VALUING VOLATILITY AND VARIANCE SWAPS FOR A NON-GAUSSIAN ORNSTEIN-UHLENBECK STOCHASTIC VOLATILITY MODEL

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Abstract. Following the increasing awareness of the risk from volatility fluctuations the markets for hedging contracts written on realised volatility has surged. Companies looking for means to secure against unexpected accumulation of market activity can find over-the-counter products written on volatility indices. Since the Black and Scholes model require a constant volatility the need to consider other models is obvious. We investigate swaps written on powers of realised volatility in the stochastic volatility model proposed by Barndorff-Nielsen and Shephard [3]. We derive a *key formula* for the realised variance and are able to represent the swap price dynamics in terms of Laplace transforms, which makes fast numerical inversion methods viable. We show an example using the fast Fourier transform and compare with the approximation proposed by Brockhaus and Long [7]

1. Introduction

A constant volatility is not able to explain the volatility clustering observed in financial markets, where periods of high activity and large price movements occur. An increasing awareness of the risk associated with the fluctuations in the market activity has led to a growing focus on stochastic volatility models. Making the volatility stochastic force the actors to consider the impact from changes in trading intensity and measures to hedge against unwanted effects. The risk from volatility movements can be hedged using financial instruments where the underlying asset is realised variance. Swaps on realised variance has been traded over the counter for several years, giving firms means to manage the perceived risk. The interest in such products indicates that actors perceives the uncertainty in the variance as a feature in the market, which they need to hedge themselves against. More recently, this has spun out to a fully fledged market for hedging and speculation in financial contracts on realised variance, like the CBOE S&P 500 Volatility Index (VIX).

The industry standard model for stock returns, the Black and Scholes model, gives no room for uncertainty in the volatility, since it is considered as a constants entity. It is well known that the model is unable to replicate the implied volatility smiles observed empirically, resulting in a flat implied volatility across strike and maturity. Clearly this is not viable when pricing contracts on realised variance and more realistic models are needed. The interest has focused on stochastic volatility models, including models with jumps in the volatility process, see for example Carr *et.al.* [8] who thoroughly investigate quadratic

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variance for infinite activity processes, more specifically the class of $CGMY$ processes. Stochastic volatility models are undeniably more complicated to work with compared to the Black and Scholes model due to the much richer structure of randomness.

We consider the problem of valuing volatility and variance swaps in the framework of the non-Gaussian Ornstein-Uhlenbeck model for stochastic volatility proposed by Barndorff-Nielsen and Shephard [3]. Instead of the constant volatility in the Black and Scholes market the volatility is stochastic and given as a mean-reverting process driven by a subordinator, *i.e.* a Lévy process with positive jumps and no continuous part. The model is able to replicate the skewness and fat tails seen in high-frequency stock returns and capture implied volatility smiles. Option pricing under the Barndorff-Nielsen and Shephard model is investigated by Nicolato and Venardos [12] and in an indifference pricing setting by Benth and Meyer-Brandis [5] and Benth and Groth [4].

Transform based option pricing methods were investigated in several papers before Carr and Madan [9] showed how to utilise the computational efficiency of the fast Fourier transform. Given the analytical form of the risk-neutral density the method is one of the swiftest numerical pricing algorithms. The drawback is that the risk-neutral density is not always available analytically. We will show that by casting the swap pricing problems in form of an (inverse) Laplace transform we may use the fast Fourier transform to simulate prices. We derive a general formula and provide an example when the stationary distribution of the Ornstein-Uhlenbeck process is Inverse Gaussian. We compare the numerical results with the approximation by Brockhaus and Long [7]. Moreover, swaptions on realised variance is also an applicable problem for the fast Fourier transform and we present a short description how to use the framework of Carr and Madan [9] to price them.

The rest of the paper is organised as follows: In the next section we review the Barndorff-Nielsen and Shephard stochastic volatility model, realised variance and swaps written on realised variance. Section 3 provides a *key formula* similar to the one found in Eberlein and Raible [10], the transform-based swap price dynamics and a subsection on options written on realised variance. Brockhaus and Long [7] suggested an approximation for the volatility swap price dynamics which is reviewed in section 4. In section 5 we give an example and compare the accuracy of the Brockhaus-Long approximation with numerical results using the fast Fourier transform on our transform-based swap price dynamics.

2. The volatility model of Barndorff-Nielsen and Shephard

The stochastic volatility model of Barndorff-Nielsen and Shephard (from now on called the BNS-model) appeared first in [3]. The BNS-model is a very flexible class of stochastic volatility models, being able to model accurately heavy tailed and skewed log-returns as well as the autocorrelation in the returns. We will present the model with some of its analytical properties being useful for our analysis of the volatility and variance swaps considered in this section and later.

