete.

16. pNA $\subset$ MIX

The purpose of this section is to show that pNA $\subset$ MIX. Since MIX is a closed subspace, it is sufficient to show that $\nu^k \in$ MIX, for each non–atomic probability measure ν and each positive integer k. As it turns out, it is just as easy to prove that $f \circ \nu \in$ MIX, when f is nondecreasing and continuously differentiable on the range [0, 1] of ν; and, notationally, it makes the proof somewhat more transparent.

From now until the end of this section, f will denote a nondecreasing, continuously differentiable function on [0, 1] with $f(0) = 0$ and $f(1) = 1$, ν and μ non–atomic probability measures on $(I, \mathcal{C})$, and $\mathcal{R}$ a measurable order; f and ν will be fixed throughout the discussion, but μ and $\mathcal{R}$ may vary as indicated in each context. The set function $f \circ \nu$ will be denoted v. $J(\alpha; \mu, \mathcal{R})$ will be as in Sec. 12, i.e., the intersection of all initial segments of μ–measure $> \alpha$. $J(\beta; \mu, \mathcal{R}) \backslash J(\alpha; \mu, \mathcal{R})$ will be denoted $(\alpha, \beta]^{\mu}_{\mathcal{R}}$, or, when no confusion can result, simply $(\alpha, \beta]^{\mu}$. Parentheses will not be repeated around $(\alpha, \beta]^{\mu}$; thus $\nu(\alpha, \beta]^{\mu}$ means $\nu((\alpha, \beta]^{\mu})$, and so on. The derivative of f will be denoted f'. Since f' is continuous on a closed interval, it is uniformly continuous there; its modulus of uniform continuity will be denoted by $\eta(\epsilon)$.

LEMMA 16.1. *Let μ and $\mathcal{R}$ be arbitrary, and let $G(\mu, \mathcal{R})$ denote the σ–field generated by all the sets $J(\alpha; \mu, \mathcal{R})$ for $\alpha \in [0, 1]$. Then for each measurable T* there is a set $T' \in G(\mu, \mathcal{R})$ such that $\mu(T' + T) = 0$, where + denotes the symmetric difference.

Proof. Recall that $T' + T = (T' \setminus T) \cup (T \setminus T')$. First let T be an initial segment $I(s, \mathcal{R})$. Set $\alpha = \mu(T)$, and let $J = J(\alpha; \mu, \mathcal{R})$, Then $J \supset T$ and by Lemma 12.15, $\mu(J) = \alpha = \mu(T)$; since $J \in G(\mu, \mathcal{R})$, the lemma is proved in this case. Now if F is the family of all sets T satisfying the conclusion of the lemma, then F is a σ–field; since it contains all the $I(s, \mathcal{R})$, it follows from (12.3) that it contains $\mathcal{C}$, and the lemma is proved.

LEMMA 16.2. Let μ be such that $\nu \ll \mu$. Then for all $\epsilon > 0$, all $\mathcal{R}$, and all α, β such that $0 \leqq \alpha < \beta \leqq 1$ and $\nu(\alpha, \beta]^{\mu}_{\mathcal{R}} < \eta(\epsilon)$, we have

$$|(\varphi^{\mathcal{R}} v)(\alpha, \beta]^{\mu}_{\mathcal{R}} - \nu(\alpha, \beta]^{\mu}_{\mathcal{R}} f'(\nu(J(\alpha; \mu, \mathcal{R})))| \leqq \epsilon \nu(\alpha, \beta]^{\mu}_{\mathcal{R}}.$$

Proof. Set $x = \nu(J(\alpha; \mu, \mathcal{R}))$, $x + \Delta x = \nu(J(\beta; \mu, \mathcal{R}))$, and hence $\Delta x = \nu(\alpha, \beta]^{\mu} \geqq 0$. Then by Lemma 12.14 and the mean value theorem,

$$\begin{aligned}(\varphi^{\mathcal{R}} v)(\alpha, \beta]^{\mu} &= (\varphi^{\mathcal{R}} v)(J(\beta; \mu, \mathcal{R})) - (\varphi^{\mathcal{R}} v)(J(\alpha; \mu, \mathcal{R})) \\ &= v(J(\beta; \mu, \mathcal{R})) - v(J(\alpha; \mu, \mathcal{R})) \\ &= f(x + \Delta x) - f(x)\end{aligned}$$

$$= \Delta x\, f'(x + \theta\Delta x)$$

$$= \Delta x\, f'(x) + \Delta x(f'(x + \theta\Delta x) - f'(x))$$

$$= \nu(\alpha, \beta]^{\mu} f'(\nu(J(\alpha; \mu, \mathcal{R})))$$

$$+ \nu(\alpha, \beta]^{\mu} (f'(x + \theta\nu(\alpha, \beta]^{\mu}) - f'(x)),$$

where $0 \leqq \theta \leqq 1$. The lemma now follows easily.

