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Vanishing Theorems and String Backgrounds

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Abstract

We show various vanishing theorems for the cohomology groups of compact hermitian manifolds for which the Bismut connection has (restricted) holonomy contained in $SU(n)$ and classify all such manifolds of dimension four. In this way we provide necessary conditions for the existence of such structures on hermitian manifolds. Then we apply our results to solutions of the string equations and show that such solutions admit various cohomological restrictions like for example that under certain natural assumptions the plurigenera vanish. We also find that under some assumptions the string equations are equivalent to the condition that a certain vector is parallel with respect to the Bismut connection.

1 Introduction

Riemannian manifolds equipped with a closed form have found many applications in various branches of mathematics and physics. In physics, the classical example is that of manifolds equipped with a closed two-form which describe gravity in the presence of a Maxwell field. More recently, Riemannian or pseudo-Riemannian manifolds M equipped with (closed) forms of any degree arise in the context of string theory. In particular in string compactifications, one investigates manifolds $\mathbb{R}^k \times M$, where M is a compact manifold of dimension $10 - k$. For most applications, the closed forms vanish and M is required to have special holonomy [15, 47]. Then much of the physical data, like the particle spectrum on $\mathbb{R}^k$, which are extracted from these compactifications, are determined in terms of the cohomological information on M, like for example Euler and Hodge numbers. Recently however, compactifications have also been considered for which the closed forms do not vanish [51, 10, 31]. Amongst the various closed forms that appear in string theory, there is a closed three-form H which is associated to the fundamental string. Such compactifications have been investigated in [51]. In this case, this three-form can be interpreted as torsion of a metric connection, ∇, on M. For compactifications of string theory with non-vanishing H, it is required that the connection with torsion ∇ has holonomy which is contained in $SU(n)$ ($n = 2, 3, 4$), G_2 or Spin(7). The case of holonomy $SU(n)$ is of particular interest because the underlying manifolds M are hermitian. The connection with torsion can then be identified with the Bismut connection [11]; for the various connections on hermitian manifolds see [52, 53]. Other applications include the geometry on the target spaces of supersymmetric one- [16] and two-dimensional sigma models [21, 33]. More recently it has been shown that the geometry on the moduli space of certain five-dimensional black holes is hyper-Kähler with torsion (HKT)[34], ie the holonomy of the Bismut connection is in $Sp(k)$ [29, 43, 32], .

In mathematics, a natural arena for the investigation of metric connections with torsion a three-form is the theory of Hermitian manifolds. Hermitian manifolds apart from the Bismut connection that has been mentioned above also admit the Chern connection (see for example [52, 53]). Both connection have holonomy which is contained in $U(n)$. The Chern connection, in addition, has curvature which is a (1,1)-form with respect to the natural complex structure and therefore it is compatible with the holomorphic structure of the tangent bundle. Consequently, many theorems regarding the non-existence of holomorphic sections have been expressed in terms of conditions on the curvature of the Chern connection. Recently, however, vanishing theorems for Dolbeault cohomology groups have been expressed in terms of conditions on the curvature of the Bismut connection [5].

In string theory, the metric g and the three-form H of a manifold M are required to satisfy the 'string equations' which are a generalization of the Einstein equations of general relativity. For this, in addition, a function ϕ is introduced on M, called dilaton, and the (type II) string equations can be written as follows:

$$\begin{aligned} Ric^g_{ij} - \frac{1}{4} H_{imn} H_j{}^{mn} + 2\nabla^g_i \partial_j \phi = 0, \\ \nabla^g_i \left(e^{-2\phi} H^{imn} \right) = 0 \ , \end{aligned} \tag{1.1}$$

where Ric^g and ∇^g are the Ricci tensor and the Levi-Civita connection of the metric g. The three-form H is closed in the context of (type II) string theory but we can take it to be any three-form. In what follows, if H is required to be closed, it will be mentioned explicitly. In fact there is a third field equation that associated with the dilaton ϕ. However the dilaton equation is implied from those of (1.1) up to a constant and so we shall not further investigate it. The above equations are also supplemented with two additional equations, called Killing spinor equations, the following:

$$\begin{aligned} \nabla\eta &= 0 \\ (d\phi - \frac{1}{6}H)\eta &= 0 , \end{aligned} \tag{1.2}$$

where η is a spinor and we have denoted with the same symbol the forms and their associated Clifford algebra element. In fact in type II string theory there is an additional set of Killing spinor equations for another spinor ϵ which are related to those in (1.2) by taking H to $-H$. The solutions of the string equations that are of interest are those for which there is a non-vanishing spinor η which satisfies the Killing spinor equations. In the heterotic string theory, the situation is somewhat different. The form H is not necessarily closed. In fact the 'sigma model anomaly cancellation at one loop' requires that dH is proportional to the difference of the first Pontriangin classes of M and a vector bundle over M with coefficient which depends on an expansion parameter called string tension. Apart from the Killing spinor equations of the heterotic string given in (1.2), there is also another one associated with the gauge sector of the theory which however does not affect most of our considerations. It will be again mentioned though in sections four and five. In both type II and heterotic strings, the string and killing spinor equations above receive higher order corrections which typically involve powers of the curvature tensor with expansion parameter the string tension. In fact the string equations for the heterotic string involve additional terms arising from the gauge sector even in the same order as that of (1.1).

