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Published on: 01 Jan 2003 - Journal of Computational Finance (Warwick Business School, Financial Econometrics Research Centre)

Topics: Monte Carlo methods for option pricing, Monte Carlo method and Valuation of options

Financial Econometrics Research Centre

WORKING PAPERS SERIES

WP02-04

Valuing Path Dependent Options in the Variance-Gamma Model by Monte Carlo with a Gamma Bridge

Claudia Riveiro and Nick Webber

Valuing Path Dependent Options in the [illegible]

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September 20, 2002

Abstract

The Variance-Gamma model has analytical formulae for the values of European calls and puts. These formulae have to be computed using numerical methods. In general, option valuation may require the use of numerical methods including PDE methods, lattice methods, and Monte [illegible]

We investigate the use of Monte Carlo methods in the Variance-Gamma model. We demonstrate how a gamma bridge process can be constructed. Using the bridge together with stratified sampling we obtain considerable speed improvements over a plain Monte Carlo method.

The method is illustrated by pricing lookback, average rate and barrier options in the Variance-Gamma model. We find the method is up to [illegible]

*Corresponding author. Claudia Ribeiro gratefully acknowledges the support of Fundação para a Ciência e a Tecnologia and Faculdade de Economia, Universidade do Porto. The paper [illegible]

[illegible]

The variance-gamma model has been investigated by a number of authors for application to option valuation (Madan and Seneta (90) [16], Madan and Milne (91) [15], Madan, Carr and Chang (98) [14], Ané (99) [4] and Carr, Geman, Madan and Yor (01) [7]). Although analytical solutions are available for European-style options, other options require the use of numerical methods. These include Monte Carlo methods, Fourier transform (FFT) methods (Carr and Madan (99) [8]), and PDE approaches (the 'method of lines'; Albanese, Jaimungal and Rubisov (01a), (01b) [1], [2]).

This paper investigates the use of Monte Carlo methods with the variance-gamma model. In particular we show how a gamma bridge may be constructed and used in conjunction with stratified sampling. We demonstrate that considerable efficiency gains are possible. If improved algorithms for sampling certain distributions become available these gains may be further improved.

The gamma bridge can be used with, augment and supplement control variate methods, importance sampling methods and other variance reduction techniques.[1]

The second section of this paper recaps the variance-gamma process and its application to option pricing. We review how Monte Carlo methods may be applied by exploiting the subordinated Brownian motion representation of the variance-gamma process. In the third section we show how a gamma-bridge can be constructed and applied. The fourth section presents numerical results and [illegible]

[illegible]

We review the variance-gamma process and its application to option pricing. We describe a 'plain' Monte Carlo method related to the subordinated Brownian [illegible]

[illegible]

A variance-gamma process X_t has three parameters: $\mu \in \mathbb{R}$, $\sigma > 0$, $\nu > 0$. It is pure jump with Lévy density $k_X(x)$,

$$k_X(x)\,\mathrm{d}x = \frac{\exp\left(\frac{\mu x}{\sigma^2}\right)}{\nu\,|x|}\exp\left(-\frac{1}{\sigma}\sqrt{\frac{2}{\nu}+\frac{\mu^2}{\sigma^2}}\,|x|\right)\mathrm{d}x \tag{1}$$

[1] Indeed a suitable delta control variate for the variance-gamma model of asset returns is the delta of the option value generated by ordinary geometric Brownian motion. Since the density function of the variance-gamma process is known, importance sampling can be applied by using shifted mean and variance parameters to sample more closely the region of [illegible]

and characteristic function

$$\mathbb{E}\left[\exp\left(iuX_t\right)\right] = \left(\frac{1}{1 - i\mu\nu u + \frac{1}{2}\sigma^2\nu u^2}\right)^{\frac{t}{\nu}}. \tag{2}$$

[illegible]

$$f_t^{VG}(x) = \frac{2\exp\left(\frac{\mu x}{\sigma^2}\right)}{\nu^{\frac{t}{\nu}}\sqrt{2\pi}\sigma\Gamma\left(\frac{t}{\nu}\right)}\left(\frac{x^2}{\frac{2\sigma^2}{\nu}+\mu^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}} K_{\frac{t}{2\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2\left(\frac{2\sigma^2}{\nu}+\mu^2\right)}\right)$$

[illegible]

where $K_v(z)$ is the modified Bessel function of the third kind,

[illegible]

X_t can be represented as a subordinated Brownian motion, $X_t = w_{h(t)}$, where w_t is Brownian motion with drift μ and variance σ^2 and $h(t)$ is a gamma process $h_t \sim \mathrm{G}\left(\frac{t}{\nu},\nu\right) \sim \nu\mathrm{G}\left(\frac{t}{\nu}\right)$. The density $f_t^h(x)$ of h_t conditional on $h(0) =$ [illegible]