LEMMA 16.3. Let μ be such that $\nu \ll \mu$. Then for each $\epsilon > 0$, there is a $\delta > 0$ such that for $\mathcal{R}$, all α, β with $0 \leqq \alpha < \beta \leqq 1$ and $\beta - \alpha < \delta$, and all $T \subset (\alpha, \beta]^{\mu}_{\mathcal{R}}$, we have

$$|(\varphi^{\mathcal{R}}\nu)(T) - \nu(T)f'(\nu(J(\alpha; \mu, \mathcal{R})))| \leqq \epsilon\,\nu(\alpha, \beta]^{\mu}_{\mathcal{R}}.$$

Proof. We will prove a somewhat stronger statement, namely the one obtained from that in the lemma by replacing the right side by $\epsilon\,\nu(T)$. Choose $\delta < 1$ such that $\mu(S) < \delta$ implies $\nu(S) < \eta(\frac{\epsilon}{2})$; this is possible because $\nu \ll \mu$. Assume $0 < \beta - \alpha < \delta$.

First let T be of the form $(\gamma, \zeta]^{\mu}$. Since $T \subset (\alpha, \beta]^{\mu}$, it follows that $\alpha \leqq \gamma \leqq \zeta \leqq \beta$; hence

(16.4) $$0 \leqq \mu(\gamma, \zeta]^{\mu} = \zeta - \gamma \leqq \beta - \alpha < \delta,$$

and hence $\nu(\gamma, \zeta]^{\mu} < \eta(\frac{\epsilon}{2})$, and similarly $\nu(\alpha, \gamma]^{\mu} < \eta(\frac{\epsilon}{2})$. Therefore by Lemma 16.2,

$$|(\varphi^{\mathcal{R}}v)(T) - \nu(T)f'(\nu(J(\gamma; \mu, \mathcal{R})))| \leqq \frac{\epsilon}{2}\ \nu(T),$$

and by the definition of $\eta(\frac{\epsilon}{2})$,

$$|f'(\nu(J(\gamma; \mu, \mathcal{R}))) - f'(\nu(J(\alpha; \mu, \mathcal{R})))| \leqq \frac{\epsilon}{2} .$$

Combining these, we obtain

$$(16.5) \qquad |(\varphi^{\mathcal{R}}v)(T) - \nu(T)f'(\nu(J(\alpha; \mu, \mathcal{R})))| \leqq \epsilon\ \nu(T).$$

Now think of α and β as fixed but of T as varying over $\mathcal{C}$; denote by $\xi(T)$ what's inside the absolute value symbol in (16.5). From $v = f \circ \nu$ and the conditions on f it follows that $v \ll \nu$; hence also $\varphi^{\mathcal{R}}v \ll \nu$ (Proposition 12.7), and so $\xi \ll \nu$. What (16.5) says is that when T is an $\mathcal{R}$–interval included in $(\alpha, \beta]^{\mu}$, then $\xi(T) < \epsilon\ \nu(T)$ and $- \xi(T) < \epsilon\ \nu(T)$. But then the same inequalities follow for arbitrary finite unions of $\mathcal{R}$–intervals included in $(\alpha, \beta]^{\mu}$, and then, by a standard approximation theorem [H_1, p. 56, Theorem D] for all T in $G(\mu, \mathcal{R})$ (with $T \subset (\alpha, \beta]^{\mu}$, of course). Now if $T \subset (\alpha, \beta]^{\mu}$ is an arbitrary measurable set, we may find T' in $G(\mu, \mathcal{R})$ with $\mu(T' + T) = 0$, and we may suppose w.l.o.g. that $T' \subset (\alpha, \beta]^{\mu}$ (otherwise intersect it with $(\alpha, \beta]^{\mu}$). From $\xi \ll \nu \ll \mu$ it then follows that $\xi(T' + T) = \nu(T' + T) = 0$; hence $\xi(T') = \xi(T)$ and

$\nu(T') = \nu(T)$, and so (16.5) is proved for all measurable sets included in $(\alpha, \beta]^{\mu}$. This proves the lemma.

LEMMA 16.6. *For given* μ, *let* ξ *be a non-negative measure with* $\xi \ll \mu$, *and let* $\{\Theta_n\}$ *be a* μ*-mixing sequence. Then*

$$\xi(\Theta_n T) \to \mu(T)\, \xi(I)$$

as $n \to \infty$, *for all measurable* T.

Proof. The Radon–Nikodym theorem gives $\xi(T) = \int_T g(t)d\mu(t)$ for some $g \geqq 0$. Suppose first that g is a characteristic function, i.e., equal to 1 on some measurable U, 0 on $I \backslash U$. Then $\xi(\Theta_n T) = \mu(U \cap \Theta_n T)$, and by the μ-mixing property this approaches

$$\mu(T)\mu(U) = \mu(T) \int_I g(t)d\mu(t) = \mu(T)\xi(I),$$

so the result is proved for characteristic functions. Hence if g is a simple function (finite linear combination of characteristic functions), it follows as well. Next, let g be an arbitrary member of $L^1(I, \mathcal{C}, \mu)$, and for given ϵ, let h be a simple function such that

$$\int_I |g(t) - h(t)| d\mu(t) < \epsilon .$$

Then

$$|\xi(\Theta_n T) - \mu(T)\xi(I)| = \left| \int_{\Theta_n T} g(t)d\mu(t) - \mu(T) \int_I g(t)d\mu(t) \right|$$