Consider the probability space $(\Omega, \mathcal{F}, P)$ and assume the asset price evolves in time as

$$\mathrm{d}S(t) = \left(\mu + \beta\sigma^2(t)\right) S(t)\,\mathrm{d}t + \sqrt{\sigma^2(t)}S(t)\,\mathrm{d}B(t), \tag{2.1}$$

where $B(t)$ is a Brownian motion, μ and β constants and $\sigma^2(t)$ follows a non-Gaussian Ornstein-Uhlenbeck process. The idea of the BNS-model is to find an Ornstein-Uhlenbeck dynamics for which the marginal distribution and the autocorrelation structure of the log-returns are modelled separately. This is achieved by assuming

$$\mathrm{d}\sigma^2(t) = -\lambda\sigma^2(t)\,\mathrm{d}t + \mathrm{d}L(\lambda t), \tag{2.2}$$

where λ is a positive constant and L is the *background driving Lévy process* to be specified. We suppose L to be a subordinator to ensure the positivity of the process $\sigma^2(t)$. We denote $\{\mathcal{F}_t\}_{t\geq 0}$ the completion of the filtration $\sigma(B(s), L(\lambda s); s \leq t)$ generated by the Brownian motion and the subordinator such that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ becomes a complete filtered probability space. The Lévy measure is denoted $\ell(\mathrm{d}z)$, and is supported on the positive real line since L is a subordinator. Since the log-returns now become scaled mixtures of normal distributions, the marginal distribution of the log-returns are modelled (indirectly) by assuming a specific stationary distribution for $\sigma^2(t)$. Given this specification, there will exist a subordinator process L such that $\sigma^2(t)$ is the solution of the Ornstein-Uhlenbeck equation (2.2). Moreover, the autocorrelation function for (the stationary) $\sigma^2(t)$ is $r(u) = \exp(-\lambda|u|)$. The reason for the unusual time scaling $L(\lambda t)$ in the dynamics for $\sigma^2(t)$ is namely the separation of the modelling of autocorrelation (i.e. the time dynamics of the volatility) and the invariant distribution (i.e. the marginal distribution for the log-returns). Note that from Itô's Formula for semimartingales it follows that for $s \leq t$

$$\sigma^2(t) = \sigma^2(s)\mathrm{e}^{-\lambda(t-s)} + \int_s^t \mathrm{e}^{-\lambda(t-u)}\,\mathrm{d}L(\lambda u).$$

A more general autocorrelation structure is obtained by a superposition of m different non-Gaussian Ornstein-Uhlenbeck processes: Let $w_k, k = 1, 2, \ldots, m$, be positive weights summing to one, and define

$$\sigma^2(t) = \sum_{k=1}^{m} w_k Y_k(t), \tag{2.3}$$

where

$$\mathrm{d}Y_k(t) = -\lambda_k Y_k(t)\,\mathrm{d}t + \mathrm{d}L_k(\lambda_k t), \tag{2.4}$$

for independent background driving Lévy processes L_k, $k = 1, \ldots, m$. We denote the corresponding Lévy measures $\ell_k(\mathrm{d}z)$, $k = 1, \ldots, m$, which all are supported on the positive real line under the assumption that the L_k's are subordinators. The autocorrelation function for the stationary $\sigma^2(t)$ then becomes

$$r(u) = \sum_{k=1}^{m} \widetilde{w}_k \exp(-\lambda_k|u|),$$

thus allowing for much more flexibility in modelling long-range dependency in log-returns. The weight functions $\widetilde{w}_k$ in autocorrelation function $r(u)$ are proportional to $w_k\mathrm{Var}(L_k)$.

As earlier literature has shown, the log-returns of financial data can be successfully modelled by the normal inverse Gaussian (NIG) distribution (see e.g. Barndorff-Nielsen and

Shephard [3] and the references therein). Following the discussion of Barndorff-Nielsen and Shephard [3], we may derive the background driving Lévy process yielding NIG-distributed log-returns by specifying the marginal law of $\sigma^2(t)$ to be generalised inverse Gaussian, $\sigma^2(t) \sim \mathrm{GIG}(\nu, \delta, \gamma)$. The density of a the $\mathrm{GIG}(\nu, \delta, \gamma)$ is

$$\frac{(\gamma/\delta)^\nu}{2K_\nu(\delta\gamma)} x^{\nu-1} \exp\left(-\frac{1}{2}\left(\delta^2 x^{-1} + \gamma^2 x\right)\right),$$

where K_ν is the Bessel function of third kind with index ν. The Lévy measure of the subordinator $L(t)$ becomes (see Thm. 2 in Barndorff-Nielsen and Shephard [3])

$$\ell(\mathrm{d}z) = z^{-1}\left\{\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\delta^{-2} z\xi\right) g_\nu(\xi)\,\mathrm{d}\xi + \lambda \max(0, \nu)\right\} \exp\left(-\frac{1}{2}\gamma^2 z\right)\,\mathrm{d}z,$$