Suppose that M is a KT manifold. The first Killing spinor equation in (1.2) requires that the spinor is parallel with respect to the Bismut connection. Manifolds that admit such parallel spinors are those for which the holonomy of the Bismut connection is a subgroup of $SU(n)$; for applications in string theory $n \leq 4$; for n=2 the existence of parallel spinors with respect to the Bismut connection leads to some constraints on the KT geometry depending also on the type of the parallel spinor [17]. The second Killing spinor equation in (1.2) imposes additional conditions which have been investigated in [51]. Here we shall not be concerned with solutions of this equation; we have already given some applications of this in [37]. In any case the second Killing spinor equation does not directly arise in the world-volume (conformal field theory) approach to strings. This is unlike the first one which is a sufficient condition for the existence of a complex structure which characterizes the relevant conformal field theories.

In the first part of the paper, we shall first establish various relations between the curvature tensors of the Chern and Bismut connections. Then we shall state various vanishing theorems, like that of Kodaira, for hermitian manifolds in terms of conditions on the curvature of the Bismut connection. We shall find that if the (restricted) holonomy of the Bismut connection is contained in $SU(n)$ and a certain condition on the torsion is satisfied, then the Kodaira-type vanishing theorem for holomorphic (p,0)-forms holds. As an application of our results we show that a certain class of balanced hermitian manifolds does not admit such KT structures. Then we shall classify all four-dimensional compact hermitian manifolds for which the Bismut connection has (restricted) holonomy contained in $SU(2)$. We shall show that such spaces are either conformal to a Calabi-Yau space or to a Hopf surface.

In the second part of the paper, we shall apply the above results to investigate the cohomological properties of the solutions of the string equations. We shall first find that under the assumption of $SU(n)$ holonomy for the Bismut connection, the string equations can be expressed in a simple form. Then we shall show that some of the cohomology groups of such spaces vanish.

This paper is organized as follows: In section two, we give various definitions of the manifolds that we shall investigate and establish our notation. In section three, we show various identities that relate the curvature of the Chern and Bismut connections. In section four, we present our main theorems and apply our results to (i) balanced hermitian manifolds and (ii) four-dimensional hermitian manifolds which admit a Bismut connection with (restricted) holonomy contained in $SU(2)$. In section five, we investigate various compact solutions to the string equations and describe certain properties of their cohomology groups. In section six, we give our conclusions.

2 Hermitian and KT Manifolds

Let (M, g, J) be a $2n$-dimensional $(n > 1)$ Hermitian manifold with complex structure J and compatible Riemannian metric g. The Kähler form Ω of (M, g, J) is defined by

$$\Omega(X, Y) = g(X, JY) \ . \tag{2.3}$$

Denote by θ the Lee form of (M, g, J),

$$\theta = d^\dagger\Omega \circ J \ , \tag{2.4}$$

where $d^\dagger$ is the adjoint of d. For a one-form α, we shall denote by $J\alpha$ the form dual to $J\alpha^\#$, where $\alpha^\#$ is the vector dual to α. Equivalently, $J\alpha = -\alpha \circ J$. Hence, $d^\dagger\Omega = J\theta$. It has been shown by Gauduchon [22] that any conformal class of Hermitian metrics on a compact manifold contains a unique (up to homothety) metric satisfying $d^\dagger\theta = 0$. This metric is called the *Gauduchon metric*.

The Bismut connection ∇ and the Chern connection D are given by

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} d^c\Omega(X, Y, Z), \tag{2.5}$$

$$g(D_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} d\Omega(JX, Y, Z), \tag{2.6}$$

respectively, where ∇^g is the Levi-Civita connection of g. Recall that $d^c = i(\overline{\partial} - \partial)$. In particular, $d^c\Omega(X, Y, Z) = -d\Omega(JX, JY, JZ)$. Both these connections on Hermitian manifolds have been known for sometime, see for example [52, 53, 11, 26].

Let T be the torsion of ∇, $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, and C be the torsion of D, respectively. It follows from the definition of the Bismut connection (2.5) that

$$\begin{aligned} \nabla g &= 0, \quad \nabla J = 0, \\ T(X, Y, Z) : &= g(T(X, Y), Z) = d^c\Omega(X, Y, Z); \end{aligned} \tag{2.7}$$

The equation (2.7) shows that $T = d^c\Omega$, $dT = dd^c\Omega = 2i\partial\bar{\partial}\Omega$. Hermitian manifolds equipped with the Bismut connection are called *Kähler with torsion* (KT). A hermitian manifold admits a *strong* KT structure if the torsion of the Bismut connection is *closed*. Hence, a KT manifold is strong, iff its Kähler form is $\partial\bar{\partial}$-closed.

From the definition (2.6) of the Chern connection, we have

$$
\begin{aligned}
Dg &= 0, \quad DJ = 0, \\
2C(X,Y,Z) : &= 2g(C(X,Y),Z) = d\Omega(JX,Y,Z) + d\Omega(X,JY,Z).
\end{aligned} \tag{2.8}
$$

The equality (2.8) yields $C(JX,Y) = C(X,JY)$ which implies [9] $C(JX,Y) = JC(X,Y)$.