$$\leqq \left| \int_{\Theta_n T} g(t)d\mu(t) - \int_{\Theta_n T} h(t)d\mu(t) \right|$$

$$+ \left| \int_{\Theta_n T} h(t)d\mu(t) - \mu(T) \int_I h(t)d\mu(t) \right|$$

$$+ \left| \mu(T) \int_I h(t)d\mu(t) - \mu(T) \int_I g(t)d\mu(t) \right|$$

$$\leqq \int_{\Theta_n T} |g(t) - h(t)| d\mu(t) + \left| \int_{\Theta_n T} h(t)d\mu(t) - \mu(T) \int_I h(t)d\mu(t) \right|$$

$$+ \mu(T) \int_I |g(t) - h(t)| d\mu(t)$$

$$< \left| \int_{\Theta_n T} h(t)d\mu(t) - \mu(T) \int_I h(t)d\mu(t) \right| + 2\epsilon .$$

Now, allowing $n \to \infty$, we find that the first term approaches 0, so the upper limit of $|\xi(\Theta_n T) - \mu(T)\xi(I)|$ must be less than 2ϵ; but since ϵ is arbitrary, this upper limit must vanish, and the lemma is proved.

COROLLARY 16.7. *Let* μ *be such that* $\nu \ll \mu$, *and let* $\{\Theta_n\}$ *be a* μ*-mixing sequence. Then*

$$\nu(S \cap \Theta_n T) \to \mu(T)\nu(S)$$

as $n \to \infty$, *for all measurable* S *and* T.

Proof. This follows immediately from Lemma 16.6 by setting $\xi(T) = \nu(S \cap T)$.

LEMMA 16.8. v has a mixing value, which equals ν.

Proof. Let μ be such that $\nu \ll \mu$. For given $\epsilon > 0$, pick a positive integer m so that $1/m$ is less than the δ provided by Lemma 16.3. Let $\{\Theta_n\}$ be a μ–mixing sequence, and $\mathcal{R}$ a measurable order. From $v \ll \nu \ll \mu$, Proposition 12.7, and Lemma 12.15, we obtain

$$\varphi(v, \Theta_n \mathcal{R})(J(0; \mu, \Theta_n \mathcal{R})) = \nu(J(0; \mu, \Theta_n \mathcal{R})) = 0.$$

Moreover, Lemma 12.16 and the fact that the Θ_n are μ–measure preserving yields

$$J(\alpha; \mu, \Theta_n \mathcal{R}) = \Theta_n J(\alpha; \mu, \mathcal{R})$$

and

$$(\alpha, \beta]^{\mu}_{\Theta_n \mathcal{R}} = \Theta_n (\alpha, \beta]^{\mu}_{\mathcal{R}}.$$

Hence by Lemma 16.3,

$$\varphi(v; \Theta_n \mathcal{R})(S) = \Sigma_{j=0}^{m-1} \varphi(v; \Theta_n \mathcal{R})(S \cap (\frac{j}{m}, \frac{j+1}{m}]^{\mu}_{\Theta_n \mathcal{R}})$$

$$= \Sigma_{j=0}^{m-1} \nu(S \cap \Theta_n (\frac{j}{m}, \frac{j+1}{m}]^{\mu}_{\mathcal{R}}) f'(\nu(\Theta_n J(\frac{j}{m}; \mu, \mathcal{R}))) + \Sigma_{j=0}^{m-1} \theta_j.$$

where $|\theta_j| < \epsilon\, \nu(\frac{j}{m}, \frac{j+1}{m}]^{\mu}_{\mathcal{R}}$, and hence

$$|\Sigma_{j=0}^{m-1} \theta_j| < \epsilon \nu(I) = \epsilon.$$

Let us for the moment ignore the error term $\Sigma_{j=0}^{m-1} \theta_j$, and allow $n \to \infty$ in the major term on the right side of the expression for $\varphi(v; \Theta_n \mathcal{R})(S)$. Then by Corollary 16.7 and the continuity of f', this major term approaches

$$\Sigma_{j=0}^{m-1} \mu(\frac{j}{m}, \frac{j+1}{m}]_{\mathcal{R}}^{\mu} \nu(S) f'(\mu(J(\frac{j}{m}; \mu, \mathcal{R}))) = \nu(S) \Sigma_{j=0}^{m-1} \frac{1}{m} f'(\frac{j}{m}),$$

which we shall call $\nu(S)\ \Sigma(m)$. It follows that both the upper and lower limits of $\varphi(v; \Theta_n \mathcal{R})(S)$ as $n \to \infty$ differ from $\nu(S)\ \Sigma(m)$ by ϵ at most. Now $\Sigma(m)$ is an approximating Riemann sum to the integral $\int_0^1 f'(x)dx$, which equals 1. Letting $m \to \infty$, we find that the upper and lower limits of $\varphi(v; \Theta_n \mathcal{R})(S)$ differ from $\nu(S)$ by less than ϵ. But since ϵ can be chosen arbitrarily small, both limits actually equal $\nu(S)$; so $\lim_{n \to \infty} \varphi(v; \Theta_n \mathcal{R})(S)$ exists and equals $\nu(S)$. If we now set $\mu_v = \nu$ in Proposition 14.1, we find that ν is the mixing value of v.