where

$$g_\nu(x) = \frac{2}{x\pi^2}\left\{J_{|\nu|}^2(\sqrt{x}) + Y_{|\nu|}^2(\sqrt{x})\right\}^{-1},$$

and $J_{|\nu|}$ and $Y_{|\nu|}$ are Bessel functions[1] of the first and second kind, respectively, with index (or order) $|\nu|$ (see e.g. Abramowitz and Stegun [1], Section 9.1). Specifying the non-Gaussian Ornstein-Uhlenbeck process $\sigma^2(t)$ with this background driving Lévy process, the log-returns will become approximately generalised hyperbolic distributed, including the cases of NIG (with $\nu = -1/2$) and hyperbolic (with $\nu = 1$). Note that the parameter α in the NIG-distribution is given as $\alpha = \sqrt{\beta^2 + \gamma^2}$.

The realised volatility $\sigma_R(T)$ over a period $[0, T]$ is defined as

$$\sigma_R(T) = \sqrt{\frac{1}{T}\int_0^T \sigma^2(s)\,\mathrm{d}s}.$$

The quadratic variation of the log-prices $\ln S(t)$ is connected to the realised volatility by the following relation:

$$[\ln S](t) := p\lim_{r\to\infty}\sum_{i=1}^{m_r}(\ln S(t_{i+1}^r) - \ln S(t_i^r))^2 = \int_0^t \sigma^2(s)\,\mathrm{d}s$$

for any sequence of partitions $t_0^r = 0 < t_1^r < \ldots < t_{m_r}^r$ with $\sup_i(t_{i+1}^r - t_i^r) \to 0$ for $r \to \infty$.

A volatility swap is a forward contract that pays to the holder the amount

$$c\left(\sigma_R(T) - \Sigma\right)$$

where Σ is a fixed level of volatility and the contract period is $[0, T]$. The constant c is a factor converting volatility surplus or deficit into money. For simplicity, we choose $c = 1$ in this paper. The fixed level of volatility Σ is chosen so that the swap has a risk-neutral price equal to zero, that is, at time $0 \le t \le T$, the fixed level is given as the conditional risk-neutral expectation (using the adaptedness of the fixed volatility level):

$$\Sigma(t, T) = \mathbb{E}_Q\left[\sigma_R(T)\,|\,\mathcal{F}_t\right] \tag{2.5}$$

[1] There is an unfortunate duplication of notation here. It is customary to denote the Bessel function Y, which we chose to keep in the faith that the reader will understand what is what from the context

where Q is an equivalent martingale measure. As can be seen, this is nothing but a forward contract written on realised volatility. As special cases, we obtain

$$\Sigma(0,T) = \mathbb{E}_Q\left[\sigma_R(T)\right]$$
$$\Sigma(T,T) = \sigma_R(T).$$

In a completely similar manner, we define a variance swap to have the price

$$\Sigma_2(t,T) = \mathbb{E}_Q\left[\sigma_R^2(T)\,|\,\mathcal{F}_t\right]. \tag{2.6}$$

To have a more compact notation, we define for $\gamma > -1$

$$\Sigma_{2\gamma}(t,T) = \mathbb{E}_Q\left[\sigma_R^{2\gamma}(T)\,|\,\mathcal{F}_t\right]. \tag{2.7}$$

Below, we shall derive pricing dynamics for swaps written on all powers of the realised volatility σ_R bigger than -2. Of course, our concern is the volatility and variance swap prices, corresponding to $\gamma = 1/2$ and $\gamma = 1$, resp. However, as we shall see below, our framework gives prices that naturally extends to any $\gamma > -1$.

3. Valuation of volatility and variance swaps using the Laplace transform

We construct martingale measures Q using the Esscher transform, following the analysis in Benth and Saltyte-Benth [6]. Assume $\theta_k(t)$, $k = 1,\ldots,m$ are real-valued measurable and bounded functions. Consider the stochastic process

$$Z^\theta(t) = \exp\Big(\sum_{k=1}^m\Big(\int_0^t \theta_k(s)\,\mathrm{d}L_k(\lambda_k s) - \int_0^t \lambda_k\psi_k(\theta_k(s))\,\mathrm{d}s\Big)\Big),$$

where $\psi_k(x)$ are the log-moment generating functions of $L_k(t)$, e.g. $\psi_k(x) = \ln\mathbb{E}[\exp(xL_k(1))]$. For many natural choices of L_k these functions are explicitly known. We refer the reader to Section 5 for one example. Let us impose an exponential integrability condition on the Lévy measure ensuring existence of moments.