Using the formula (see e.g. [40])

$$
(\nabla^g_X \Omega)(Y,Z) = -g\left((\nabla^g_X J)Y, Z\right) = -\frac{1}{2}\left(d\Omega(X,JY,JZ) - d\Omega(X,Y,Z)\right) ,
$$

we get the expressions

$$
\theta(X) = d^\dagger \Omega(JX) = -\frac{1}{2}\sum_{i=1}^{2n} T(JX,e_i,Je_i) = \frac{1}{2}\sum_{i=1}^{2n} C(JX,e_i,Je_i). \tag{2.9}
$$

Here and henceforth $e_1, e_2, ..., e_{2n}$ is an orthonormal basis of the tangential space.

Let $(M, g, (J_a), a = 1,2,3)$ be a 4n-dimensional hyper-Kähler manifold with torsion (HKT) [34]. We denote with Ω_a and θ_a the Kähler form and the Lee form of the complex structure J_a, respectively. For HKT manifolds, the Bismut connections of the three hermitian structures $(g, J_a), a = 1,2,3$ associated with the hypercomplex structure $J_a, a = 1,2,3$ coincide and this condition is equivalent to [34, 29, 26, 30]

$$
d_1\Omega_1 = d_2\Omega_2 = d_3\Omega_3 ,
$$

where $d_a\Omega_a(X,Y,Z) := -d\Omega_a(J_aX, J_aY, J_aZ), a = 1,2,3$. In particular, if one of the hermitian structures $(g, J_a), a = 1,2,3$ of M is Kähler, then the other two are also Kähler and so M is a hyper-Kähler manifold. The torsion T is $(2,1)+(1,2)$-form with respect to each complex structure $J_a, a = 1,2,3$. This leads to the equality of the three Lee forms [28, 36], i.e. $\theta_1 = \theta_2 = \theta_3 := \theta$.

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature tensor of type (1,3) of the Bismut connection ∇. We denote the curvature tensor of type (0,4) $R(X,Y,Z,V) = g(R(X,Y)Z,V)$ by the same letter. We shall denote the curvature of the Chern and the Levi-Civita connections with K and R^g, respectively.

The Ricci tensor Ric and the Ricci form ρ of the Bismut connection ∇ are defined by

$$
Ric(X,Y) = \sum_{i=1}^{2n} R(e_i,X,Y,e_i), \quad \rho(X,Y) = \frac{1}{2}\sum_{i=1}^{2n} R(X,Y,e_i,Je_i) .
$$

The two Ricci forms ρ^D and κ associated with the Chern connection are defined by

$$
\rho^D(X,Y) = \frac{1}{2}\sum_{i=1}^{2n} K(X,Y,e_i,Je_i), \quad \kappa(X,Y) = \frac{1}{2}\sum_{i=1}^{2n} K(e_i,Je_i,X,Y).
$$

The (1,1)-form ρ^D represents the first Chern class of the manifold M. The (1,1)-form κ is called sometimes 'the mean curvature' of the holomorphic tangent bundle $T^{1,0}M$ with the hermitian metric induced by g [38]. Vanishing theorems for holomorphic (p,0)-forms $p = 1,2,\dots,n$, which we shall use later, are expressed in terms of the non-negativity of κ [39, 23].

The two Ricci forms ρ and ρ^D are related by [5]

$$\rho^D = \rho + d(J\theta). \tag{2.10}$$

This can be most easily proved by computing first the trace of the Chern and Bismut connections with J and then comparing the two expressions using the definition of θ.

We shall adopt the conventions of [5] to denote the trace of ρ with J by b and the conventions of [24] to denote the trace of κ with J, which is equal to the trace of ρ^D, by $2u$, ie

$$\begin{aligned} b &= \sum_{j=1}^{2n} \rho(Je_j, e_j), \\ 2u &= \sum_{j=1}^{2n} \rho^D(Je_j, e_j) = \sum_{j=1}^{2n} \kappa(Je_j, e_j) \ . \end{aligned} \tag{2.11}$$

The standard scalar curvature $Scal^\nabla$ of the Bismut connection ∇, is defined by

$$Scal^\nabla = \sum_{j=1}^{2n} Ric(e_j, e_j) \ .$$

The trace of the exterior derivative dT of the torsion 3-form with J, we denote by λ^Ω, i.e.

$$\lambda^\Omega(X,Y) = \sum_{i=1}^{2n} dT(X,Y,e_i,Je_i). \tag{2.12}$$

Note that λ^Ω is an (1,1)-form with respect to J since dT is a (2,2)-form.

In what follows the following definition seem to be useful

Definition A KT (resp. HKT) manifold is said to be *almost strong KT* (resp. *almost strong HKT*) manifold if $\lambda^\Omega = 0$ (resp. $\lambda^{\Omega_1} = 0$).

Note that for HKT manifolds $\lambda^{\Omega_1}(J_1., .) = \lambda^{\Omega_2}(J_2., .) = \lambda^{\Omega_3}(J_3., .)$ since the four-form dT is of type (2,2) with respect to each of the three complex structures (see e.g. [36]). Clearly, every strong KT manifold is almost strong KT. If a 2n-dimensional KT manifold is locally conformally Kähler, then $T = \frac{1}{n-1} J\theta \wedge \Omega$. In such case, a straight forward computation reveals that

$$(n-1)\lambda^\Omega = (4-2n)(dJ\theta + \theta \wedge J\theta + |\theta|^2\Omega) - 2d^\dagger\theta\Omega, \tag{2.13}$$

where $|.|^2$ is the usual tensor norm induced by g.