PROPOSITION 16.9. pNA $\subset$ MIX, and the unique value on pNA equals the mixing value there.

Proof. That pNA $\subset$ MIX follows from Lemma 16.8 and Proposition 15.4; the rest follows from Proposition 15.6 and the uniqueness of the value on pNA. This completes the proof.

Note that we have supplied an independent proof of the existence of a value on pNA (cf. Theorem A, Sec. 3). Mixing values can also be used to give an independent proof of the second sentence of Theorem B, by showing that the mixing value satisfies (3. 1). Indeed, when μ is one–dimensional, then the formula (3. 1) is equivalent to Lemma 16.8. When μ is multidimensional and f is a polynomial, then (3. 1) follows from the one–dimensional case by using Lemma 7.2, the linearity of the mixing value (Proposition 15. 6), and the linearity of the right side of (3. 1). In general, when f is continuously differentiable on the range of μ, then (3. 1) follows from the polynomial case by using the continuity of the mixing value in the variation norm (Corollary 15. 3), the continuity of the variation norm in the C^1–norm (7. 5), the continuity of the right side of (3. 1) in the C^1–norm, and the density of the polynomials in C^1 (Lemma 7. 4).

17. AN ALTERNATIVE PROOF FOR EXAMPLE 5.8.

In this section we will prove that the set function of Example 5.8 is not in MIX; this proves, a fortiori, that it is not in pNA, and it also proves that not every set function in AC is in MIX.

It is convenient to let the underlying space I be given by

$$I = \times_{i=-\infty}^{\infty} J_i,$$

where J_i is a copy of the 2–point space $\{0, 1\}$, and to let $\mathcal{C}$ be the standard product σ–field.

Define λ_i on J_i by

$$\lambda_i(\{0\}) = \lambda_i(\{1\}) = \tfrac{1}{2},$$

and let λ be the product measure of the λ_i on I. Let

$$I^0 = (\times_{i=-\infty}^{-1} J_i) \times \{0\} \times (\times_{i=1}^{\infty} J_i)$$

$$I^1 = (\times_{i=-\infty}^{-1} J_i) \times \{1\} \times (\times_{i=1}^{\infty} J_i);$$

then $I^0 \cap I^1 = \emptyset$, $I^0 \cup I^1 = I$. Define λ^0 and λ^1 by

$$\lambda^0(S) = \lambda(S \cap I^0)$$
$$\lambda^1(S) = \lambda(S \cap I^1),$$

let $\mu = 2\lambda^0 - 2\lambda^1$, and let $v(S) = |\mu(S)|$. It is easily verified that this situation is entirely isomorphic to that described in Example 5.8.

Now arrange the integers in the order $(0, 1, -1, 2, -2, \ldots)$. Let $\mathcal{R}$ be the lexicographic order on I w.r.t. the above order on the indices; that is, $x \mathcal{R} y$ if and only if $x_i = 1$ and $y_i = 0$ when i is the first index—in the above order—for which $x_i \neq y_i$. In the order $\mathcal{R}$, all of I^0 comes "before" all of I^1, and it follows easily that

$$\varphi(v; \mathcal{R}) = 2\lambda^0 - 2\lambda^1 = \mu ,$$

since this is so on the initial segments.

Next, let Θ be the "bilateral 1-shift" on I, i.e.,

$$(\Theta x)_i = x_{i+1}.$$

In the order $\Theta\mathcal{R}$, we first get half of I^0, then half of I^1, then another half of I^0, and finally another half of I^1. It follows that again

$$\varphi(v; \Theta\mathcal{R}) = 2\lambda^0 - 2\lambda^1 = \mu .$$

In the order $\Theta^n\mathcal{R}$, the space I is divided into 2^{2n} segments of equal λ-measure, in which the odd-numbered segments are contained in I^0 and the even-numbered segments in I^1. Thus we always have

$$\varphi(v; \Theta^n\mathcal{R}) = 2\lambda^0 - 2\lambda^1 = \mu$$

for the initial segments, and hence for all measurable sets. Since the powers of Θ form a λ-mixing sequence [H_2], and $v \ll \lambda$, it follows that the mixing value, if it exists, must be μ. But now if we start with the order $\mathcal{R}'$ that is the "reverse" of $\mathcal{R}$—i.e., $x \mathcal{R}' y$ if and only if $y \mathcal{R} x$—and again apply the sequence $\{\Theta, \Theta^2, \ldots\}$, then we find that

$$\varphi(v; \Theta^n\mathcal{R}') = 2\lambda^1 - 2\lambda^0 = -\mu ;$$

hence the mixing value, if it exists, must be $-\mu$. Since $\mu \neq 0$, this shows that the mixing value does not exist.

18. THE ASYMPTOTIC VALUE

In this section we shall construct a closed symmetric subspace of BV on which it is possible to define a value by means of approximation by finite games. This definition is essentially that of Kannai $[K_2]$,* and most of this section is meant less as an original contribution than as an indication as to how his work relates to that described up to now.