Condition (**L**): There exist a constant $\kappa > 0$ such that the Lévy measure satisfies the integrability condition

$$\int_1^\infty e^{z\kappa}\ell_k(\mathrm{d}z) < \infty.$$

The processes $Z^\theta(t)$ are well-defined under natural exponential integrability conditions on the Lévy measures ℓ_k which we assume to hold. That is, they are well defined for $t \in [0,T]$ if condition (**L**) holds for $\kappa = \sup_{k=1,..,m,s\in[0,T]}|\theta_k(s)|$. Introduce the probability measure

$$Q^\theta(A) = \mathbb{E}[1_A Z^\theta(\tau_{\max})],$$

where 1_A is the indicator function and $\tau_{\max}$ is a fixed time horizon including all the trading times. We denote the expectation under the probability Q^θ by $\mathbb{E}_\theta[.]$. By using the time varying θ's we have a flexible class of martingale measures Q^θ of which we shall call θ the "market price of risk".

The following key formula for $\sigma_R^2(T)$ is useful when deriving explicit pricing formulas for the swaps in terms of Fourier transforms:

Lemma 3.1. *Let $z \in \mathbb{C}$ and $\theta_k : \mathbb{R}_+ \longrightarrow \mathbb{R}, k = 1, \ldots, m$ be real-valued measurable functions. Suppose condition* **(L)** *is satisfied and well defined for $|Re(z)| < [\frac{\lambda_k^{-1}}{T}(1 - e^{-\lambda_k(T-s)})]^{-1}\kappa$ for all k, where $\kappa = \sup_{k=1,..,m,s\in[0,T]} |\theta_k(s)|$. Then*

$$\mathbb{E}_\theta\left[e^{z\sigma_R^2(T)}\,|\,\mathcal{F}_t\right] = \exp\left(\sum_{k=1}^m \lambda_k\left(\int_t^T \psi_k\left(\frac{z\omega_k}{\lambda_k T}(1-e^{-\lambda_k(T-s)})+\theta_k(s)\right) - \psi_k(\theta_k(s))\,\mathrm{d}s\right)\right)$$
$$\times\exp\left(\frac{z}{T}\left(t\sigma_R^2(t)+\sum_{k=1}^m \frac{1}{\lambda_k}(1-e^{-\lambda_k(T-t)})\omega_k Y_k(t)\right)\right).$$

Proof. From Bayes' Formula it follows

$$\begin{aligned}\mathbb{E}_\theta\left[\exp\left(z\sigma_R^2(T)\right)\,|\,\mathcal{F}_t\right] &= \mathbb{E}\left[\exp\left(\sum_{k=1}^m \frac{z\omega_k}{T}\int_0^T Y_k(s)\,\mathrm{d}s\right)\frac{Z^\theta(T)}{Z^\theta(t)}\,\Big|\,\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\exp\left(\sum_{k=1}^m\left(\frac{z\omega_k}{T}\int_0^T Y_k(s)\,\mathrm{d}s+\int_t^T\theta_k(s)\,dL_k(\lambda_k s)\right)\right)\,\Big|\,\mathcal{F}_t\right]\\ &\quad\times\exp\left(\sum_{k=1}^m -\lambda_k\int_t^T\psi_k(\theta_k(s))\,\mathrm{d}s\right).\end{aligned}$$

Since $\sigma^2(s)$ is $\mathcal{F}_s$-adapted, we have

$$\begin{aligned}\mathbb{E}_\theta\left[\exp\left(z\sigma_R^2(T)\right)\,|\,\mathcal{F}_t\right] &= \mathbb{E}\left[\exp\left(\sum_{k=1}^m\left(\frac{z\omega_k}{T}\int_t^T Y_k(s)\,\mathrm{d}s+\int_t^T\theta_k(s)\,dL_k(\lambda_k s)\right)\right)\,\Big|\,\mathcal{F}_t\right]\\ &\quad\times\exp\left(\sum_{k=1}^m\left(\frac{z\omega_k}{T}\int_0^t Y_k(s)\,\mathrm{d}s-\lambda_k\int_t^T\psi_k(\theta_k(s))\,\mathrm{d}s\right)\right).\end{aligned}$$

To this end, recall from the dynamics of Y_k that

$$\lambda_k\int_t^T Y_k(s)\,\mathrm{d}s = -Y_k(T)+Y_k(t)+\int_t^T \mathrm{d}L_k(\lambda_k s),$$

and invoking the explicit expression for $Y_k(T)$ yields

$$\int_t^T Y_k(s)\,\mathrm{d}s = \frac{1}{\lambda_k}Y_k(t)\left(1-\mathrm{e}^{-\lambda_k(T-t)}\right)+\frac{1}{\lambda_k}\int_t^T\left(1-\mathrm{e}^{-\lambda_k(T-s)}\right)\,\mathrm{d}L_k(\lambda_k s).$$