A four-dimensional KT manifold admits an almost strong KT structure if and only if it admits a strong KT. Indeed, (2.13) for $n = 2$ gives $\lambda^\Omega = -2d^\dagger\theta\Omega$. In addition, in four dimensions

$$T = - * \theta = J\theta \wedge \Omega \ . \tag{2.14}$$

Hence if $\lambda^\Omega = 0$, then $d^\dagger\theta = 0$ which in turn implies that $dT = 0$.

3 Curvature Identities

In this section we shall establish various identities for the curvatures of the Levi-Civita, Chern and Bismut connections. In particular we have the following:

Proposition 3.1 *Let (M, g, J, ∇) be a KT manifold. The following identities hold*

$$Ric^g(X,Y) = Ric(X,Y) + \frac{1}{2}d^\dagger T(X,Y) + \frac{1}{4}\sum_{i=1}^{2n} g\left(T(X,e_i), T(Y,e_i)\right), \tag{3.15}$$

$$\rho(X,Y) = Ric(X,JY) + (\nabla_X\theta)JY + \frac{1}{4}\lambda^\Omega(X,Y). \tag{3.16}$$

$$b = Scal^\nabla - 3d^\dagger\theta - 2|\theta|^2 + \frac{1}{3}|T|^2 \tag{3.17}$$

Proof: Since the torsion is a three-form, we have

$$(\nabla^g_X T)(Y,Z,U) = (\nabla_X T)(Y,Z,U) + \frac{1}{2}\underset{XYZ}{\sigma}\{g(T(X,Y),T(Z,U)\}. \tag{3.18}$$

Here and henceforth $\underset{XYZ}{\sigma}$ denote the cyclic sum of X, Y, Z.

Using (2.5), we can express the curvature R^g of the Levi-Civita connection in terms of that of the Bismut connection R as follows:

$$\begin{aligned} R^g(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{2}(\nabla_X T)(Y,Z,U) + \frac{1}{2}(\nabla_Y T)(X,Z,U) \\ &- \frac{1}{2}g(T(X,Y),T(Z,U)) \\ &- \frac{1}{4}g(T(Y,Z),T(X,U)) - \frac{1}{4}g(T(Z,X),T(Y,U)). \end{aligned} \tag{3.19}$$

Taking the trace of (3.19), using (3.18) and the fact that T is a three-form, we get (3.15).

Further, the exterior derivative dT of T is given in terms of ∇ by

$$\begin{aligned} dT(X,Y,Z,U) &= \underset{XYZ}{\sigma}\{(\nabla_X T)(Y,Z,U) + 2g(T(X,Y),T(Z,U)\} \\ &- (\nabla_U T)(X,Y,Z). \end{aligned} \tag{3.20}$$

The first Bianchi identity for ∇ together with (3.20) yields

$$\begin{aligned} \underset{XYZ}{\sigma}R(X,Y,Z,U) = dT(X,Y,Z,U) &+ (\nabla_U T)(X,Y,Z) \\ &- \underset{XYZ}{\sigma}\{g(T(X,Y),T(Z,U)\}. \end{aligned} \tag{3.21}$$

Next we take the trace of (3.21) and of (3.20) with J taking into account (2.9). Then the first equation is multiplied with two and it is added to the second yielding

$$\begin{aligned} 4\rho(X,Y) &+ 2Ric(Y,JX) - 2Ric(X,JY) \\ &= \lambda^\Omega(X,Y) + 2(\nabla_X\theta)JY - 2(\nabla_Y\theta)JX. \end{aligned} \tag{3.22}$$

In addition taking the trace of (3.21) with J , and using (2.9) and $R \circ J = J \circ R$, we obtain

$$
\begin{aligned}
(3.23) \qquad Ric(Y, JX) + Ric(X, JY) &= \sum_{i=1}^{2n} (R(X, Je_i, e_i, Y) - R(e_i, Y, X, Je_i)) \\
&= -(\nabla_X \theta)JY - (\nabla_Y \theta)JX.
\end{aligned}
$$

The equation (3.16) in the proposition follows from (3.22) and (3.23).

Finally, the equation (3.17) is a consequence of (3.16) and the following identity

$$
(3.24) \qquad \sum_{i=1}^{2n} \lambda^{\Omega}(e_i, Je_i) = 8|\theta|^2 + 8d^{\dagger}\theta - \frac{4}{3}|T|^2.
$$

shown in [5]. We remark that the above equation (3.24) can be derived by taking the trace of (3.20) twice with J. **Q.E.D.**

It is straightforward using proposition 3.1 to demonstrate the following:

Corollary 3.2 *On a KT manifold the Ricci tensor and the Ricci form satisfy the following relations*

$$
(3.25) \qquad Ric(X, Y) - Ric(Y, X) = -d^{\dagger}T(X, Y),
$$

$$
(3.26) \qquad Ric(JX, JY) - Ric(Y, X) = -(\nabla_{JX}\theta)JY + (\nabla_Y \theta)X,
$$

$$
(3.27) \qquad \rho(JX, JY) - \rho(X, Y) = d^{\dagger}T(JX, Y) - d^{\nabla}\theta(JX, Y),
$$

where d^{∇} *is the exterior differential with respect to* ∇ *given by* $d^{\nabla}\theta(X, Y) = (\nabla_X \theta)Y - (\nabla_Y \theta)X$.