A partition Π of the underlying space is a finite family of disjoint subsets whose union is the whole space; a partition Π_2 is a refinement of another partition Π_1 if each member of Π_1 is a union of members of Π_2. A partition is called measurable if each of its members is measurable. A sequence $(\Pi_1, \Pi_2, \ldots)$ of partitions is said to be decreasing if Π_{i+1} is a refinement of Π_i for each i; and separating if for all s, $t \in I$ with $s \neq t$, there is an i such that s and t are in different members of Π_i. A decreasing separating sequence of measurable partitions is called an admissible sequence.

If u is a game with finitely many players, we will denote by φu the Shapley value of u $[S_1]$, considered as a measure on the set of players of u.

If v is a set function and Π a measurable partition of the underlying space, let v_Π be the finite game whose players are the members of Π, given by

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*There is a small difference due to the fact that $[K_2]$ works only with set functions that are absolutely continuous w.r.t. a given measure.

$$v_{\Pi}(\Xi) = v(\cup_{j \in \Xi} j)$$

for all $\Xi \subset \Pi$. Now let T be a measurable subset of I, and let $\mathscr{P} = (\Pi_1, \Pi_2, \ldots)$ be a decreasing sequence of measurable partitions whose first term Π_1 is the partition $\{T, I \setminus T\}$. For each i, let

$$T_i = \{j \in \Pi_i : j \subset T\};$$

T is a coalition in the original infinite game v, and T_i is the corresponding coalition in the corresponding finite game v_{Π_i}. If the numbers $(\varphi v_{\Pi_i})(T_i)$ approach a limit as $i \to \infty$, then this limit will be denoted $(\varphi_{\mathscr{P}} v)(T)$. If the limit exists for all admissible $\mathscr{P}$ starting with $\{T, I \setminus T\}$, and is independent of the choice of such $\mathscr{P}$, then we will denote it by $(\varphi v)(T)$. If that is the case for all measurable T, then we will call the set function φv the asymptotic value of T. Note that if φv exists, it is necessarily finitely additive. The set of all $v \in BV$ having an asymptotic value* will be denoted ASYMP.

For given v, the asymptotic value, if it exists, is clearly unique. The main theorem of this section is that the asymptotic value does in fact exist for each member of pNA, and that it coincides with the value on pNA. On the way to this theorem, we will prove that ASYMP is a closed symmetric linear subspace of BV, and that the operator φ that

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*There are also set functions outside BV that have asymptotic values, for example, the finite but unbounded finitely-additive "measures".

associates to each v its asymptotic value φv is a value on ASYMP. Our proof will be independent of the proof of Theorems A and B given in Part I, and indeed it will enable us to give yet another proof of the existence of a value on pNA and of the formula (3. 1).

Throughout the remainder of this section, when $v \in$ ASYMP, we will always use φv to denote the asymptotic value of v.

It will be useful to extend the notion of the variation norm to finite games; this is done in the obvious way. Clearly we then have

$$\|v_{\Pi}\| \leqq \|v\|$$

for any measurable partition Π. If u is a finite game with player space J, note that the value φu, as a measure on J, is itself a finite game. Then from the expression $[S_1]$ for the value of a finite game in terms of random orderings of the players, we obtain

$$\|\varphi u\| \leqq \|u\| .$$

Finally, note that

$$\| \varphi u \| = \Sigma_{j \in J} |(\varphi u)(\{j\})| ,$$

since φu is a measure on J.

<u>PROPOSITION 18. 1.</u> $\|\varphi v\| \leqq \|v\|$ <u>for all</u> $v \in$ ASYMP.

<u>Proof.</u> For a given $\epsilon > 0$, let Ω be a chain

$$\emptyset = S_0 \subset S_1 \subset \ldots \subset S_m = I$$

such that

$$\|\varphi v\|_{\Omega} \geqq \|\varphi v\| - \epsilon .$$

Let $U_j = S_j \setminus S_{j-1}$, let Π_1 be the partition $(U_1, \ldots, U_m)$, and let $\mathcal{P}$ be an admissible sequence $(\Pi_1, \Pi_2, \ldots)$ starting with this Π_1. For each $T \in \Pi_1$ and each i, let $T_i = \{ j \in \Pi_i : j \subset T \}$. Then

$$(18.2) \qquad \|\varphi v\| - \epsilon \leqq \|\varphi v\|_{\Omega} = \Sigma_{T \in \Pi_1} |(\varphi v)(T)|$$
$$= \Sigma_{T \in \Pi_1} |\lim_{i \to \infty} (\varphi v_{\Pi_i})(T_i)|$$
$$= \Sigma_{T \in \Pi_1} \lim_{i \to \infty} |(\varphi v_{\Pi_i})(T_i)|$$
$$= \lim_{i \to \infty} \Sigma_{T \in \Pi_1} |(\varphi v_{\Pi_i})(T_i)|,$$

since the sum is finite. Moreover, for fixed i, we have

$$\Sigma_{T \in \Pi_i} |(\varphi v_{\Pi_i})(T_i)| = \Sigma_{T \in \Pi_1} |\Sigma_{j \in T_i} (\varphi v_{\Pi_i})(\{j\})|$$
$$\leqq \Sigma_{T \in \Pi_1} \Sigma_{j \in T_i} |(\varphi v_{\Pi_i})(\{j\})|$$
$$= \Sigma_{j \in \Pi_i} |(\varphi v_{\Pi_i})(\{j\})|$$
$$= \|\varphi v_{\Pi_i}\| \leqq \|v_{\Pi_i}\| \leqq \|v\| .$$

Combining this with (18.2) we get $\|\varphi v\| - \epsilon \leqq \|v\|$, and by letting $\epsilon \to 0$ we complete the proof of Proposition 18.1.