[illegible]

[illegible]

$$f_t^{VG}(x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu g}{\sigma\sqrt{g}}\right)^2\right)\frac{g^{\frac{t}{\nu}-1}\exp\left(-\frac{g}{v}\right)}{\nu^{\frac{1}{\nu}}\Gamma\left(\frac{t}{\nu}\right)}\mathrm{d}g. \tag{6}$$

The variance gamma process can be used to model stock price returns. Let S_t be the stock price at time t. In this exposition we assume that the stock pays no dividends. We take the state space Ω to be the path space of X_t equipped with the filtration induced by the variance-gamma process. Following Madan, [illegible] and Keller (95) [12], *et cetera*, we model a stock price process S_t under the [illegible]

[illegible]

where X_t is a variance-gamma process, r is the short rate, a constant, and the presence of the compensator ϖ, defined by $\mathrm{e}^{\varpi} = \mathbb{E}\left[\exp\left(X_1\right)\right]$, ensures that $S_t\mathrm{e}^{-rt}$ is a martingale under the measure associated with the accumulator account numeraire. From (2) we have

[illegible]

The log price relative $z_t = \ln\left(S_t/S_0\right)$ of S_t has density $f_t^z(x) = f_t^{VG}(x')$ where [illegible]

[2] In general there is no unique martingale measure for a Lévy process. Since we focus on numerical solutions for processes of the form (7) we do not pursue this issue further.

[illegible]

[illegible]

Suppose an option has payoff $H_T \equiv H_T(\omega)$ at time T, where H_T may depend on the state $\omega \in \Omega$. Under the martingale measure F associated with the [illegible]

[illegible]

In this section we recall how (9) can be solved using plain Monte Carlo. A standard reference for applications of Monte Carlo methods in finance is Jäckel (02) [13].

[illegible]

$$c_t = \mathbb{E}\left[H_T \mathrm{e}^{-r(T-t))}\right] = \mathrm{e}^{-r(T-t)} \int_\Omega H_T(\omega)\, \mathrm{d}F(\omega). \tag{10}$$

The integral can be approximated by constructing a set $\{\widehat{\omega}^m\}_{m=1,\ldots,M}$ of discrete sample paths randomly selected under a measure $\widehat{F}$, a discrete approximation to the measure F. Then the approximation $\widehat{c}_t$ to c_t is

[illegible]

Discrete sample paths for a subordinated Brownian motion, $X_t = w_{h(t)}$, can be constructed by first constructing discrete sample paths for the subordinator $h(t)$ and then sampling the process w_t at times determined by the paths found for $h(t)$.

We construct discrete sample paths for X_t over the period $[0, T]$ with N time steps at times $0 = t_0 < t_1 < \ldots < t_N = T$. First we construct a discrete sample path $\left\{\widehat{h}_n\right\}_{n=0,\ldots,N}$ for $h(t)$. Set $\widehat{h}_0 = 0$. Iteratively, $\widehat{h}_{n+1} - \widehat{h}_n$ is a random increment in h over the interval $\Delta t_n = t_{n+1} - t_n$. For the variance-gamma process, $\Delta\widehat{h}_n = \widehat{h}_{n+1} - \widehat{h}_n \sim \mathrm{G}\left(\frac{\Delta t_n}{\nu}, \nu\right)$ and is easy to simulate. Given the path $\left\{\widehat{h}_n\right\}_{n=0,\ldots,N}$, and setting $\widehat{w}_0 = 0$, we set iteratively $\Delta\widehat{w}_n = \widehat{w}_{n+1} - \widehat{w}_n \sim \mathrm{N}\left(\mu\Delta\widehat{h}_n, \sigma^2\Delta\widehat{h}_n\right)$. The path $\widehat{w} = \{\widehat{w}_n\}_{n=0,\ldots,N}$ is a discrete approximation to [illegible]

[illegible]

We may assume elsewhere that the time step $\Delta t = \frac{T}{N}$ is a constant. This assumption is for simplicity only and may be relaxed trivially.

Process

The plain Monte Carlo estimate $\widehat{c}_t$ converges to c_t as M and N go to infinity. However, convergence to within an error bound may be very slow.[3] An effective Monte Carlo method requires effective speed-ups. When simulating a Brownian motion, a Brownian bridge is often used in conjunction with stratified sampling. This technique helps to ensure that the set $\{\widehat{w}^m\}_{m=1,\dots,M}$ of discrete sample paths is drawn more evenly under the measure $\widehat{F}$. In this section we review the method of stratified sampling and the use of a bridge for a stochastic process and its application to a Wiener process. We describe the construction of the gamma bridge and its application to the variance-gamma process. The algorithm, 'bridge' Monte Carlo, is presented.