It is worth pointing out that the Ricci tensor of the Bismut connection is not symmetric in general. Moreover as consequence of (3.25), the Ricci tensor of a linear connection with totally skew-symmetric torsion is symmetric if and only if the torsion 3-form is co-closed.

The next proposition is our second technical result.

Proposition 3.3 *On a Hermitian manifold the following identity holds*

$$
(3.28) \qquad \kappa(JX, Y) = \rho^{1,1}(JX, Y) + < i_X C, i_Y C > -\frac{1}{4}\lambda^{\Omega}(JX, Y),
$$

where $\rho^{1,1}$ *is the (1,1)-part of the Bismut-Ricci form,* $(i_X C)(Y, Z) := C(X, Y, Z)$ *and* $<,>$ *denotes the usual scalar product on tensors induced by* g.

Proof: The above relation can be most easily established in local holomorphic coordinates $\{z^{\alpha}\}, \alpha = 1, ..., n$. The torsion of the Chern connection is

$$
C_{\alpha\beta\bar{\gamma}} = id\Omega_{\alpha\beta\bar{\gamma}} = \partial_{\alpha} g_{\beta\bar{\gamma}} - \partial_{\beta} g_{\alpha\bar{\gamma}}
$$

and so

$$
\bar{\partial}\partial\Omega = i\left(\partial_{\bar{\delta}} C_{\gamma\alpha\bar{\beta}} - \partial_{\bar{\beta}} C_{\gamma\alpha\bar{\delta}}\right) dz^{\gamma} \wedge dz^{\bar{\delta}} \wedge dz^{\alpha} \wedge dz^{\bar{\beta}}.
$$

(We have used the Einstein summation conventions.) The latter equation, using the properties of the Chern connection, can be rewritten as

$$
(3.29) \qquad \partial\bar{\partial}\Omega = -i\left(D_{\gamma} C_{\bar{\delta}\bar{\beta}\alpha} - D_{\alpha} C_{\bar{\delta}\bar{\beta}\gamma} + C^{\bar{s}}_{\bar{\delta}\bar{\beta}} C_{\gamma\alpha\bar{s}}\right) dz^{\alpha} \wedge dz^{\bar{\beta}} \wedge dz^{\gamma} \wedge dz^{\bar{\delta}}.
$$

Taking the trace of first Bianchi identity of the Chern connection,

$$K_{\alpha\bar{\beta}\gamma\bar{\lambda}} - K_{\gamma\bar{\beta}\alpha\bar{\lambda}} = -D_{\bar{\beta}} C_{\alpha\gamma\bar{\lambda}}$$

with $g^{\gamma\bar{\lambda}}$ and after some computation, one finds (see e.g. [9])

$$i\rho^D_{\alpha\bar{\beta}} - i\kappa_{\alpha\bar{\beta}} = -D_{\bar{\beta}}\theta_\alpha + D^{\bar{s}} C_{\bar{s}\bar{\beta}\alpha}. \tag{3.30}$$

Next take the trace of (3.29) and use (2.9) to get

$$i(\partial\bar{\partial}\Omega)_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{\gamma\bar{\delta}} = D^{\bar{s}} C_{\bar{s}\bar{\beta}\alpha} + D_\alpha \theta_{\bar{\beta}} + C_\alpha^{\ \bar{\mu}s} C_{\bar{\beta}\bar{\mu}s}. \tag{3.31}$$

Combining (3.30) and (3.31), we find

$$i\rho^D_{\alpha\bar{\beta}} - i\kappa_{\alpha\bar{\beta}} = -D_{\bar{\beta}}\theta_\alpha - D_\alpha\theta_{\bar{\beta}} - C_\alpha^{\ \bar{\mu}s} C_{\bar{\beta}\bar{\mu}s} + i(\partial\bar{\partial}\Omega)_{\alpha\bar{\beta}\gamma\bar{\lambda}} g^{\gamma\bar{\lambda}}. \tag{3.32}$$

Taking the (1,1)-part of (2.10), we derive

$$i\rho^D_{\alpha\bar{\beta}} = i\rho_{\alpha\bar{\beta}} - D_\alpha\theta_{\bar{\beta}} - D_{\bar{\beta}}\theta_\alpha, \tag{3.33}$$

since $d(J\theta)_{\alpha\bar{\beta}} = i(\partial_\alpha\theta_{\bar{\beta}} + \partial_{\bar{\beta}}\theta\alpha) = i(D_\alpha\theta_{\bar{\beta}} + D_{\bar{\beta}}\theta_\alpha)$. We obtain (3.28) from (3.33) and (3.32). **Q.E.D.**

4 Vanishing Theorems and $SU(n)$ Holonomy

Suppose that the (restricted) holonomy of the Bismut connection ∇ of a KT manifold is contained in $SU(n)$, ie $\mathrm{hol}(\nabla) \subseteq SU(n)$ for short. This holonomy condition imposes certain constraints on the geometry, cohomology groups and topology of KT manifolds which are similar to those found in the context of compact Calabi-Yau spaces. In particular, if $\mathrm{hol}(\nabla) \subseteq SU(n)$ of a KT manifold M, then the Bismut Ricci form vanishes,

$$\rho = 0. \tag{4.34}$$

If in addition M is compact the condition (4.34) implies that the first Chern class $c_1(M) = 0$, since ∇ gives rise to a flat unitary connection on the canonical line bundle K.

