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Values of p-adic L-functions and a p-adic Poisson kernel

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Introduction

In [MTT], the author and his colleagues conjectured a precise relationship between the special values of p-adic L-functions associated to weight two elliptic modular forms of level pN and the p-adic periods of the fiber at p of the corresponding modular curve. Since this conjecture was closely related to a difference in the order of vanishing of a p-adic and classical L-function, we called it the "Exceptional Zero Conjecture." Numerical evidence for this conjecture was reported in [MTT] and [T], and a related, "refined", conjecture was studied in [MT].

Based on computational evidence (described in [MTT, Section 15]), we believed that it ought to be possible to make a similar, precise conjecture for forms of weight greater than two. At that time, we were unable to formulate such a conjecture.

In the first part of this work, we fill much, but not all of this gap, by making a precise conjecture (see Conjecture 1 below) for a large class of forms of even weight – namely, those forms which are lifts, by Jacquet-Langlands, from indefinite quaternion algebras. This case is accessible by means of the p-adic uniformization theory for Shimura curves due to Cerednik ([Cer]). For a discussion of how our new conjecture fits into the set of already existing ones, see the Remark at the end of Section 1 below.

The second portion of this work is devoted to describing some new results in the rigid analysis of the p-adic upper half plane $\mathscr{H}_p$. In particular, we produce an explicit inverse to a map studied by Drinfeld and Schneider. They attached a type of harmonic function (called a harmonic cocycle) on the tree $\mathscr{T}$ of $SL_2(\mathbb{Q}_p)$ to a rigid analytic modular form f on $\mathscr{H}_p$; our map recovers the rigid function f from this harmonic cocycle as an integral of a certain measure on the limit set $\mathbb{P}^1(\mathbb{Q}_p)$ of $\mathscr{H}_p$. We call this integral the p-adic Poisson kernel, since it constructs an analytic function on $\mathscr{H}_p$ having prescribed boundary values.

The final section of the paper applies the p-adic Poisson kernel to the computation of the p-adic periods which enter into our generalized Exceptional Zero

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Conjecture. To the many weight two calculations described in [MTT], [T], and [MT], we add a calculation which shows that the conjecture holds mod p^5 for the prime $p = 3$ and the modular form f of weight four and level six.

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Preliminaries

Let B be a definite quaternion algebra over $\mathbb{Q}$ of discriminant N and let p be a prime which splits B. Fix an isomorphism

$$\iota : B \otimes \mathbb{Q}_p \to M_2(\mathbb{Q}_p) .$$

Let $\mathcal{O}$ be a maximal $\mathbb{Z}[\frac{1}{p}]$-order in B (all such are conjugate by the strong approximation theorem) and let Γ be a congruence subgroup of $\mathcal{O}^\times$. The map ι gives us a two dimensional, p-adic representation of Γ. We assume further that $\iota(\Gamma) \subset SL_2(\mathbb{Q}_p)$, and we will identify Γ with its image under ι. Fix a coordinate z on $\mathbb{P}^1$ so that $SL_2(\mathbb{Q}_p)$ acts on $\mathbb{P}^1$ through linear fractional transformations in z. Let $\mathscr{H}_p = \mathbb{C}_p - \mathbb{Q}_p$ be the p-adic upper half plane, viewed as a rigid analytic subspace of $\mathbb{P}^1$, and let $\mathscr{T}$ be the tree of $\mathrm{SL}_2(\mathbb{Q}_p)$. Fix a reduction map from $\mathscr{H}_p$ to $\mathscr{T}$ compatible with the various group actions. For a detailed discussion of these constructions, see [G], [D2], or [M].

Definition 1. Let M be an abelian group. An M-valued function c on the edges of $\mathscr{T}$ is called a *harmonic cocycle* if, for all vertices v of $\mathscr{T}$,

$$\sum_{e \mapsto v} c(e) = 0$$

where the sum is over the oriented edges e of $\mathscr{T}$ meeting v.

We will be interested in a particular coefficient module. Let $P_k(E)$ be the $k + 1$ dimensional vector space of degree k polynomials in T over a field E. We abbreviate $P_k(\mathbb{C}_p)$ by P_k. Let $SL_2(\mathbb{Q}_p)$ act on $f(T) \in P_k(E)$ on the left by the rule

$$\gamma \cdot f(T) = (bT + d)^k f\left(\frac{aT + c}{bT + d}\right) .$$

Definition 2. Let $C_{\mathrm{har}}(\Gamma, k)$ be the set of harmonic cocycles c with values in $P_{k-2}(\mathbb{C}_p)$ satisfying

$$c(\gamma e)(T) = \gamma \cdot c(e)(T) .$$

Let us introduce modular forms into our discussion.

Definition 3. Let $S_k(\Gamma)$ be the space of rigid analytic modular forms on $\mathscr{H}_p$ of weight k for Γ over $\mathbb{C}_p$. Thus,

$$S_k(\Gamma) = \{f : f \text{ rigid analytic on } \mathscr{H}_p \text{ and } f(\gamma z) = (cz + d)^k f(z)\} .$$

Here z is our fixed parameter on $\mathscr{H}_p$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL_2(\mathbb{Q}_p) .$$

We will be considering only modular forms of even weight in this paper. The forms of weight $2k$ are naturally identified with global sections of the sheaf of k-differentials on the curve $\mathscr{H}_p/\Gamma$. By Cerednik's theorem ([Cer] and [D2]), this curve is (the fiber at p of) a Shimura curve. Therefore the space $S_k(\Gamma)$ can be identified with a corresponding space of classical modular forms coming from the indefinite quaternion algebra of discriminant Np. In particular, we know that such forms exist.

Associated to a modular form $f \in S_k(\Gamma)$ is a harmonic cocycle, defined by Drinfeld and Schneider.