3.1 Stratified Sampling

Initially suppose that the payoff function H_T depends solely on the value X_T of a state variable at time T, with distribution function $F_T^X(x)$ and density function $f_T^X(x)$. Then

$$
\begin{aligned}
&= \mathrm{e}^{-r(T-t)} \int H_T(x)\, f_t^X(x)\, \mathrm{d}x && (13)\\
&\sim \mathrm{e}^{-r(T-t)} \frac{1}{M} \sum^{M} H_T\left(\widehat{X}^m\right) && (14)
\end{aligned}
$$

where $\widehat{X}^m$ is drawn from the distribution F_T^X.

A stratified sample of size M from F_T^X is one in which the mth draw, $\widehat{X}^m$, is constructed to lie in the mth quantile band, $\left[\frac{m-1}{M}, \frac{m}{M}\right]$, for $1 \le m \le M$. Given a sample $\{\widehat{v}^m\}_{m=1,\dots,M}$ drawn from $\mathrm{U}[0,1]$, the set $\{\widehat{u}^m\}_{m=1,\dots,M}$, where $\widehat{u}^m =$

stratification method is to sample $\mathrm{U}[0,1]$ using a low discrepancy sequence. If the function $\left(F_T^X\right)^{-1}$ is known, and given a stratified sample $\{\widehat{u}^m\}_{m=1,\dots,M}$ of the uniform distribution $\mathrm{U}[0,1]$, then the set $\left\{\left(F_T^X\right)^{-1}(\widehat{u}^m)\right\}_{m=1,\dots,M}$ is a stratified sample from the distribution F_T^X. We use this technique, the inverse

An option value constructed using a stratified sample may have an actual standard deviation significantly less the size of that of a value found using plain

[4] This is because a stratified sample has autocorrelation. We find the actual standard

If H_T depends on an entire sample path, as is the case for an average rate option, a set of sample paths may be found by first finding a stratified sample $\left\{\widehat{X}^m\right\}_{m=1,\dots,M}$ from the terminal time, and then constructing a path $0 = \widehat{X}_0^m < \widehat{X}_1^m < \ldots < \widehat{X}_N^m = \widehat{X}_m$ so that each $\widehat{X}_n^m$ has the correct conditional distribution. This set of paths will sample from $\widehat{F}$ more evenly than a sample without stratification. We call the path $\left\{\widehat{X}_n^m\right\}_{n=0,\dots,N}$, constructed from $\widehat{X}_0^m$ and $\widehat{X}_N^m$, an X-bridge. Intermediate points $\widehat{X}_n^m$ are constructed by sampling from a bridge distribution, defined and described in the next section. This sampling may also be stratified, leading to improved sampling at the intermediate [illegible]

3.2 A Bridge for a Wiener Process

Suppose that $x \sim F_x$ and $y \sim F_y$ are random variables with distributions F_x and F_y, densities f_x and f_y and joint density function $f_{x,y}$. Set $z = x + y$ with density f_z. We are interested in the conditional distribution of $x \mid z$. Write $f_{x|z}$ [illegible]

$$f_{x|z}(x) = \frac{f_{x,y}(x, z - x)}{f_z(z)}. \tag{15}$$

[illegible]

When x and y represent increments in a Markov stochastic process their densities will depend on the time increment. For instance, for a Wiener process w_t where $x = w_{t_j} - w_{t_i}$ is a random increment between times t_i and t_j and $y = w_{t_k} - w_{t_j}$ is a random increment between times t_j and t_k, then $x \sim \mathrm{N}\left(0, \sigma_x^2\right)$, $y \sim \mathrm{N}\left(0, \sigma_y^2\right)$, $z \sim \mathrm{N}\left(0, \sigma_z^2\right)$ where $\sigma_x^2 = t_j - t_i$, $\sigma_y^2 = t_k - t_j$ and $\sigma_z^2 = t_k - t_i$. [illegible]

[illegible]

where

[illegible]

Hence, given $z = w_{t_k} - w_{t_i}$, $x = w_{t_j} - w_{t_i}$ is normally distributed, $x \sim$

$\mathrm{N}\left(za, b^2\right) \sim \mathrm{N}\left(z\frac{t_j - t_i}{t_k - t_i}, \frac{(t_j - t_i)(t_k - t_j)}{(t_k - t_i)}\right)$, so that

[illegible]

[illegible]

[illegible]

[illegible]
$\tau_y = t_k - t_j$ and $\tau_z = t_k - t_i$. Let x be an increment in g_t over the period
[illegible]
$g_{t_j} - g_{t_i} \sim \mathrm{G}\left(\frac{\tau_x}{\nu}, v\right)$, $y = g_{t_k} - g_{t_j} \sim \mathrm{G}\left(\frac{\tau_y}{\nu}, v\right)$ and $z = g_{t_k} - g_{t_i} \sim \mathrm{G}\left(\frac{\tau_z}{\nu}, v\right)$. The conditional density $f_{x|z}(x)$ is

[illegible]

[illegible]

[illegible]
parameters $\frac{\tau_x}{\nu}$ and $\frac{\tau_y}{\nu}$. Given g_{t_k} and g_{t_i}, $\frac{x}{z} = \frac{g_{t_j} - g_{t_i}}{g_{t_k} - g_{t_i}}$ has a beta distribution so

[illegible]

for $\beta_{t_j} \sim \mathrm{B}\left(\frac{t_j - t_i}{\nu}, \frac{t_k - t_j}{\nu}\right)$. This is the bridge distribution for a gamma process.