Recall that for $m > 0$ the m-th plurigenenus of a compact complex manifold (M, J) is defined by $p_m(J) = \dim H^0(M, \mathcal{O}(K^m))$.

Theorem 4.1 *Let (M, g, J, ∇) be a closed 2n-dimensional (non-Kähler) KT manifold with the trace of the Bismut Ricci form to satisfy*

$$b > -|C|^2 + \frac{1}{2}h$$

where $2h = \sum_{i=1}^{2n} \lambda^\Omega(Je_i, e_i)$. Then the plurigenera $p_m(J) = 0, \quad m > 0$.

Proof: To prove the statement we shall apply the Gauduchon's plurigenera theorem [22, 24]. For this, it is sufficient to show that

$$\int_M u_G \, dV_G > 0 \ , \tag{4.35}$$

where u_G and dV_G are the trace of the Chern Ricci form with J (2.11) and the volume of M with respect the Gauduchon metric g_G of the given hermitian structure (g, J), respectively.

Let $g = e^f g_G$ be a conformal transformation relating g to the Gauduchon metric g_G. For the functions u and u_G corresponding to g and g_G, respectively we have (see e.g.[24])

$$2e^f u = 2u_G + n(n-1) < \theta_G, df >_G + n\triangle_G f, \tag{4.36}$$

where $\triangle_G f$ is the usual Laplacian of f and all terms in the right hand side are taken with respect to the Gauduchon metric g_G.

Taking the trace in (3.28), we get

$$2u = b + |C|^2 - \frac{1}{2}h \ . \tag{4.37}$$

Substituting (4.36) into (4.35), integrating over M and using $d^\dagger\theta_G = 0$ and (4.37), we obtain

$$\int_M u_G\, dV_G = \frac{1}{2}\int_M e^f(b + |C|^2 - \frac{1}{2}h)\, dV_G > 0$$

which is positive under the hypothesis of the theorem. **Q.E.D.**

An immediate corollary to the above theorem is the following:

Corollary 4.2 *The plurigenera $p_m(J) = 0, m > 0$ if any one of the following conditions holds*

(i) M is an almost strong (non Kähler) KT manifold with $\mathrm{hol}(\nabla) \subseteq SU(n)$*;*

(ii) $\mathrm{hol}(\nabla) \subseteq SU(n)$ *and* $|C|^2 - \frac{1}{2}h > 0$.

Proof: If the (restricted) holonomy of the Bismut connection is in $SU(n)$, then $b = 0$. In addition under the assumption of (i), $h = 0$ and the corollary follows from the theorem above. Case (ii) is straightforward. **Q.E.D.**

We remark that case (i) of the corollary is applicable to the type II strings since the relevant manifolds there are strong KT with (restricted) holonomy of the Bismut connection contained $SU(n)$. The case (ii) of the corollary is applicable to heterotic strings. For heterotic strings the 'sigma model anomaly cancellation' requires that

$$dH = dT = \mu(p_1(TM) - p_1(E)) \ , \tag{4.38}$$

where $p_1(TM)$ and $p_1(E)$ are the Pontriangin classes of the tangent and a vector bundle E over M in some constant μ which depends on the string tension; the vector bundle E is associated with the gauge sector of the heterotic string. Both these classes are taken with respect to Einstein-Hilbert connections; For these connections the associated curvature is a (1,1) form with respect to the complex structure J of M and its trace with J vanishes. In fact the connection $\tilde\nabla$ on TM is given by setting T to $-T$ in the definition (2.5) of the Bismut connection. Observe that $R(X,Y,Z,W) = \tilde R(Z,W,X,Y) + \frac{1}{2}dT(X,Y,Z,W)$ and note that at the zeroth loop order $dT = 0$. Therefore

$$h = \frac{\mu}{4}\left(|\tilde R|^2 - |F|^2\right) \ ,$$

where F is the Einstein Hilbert curvature of the vector bundle E.

Other applications of the various curvature identities of the previous section are the following Bochner- Kodaira vanishing theorem:

Theorem 4.3 *Let (M, g, J, ∇) be a compact 2n-dimensional KT manifold with non-negative quadratic form*

$$<< X, Y >>= \rho^{1,1}(JX, Y) + < i_X C, i_Y C > -\frac{1}{4}\lambda^{\Omega}(JX, Y) \ .$$

Then

i) every holomorphic (p,0)-form, $p = 1, 2, \ldots, n$ is parallel with respect to the Chern connection;

ii) if moreover $<<, >>$ is positive definite at only one point, then the Dolbeault cohomology groups $H^0(M, \Lambda^p) = 0, p = 1, 2, \ldots, n$.