Definition 4. If $f \in S_k(\Gamma)$ we define (following [S])

$$c_f(e) = \sum_{i=0}^{k-2} \operatorname{Res}_e(z^i f\,dz)\binom{k-2}{i} T^i$$

where Res denotes the p-adic annular residue. The modular property of f, together with the p-adic residue theorem, imply that $c_f \in C_{\mathrm{har}}(\Gamma, k)$.

In fact, the harmonic cocycle c_f determines the form f. We supply our own proof of this in Corollary 11 below, but the result is due to Drinfeld and Schneider.

1. p-adic periods and L-functions

Construction of an $\mathscr{L}$ invariant for rigid analytic modular forms. In this section we will define a "period" associated to a modular form f for Γ. This period generalizes the $\mathscr{L}$ invariant for weight two modular forms defined in [MTT]. It is constructed as the "ratio" of two maps from $S_k(\Gamma)$ to $H^1(\Gamma, P_{k-2})$. The first of these maps, which was introduced by Schneider, is derived from the tree $\mathscr{T}$.

Definition 5. For $f \in S_k(\Gamma)$, let as before c_f be the associated element of $C_{\mathrm{har}}(\Gamma, k)$. Let v be a vertex of $\mathscr{T}$. Define

$$\psi_f^v(\gamma) = \sum_{v \to \gamma v} c_f(e) ,$$

where the sum is over the edges in $\mathscr{T}$ joining v to γv.

It is simple to verify that ψ_f^v is a 1-cocycle for Γ with coefficients in P_{k-2}, and that the class ψ_f of ψ_f^v is independent of v. Thus one obtains a map $\psi\colon S_k(\Gamma) \to H^1(\Gamma, P_{k-2})$.

Theorem 1 ([S], [SS], [dS]). *The map ψ is an isomorphism.*

For the second map, which was studied extensively in [dS], we exploit Coleman's p-adic integral. ([Cl])

Definition 6. Fix the branch $\log_p$ of the p-adic logarithm such that $\log_p p = 0$. Let Q be a point in $\mathscr{H}_p$, and let $f \in S_k(\Gamma)$. Define

$$\lambda_f^Q(\gamma) = \sum_{i=0}^{k-2} \left(\int_Q^{\gamma Q} z^i f\, dz \right) \binom{k-2}{i} T^i$$

where the integral is the branch of Coleman's p-adic integral associated to this choice of logarithm.

This map, too, is a cocycle; we summarize its properties in the following lemma.

Lemma 7. *λ_f^Q is a 1-cocycle for Γ, taking values in P_{k-2}. It depends on Q only up to a coboundary, and therefore defines an element λ_f of $H^1(\Gamma, P_{k-2})$.*

Proof. These properties are formal consequences of the properties of the integral. □

Therefore we have constructed a second map λ: $S_k(\Gamma) \to H^1(\Gamma, P_{k-2})$. The Hecke algebra **T** for Γ acts naturally on the spaces $C_{\text{har}}(\Gamma, k)$ and $H^1(\Gamma, P_{k-2})$. Since this action is through the action of $GL_2(\mathbb{Q}_p)$, it commutes with each of the maps ψ and λ.

Theorem 2. *Let ψ^{new} and λ^{new} denote the restrictions of ψ and λ to the new part of $S_k(\Gamma)$. Then there exists a unique element $\mathscr{L} \in \mathbf{T} \otimes \mathbb{Q}_p$ such that*

$$\lambda^{\text{new}} = \mathscr{L} \psi^{\text{new}} .$$

$\mathscr{L}$ is independent of the choice of imbedding ι: $B \to M_2(\mathbb{Q}_p)$.

Proof. By Cerednik's interchange of invariants theorem (see [Cer]), $S_k(\Gamma)$ is isomorphic as a **T**-module to a space of modular forms for the indefinite quaternion algebra with discriminant Np. Therefore the new part $S_k(\Gamma)^{\text{new}}$ of $S_k(\Gamma)$ satisfies the multiplicity one theorem, as do the corresponding spaces $C_{\text{har}}(\Gamma, k)^{\text{new}}$ and $H^1(\Gamma, P_{k-2})^{\text{new}}$. This implies that all of these modules have rank one over $\mathbf{T} \otimes \mathbb{Q}_p$. Since ψ is an isomorphism (see Corollary 11 below), a Hecke operator $\mathscr{L}$ with the stated property must exist. If ι is replaced by a different ι', then we know that $\iota' = \alpha \iota \alpha^{-1}$ for some $\alpha \in GL_2(\mathbb{Q}_p)$. Tracing through the definitions, we see that the cohomology classes ψ and λ are replaced by $\alpha \cdot \psi$ and $\alpha \cdot \lambda$. As a result, the invariant $\mathscr{L}$ is unchanged. □

The invariant $\mathscr{L}$ we have constructed can be derived from the p-adic period matrix of the Jacobian of $\mathscr{H}_p/\Gamma$ when f has weight two. Suppose that in this case we let X be the sublattice of $S_2(\Gamma)^{\text{new}}$ consisting of new forms f such that c_f has integer coefficients. It then follows from the theory of Drinfeld-Manin ([DM]) that if $f \in X$ then the differential form $\omega_f = f(z)\, dz$ on $\mathscr{H}_p$ is logarithmic; that is, there is a function $u(f)$ on $\mathscr{H}_p$ such that

$$\omega_f = \frac{du(f)}{u(f)}.$$

The pairing

$$X \times \Gamma^{ab} \to \mathbb{Q}_p^*$$

defined by $[\omega_f, \gamma] = u(f)(\gamma z)/u(f)(z)$ then leads, via the process described in