Thus,

$$\begin{aligned}\mathbb{E}_\theta\left[e^{\left(z\sigma_R^2(T)\right)}\Big|\,\mathcal{F}_t\right] &= \mathbb{E}\left[\exp\left(\sum_{k=1}^m\left(\int_t^T\frac{z\omega_k}{\lambda_k T}\left(1-\mathrm{e}^{-\lambda_k(T-s)}\right)+\theta_k(s)\,\mathrm{d}L_k(\lambda_k s)\right)\right)\,\Big|\,\mathcal{F}_t\right]\\ &\quad\times\exp\left(\frac{z}{T}t\sigma_R^2(t)+\sum_{k=1}^m\left(\frac{z\omega_k}{T\lambda_k}\left(1-\mathrm{e}^{-\lambda_k(T-t)}\right)-\lambda_k\int_t^T\psi_k(\theta_k(s))\,\mathrm{d}s\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\sum_{k=1}^{m} \lambda_k \left(\int_t^T \psi_k\left(\frac{z\omega_k}{\lambda_k T}(1-\mathrm{e}^{-\lambda_k(T-s)}) + \theta_k(s)\right) - \psi_k(\theta_k(s))\,\mathrm{d}s\right)\right) \\
&\quad \times \exp\left(\frac{z}{T}\left(t\sigma_R^2(t) + \sum_{k=1}^{m}\frac{1}{\lambda_k}(1-\mathrm{e}^{-\lambda_k(T-t)})\omega_k Y_k(t)\right)\right),
\end{aligned}
$$

where we have used the independent increment property of the subordinator. Hence, the proof is complete. □

We remark that a related formula can be found in Eberlein and Raible [10], with a further generalization in Nicolato and Venardos [12].

Applying the key formula in Lemma 3.1, we are now in the position to derive representations of the swap price dynamics in terms of Laplace transforms. The details are given in the next Proposition:

Proposition 3.2. *For every $\gamma > -1$ and any $c > 0$ satisfying $c < [\frac{\lambda_k^{-1}}{T}(1-e^{-\lambda_k(T-s)})]^{-1}\kappa$ for all k, where $\kappa = \sup_{k=1,..,m,s\in[0,T]} |\theta_k(s)|$, it holds*

$$
\begin{aligned}
\Sigma_{2\gamma}(t,T) = \frac{\Gamma(\gamma+1)}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} & z^{-(\gamma+1)}\Psi_\theta(t,T,z) \\
& \times \exp\left(\frac{z}{T}\left(t\sigma_R^2(t) + \sum_{k=1}^{m}\frac{\omega_k Y_k(t)}{\lambda_k}(1-e^{-\lambda_k(T-t)})\right)\right)\,\mathrm{d}z\,,
\end{aligned}
$$

where

$$
\Psi_\theta(t,T,z) = \exp\left(\sum_{k=1}^{m}\lambda_k\left(\int_t^T \psi_k\left(\frac{z\omega_k}{\lambda_k T}\left(1-e^{-\lambda_k(T-s)}\right)+\theta_k(s)\right) - \psi_k(\theta_k(s))\,\mathrm{d}s\right)\right).
$$

Proof. We know from the theory of Laplace transforms that

$$
x^\gamma = \frac{\Gamma(\gamma+1)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} z^{-(\gamma+1)}\mathrm{e}^{zx}\,\mathrm{d}z\,,
$$

for any $c > 0$ and $\gamma > -1$. Thus, under the conditions of the Proposition making the moment generating function well-defined, we have

$$
\Sigma_{2\gamma}(t,T) = \frac{\Gamma(\gamma+1)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} z^{-(\gamma+1)}\mathbb{E}_\theta\left[\exp\left(z\sigma_R^2(T)\right) \mid \mathcal{F}_t\right]\,\mathrm{d}z\,.
$$

Applying the Key Formula in Lemma 3.1 gives the desired result. □

We remark that the expression for the swap prices in the Proposition above is suitable for numerical calculations based on the fast Fourier transform (FFT) or other fast numerical inversion techniques for the Laplace transform. This will be the topic in Section 5.

The variance swap price has an explicit expression, which is stated in the Proposition below.

Proposition 3.3. *The variance swap has a price given by the following expression:*

$$\Sigma_2(t,T) = \frac{t}{T}\sigma_R^2(t) + \sum_{k=1}^{m} \frac{\omega_k}{T\lambda_k}\left(1 - e^{-\lambda_k(T-t)}\right)Y_k(t) +$$

$$+ \sum_{k=1}^{m} \left[\frac{\omega_k}{T}\int_t^T \psi_k'(\theta_k(s))(1 - e^{-\lambda_k(T-s)})\,\mathrm{d}s\right]. \tag{3.1}$$

Proof. We can prove this directly by using $z \in \mathbb{R}$, differentiating with respect to z in the Key Formula in Lemma 3.1 and then let $z = 0$. □

Observe that the swap prices $\Sigma_{2\gamma}$ at time t are dependent both on the current level of the variance $\sigma^2(t)$ and the realised variance $\sigma_R^2(t)$. Based on this, we can go further and price options written on the swaps.