Proof: To show this, we apply the vanishing theorem for holomorphic (p,0)-forms on compact Hermitian manifold [39, 23]. According to this general result, it is sufficient to show that the 'mean curvature' κ is non-negative. The non-negativity of κ follows from Proposition 3.3, formula (3.28) and the hypothesis of the theorem.

Q.E.D.

Two immediate corollaries to the above theorem are the following:

Corollary 4.4 *Let (M, g, J, ∇) be a compact almost strong 2n-dimensional KT manifold with non-negative (1,1) part $\rho^{1,1}$ of the Bismut Ricci form ρ. Then:*

i) every holomorphic (p,0)-form, $p = 1, 2, \ldots, n$ is parallel with respect to the Chern connection;

ii) if moreover $\rho^{1,1}$ is positive definite at only one point, then the Dolbeault cohomology groups $H^0(M, \Lambda^p) = 0, p = 1, 2, \ldots, n$.

Corollary 4.5 *Let (M, g, J, ∇) be a compact 2n-dimensional KT manifold equipped with a Bismut connection which has (restricted) holonomy contained in $SU(n)$ and the quadratic form*

$$<< X, Y >>=< i_X C, i_Y C > -\frac{1}{4}\lambda^{\Omega}(JX, Y) \ ,$$

is non-negative. Then every holomorphic (p,0)-form, $p = 1, 2, \ldots, n$ is parallel with respect to the Chern connection.

The proof of the above two corollaries follow from that of the main theorem. The first corollary applies to the case of type II string theory while the second applies to heterotic string theory. In the former case, if in addition the (restricted) holonomy of the Bismut connection is contained in $SU(n)$, then existence of holomorphic (p,0) forms depends on whether at some point in M $< i_X C, i_Y C >$ is strictly positive. In the heterotic string case, it also depends on the positivity properties of λ^{Ω} which at 'one loop' is

$$\lambda^{\Omega}(X, Y) = \mu\left(\sum_{i=1}^{2n} (p_1(TM) - p_1(E))(X, Y, e_i, Je_i)\right) \ .$$

Q.E.D..

Another application of the Theorems 4.1 and Theorem 4.3 above is in context HKT manifolds. For such spaces the holonomy of the Bismut connection is contained in $Sp(k) \subset SU(2k)$, $(n = 2k)$. In particular we have the following:

Theorem 4.6 *Let* $(M, g, J_a, a = 1, 2, 3, \nabla)$ *be a compact almost strong 4n-dimensional HKT which is not hyperKähler. Then*

i) the plurigenera of a complex structure $J_a, a = 1, 2, 3$ *is* $p_m(J_a) = 0, \quad m > 0$*;*

ii) every holomorphic with respect to a complex structure $J_a, a = 1, 2, 3$ *(p,0)-form ,* $p = 1, 2, \dots, 2n$ *is parallel with respect to the corresponding Chern connection of* (g, J_a)

We note that on a compact HKT $p_m(J_a) \in \{0, 1\}, a = 1, 2, 3$ as it is shown in [5].

4.1 Balanced Hermitian Manifolds

Definition Balanced Hermitian manifolds are Hermitian manifolds with co-closed Kähler form or equivalently with vanishing Lee form.

Such manifolds have been intensively studied in [42, 1, 2, 3]; in [23] they are called semi-Kähler manifolds of special type. This class of manifolds includes the class of Kähler manifolds but also many important classes of non-Kähler manifolds, such as: complex solvmanifolds, twistor spaces of oriented Riemannian 4-manifolds, 1-dimensional families of Kähler manifolds (see [42]), some compact Hermitian manifolds with flat Chern connection (see [23]), twistor spaces of quaternionic Kähler manifolds [48, 4], manifolds obtained as modification of compact Kähler manifolds [1] and of compact balanced manifolds [2] (see also [3]). Some vanishing theorems for balance manifold are given in [19, 20].

An application of the previous vanishing theorems is in the context of balanced hermitian manifolds. For this consider a hermitian manifold (M, g, J) for which the Lee form θ is exact; such manifolds solve the second Killing spinor equation in (1.2) and have been investigated in [37]. If in addition M is compact, then a direct consequence of the Gauduchon theorem is that (M, g_G, J) is balanced, where g_G is the Gauduchon metric. For the existence of almost strong KT structures on a balanced hermitian manifolds the following corollary holds:

Corollary 4.7 *Let* (M, g, J) *be a 2n-dimensional compact hermitian manifold for which the* $\mathrm{hol}(\nabla) \subseteq SU(n)$*. If in addition the Lee form of* (M, g, J) *is exact, and so* (M, g_G, J) *is balanced, then the complex manifold* (M, J) *does not admit any almost strong KT structure with* $\mathrm{hol}(\nabla) \subseteq SU(n)$ *. In particular if such a structure do exist, then it is Kähler and* (M, J) *is a Calabi-Yau space.*

Proof: This is a direct consequence of a statement shown in [37]. Under the same assumptions as those in the hypothesis of the corollary, it has been shown in [51] (see also [37]), that (M, J) admits a globally defined holomorphic $(n, 0)$ form $\tilde{\epsilon}$ and so $p_1 = h^{n,0} \geq 1$. This form can be expressed in terms of the ∇-parallel (n,0)-form ϵ as

$$\tilde{\epsilon} = e^{-f} \epsilon$$

where $\theta = df$.