[MTT, Section 11], to the p-adic uniformization of the new part of the Jacobian of $\mathscr{H}_p/\Gamma$. If we let $A(f)$ be the factor of $\mathrm{Jac}(\mathscr{H}_p/\Gamma)$ cut out by the Hecke orbit of f and restrict the pairing $[\cdot,\cdot]$ to the corresponding sublattices of X and Γ^{ab}, then we obtain from [M] the formulas:

$$\psi_f(\gamma) = \mathrm{ord}_p[\omega_f, \gamma]$$
$$\lambda_f(\gamma) = \log_p[\omega_f, \gamma] .$$

Comparing with [MTT, Section 11], we see that ψ and λ are the maps denoted there by α and β respectively, and that the $\mathscr{L}$ we define in this paper is formally the same as that defined in [MTT]. This invariant has been computed in various situations, as described in [MTT] and [T]. In the simplest case of all, where $\mathrm{Jac}(\mathscr{H}_p/\Gamma) = A(f)$ is a Tate elliptic curve, we have

$$\mathscr{L} = \frac{\log_p(q)}{\mathrm{ord}_p(q)} \cdot 1_T$$

where q is the p-adic period of $A(f)$.

In [dS], de Shalit constructs abelian varieties associated to higher weight rigid analytic modular forms, and shows that the higher weight invariant $\mathscr{L}$ can be derived from these abelian varieties in an analogous way.

The exceptional zero conjecture. Suppose now that $f \in S_k(\Gamma)$ is an eigenform for $\mathbf{T}$. We attach a p-adic L-function to f by a rather circuitous procedure. It follows from the theory of interchange of invariants (see [Ce] and [D2]) that the newforms which arise in some $S_k(\Gamma)$ are precisely the newforms for the indefinite quaternion algebra with discriminant Np. Therefore, we interchange invariants to obtain a complex analytic automorphic form f_∞ on this indefinite quaternion algebra which has the same Hecke eigenvalues as f. This form then lifts, via the Jacquet-Langlands theory ([JL]), to an elliptic modular form f_E of some level M, where $p \parallel M$.

Lemma 8. *Let $a_p(f)$ be the coefficient of q^p in the Fourier expansion of f_E. Then* $\mathrm{ord}_p(a_p(f)) = (k-2)/2$, *and, in the terminology of* [MTT], *$a_p(f)$ is the unique "allowable p-root" for f_E. There is a unique p-adic L-function $L_p(f_E, a_p(f), \chi, s)$ associated to f_E, and this function L_p is exceptional at the central point $j = (k-2)/2$.*

Proof. This follows from [MTT, p. 16] and [MTT, p. 21]. □

Let χ be a primitive Dirichlet character of conductor m, and let $\Lambda^{\mathrm{sign}(x)}(f_E, \chi, n)$ be the special value

$$\Lambda^{\pm}(f_E, \chi, n) = \frac{n!}{(-2\pi i)^n} \frac{m^{n+1}}{\tau(\bar{\chi})} L(f_{\bar{\chi}}, \cap + 1)$$

of the classical L-function associated to f twisted by χ (See [MTT, Section 8].) Both L_p and Λ take values in the sub-lattice of $\mathbb{C}$ generated by the modular symbols for f_E, tensored with $\mathbb{Q}_p$. This is a finitely generated $\mathbb{Z}$ module on which $\mathbf{T}$ acts. Therefore $\mathscr{L}$ acts naturally on Λ and L_p. We make the following conjecture, extending the "Exceptional Zero Conjecture" of [MTT, Section 14] to modular forms of even weight which are lifts from quaternion algebras.

Conjecture 1. *Let $f \in S_k(\Gamma)$ be a newform for Γ, and f_E an elliptic form obtained from f by interchange of invariants followed by the Jacquet-Langlands lift. Let χ be a primitive quadratic Dirichlet character such that $e_p(a_p(f), (k-2)/2, \chi) = 0$, where e_p is the p-adic multiplier discussed in* [MTT]. *Then*

$$\frac{d}{ds} L_p(f_E, a_p, \chi x_p^{(k-2)/2}, s)|_{s=0} = \mathscr{L} \Lambda^{\pm}(f_E, \chi, (k-2)/2) .$$

We point out that f and f_E are each only defined up to non-zero constant multiples (by respectively elements of $\mathbb{C}_p$ and $\mathbb{C}$!) but that this dependence is cancelled out by the form of the conjecture. Also, there may be some non-quadratic characters χ such that $e_p(a_p(f), (k-2)/2, \chi)$ vanishes; quite possibly the conjectured equality still holds for these but since we have no numerical evidence one way or the other we are reluctant to say anything about them.

Remark. Conjecture 1 is consistent with the conjectures of [MTT] and [MT]. Indeed, as we have seen, the invariant $\mathscr{L}$ for forms of weight two depends on the p-adic period lattice of the abelian variety $A(f)$ cut out by f from the Jacobian of a Shimura curve. Ribet proved (see [R]) that, when Γ is the principal congruence subgroup, then $A(f)$ is isogenous to the abelian variety $A(f_E)$ cut out of the Jacobian of a modular curve by f_E. The restriction to the principal congruence subgroup can almost certainly be removed by appealing to the theory of Jacquet-Langlands and Faltings' Isogeny Theorem. Therefore the $\mathscr{L}$ we compute in this paper is the same as that computed in [MTT], and so our conjecture implies the weight two conjecture of [MTT] for those forms which come from quaternion algebras. In addition, our conjecture implies that the p-adic L-functions associated to higher weight modular forms coming from quaternion algebras are of "local type." Ironically, although considerable numerical evidence has been amassed for the weight two conjectures, as reported in [MTT], [MT] and [T], none of this evidence applies directly to our Conjecture 1 since all of the forms considered in earlier calculations had prime level. Most significantly, in [MTT] we were led to ask for the existence of an $\mathscr{L}$ invariant for modular forms of higher weight based on calculations for the forms of weight four and level five, and of weight six and level three which suggested that the L-functions for these forms had exceptional zeros of local type. Although this paper sheds no light on these forms, Coleman, in [C2], has constructed an invariant for elliptic modular forms of arbitrary weight which is probably the same as our $\mathscr{L}$ when it is defined, and which could well fill the role of $\mathscr{L}$-invariant in the remaining cases not covered by our construction.