This result is intuitive. A beta variate takes values in the interval $[0, 1]$. The gamma process is an increasing process. Given the increment z over the period $[t_i, t_k]$, the beta distribution samples from the proportion of this increment achieved by time t_j.

3.4 Application of the Gamma Bridge

To apply bridge Monte Carlo we use the subordinator representation of the variance-gamma process and stratify it at the terminal time. We construct a bridge to the points we have constructed. The bridge may itself be stratified at intermediate times.

Suppose $X_t = w_{h(t)}$ for a Brownian motion $w_t \sim \mathrm{N}\left(\mu t, \sigma^2 t\right)$ and subordinator $h_t \sim \mathrm{G}\left(\frac{t}{\nu}, \nu\right)$. Time is discretised into N time steps, $0 = t_0, t_1, \ldots, t_N = T$, up to the terminal time T. For a sample of size M we construct:

[illegible]

[illegible]

The set of paths $\widehat{w}^i$, $i = 1, \ldots, M$, is a stratified bridge sample from the path space of the Lévy process X_t.

Note that we stratify both h_t and $w_{h(t)}$ at the terminal time. We found experimentally that full speed-ups were not achieved unless both the subordinator and the Brownian motion were stratified. In the sequel we always stratify both h_t and $w_{h(t)}$.

3.5 Stratifying the Variance-Gamma Bridge

For $t_i < t_k$, given g_{t_i} and g_{t_k}, the value g_{t_j} of a gamma process at an intermediate time t_j is generated as $g_{t_j} = g_{t_i} + \beta_{t_j} (g_{t_k} - g_{t_i})$ for $\beta_{t_j} \sim \mathrm{B}\left(\frac{t_j - t_i}{\nu}, \frac{t_k - t_j}{\nu}\right)$. A stratified sample for β_{t_j} yields a stratified sample for g_{t_j}. We obtain such a sample by inverse transform from a stratified sample $\widehat{u}_n$, $n = 1, \ldots, M$ of the unit interval. Set $\beta_{t_j}^n = B^{-1}_{\frac{t_j - t_i}{\nu}, \frac{t_k - t_j}{\nu}} (\widehat{u}_n)$, where $B^{-1}_{\alpha,\beta}$ is the inverse of the [illegible]

If the gamma process is stratified at time t_j then we also stratify w at time [illegible]

We present in the next section comparisons of bridge Monte Carlo when stratified at different numbers of times. To stratify at K times, where $K = 2^P$ and $N = QK$ for integer P and Q, we first compute a stratified sample of points h_N^i and w_N^i, $i = 1, \ldots, M$, at time t_N. We then stratify successively at times $\frac{1}{2}t_N$, $\frac{3}{4}t_N$, $\frac{1}{4}t_N$, $\frac{7}{8}t_N$, $\frac{5}{8}t_N$, $\frac{3}{8}t_N$, $\frac{1}{8}t_N$, and so on, until all times $\frac{k}{K}t_N$, $k =$ [illegible]
are filled in using ordinary random draws from beta and normal distributions. We stratify by constructing a stratified sample from a $2K$-dimensional unit hypercube. For $K \leq 2$ it may be plausible to use a Monte Carlo stratified sample. In the numerical section, except where noted, we use low discrepancy [illegible]

[illegible]

We first benchmark the bridge Monte Carlo method against European call option values. Then we use the bridge to value average rate, lookback and barrier options and compare the results to those found with plain Monte Carlo. The performance of the bridge under various degrees of stratification is investigated.

All the path dependent instruments mature in one year. We investigate reset frequencies from quarterly to approximately daily. We find that the bridge method benchmarks very accurately, achieving efficiency gains of a factor of 50 for European calls with one year to maturity. For path dependent options with daily resets and 16 stratification times we achieve gains of a factor of 130 for lookback options and 380 for average rate options. Gains are also found for barrier options but these may be significantly less. For our examples one stratification (at the terminal time) is ten to twenty times faster than plain Monte Carlo, but further stratification does not always bring further efficiency gains.

[illegible]

We require algorithms for generating uniform, normal, gamma and beta random variates.