3.1. **Options.** Let f be a real-valued measurable function with at most linear growth. Then the fair price $C(t)$ at time t of an option price paying $f(\Sigma_{2\gamma}(\tau, T))$ at exercise time $\tau > t$ is given by

$$C(t) = e^{-r(\tau-t)}\mathbb{E}_\theta[f(\Sigma_{2\gamma}(\tau,T))\,|\,\mathcal{F}_t],$$

where $\Sigma_{2\gamma}(\tau, T)$ is given in Proposition 3.2, with $T > \tau$.

For the variance swap the explicit solution in Proposition 3.3 leads to a formulation of the option pricing problem where the fast Fourier transform is applicable. We focus our discussion on call options. Using the approach by Carr and Madan [9] we can formulate the price of a call option as an inverse Fourier transform in the strike price K. Let $\widetilde{K} = \ln(K)$ be the log of the strike price. After introducing an exponential damping to get a square integrable function we can represent the price of the option as

$$C(t) = \frac{\exp(-\alpha\widetilde{K})}{\pi}\int_0^\infty \mathrm{e}^{-\mathrm{i}v\widetilde{K}}\Phi(v)\,\mathrm{d}v \tag{3.2}$$

where

$$\Phi(v) = \int_{-\infty}^{\infty} \mathrm{e}^{iv\widetilde{K}}\mathbb{E}_\theta\left[\mathrm{e}^{-r(\tau-t)}\mathrm{e}^{\alpha\widetilde{K}}\left(e^{\Sigma_2(\tau,T)} - \mathrm{e}^{\widetilde{K}}\right)^+ \,|\, \mathcal{F}_t\right]\,\mathrm{d}\widetilde{K}.$$

Using the explicit expression for the variance swap (3.3), the explicit solution for the non-Gaussian Ornstein-Uhlenbeck processes $Y_k(t)$ and the independent increments property of the subordinators we get that

$$\begin{aligned}\Phi(v) = &\frac{\mathrm{e}^{-r(\tau-t)}}{(\alpha+1)(\alpha+1+\mathrm{i}v)}\\ &\times \exp\left((1+\alpha+\mathrm{i}v)\sum_{k=1}^{m}\frac{\omega_k Y_k(t)}{\lambda_k T}\left(\tau + (1-\tau)\mathrm{e}^{-\lambda_k(\tau-t)} - \mathrm{e}^{-\lambda_k(T-t)}\right)\right)\\ &\times \exp\left((1+\alpha+\mathrm{i}v)\left(\frac{\tau}{T}\sigma_R^2(t) + \sum_{k=1}^{m}\frac{\omega_k}{T}\int_\tau^T \psi_k'(\theta_k(s))(1 - \mathrm{e}^{-\lambda_k(T-s)})\,\mathrm{d}s\right)\right)\end{aligned}$$

$$\times \exp\left(\sum_{k=1}^{m}\lambda_k\int_t^{\tau}\psi_k\left(\frac{\omega_k}{\lambda_k T}(1+\alpha+\mathrm{i}v)\left(\tau+(1-\tau)\mathrm{e}^{-\lambda_k(\tau-s)}-\mathrm{e}^{-\lambda_k(T-s)}\right)\right)\mathrm{d}s\right),$$

where we recall $\psi_k(\cdot)$ to be the log-moment generating functions of the subordinators L_k. The option price is then possible to calculate using fast Fourier transform of the integral in (3.2) following the outline in Carr and Madan [9].

4. An approximation of the volatility swap price dynamics

We have seen above how we can apply techniques based on the Laplace transform to derive formulas for the swap price dynamics. An alternative approach for volatility swaps is to derive an approximation from a second-order Taylor expansion of the function $\sqrt{x}$. This was suggested by Brockhaus and Long [7], and we now elaborate on this approximation for the BNS-model. Below we derive the approximate volatility swap price dynamics, and analyse the error made with this method in section 5.

The following Proposition holds true:

Proposition 4.1. *The volatility swap price dynamics can be expressed by*

$$\Sigma(t,T)=\frac{1}{2}\sqrt{\Sigma_2(0,T)}+\frac{\Sigma_2(t,T)}{2\sqrt{\Sigma_2(0,T)}}-\frac{\Sigma_4(t,T)-2\Sigma_2(0,T)\Sigma_2(t,T)+\Sigma_2^2(0,T)}{8\Sigma_2^{3/2}(0,T)}+R(t,T)\,,$$

where

$$R(t,T)=\frac{1}{32}\mathbb{E}_\theta\left[\frac{\left(\sigma_R^2(T)-\Sigma_2(0,T)\right)^3}{\left(\Sigma_2(0,T)+\Theta\left(\sigma_R^2(T)-\Sigma_2(0,T)\right)\right)^{5/2}}\,|\,\mathcal{F}_t\right],$$

and Θ *is a random variable such that* $0<\Theta<1$.