2. A p-adic Poisson kernel

Construction of the kernel. We turn in this section to analysis on the p-adic upper half plane. We will describe a technique for explicitly constructing rigid analytic modular forms, and show how this technique can be applied to obtain a formula for the map λ entering into the construction of $\mathscr{L}$.

As is well known, there is a natural correspondence between the oriented edges e of $\mathscr{T}$ and compact open subsets U of the ends of $\mathscr{T}$. Drinfeld (in weight 2, see

[D1]) and Schneider (in higher weight, see [S]) have shown how to obtain a p-adic distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ from a harmonic cocycle with values in P_{k-2}. We will extend Schneider's construction. In the following, we denote with x the restriction of the fixed parameter z to the set $\mathbb{P}^1(\mathbb{Q}_p)$, so that x identifies $\mathbb{P}^1(\mathbb{Q}_p)$ with $\mathbb{Q}_p \cup \{\infty\}$.

Let A_k denote the set of $\mathbb{C}_p$-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic except for a pole at ∞ of order at most $k-2$. Let A_{cs} be the set of compactly supported functions on $\mathbb{Q}_p$. Choose $c \in C_{\mathrm{har}}(\Gamma, k)$, and write

$$c(e) = \Sigma\, c_i(e) \binom{k-2}{i} T^i .$$

Let $\tilde{\mu}$ be the distribution associated to c as defined by Schneider in [S]. Recall that this integral is defined for elements $F \in A_{cs}$ and compact open subsets $U \in \mathbb{Q}_p$. If $U(e)$ is the compact open in $\mathbb{Q}_p$ corresponding to the edge e of $\mathscr{T}$, then the fundamental relation defining $\tilde{\mu}$ is

$$\int_{U(e)} x^i \, d\tilde{\mu} = c_i(e) \quad i = 0, \ldots, k-2 ,$$

and $\tilde{\mu}$ is extended to A_{cs} by the methods of Vishik and Amice-Velu. We extend the definition of $\tilde{\mu}$ so that a larger class of functions can be integrated against it. The statement and proof of the following proposition were copied, with only slight changes, from the statement and proof of the theorem of Amice-Velu and Vishik given in [MTT, p. 13].

Proposition 9. *Let $c \in C_{\mathrm{har}}(\Gamma, k)$ and let $\tilde{\mu}$ be the associated measure. Then there exists a unique extension μ of $\tilde{\mu}$ to A_k characterized by the following properties:*

(1) *$\int_U F\, d\mu = \int_U F\, d\tilde{\mu}$ if $U \subset \mathbb{Q}_p$.*

(2) *$\int_U F\, d\mu$ is finitely additive in U and linear in F for all $F \in A_k$ and $U \subset \mathbb{P}^1(\mathbb{Q}_p)$.*

(3) *c determines the integrals of polynomials of low degree:*

$$\int_{U(e)} x^i \, d\mu = c_i(e) \quad i = 0, \ldots, k-2 .$$

(4) *There exists a constant C such that if $\infty \in U(e)$, $0 \notin U(e)$, and $n \leqq k-2$, we have*

$$\int_{U(e)} x^n d\mu \leqq C\rho^{-n+(k-2)/2}$$

while if $a \in U(e) \subset \mathbb{Q}_p$ and $n \geqq 0$ then

$$\int_{U(e)} (x-a)^n d\mu \leqq C\rho^{n-(k-2)/2} .$$

Here $\rho = \sup_{u \in U(e)} |1/u|$ if $\infty \in U(e)$ and $\rho = \sup_{u, v \in U(e)} |u - v|$ if $\infty \notin U(e)$.

(5) *Suppose $\infty \in U(e)$ and that $F(x) = \sum_{n=k-2}^{-\infty} a_n x^n$ is a Laurent expansion for F which converges on the set $U(e) - \{\infty\}$. Then*

$$\int_{U(e)} F d\mu = \Sigma\, a_n \int_{U(e)} x^n d\mu .$$

If $\infty \notin U(e)$, λ *is a center for* $U(e)$, *and* $F(x) = \sum_{n=0}^{\infty} a_n(x-\lambda)^n$ *is a convergent Taylor series for* F *on* $U(e)$, *then*

$$\int_{U(e)} F d\mu = \Sigma a_n \int_{U(e)} (x-\lambda)^n d\mu .$$

Proof. The proof of this proposition is a slightly modified form of the proof of the theorem of Amice-Velu and Vishik given in [MTT, p. 13]. When $\infty \notin U$, the construction is exactly as in [S]. The key point in the extension is to show that the estimate in (4) holds in our more general setting. If $\infty \notin U(e)$, this is essentially done in [S, p. 228]. Suppose therefore that $\infty \in U(e)$. Let $\bar{U}$ be the complement of $U(e)$ in $\mathbb{P}^1(\mathbb{Q}_p)$. Then, by [S, op. cit.], we have

$$\begin{aligned} \int_{U(e)} x^i d\mu &= -\int_{\bar{U}} x^i d\tilde{\mu}, \\ &= -c_i(e), \\ &\leqq C(1/\rho)^{i-(k-2)/2}, \\ &\leqq C\rho^{-i+(k-2)/2} \end{aligned}$$

where we have used the fact that the radius of $\bar{U}$ is $1/\rho$. Once this estimate is obtained, the remainder of the existence proof follows [MTT]. □

Once existence of the integral is proven, the following lemma is an easy consequence.