Uniform variates are generated using a VBA version of ran2 from Numerical Recipes (92) [18]. All normal variates were generated by inverse transform. N^{-1}, the inverse of the normal distribution function, is computed using Applied Statistics Algorithm 111 [3] downloadable from lib.stat.cmu.edu/apstat/111.

To generate gamma variates directly we use the Best (83) and Best (78) algorithms as described in Devroye (86) [10]. To compute the inverse of the gamma distribution function, G^{-1}, to use with the inverse transform method for stratified sampling, we use the algorithm of DiDonato and Morris (87) [11], downloadable from www.netlib.org/toms/654. It uses an iterative method to [illegible]

Beta variates from the distribution $\mathrm{B}(\alpha,\beta)$ are generated directly by Cheng's method if $\min(\alpha,\beta) < 1$, Johnk's method if $\max(\alpha,\beta) < 1$, by Atkinson and Whittaker's method if $\min(\alpha,\beta) < 1 < \max(\alpha,\beta)$, and by ratio of gammas otherwise.[5] For stratified sampling, the inverse of the beta distribution function, B^{-1}, is computed using an algorithm due to Moshier (00) [17]. This algorithm uses an iterative method to solve for $B_{\frac{\tau_x}{\nu},\frac{\tau_y}{\nu}}\left(\beta^n_{t_j}\right) - \widehat{u}_n = 0$. We shall see that this particular procedure is relatively slow compared, for instance, to computing N^{-1}. Should faster algorithms emerge to compute $B^{-1}_{\frac{\tau_x}{\nu},\frac{\tau_y}{\nu}}$ (or indeed $G^{-1}_{\frac{\tau_y}{\nu},v}$) then the efficiency gains to the algorithm would be even greater than those we [illegible]

For low discrepancy sampling we use a Sobol' sequence based on Bratley and Bennett (88) [6]. Code is downloadable from www.netlib.org/toms/659. The

[5] See [9] and [10]. Johnk's method sometimes fails when both α and β are small. In these cases we revert to the ratio of gammas method.

Maturity	Explicit	Monte Carlo		
		Plain	**Stratified**	**Gain**
0.25	3.4742	3.4548 (0.035)	3.4748 (0.003)	25
0.5	6.2406	6.2414 (0.056)	6.2401 (0.004)	35
0.75	8.6909	8.6928 (0.077)	8.6955 (0.005)	43
1	10.9815	10.9381 (0.099)	10.9721 (0.006)	50
Time:	–	0.3	1.4	

Table 1: Comparison of Plain and Stratified Monte Carlo: Calls, one time step.

code generates low discrepancy samples from a unit hypercube of dimension at most 39. Since bridge Monte Carlo uses two low discrepancy coordinates at each stratified time, we are constrained to have at most 18 stratification times.

[redacted]

We value European calls and compare values obtained from an analytic formula [redacted] with stratified Monte Carlo, taking 10,000 samples directly from the terminal distribution for each maturity ($M = 10{,}000$, $N = 1$). Explicit values are computed using the analytical formula of Madan, Carr and Chan (98) [14]. Standard deviations are shown in brackets.[7,8] Tables 1 and 2 use 'ordinary' [redacted]

For each option in table 1 plain Monte Carlo took about 26 seconds for a hundred replications. Stratified Monte Carlo took about 144 seconds.

The final column of table 1 gives the efficiency gain of the stratified Monte [redacted]

$$E_{AP} = \frac{\sigma_P^2 t_P}{\sigma_A^2 t_A}. \tag{20}$$

[6] Call values are for options on an asset with initial value $S_0 = \mathbf{100}$, exercise price $X = \mathbf{101}$ and riskless rate $r = \mathbf{0.1}$. Parameters of the variance-gamma process are $\mu = -\mathbf{0.1436}$, $\sigma = \mathbf{0.12136}$, $\nu = \mathbf{0.3}$ (based upon Madan, Carr and Chan (98) [14]). Maturities, in years, [redacted]

[7] For plain Monte Carlo the standard deviation is approximately equal to the standard error. For bridge Monte Carlo, in all the tables of this section, the true standard deviation is found from a hundred replications of the Monte Carlo procedure.

[8] All programmes were written in Visual Basic 6.0 and were run on an 800 Mhz PC.

[illegible]

Table 2: Stratified Monte Carlo: Standard deviation against sample paths: [illegible]

Under the assumption that standard deviation scales inversely with the square root of the number of sample paths M, and that time taken is proportional to M, then E_{AP} is the multiple of the time the plain method takes to achieve a particular standard deviation compared to the alternative method.