<u>COROLLARY 18.3.</u> <u>If</u> $v \in$ ASYMP, <u>then</u> $\varphi v \in$ FA.

<u>Proof</u>. It is only necessary to show that φv is bounded, and this follows from Proposition 18.1.

PROPOSITION 18.4. ASYMP is a closed linear subspace of BV.

Proof. That ASYMP is a linear subspace is immediate. Suppose now that $v^m \in$ ASYMP and that $\|v^m - v\| \to 0$. Then by Proposition 18.1, φv^m is a Cauchy sequence, and hence has a limit ν in the variation norm; by Corollary 18.3 and Proposition 4.4, $\nu \in$ FA. For given $\epsilon > 0$, choose v^m so that both $\|v^m - v\| < \epsilon$ and $\|\varphi v^m - \nu\| < \epsilon$. Now let $\mathcal{P}$ be an admissible sequence $(\Pi_1, \Pi_2, \ldots)$ starting with $\{T, I\backslash T\}$, and for each i let $T_i = \{j \in \Pi_i : j \subset T\}$. Then

$$|(\varphi v_{\Pi_i})(T_i) - (\varphi v^m_{\Pi_i})(T_i)| \leqq \|\varphi(v_{\Pi_i} - v^m_{\Pi_i})\| \leqq \|v_{\Pi_i} - v^m_{\Pi_i}\|$$
$$\leqq \|v - v^m\| \leqq \epsilon\ ;$$

hence

$$\begin{aligned}|(\varphi v_{\Pi_i})(T_i) - \nu(T)| &\leqq |(\varphi v_{\Pi_i})(T_i) - (\varphi v^m_{\Pi_i})(T_i)| \\ &\quad + |(\varphi v^m_{\Pi_i})(T_i) - (\varphi v^m)(T)| \\ &\quad + |(\varphi v^m)(T) - \nu(T)| \\ &\leqq \epsilon + |(\varphi v^m_{\Pi_i})(T_i) - (\varphi v^m)(T)| + \|\varphi v^m - \nu\| \\ &\leqq 2\epsilon + |(\varphi v^m_{\Pi_i})(T_i) - (\varphi v^m)(T)|.\end{aligned}$$

Letting $i \to \infty$ and using $v^m \in$ ASYMP, we deduce that

$$\limsup_{i\to\infty} |(\varphi v_{\Pi_i})(T_i) - \nu(T)| \leqq 2\epsilon.$$

Hence the lim sup vanishes, hence the limit exists and vanishes, and hence ν is the asymptotic value of v. This completes the proof of Proposition 18.4.

PROPOSITION 18.5. ASYMP is symmetric, and φv is a value on ASYMP.

Proof. The linearity of φ is obvious. The normalization condition (2.3) follows easily from the corresponding condition for finite games $[S_1]$. Positivity is then a consequence of Proposition 18.1 and Proposition 4.6. The proof that ASYMP and φ are symmetric is straightforward; the details will be omitted. This completes the proof.

PROPOSITION 18.6. pNA $\subset$ ASYMP, and the unique value on pNA equals the asymptotic value there.

Proof. Once we have proved pNA $\subset$ ASYMP, the rest of the proposition is, of course, an immediate consequence of Proposition 18.5 and the uniqueness of the value on pNA. Thus it remains only to prove pNA $\subset$ ASYMP.

For $\nu \in NA^+$, let us define a ν–admissible sequence to be a decreasing sequence of measurable partitions that is ν–dense, i.e., such that every measurable set in I can be approximated in ν–measure by a member of the field $\mathfrak{F}(\mathcal{P})$ (not the σ–field) generated by the members of all the Π_i. Every admissible sequence $\mathscr{P}$ is then ν–admissible, for all ν. Indeed, $\mathfrak{F}(\mathscr{P})$ generates $\mathscr{C}$, since $\mathscr{P}$ is separating [M, p. 139, Theorem 3.3]; this implies the ν–denseness [H_1, p. 56, Theorem D].