Lemma 10. *Let* μ *be the measure associated to* $c \in C_{\text{har}}(\Gamma, k)$ *on* $P^1(\mathbb{Q}_p)$. *It has the properties*:

(1) *For* $\gamma \in \Gamma$ *we have* $\gamma^* \mu = (cx+d)^{k-2} \mu$.

(2) *For* $i = 0, \ldots, k-2$ *we have*

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} x^i d\mu(x) = 0 .$$

Proof. The first property follows from the Γ-equivariance of c, while the second follows from property (2) of μ and the fact that c is harmonic. □

Theorem 3. *Let* $c \in C_{\text{har}}(\Gamma, k)$ *and let* μ *be the corresponding measure. Then*

$$F(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{(z-x)} d\mu(x)$$

is a rigid analytic modular form of weight k, *and* $c_F = c$.

This theorem provides an explicit inverse, by means of a p-adic integral, to the map from rigid analytic modular forms to harmonic cocycles constructed by Drinfeld ([D1]) and Schneider ([S]). We call $1/(z-x)$ the Poisson kernel for the p-adic upper half plane, since it constructs from the boundary measure μ a rigid analytic function on $\mathscr{H}_p$ with μ for its "boundary values."

Proof of Theorem 3. It is clear that $1/(z-x)$ belongs to A_k for all $k \geqq 0$, and therefore we know from Proposition 9 that, for any $z \in \mathbb{C}_p - \mathbb{Q}_p$, the function $1/(z-x)$ on $\mathbb{P}^1(\mathbb{Q}_p)$ can be integrated against $d\mu$. We conclude that F is a well defined function of z. A simple calculation shows that, for $\gamma \in SL_2(\mathbb{Q}_p)$,

$$\frac{(cx+d)^{k-2}}{\gamma z - \gamma x} = \frac{(cz+d)^k}{z-x} + H_{k-2}(x)$$

where $H_{k-2}(x)$ is a polynomial in x of degree at most $k-2$. From the definition of F and Lemma 10(2), we obtain the modular relation for F:

$$F(\gamma z) = (cz+d)^k F(z) .$$

To show that F is rigid analytic, let A be a connected affinoid domain in $\mathscr{H}_p$. We may assume that A is the complement in $\mathbb{P}^1$ of finitely many open discs $B_1, \ldots, B_m$ with $\mathbb{Q}_p$-rational centers $a_1, \ldots, a_m$ and radii $\rho_1, \ldots, \rho_m$. Furthermore, each open disc B_i meets the limit set $\mathbb{P}^1(\mathbb{Q}_p)$ in a compact open set U_i, and the U_i form a covering of $\mathbb{P}^1(\mathbb{Q}_p)$. Therefore it suffices to consider

$$F_i(z) = \int_{U_i} \frac{1}{z-x}\, d\mu(x) .$$

Expand $1/(z-x)$ at a_i (making suitable adjustments if $a_i = \infty$) and apply part (5) of Proposition 9 to obtain

$$F_i(z) = \sum_{n=0}^{\infty} \frac{1}{(z-a_i)^{n+1}} \int_{U_i} (x-a_i)^n d\mu(x).$$

Part (4) of the proposition shows that this series expansion for F_i converges uniformly on the complement of B_i. Since $F = \Sigma F_i$, we have shown that F is rigid analytic on A. Since affinoids like A cover $\mathscr{H}_p$, F is globally rigid analytic. Even more, inspecting the series expansion for F and recalling the definition of the annular residue shows at once that, if $e(i)$ is the edge of $\mathscr{T}$ corresponding to the boundary of the open ball B_i, then

$$\mathrm{Res}_{e(i)}(z-a_i)^n F\, dz = \int_{U_i} (x-a_i)^n d\mu(x)$$

and applying the definitions of c_F and μ we see immediately that $c_F = c$. □

We have already made use of the following result. Originally proved by Drinfeld ([D1]) and Schneider ([S]) in the weight two and higher weight cases respectively, we include this proof to illustrate Theorem 3.

Corollary 11. *The map* $f \mapsto c_f : S_k(\Gamma) \to C_{\mathrm{har}}(\Gamma, k)$ *is an isomorphism.*

Proof. The case $k = 2$ has been well studied and was first obtained by Drinfeld-Manin ([DM].) Therefore we assume $k > 2$. It suffices to consider the case where Γ is free, since the general Γ has a free subgroup of finite index, and one may descend from the free case to the general case. By the Riemann-Roch theorem, the space of modular forms of weight k in such a case has dimension $(k-1)(g-1)$ where g is the genus of $\mathscr{H}_p/\Gamma$. Since $f \mapsto c_f$ is surjective by Theorem 3, we must have the inequality

$$\dim C_{\mathrm{har}}(\Gamma, k) \leqq (k-1)(g-1) .$$

On the other hand, an element c of $C_{\mathrm{har}}(\Gamma, k)$ is determined by specifying $k-1$

numbers for each element of a Γ-fundamental set of edges of $\mathcal{T}$. Requiring that c be harmonic means that each vertex in a fundamental set of vertices imposes $k-1$ linear conditions on c. We conclude that

$$\dim C_{\mathrm{har}}(\Gamma, k) \geqq (k-1)(E-V)$$

where E and V are respectively the number of edges and vertices in $\mathcal{T}/\Gamma$. Notice that the integer $V-E$ is the Euler characteristic of $\mathcal{T}/\Gamma$, which is $1-g$. Since the two inequalities force equality, we conclude that $\dim C_{\mathrm{har}}(\Gamma, k) = \dim S_k$, and therefore $f \mapsto c_f$ is an isomorphism. □

An integral formula for p-adic periods. We apply Theorem 3 to obtain a new formula for the Coleman integrals which enter into the definition of the map λ_f defined in Section 1. The formula we prove is quite simple, and interesting for aesthetic reasons; however, its main application comes in computing the invariant $\mathcal{L}$. We will discuss this in detail later.