Proof. For a positive random variable X, a second-order Taylor approximation of $\sqrt{X}$ around $\mathrm{E}_\theta[X]$ with remainder term gives

$$\begin{aligned}\sqrt{X}&=\sqrt{\mathbb{E}_\theta\left[X\right]}+\frac{1}{2\sqrt{\mathbb{E}_\theta[X]}}\left(X-\mathbb{E}_\theta[X]\right)-\frac{1}{8}\frac{1}{\mathbb{E}_\theta[X]^{3/2}}\left(X-\mathbb{E}_\theta[X]\right)^2+R_X\\&=\frac{1}{2}\sqrt{\mathbb{E}_\theta[X]}+\frac{1}{2}\frac{X}{\sqrt{\mathbb{E}_\theta[X]}}-\frac{1}{8}\frac{\left(X-\mathbb{E}_\theta[X]\right)^2}{\mathbb{E}_\theta[X]^{3/2}}+R_X\,,\end{aligned}$$

where the remainder term is

$$R_X=\frac{1}{32}\frac{\left(X-\mathbb{E}_\theta[X]\right)^3}{\left(\mathbb{E}_\theta[X]+\Theta(X-\mathbb{E}_\theta[X])\right)^{5/2}}\,.$$

Thus, letting $X=\sigma_R^2(T)$, and taking conditional expectation together with the definition of $\Sigma_{2\gamma}$, yields the result. □

With the dynamics of $\Sigma_4(t,T)$ given by Proposition 3.2, we can derive an approximative dynamics of the volatility swap price $\Sigma(t,T)$ based on the expression in Proposition 4.1 by ignoring the $R(t,T)$-term. How good this approximation is depends of course on the size of the remainder. We analyse the remainder term numerically in the next section.

5. Numerical studies of volatility and variance swaps

In the previous sections we have seen how the price of swaps written on all powers of realised volatility can be expressed as an inverse Laplace transform. This representation opens up for numerical solution using some inversion technique, such as the fast Fourier transform (FFT). In this section we show how to utilise the computational power of the FFT to evaluate swap prices and give a few numerical examples.

The fast Fourier method is a computationally efficient way to do the discrete Fourier transform

$$\omega(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j), \text{ for } k = 1, \ldots, N, \tag{5.1}$$

when N is a power of 2, reducing the number of multiplications from order N^2 to $N \ln_2(N)$. The use of the fast Fourier transform for option pricing was investigated by Carr and Madan [9]. The possibility to use pre-implemented and optimised versions of the algorithm from software packages, together with its speed and simplicity, makes it a competitive method. The only requirement is that we know the characteristic function of the density analytically.

Proposition 3.2 gives the price of a swap as the inverse Laplace transform of a function on a form suitable for the (inverse) fast Fourier transform. To begin with we need to discretise both z and σ_R and approximate the integral with a finite sum. As we see from the formula we actually need to discretise $\widetilde{\sigma}^2 := \sigma_R^2 \times t/T$, hence we get a time scaling of the output variable. Since FFT are restricted by sampling constraints this have the undesirable consequence that if t is small compared to T we get few data points in the domain of interest. To make the best use of the computational efficiency we let N be a power of 2 and choose $\Delta\widetilde{\sigma}^2$ sufficiently small. The discretised variable is then $\widetilde{\sigma}^2(j) = \Delta\widetilde{\sigma}^2 * (j-1)$. To rewrite the sum in the standard form of the fast Fourier transform it requires that

$$\Delta z = \frac{2\pi}{N\Delta\widetilde{\sigma}^2}$$

and $z(k) = c + i\Delta z(k-1)$. Applying this discretisations gives us a summation of the form (5.1).

The background driving Lévy processes L_k have to be specified to get the log-moment generating functions explicitly. The standard approach is to specify a stationary distribution of the Ornstein-Uhlenbeck process and then derive the log-moment generating function for the Lévy process from the distribution. Two popular distributions are the inverse Gaussian and variance-gamma, see Barndorff-Nielsen and Shephard [2], Carr and Madan [9], Nicolato and Venardos [12]. Here we only consider the inverse Gaussian distribution, and in this case the log-moment generating function is given by Nicolato and Venardos [12] as

$$\psi(\theta) = \theta\delta(\gamma^2 - 2\theta)^{1/2}.$$

After rewriting the integrand to simplify the simulations we implement it using Matlab's predefined command for applying FFT.