We note efficiency gains of 50 for one year maturity options. The efficiency [illegible]

Table 2 compares stratified Monte Carlo with different numbers of sample paths. The $M = 10,000$ column is repeated from table 1. In each case the gamma and the normal variates have an equal degree of stratification. For instance, for $M = 10,000$, there are a hundred buckets in each dimension of the stratified 2-dimensional unit hypercube. With only 400 sample paths the standard deviation of the stratified Monte Carlo method is significantly less than that of the plain Monte Carlo in table 1 and takes less than a quarter of [illegible]

Now we investigate how using bridge Monte Carlo can increase the efficiency of valuing options of different maturities simultaneously. We evolve the variance-gamma process out to 1 year in four time steps of 0.25 years each, and value simultaneously the benchmark option maturing at the conclusion of each time [illegible]

Table 3 shows benchmarked call values computed with plain Monte Carlo and with bridge Monte Carlo stratified at (i) step 4, the terminal time, (ii) steps 2 and 4, (iii) steps 1, 2, 3 and 4, so that $K = 1$, 2 and 4 for the three cases. In this table $M = 100,000$ for each option. Stratification is by low discrepancy sampling. At the stratification times, stratification is entirely deterministic so no standard deviation can be reported. Table 4 displays the efficiency gains [illegible]

We see that stratification gives very significant efficiency gains for options of all maturities. Additional stratification may give greater gains for options of

[9] Efficiency gains decrease as M decreases because of fixed set-up times in the implementation of the Monte Carlo algorithm.

Maturity	Explicit	Plain MC	Bridge MC: Stratified at steps: $K=1$	$K=2$	$K=4$
0.25	3.4742	3.4756 (0.034)	3.4879 (0.011)	3.4858 (0.009)	3.4738 (–)
0.5	6.2406	6.1712 (0.061)	6.2440 (0.015)	6.2409 (–)	6.2409 (–)
0.75	8.6909	8.6156 (0.088)	8.6811 (0.026)	8.6860 (0.011)	8.6907 (–)
1	10.9815	10.8766 (0.10)	10.9813 (–)	10.9813 (–)	10.9813 (–)
Time:	–	0.93	0.25	0.40	1.3

Table 3: Comparison of Plain and Stratified Monte Carlo: Calls, one to four [illegible]

	Efficiency Gains	
Maturity	$K=1$	$K=2$
0.25	35	33
0.5	61	–
0.75	42	147
1	–	–

Table 4: Bridge Monte Carlo: Efficiency Gains over Plain Monte Carlo.

[illegible]

As a further example we compute Black-Scholes implied volatilities for options maturing at times $\frac{n}{N}$ for $n = 1, \ldots, N$, $N = 64$. Figure 1 compares Black-Scholes implied volatilities of variance-gamma calls for bridge Monte Carlo stratified only at the terminal time against stratification every 4 time steps. The sample size is $M = 1,000$. Option prices for each maturity are computed together. With 16 stratification times (and one replication) the programme takes 9.2 seconds to run. When stratifying only at the terminal time the programme takes 2.0 seconds. Pricing is improved with greater stratification, even taking the increased run time into account, particularly at longer maturity times.

We conclude that bridge Monte Carlo benchmarks well to European calls, achieving superior accuracy to plain Monte Carlo. We now value path dependent options in the variance-gamma framework.

[illegible]

We value one year average rate, lookback and barrier options with various numbers of reset times up to final maturity, comparing the results to plain Monte [illegible]

We report results for plain Monte Carlo with $M = 1,000,000$ sample paths. The results for bridge Monte Carlo are for $M = 10,000$. Actual standard

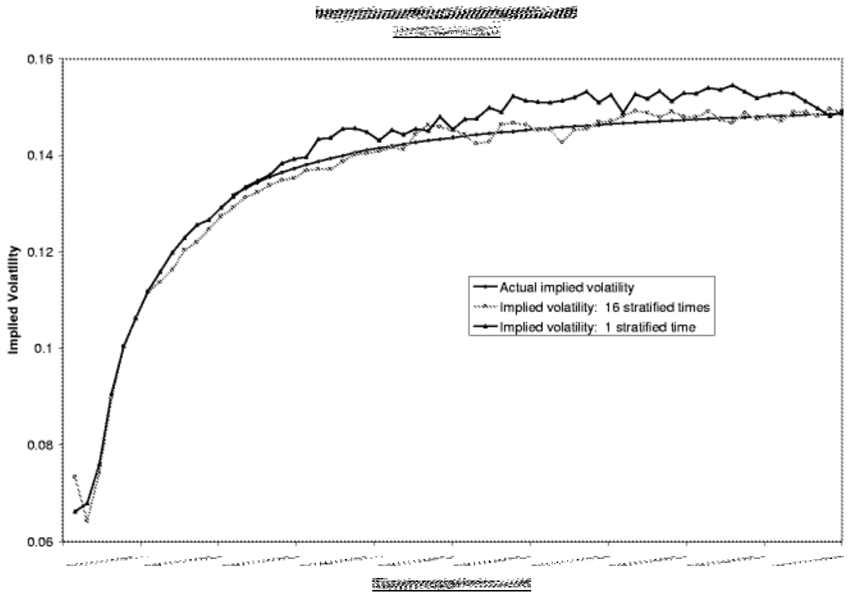

Figure 1: Comparison of implied volatilities: 1 and 16 stratification times

deviations, based on 100 replications of the full Monte Carlo procedure, are shown in round brackets. Times in seconds for a single replication are shown in square brackets. For the options investigated, the standard error and actual standard deviation of plain Monte Carlo are very similar, so only the standard