Theorem 4. *Let $f \in S_k(\Gamma)$ and let μ be the associated measure. Let $Q \in \mathbb{C}_p - \mathbb{Q}_p$ be a point in $\mathcal{H}_p$. Then*

$$\int_Q^{\gamma Q} z^i f\,dz = \int_{\mathbb{P}^1(\mathbb{Q}_p)} x^i \log_p \frac{x-\gamma Q}{x-Q}\,d\mu(x)$$

for $0 \leqq i \leqq k-2$. Here the integral on the left is the Coleman integral, while that on the right is the integral associated to μ.

Proof. We compute formally, using Proposition 9, and interchanging the order of integration:

$$\begin{aligned}\int_Q^{\gamma Q} z^i f\,dz &= \int_Q^{\gamma Q} \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{z^i}{z-x}\,d\mu(x)\,dz\\ &= \int_Q^{\gamma Q} \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{x^i}{z-x}\,d\mu(x)\,dz\\ &= \int_{\mathbb{P}^1(\mathbb{Q}_p)} \int_Q^{\gamma Q} \frac{x^i}{z-x}\,dz\,d\mu(x)\\ &= \int_{\mathbb{P}^1(\mathbb{Q}_p)} x^i \log_p \frac{x-\gamma Q}{x-Q}\,d\mu(x)\,.\end{aligned}$$

We need only justify the interchange of the two integrations; for simplicity we assume $i=0$, although the argument is the same in any case.

Choose an affinoid subregion A of $\mathcal{H}_p$ so that Q and γQ belong to the interior of A – that is, neither of them lies on one of the boundary annuli of A. Let B_i, $i = 1, \ldots, n$, be the bounding discs of A, with centers a_i. As in the proof of Theorem 3, write $f = \Sigma f_i$ where each f_i converges on the complement of B_i. For each f_i, we have

$$f_i = \sum_{n=1}^{\infty} \frac{c_n}{(z-a_i)^n}$$

where

$$c_n = \int_{B_i \cap \mathbb{P}^1(\mathbb{Q}_p)} (x - a_i)^{n-1} \, d\mu(x) .$$

The indefinite Coleman integral of f_i, on the interior of A, is just the formal integral, so

$$\int f_i \, dz = c_1 \log_p(z - a_i) + \sum_{n=2}^{\infty} \frac{1}{1-n} \frac{c_n}{(z - a_i)^{n-1}} .$$

On the other hand, we may expand $\log_p(z - x)$ in a Taylor series at $x = a_i$ to obtain

$$\int_{B_i \cap \mathbb{P}^1(\mathbb{Q}_p)} \log_p(z - x) d\mu(x) = \int_{B_i \cap \mathbb{P}^1(\mathbb{Q}_p)} \left[\log_p(z - a_i) + \sum_{n=2}^{\infty} \frac{1}{1-n} \left(\frac{x - a_i}{z - a_i} \right)^{n-1} \right] d\mu(x).$$

However, one of the properties of $d\mu$ is that we may move integrals past sums of this form (see Proposition 9 above). Therefore the integrals may be interchanged. □

3. Evidence for the exceptional zero conjecture

We have chosen to test Conjecture 1 for the prime $p = 3$ and the modular form

$$\begin{aligned} f(z) &= (\eta(z)\eta(2z)\eta(3z)\eta(6z))^2 \\ &= q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 + \ldots , \end{aligned}$$

which has weight four and level six. A number of characteristics make the form f worth studying:

(1) It has weight greater than two, and therefore provides information about the new conjecture.

(2) It is of low level, so the modular symbols calculations for f are relatively simple.

(3) It has rational Fourier coefficients, which also simplifies the computations.

(4) It is a lift from a quaternion algebra (since $6 = 2 \cdot 3$), and so is covered by our theory.

(5) The definite quaternion algebra obtained by interchange of invariants at 3 for f is the Hamilton quaternions, which among other nice properties have class number one.

(6) The Galois representation corresponding to f is "inertially large" in the sense of [MTT, Section 15]. This makes f qualitatively different from the weight two forms which have been examined numerically.

In the end, we show that the equality conjectured in Conjecture 1 holds for two "exceptional" characters χ, to 5 place of 3-adic accuracy each. This, admittedly, is not an overwhelming quantity of data, although the author finds it thoroughly convincing, being by nature an optimist. The calculations we describe were done by programs written in a symbolic manipulation language (Maple) on a relatively small computer (a SUN 3/260). The convenience and simplicity of this programming environment carries with it a steep price in efficiency; the two cases we report in this article were the largest calculations we could handle before running out of computer memory. We will report on the purely computational aspects of this project elsewhere.

There are two main steps to testing the exceptional zero conjecture for f. The first step is to compute the modular symbols, classical special value, and p-adic L-function for f. The techniques used here are exactly those of [MTT]. We present some potentially useful numerical data on this subject in the appendix.

The second, much more interesting, step in checking the conjecture is the computation of $\mathscr{L}$, and it is here that the new results in Theorems 3 and 4 play a crucial role, making possible the evaluation of the Coleman integrals which enter into the definition of $\mathscr{L}$.