α	β	μ	δ
233.0	5.612	-5.331×10^{-4}	0.0370

TABLE 1. Estimated parameters for the NIG-distribution

	λ	ω
OU_1	0.9127	0.9224
OU_2	0.0262	0.0776

TABLE 2. Estimated parameters for the decay rates and weights of the OU-processes

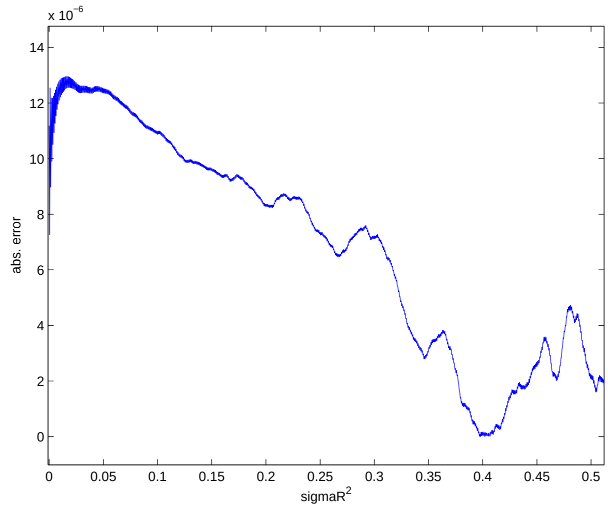

FIGURE 1. Absolute error between the explicit and FFT-solution of the variance swap price as a function of σ_R.

When specifying the stationary distribution of the Ornstein-Uhlenbeck process to be inverse Gaussian the log-returns of the stock will be approximately normal inverse Gaussian distributed. We use parameters for the normal inverse Gaussian distribution estimated by Lindberg [11] for the Swedish company AstraZeneca. The parameters are estimated based on daily log-returns over the period August 1, 2003 to June 1, 2004, see Table 1. Following the analysis of Lindberg [11] we assume that we have the superposition of two Ornstein-Uhlenbeck processes, both with inverse Gaussian law. The rates of decay and weights were also estimated at the same time, see Table 2. Left unknown are estimates of the current level of variance for both processes. For the purpose of illustration we choose these in such a way that multiplied with the weights and added they equal the variance of the NIG distribution. With the parameters in Table 1 we get that the variance of the NIG distribution is 1.59×10^{-4} and for the numerical tests we then let $Y_1(t) = 1.66 \times 10^{-4}$ and $Y_2(t) = 7.5 \times 10^{-5}$.

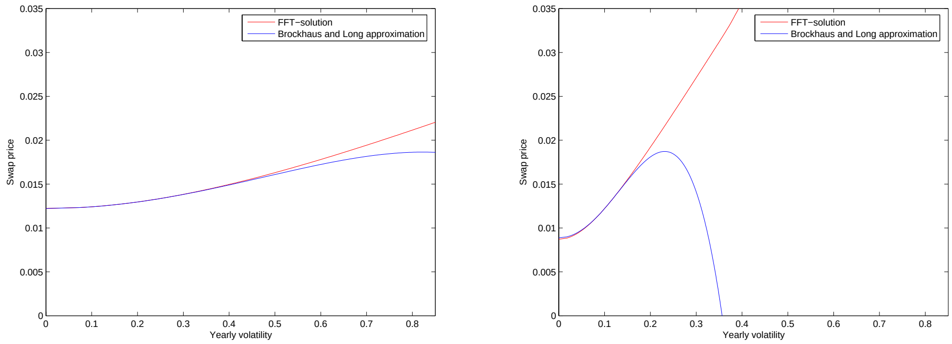

FIGURE 2. Comparison between the Brockhaus and Long approximation and the FFT-solution for the volatility swap price as a function of yearly volatility. Left:$t = 1, T = 31$, Right: $t = 31, T = 61$

The variance swap has the explicit solution given in Proposition 3.3 which we use as a benchmark for the FFT-method. We use 2^{15} points which gives a good tradeoff between speed and accuracy, and we can choose the step size to be $\Delta\widetilde{\sigma}^2 = 0.0005$. We let $t = 31$ and $T = 61$ and plot the difference between the explicit solution and the result from the FFT-method. Figure 1 shows that we have a absolute error in the order of 10^{-5} or below for the simulation. We account the error in the prices to the precision of the FFT-algorithm. Using another set of times, $t = 1, T = 31$, gives similar results but with less data points in the domain of interest because of the unfortunate time scaling of the output variable.

Turning to the volatility swap we now want to compare the FFT method with the approximation of Brockhaus and Long discussed in previous section. The approximation requires values for the variance swap prices, both for time zero and t. We use the explicit solution (3.1) for the variance swap prices, including the case $t = 0$, as calculated above. We simulate for the same two sets of times, first $t = 1, T = 31$ and second $t = 31, T = 61$ and plot the resulting price lines for the two methods. As seen in figure 2 the Brockhaus and Long approximation is reasonable for values close to the expected value of the realised variance at time zero, which is approximately 0.1. When the realised variance σ_R^2 approaches higher values the approximation is increasingly poor. We notice that the Brockhaus and Long method performs better when the fraction t/T is small. This is related to the values of the variance swap being smaller which makes the Taylor expansion less sensible.

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