Each option is priced under varying numbers of reset times per year, from 4 to 256, corresponding to quarterly up to approximately daily reset frequencies. The number of times steps is equal to the number of reset times. With N reset times, resets are at times $\frac{1}{N}, \frac{2}{N}, \ldots, 1$. Bridge Monte Carlo is implemented with from 1 to 16 stratification times. With K stratification times, stratifications are at times $\frac{1}{K}, \frac{2}{K}, \ldots, 1$. When the number of stratification times equals the number of reset times, the method is fully low discrepancy and non-stochastic. Results in this case are based on a single replication and no standard deviation is reported. For options with 4, 8 and 16 resets we 'benchmark' by pricing using fully low discrepancy sampling with $M = 1,000,000$ sample paths. We note that convergence in M for fully low discrepancy methods is not uniform.

4.3.1 Average rate options

Table 5 shows results for average rate call options. The payoff at time one is $H_T = \max(A - X, 0)$ where A is the arithmetic average of the asset value at each reset time and $X = 101$ is the exercise price.

We see that the standard deviation decreases significantly with each additional level of stratification. Computation times also increase. Doubling the number of stratification times roughly halves the standard deviation but only approximately doubles the computation time. This means that each additional level of stratification is approximately doubling the efficiency gain. These are [illegible]

Efficiency gains are most pronounced for options with greater numbers of reset times, but even the quarterly reset option with two stratification times is 7 times faster than plain Monte Carlo. For the daily reset case ($N = 256$) using [illegible] Monte Carlo. We have no reason to suppose that efficiency gains would not [illegible]

[illegible]

Table 7 shows results for lookback call options. The payoff at time one is $H_T = \max(S_T - M, 0)$ where M is the minimum of the asset values at each [illegible]

For lookbacks we see that increasing the number of stratification times brings increasing efficiency gains, although less so than for the average rate option. The gains are most pronounced for the daily reset lookback with an efficiency gain of 129 with 16 stratification times. Efficiency gains are greater for options with [illegible]

[illegible]

We report results for pricing of up-and-in call options. Pricing for down-and-out barrier options was also investigated but as we found similar results to the [illegible]

'Out'-type barrier options may be given a zero payoff immediately that an asset value is generated that has hit the barrier, and no further asset values along that sample path need be generated. An analogous speed-up is possible for 'In'-type barrier options. Once the barrier is hit the bridge Monte Carlo method requires no further asset values to be generated since the bridge has already generated the terminal asset value. The plain Monte Carlo method requires the generation of one further asset value, for the terminal time.

Tables 9 shows results for an up-and-in call option. The payoff at time one is $H_T = \max(S_T - X, 0)$ where X is the exercise price, and where the payoff is conditional on the asset value exceeding the barrier level B on at least one reset time. The table gives results for $B = 120$. Table 10 gives efficiency gains for these options. We see that a single stratification, at the terminal time, gives efficiency gains from around 12 up to 23 for daily reset options. Additional stratification does not necessarily increase the efficiency, although there are gains for the daily reset option. On the whole, efficiencies do not decrease by much, implying that there is likely to be little loss from additional stratification.

Investigation showed that efficiency gains are greater when the barrier level B is further away from the initial asset value S_0. When B is close to S_0 the gains are considerably less. For down-and-out options with $B = 100$, efficiency gains of around 2 were obtained for one level of stratification, at the terminal time. However, further stratifications do not lead to further gains. On the contrary, for all except the option with 256 reset times, the efficiency gains diminish.

We attribute the pricing behaviour of barrier options to two factors. The first is that the values of barrier options depend on the path of asset values only through the hitting, or otherwise, of a barrier level; payoffs are not computed directly from intermediate asset values. In this sense barrier options are 'less' path dependent than average rate or lookback options whose payoffs depend directly upon intermediate asset values and whose valuation benefits from a more sophisticated sampling at intermediate times. The second factor is the efficiency of the algorithms used to compute the inverse of the beta distribution function. In fact, from table 5, for instance, we can estimate that (for $M =$ [illegible] about 0.2 seconds, the gamma stratification step at the terminal time takes about 0.6 seconds and a beta stratification step takes about 5 seconds. Were a beta stratification step to be as fast as the gamma stratification step a daily [illegible]

5 Conclusions

We have shown how a gamma bridge may be used in conjunction with stratified sampling in the variance-gamma model to give much improved Monte Carlo estimates of option values, both for benchmark calls and for various path dependent options. We find efficiency gains of a factor of around 380 for average rate options and 130 for lookback options. There are also gains for barrier options [illegible]

The use of the bridge Monte Carlo technique should be considered whenever [illegible] of barrier options it appears that the greater the effective degree of path dependence, the greater are the efficiency gains due to the use of bridge Monte Carlo.