Let μ be a vector of non–atomic probability measures, let $f \in C^1([0,1]^n)$, where n is the dimension of μ, and let $v = f \circ \mu$. Kannai [K_2] proved the following: If $\nu \in NA^+$ is such that $\mu_i \ll \nu$ for all i, and $\mathscr{P}$ is a ν–admissible sequence, then* for all coalitions T, $(\varphi_{\mathscr{P}} v)(T)$ exists and equals

$$(18.7) \qquad \Sigma_{i=1}^{n} \mu_i(T) \int_0^1 f_i(t\mu(I))dt.$$

Since an appropriate ν can always be found (e.g., $\nu = \Sigma_i \mu_i$) and since every admissible sequence is ν–admissible, it follows that $(\varphi_{\mathscr{P}} v)(T)$ exists for all T and all admissible $\mathscr{P}$, i.e., $v \in$ ASYMP; furthermore $(\varphi v)(T)$ is given by (18.7). Since in particular f can be taken to be a polynomial, and since ASYMP is closed (Proposition 18.4), it follows that pNA $\subset$ ASYMP. This completes the proof of Proposition 18.6.

Note that we have again supplied an independent proof of the existence of a value on pNA (cf. Theorem A, Sec. 3; also the end of Sec. 16). Asymptotic values can also be used to give an independent proof of the second sentence of Theorem B, by showing that the asymptotic

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*Kannai restricted himself to the case in which I = [0, 1] and $\nu = \lambda$, but this obviously implies the above formulation.

value satisfied (3. 1). Indeed, as in Sec. 7 we may assume that the range of μ has full dimension. Furthermore, we may assume that the μ_i are nonnegative, and in fact probability measures. Finally, by Whitney's theorem [W] we may extend f from the range of μ to $[0, 1]^n$, and we can then apply (18. 7); then (3. 1) follows at once.

19. THE DIAGONAL PROPERTY

Let Q be a symmetric subspace of BV, φ a value on Q. We shall say that the pair (Q, φ) enjoys the diagonal property if for all $v \in Q$ such that

(19.1) there is a positive integer k, a k-dimensional vector ζ of non-atomic probability measures, and a neighborhood U in E^k of the diagonal $[0, \zeta(I)]$ such that if $\zeta(S) \in U$ then $v(S) = 0$,

we have $\varphi v = 0$. Strictly speaking, the diagonal property is a property of the pair (Q, φ), but when no confusion can result we will speak of it as applying to Q or φ alone.

In this section we will prove that all the values we have defined up to now, in both Parts I and II, possess the diagonal property. This is a remarkable fact, because it shows that the value is completely determined by the behavior of v near any diagonal. The property was already hinted at in (3.3) and (7.7).

It is an open question whether or not the diagonal property follows from the definition of value, i.e., whether it holds for all pairs (Q, φ).

PROPOSITION 19.2. The unique value on bv'NA enjoys the diagonal property.

Proof. Apply Proposition 8.32 to deduce that if $v \in bv'NA$ satisfies (19.1), then $v \in pNA$. Then apply Proposition 7.6 with $f = 0$ to deduce that $\varphi v = 0$. This completes the proof of Proposition 19.2.

Let us define DIAG to be the set of all $v \in BV$ satisfying (19.1). Clearly, the diagonal property on (Q, φ) is equivalent to the condition that $\varphi v = 0$ for all $v \in Q \cap$ DIAG. Note that DIAG is a symmetric subspace of BV.

PROPOSITION 19.3. DIAG $\cap$ AC $\subset$ MIX, and the mixing value, regarded as a value on MIX, enjoys the diagonal property.

Proof. Let $v \in$ DIAG $\cap$ AC. Let k, ζ, and U correspond to v in accordance with (19.1), and let

$$\mu_v = \Sigma_{i=1}^{k} \zeta_i.$$

Let μ be a probability measure such that $\mu_v \ll \mu$, let $\{\Theta_1, \Theta_2, \ldots\}$ be a μ-mixing sequence, and let $\mathcal{R}$ be a measurable order.

Now pick a neighborhood V of the diagonal $[0, \zeta(I)]$ with $V \subset U$, and a number $\epsilon > 0$ such that an "ϵ-thickening" of V is still in U, i.e.,

(19.4) $\quad x \in V$ and $\|y - x\| < \epsilon \quad \Rightarrow \quad y \in U.$

Next, pick a number $\delta > 0$ such that for all $S \subset I$,

(19.5) $\quad \mu(S) < \delta \quad \Rightarrow \quad \|\zeta(S)\| < \epsilon;$

this is possible because $\zeta_i \ll \mu_v \ll \mu$ for each i. Finally, pick a finite sequence $s_1, \ldots, s_m$ in I such that

$$s_m \mathcal{R}\, s_{m-1} \mathcal{R} \ldots \mathcal{R}\, s_1$$

and for every s in I there is an s_j such that

$$\|\mu(I(s; \mathcal{R})) - \mu(I(s_j; \mathcal{R}))\| < \delta; \tag{19.6}$$

this is possible because μ is non–atomic (see also Lemmas 12.15 and 12.14).