The form f is a lift from the indefinite quaternion algebra with discriminant 6. Since we are interchanging invariants at 3, the definite quaternion algebra for f has discriminant 2, and is therefore the Hamilton quaternions $\mathbb{H}$. Let $u \in \mathbb{Q}_3$ be the unique element such that $u^2 = -2$ and $u \equiv 1 \pmod 3$. We define $\iota : \mathbb{H} \to M_2(\mathbb{Q}_3)$ by setting

$$\iota(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\iota(j) = \begin{pmatrix} u & 1 \\ 1 & -u \end{pmatrix} .$$

We now view $\mathbb{H}$ as a subalgebra of $M_2(\mathbb{Q}_3)$, and let $\mathcal{O}$ denote the maximal order of $\mathbb{H}$ viewed inside $M_2(\mathbb{Q}_3)$. Identify the vertices of the tree $\mathscr{T}$ with elements of the coset space $GL_2(\mathbb{Z}_3)/GL_2(\mathbb{Q}_3)$. If v is such a vertex, then $\gamma \in GL_2(\mathbb{Q}_3)$ acts on the left via $\gamma \cdot v = v\gamma^{-1}$. Let

$$\Gamma = \{\gamma \in \mathcal{O}[\tfrac{1}{3}] : \gamma\bar{\gamma} = 1\} ,$$

and let

$$\Gamma(2) = \{\gamma \in \Gamma : \gamma \equiv 1 \pmod 2\} .$$

We rely on [G] for a wealth of useful information about the groups Γ and $\Gamma(2)$. Summarizing what we learn there, we find that $\Gamma(2)$ is a free group on 3 generators, of index 24 in Γ. Let $U = \mathbb{H}(\mathbb{Z}/2\mathbb{Z})^*$. U is isomorphic to A_4, and there is an exact sequence

$$1 \to \Gamma(2) \to \Gamma \to U \to 1 .$$

This exact sequence splits. Indeed, let $\rho = (1 + i + j + k)/2$. Then i and ρ belong to Γ, and together generate a copy of A_4 which maps onto U.

Lemma 12. *Let*

$$q_\infty = i + j + k ,$$

$$q_0 = i - j - k ,$$

$$q_1 = i - j + k ,$$

$$q_2 = i + j - k .$$

The edges $e_i = [1, q_i]$ *for* $i \in \{\infty, 0, 1, 2\}$ *are a* $\Gamma(2)$ *fundamental set in* $\mathcal{T}$. *The action of* U *on these edges is the standard permutation representation of* A_4.

Proof. See [G, Chapter 9.] □

Proposition 13. *Define a function* c *on the edges of* $\mathcal{T}$ *by setting*

$$c(e_\infty) = (u-1) + (-2-u)T + T^2$$

$$c(\gamma e_\infty) = \gamma \cdot c(e_\infty)$$

for $\gamma \in \Gamma$. *Then* c *spans* $C_{\mathrm{har}}(\Gamma, 4)$.

Proof. We know that $C_{\mathrm{har}}(\Gamma, 4)$ has dimensional 1 over $\mathbb{Q}_3$, since the space of modular forms of weight 4 and level 6 is 1-dimensional. Alternatively, we know that $C_{\mathrm{har}}(\Gamma(2), 4)$ is 6 dimensional since $\mathcal{H}_p/\Gamma(2)$ is of genus 3 and $\Gamma(2)$ is free. It is then not hard to check that, as a representation of U, $C_{\mathrm{har}}(\Gamma(2), 4)$ is isomorphic to $E \otimes E$ where E is the standard permutation representation of A_4 on the edges E in the fundamental set for $\Gamma(2)$ described above. But $E \otimes E$ contains one copy of the trivial representation. Notice that the stabilizer of e_∞ is the subgroup R of U generated by ρ, and that R fixes $c(e_\infty)$. This is sufficient to guarantee that c is well-defined. One can further determine from this that, if we view the coefficients of $c(e_\infty)$ as defining points in a 3-space, then the U orbit of $c(e_\infty)$ consists of the four vertices of a tetrahedron in that 3-dimensional space, centered at the origin. This guarantees that c is harmonic. □

Corollary 14. *Let* B *be the compact open in* $\mathbb{P}^1(\mathbb{Q}_3)$ *consisting of points* x *such that* $|x|_p < 1$. *The measure* μ *on* $\mathbb{P}^1(\mathbb{Q}_3)$ *associated to the newform* f *is characterized by the invariance property* $\gamma^* \mu = (cx+d)^2 \mu$ *and by the fundamental integrals*

$$\int_B d\mu = 1 ,$$

$$\int_B x\, d\mu = 1 + \frac{u}{2} ,$$

$$\int_B x^2 d\mu = u - 1 .$$

Proof. The edge $e_0 = [1, q_0]$ corresponds to the open set B. We know that $i \cdot [1, q_\infty] = [i, q_\infty i^{-1}] = [1, q_0]$ and therefore by invariance of c we compute

$$c(e_0) = 1 + (2+u)T + (u-1)T^2 .$$

This gives the claimed fundamental integrals. Since the Γ translates of B are a basis of compact opens in $\mathbb{P}^1(\mathbb{Q}_3)$, these together with the invariance property determine μ. □

We obtain from the cocycle c and the measure μ two cocycles ψ and λ, with coefficients in $P_{k-2}(\mathbb{C}_p)$ as described in section 2. Our goal is to compare the two cohomology classes in $H^1(\Gamma, P_2(\mathbb{C}_3))$ which they represent. We will summarize a few algebraic facts which will simplify this comparison.

Lemma 15. *Let* $q = q_2 q_1$. *Then* Γ *is generated by* i, ρ, *and* q. *Any class* v *in* $H^1(\Gamma, P_2)$ *is represented by a unique cocycle* n *such that* $n(i) = 0$ *and* $n(\rho) = 0$.