Bridge Monte Carlo may be used to maximum effect if an efficient algorithm is available to compute the inverse of the bridge distribution function. Very good algorithms exist to compute the inverse of the normal distribution. Further efficiency gains would be possible for the gamma bridge if improved algorithms for the computation of the functions G^{-1} and particularly for B^{-1} were available. We would then expect to see that stratifications at intermediate times for barrier options with asset values close to the barrier level could lead to much greater efficiency gains.

In principle the bridge Monte Carlo method is widely applicable, but its ease of application depends upon the nature of the conditional distribution function at intermediate times, and on the efficiency of available algorithms to compute the inverse of that distribution function.

For the variance-gamma process the use of the gamma-bridge is strongly recommended for appropriate applications.

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Table 5: Average Rate Call Options: Comparison of Plain and Bridge Monte Carlo

Average rate call options: Efficiency gains.						
K	**4 resets**	**8 resets**	**16 resets**	**32 resets**	**64 resets**	**256 resets**
1	1.7	2.7	3.5	3.1	2.4	4.5
2	7.2	8.2	7.5	8.6	11	15
4	–	10	14	18	32	53
8	–	–	34	42	60	134
16	–	–	–	115	136	383

Table 6: Average Rate Call Options: Efficiency Gains for Bridge Monte Carlo over Plain Monte Carlo

[illegible]

Table 7: Lookback Call Options: Comparison of Plain and Bridge Monte Carlo

Lookback call options: Efficiency gains.						
K	**4 resets**	**8 resets**	**16 resets**	**32 resets**	**64 resets**	**256 resets**
1	5.3	4.4	6.9	5.5	7.2	9.0
3	11	11	13	13	11	20
4	–	9.2	11	20	28	40
8	–	–	16	33	42	59
16	–	–	–	40	45	129

Table 8: Lookback Call Options: Efficiency Gains for Bridge Monte Carlo over [illegible]

Up-and-In barrier call options: Times and standard deviations.						
K	**4 resets**	**8 resets**	**16 resets**	**32 resets**	**64 resets**	**256 resets**
0	7.0347 (0.011) [85.2]	7.2316 (0.011) [169.0]	7.3727 (0.011) [315.4]	7.4765 (0.011) [597.5]	7.5351 (0.011) [1157]	7.5851 (0.011) [4507]
1	7.0094 (0.018) [2.1]	7.2556 (0.024) [2.8]	7.4214 (0.028) [4.1]	7.4392 (0.028) [6.6]	7.5155 (0.027) [11.3]	7.5433 (0.027) [32.5]
2	7.0283 (0.014) [3.6]	7.2211 (0.019) [4.4]	7.4261 (0.023) [5.7]	7.4640 (0.026) [8.1]	7.5607 (0.024) [12.8]	7.5608 (0.025) [34.0]
4	7.0261 (–) [11.8]	7.2220 (0.014) [12.6]	7.4015 (0.017) [13.9]	7.4826 (0.021) [16.3]	7.5713 (0.018) [21.0]	7.6147 (0.017) [42.1]
8	–	7.2255 (–) [28.0]	7.4021 (0.013) [29.3]	7.5134 (0.018) [31.8]	7.5316 (0.014) [36.4]	7.6206 (0.015) [57.8]
16	–	–	7.3772 (–) [58.7]	7.4905 (0.008) [61.1]	7.5456 (0.011) [65.8]	7.5941 (0.013) [87.0]
Bench-mark	7.0268 (–) [1180]	7.2348 (–) [2801]	7.3857 (–) [5860]	–	–	–

Table 9: Up-and-In Barrier Call Options: Comparison of Plain and Bridge Monte Carlo. B = 120

Up-and-In barrier call options: Efficiency gains.						
K	**4 resets**	**8 resets**	**16 resets**	**32 resets**	**64 resets**	**256 resets**
1	15.2	12.7	11.9	14.0	17.0	23.0
2	14.6	12.9	12.7	13.2	19.0	25.7
4	–	8.3	9.5	10.1	20.6	44.8
8	–	–	7.7	7.0	19.6	41.9
16	–	–	–	18.5	17.6	37.1

Table 10: Up-and-In Barrier Call Options: Efficiency Gains for Bridge Monte [illegible]

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