For each j, apply Lemma 16.6 with $\xi = \zeta_i$ and $S = I(s_j; \mathcal{R})$, obtaining

$$\zeta(\Theta_n I(s_j; \mathcal{R})) \to \mu(I(s_j; \mathcal{R}))\, \zeta(I)$$

as $n \to \infty$. It follows that $\zeta(\Theta_n I(s_j; \mathcal{R})) \in V$ for all sufficiently large n, say for $n > n_0$; and since there are only finitely many j, we may choose n_0 independent of j (though not of $\mathcal{R}$ or of μ). But then for $n > n_0$, it follows from (19.4), (19.5), and (19.6) and the fact that Θ_n preserves μ–measure, that $\zeta(\Theta_n I(s; \mathcal{R})) \in U$ for all s. In particular, we may substitute $\Theta_n^{-1} s$ for s, and deduce that

$$\zeta(I(s; \Theta_n \mathcal{R})) = \zeta(\Theta_n I(\Theta_n^{-1} s; \mathcal{R})) \in U$$

whenever $n > n_0$. Applying (12.2), we deduce that for all s and all $n > n_0$,

$$\varphi(v; \Theta_n \mathcal{R})(I(s; \Theta_n \mathcal{R})) = v(I(s; \Theta_n \mathcal{R})) = 0.$$

But now for fixed $n > n_0$, $\varphi(v; \otimes_n \mathcal{R})$ is a measure (Proposition 12.7), and since it vanishes on each initial segment $I(s; \otimes_n \mathcal{R})$, it must, by (12.3), vanish identically. This completes the proof of Proposition 19.3.

PROPOSITION 19.7. DIAG $\subset$ ASYMP, and the asymptotic value, regarded as a value on ASYMP, enjoys the diagonal property.

Proof. Let $v \in$ DIAG, and let k, ζ, and U correspond to v in accordance with (19.1). Let V be an open neighborhood of the diagonal with $\bar{V} \subset U$. Choose a positive integer ℓ such that

$$(p-1)e/\ell \leqq y \leqq p\, e/\ell \quad \Rightarrow \quad y \in V$$

for all integers p with $1 \leqq p \leqq \ell$. Then there exist open neighborhoods $V_0, V_1, \ldots, V_{\ell-1}, V_\ell$ of $0, e/\ell, \ldots, (\ell-1)e/\ell$, e respectively such that

(19.8) $$x \in V_\ell,\ z \in V_{\ell+1},\ x \leqq y \leqq z \quad \Rightarrow \quad y \in U.$$

Now let $\mathcal{P}$ be an admissible sequence. Let $\Pi = \Pi_m$ in $\mathcal{P}$ have $n = n(m)$ members $A_1, \ldots, A_n$; the A_j are the players of the finite game v_Π. For each p with $1 \leqq p \leqq \ell$, choose a positive integer $s_p = s_p(m)$ so that

(19.9) $$s_p/n \to p/\ell$$

as $m \to \infty$. For a fixed m and p, let us now consider a finite probability space consisting of all the subsets S of Π having exactly s_p members, where every such S has the same probability $1/\binom{n}{s_p}$. For each $i = 1, \ldots, k$, define a random variable $X^{(i)}$ by

$$X^{(i)} = \Sigma_{j \in S}\, \alpha_j^i$$

where $\alpha_j^i = \zeta_i(A_j)$; then

$$E(X^{(i)}) = s_p/n.$$

Kannai [K_2, bottom of p. 57 ff.] proved* that if

(19.10) $$\max|\alpha_j^i| \to 0$$

as $m \to \infty$. Note that in our case condition (19.10) is satisfied, since $\wp$ is admissible, hence ζ_i–admissible for each i (see Sec. 8).

Now for a fixed m, let us consider the probability space of all orders $\mathcal{R}$ on the space Π of players A_j, in which all such orders are assigned the same probability $1/n!$. For each order $\mathcal{R}$ and each h with $1 \leqq h \leqq n$ we may define S(h) to be the set consisting of the first h elements of Π in the order $\mathcal{R}$. Then each subset S of Π containing exactly h members has the same probability $1/\binom{n}{h}$ of being the set S(h). Now, setting $h = s_p$, we deduce from (19.9) and (19.11) that for

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*Kannai has n − 1 and s − 1 where we have n and s_p.

given $\epsilon > 0$, if m is chosen sufficiently large, then

$$\text{Prob}\,\{\zeta(\cup S(s_p)) \in V_p \text{ for all } 0 \leqq p \leqq \ell\} > 1 - \epsilon;$$

here $\cup S(s_p)$ denotes the union of the members of $S(s_p)$ (recall that $S(s_p)$ is a subset of Π, i.e., a family of subsets of I). But by (19.8), it follows from

$$\zeta(\cup S(s_p)) \in V_p \text{ for all } 0 \leqq p \leqq \ell$$

that

$$\zeta(\cup S(h)) \in U \text{ for all } 1 \leqq h \leqq n;$$

and by the definition of DIAG, it then follows that

$$v(\cup S(h)) = 0 \text{ for all } 1 \leqq h \leqq n.$$

Now the variation of v over the chain

$$\emptyset = S(0) \subset S(1) \subset \ldots \subset S(n) = I$$

is at most $\|v\|$ for <u>any</u> order $\mathcal{R}$; and we have just seen that it is 0 with probability $> 1 - \epsilon$. Hence from the random-order definition of the value for finite games [S_1] it follows that for sufficiently large m,

$$\|\varphi v_{\Pi_m}\| \leqq \epsilon \|v\|.$$

From this it follows easily that the asymptotic value exists and vanishes identically, as was to be proved.

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