Proof. The first statement follows from [G]. To obtain the second, let v be any class represented by a cocycle m. It is not hard to verify that there is a unique solution a to the equations

$$(1-\rho)\cdot a = m(\rho)\,,$$
$$(1-i)\cdot a = m(i)\,.$$

If we let $d_q(\gamma) = (1-\gamma)\cdot a$, then $m - d_a$ represents v and has the same properties. □

Let v_1 be the vertex of $\mathcal{T}$ corresponding to the identity element of $\mathbb{H}$. Since i and ρ fix v_1, the cocycle ψ^{v_1} has the properties in the lemma. From the definition and our explicit calculation of c, we compute:

$$\psi^{v_1}(q) = \tfrac{1}{3}(-2-5u+2(2-u)T+(2-3u)T^2)\,.$$

This information determines the map ψ completely.

Of course, the hard part is the computation of λ. Knowing the measure μ, we must approximate the integrals in Theorem 4. Our package of computer programs tells us that, if the base point Q is chosen so that $z(Q) = 1 - 3\sqrt{-1}$, then

$$\lambda(q) \equiv -53 - 737T + 880T^2 \pmod{3^7}$$
$$\lambda(\rho) \equiv -270 - 783T + 981T^2 \pmod{3^7}$$
$$\lambda(i) \equiv 372 - 387T - 372T^2 \pmod{3^7}\,.$$

For comparison purposes with ψ, we normalize this λ to obtain an equivalent cocycle λ' which satisfies $\lambda'(\rho) \equiv \lambda'(i) \equiv 0 \pmod{3^5}$. Evaluating the normalized cocycle λ' on q and comparing with ψ, we obtain the relation

$$\lambda'(q) \equiv 10\psi^{v_1}(q) \quad \pmod{3^5}\,,$$

where the unfortunate presence of denominators at various stages of the calculation of λ' reduces our accuracy from 3^7 to 3^5. In any case, we conclude that:

$$\mathcal{L} \equiv 10\cdot 1_T \quad \pmod{3^5}\,.$$

This is precisely the predicted value from the L-function calculations which we report in the appendix, in accord with Conjecture 1.

Appendix

Summary of numerical data for modular symbol computations. We are studying the modular form of level 6 and weight 4. To compute the associated p-adic L-function, we apply the definitions of [MTT]. The "raw data" for the computation is the set of basic modular symbols denoted by $\Phi(f, P, r)$ in [MTT, Section 1] and by $[r, P(z)]$ in the remainder of this work. The lattice Ω of modular symbols is spanned by the 12 fundamental symbols $[r, z^j]$ for $0 \leqq j \leqq 2$ and $r \in \{0, 1/2, 1/3, -1/3\}$. Let $\Omega^{\pm}$ denote the quotient lattices of Ω subject to the respective relations $[-r, P(-z)] = \pm[r, P(z)]$. Denote by $[1/2, z^2]^{\pm}$ the image of the symbol $[1/2, z^2]$ in each of these quotient lattices. If we identify

$$\Omega \otimes \mathbb{C}_3 = (\Omega^+ \oplus \Omega^-) \otimes \mathbb{C}_3,$$

it turns out that $[1/2, z^2]^{\pm}$ spans the corresponding factor space. Every basic symbol $[r, P]^{\pm}$ is therefore a multiple of one of these two symbols. These multiples are given in table 1.

Table 1. Fundamental modular symbols of weight four for $\Gamma_0(6)$

	$[0,1]$	$[0,z]$	$[0,z^2]$	$[\frac{1}{2},1]$	$[\frac{1}{2},z]$	$[\frac{1}{2},z^2]$	$[\frac{1}{3},1]$	$[\frac{1}{3},z]$	$[\frac{1}{3},z^2]$
+	0	$-\frac{1}{2}$	0	0	1	1	-6	$-\frac{3}{2}$	0
−	-2	0	$\frac{1}{3}$	6	3	1	4	3	$\frac{4}{3}$

Numerical values for the modular symbols $[r, P] = \Phi(f, r, P)$ in terms of the basic symbols $[\frac{1}{2}, z^2]^{\pm}$

Table 2. Algebraic special values $\Lambda^{\pm}(\chi_m, s)$

m	$s=0$	$s=1$	$s=2$
1	-2^-	$-\frac{1}{2}^+$	$\frac{1}{3}^-$
5	-240^-	-36^+	40^-
-7	-216^+	-40^-	36^+
29	-19440^-	-900^+	3240^-
-31	8856^+	-360^-	-1476^+

Numerical values in terms of fundamental symbols $[\frac{1}{2}, z^2]^{\pm}$ depending on the sign of χ_m, as indicated

Table 3. Values of $L_3'(\chi_m x_3, 0)$

m	$L_3'(\chi_m x_3, 0)$	
5	-360^+	$(\text{mod } 3^6)$
-7	-400^-	$(\text{mod } 3^6)$

Numerical values in terms of fundamental symbols $[\frac{1}{2}, z^2]^{\pm}$, depending on the sign of χ_m, as indicated

Knowing the numbers in Table 1, we can compute the algebraic special values $\Lambda^{\pm}(\chi_m)$ (which occurs on the "right hand side" of the equation in Conjecture 1) using the formulas in [MTT]. Table 2 summarizes the results of these calculations, listing the algebraic special values $\Lambda(\chi)$ for a few characters χ.

The final piece of information we need to check the conjecture is the value of the derivative of the p-adic L-function for an exceptional character χ (the "left hand side" of the equation in Conjecture 1.) Using the results in [MTT], we determine that a primitive, even, Dirichlet character χ of prime conductor M is exceptional for f if and only if $m \equiv 5 \pmod{24}$, while an odd character is exceptional if $m \equiv -7 \pmod{24}$. In Table 3 we give the corresponding values of the derivative of the p-adic L-function for two exceptional characters.

Comparing the numbers in table 3 with those in table 2, we see that

$$L_p'(\chi_m x_3, 0) \equiv 10\Lambda^{\pm}(\chi_m, 1) \pmod{3^5}$$

so that these tables also predict that $\mathscr{L} \equiv 10 \pmod{3^5}$.

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