Next suppose that there exists a almost strong KT structure on the complex manifold (M, J) satisfying the conditions of the corollary. A direct application of corollary 4.2 reveals that $p_1 = 0$. Therefore no such form can exist unless the torsion of the Chern connection vanishes and the manifold is Calabi-Yau. **Q.E.D**

4.2 Four-dimensional KT Manifolds

In four dimensions, compact KT manifolds equipped with a Bismut connection with (restricted) holonomy contained in $SU(2)$ can be classified. In particular we shall show that such manifolds are conformal either to a Calabi-Yau space or to a Hopf surface. This generalizes the result of [49].

We begin by assuming that the (restricted) holonomy of the Bismut connection of a KT manifold M is contained in SU(2)=Sp(1). Locally this is equivalent to the existence of a HKT structure. Indeed, there exist (locally) additional two ∇-parallel almost complex structures which together with J satisfy the relations of imaginary quaternions. The torsion T is (1,2)+(2,1) form with respect to each of these new almost complex structures because the (3,0) and (0,3) parts of a three form on a four-dimensional manifold vanish identically. This implies that both new almost complex structures are integrable since their associated Nijenhuis tensor vanishes. Therefore any KT manifold with (restricted) holonomy contained in $SU(2)$ admits a (local) HKT structure.

If the Lee form $d\theta = 0$ then the HKT structure is locally conformally equivalent to a hyper Kähler. There are examples of local HKT structures with non closed Lee form (see e.g. [44, 46]).

On the other hand, the existence of a HKT structure on a four manifold is equivalent to the existence of a hypercomplex one [28]. This can be seen as follows: Given a hypercomplex manifold (M, J_r) equipped with a Riemannian metric g, we can find a metric h on M which is trihermitian by averaging over the complex structures, i.e.

$$h(X, Y) = g(X, Y) + \sum_{r=1}^{3} g(J_r X, J_r Y) \ .$$

It can be easily seen that h is a Riemannian metric on M with the desirable property. Using the integrability of the complex structures and the fact that (3,0) and (0,3) forms vanish identically in four dimensions, one can show that (M, h, J_r) is an HKT manifold.

It is a well known consequence of the integrability theorem in [8] that the self-dual part of the Weyl tensor of a four-dimensional Riemannian manifold (M, g) must vanish if there exists a (local) hypercomplex structure on M (see e.g. [28] and references there). The converse is not true. For example, the complex projective space taken with the reverse orientation is anti-self-dual and does not admit any local hypercomplex structure [18], see also [7] and references there. More precisely, we have

Proposition 4.8 *A four-dimensional Hermitian manifold (M, g, J) admits a anti-self-dual Weyl tensor if and only if the Bismut Ricci form is anti-self-dual.*

Proof: A two form on a hermitian surface is anti-self-dual if it is of type (1,1) and trace-free. Applying (2.14) to (3.27) we compute $\rho^{(2,0)+(0,2)} = -d\theta_+$, where the subscript $(+)$ denotes the self-dual part. Hence, the Bismut Ricci form is of type (1,1) iff the Lee form θ is anti-self-dual. It is shown in [5] that the trace $b = k$ where k is the conformal scalar curvature determined by the trace of the self-dual Weyl tensor W_+, ie $k =< 3W^+(\Omega), \Omega >$. Proposition 1 in [12] tells us that $W_+ = 0$ on a hermitian surface iff $k = 0$ and $d\theta$ is anti-self-dual. **Q.E.D.**

In the compact four-dimensional case the situation is different since there are compact complex surfaces with local hypercomplex structures which does not admit any global one. This can be seen in the example of Hopf surfaces explain below.

A Hopf surface HS is by definition a compact complex surface whose universal covering is $\mathbb{C}^2 - \{0\}$. It was shown by Kodaira [41] that the fundamental group $\pi_1 \tilde{=} \mathbb{Z} \otimes \mathbb{Z}_n$ and the second Betti number $b_2 = 0$. A primari Hopf surface is a Hopf surface with $\pi_1 \tilde{=} \mathbb{Z}$. Primary Hopf surfaces are diffeomorphic to $S^1 \times S^3$. Every Hopf surface is finitely covered by a primary one. A subclass of primary Hopf surface are those Hopf surfaces HS_o which admit a hermitian metric locally conformally equivalent to a flat Kähler metric. The fundamental group of a primary locally conformally flat Hopf surface is generated by [25] $\Gamma : (z, w) \to (az, b^2 aw), a, b \in \mathcal{C}, |b| = 1 < |a|$. Such a Hopf surface has one hypercomplex structure exactly when $b = \pm 1$ or when $ab \in \mathbb{R}$ and it admits exactly two hyperhermitian structures when both a and b are real [25]. In view of the classification of compact hyperhermitian complex surfaces [13] these are the only compact non Calabi-Yau HKT structures.

Theorem 4.9 *Let (M^4, g, J) be a compact Hermitian surface. The restricted holonomy of the Bismut connection is contained in SU(2) if and only if (M, J) is a Calabi-Yau or (M, g_G, J) with the Gauduchon metric g_G is locally isometric (up to homothety) to $\mathbb{R} \times S^3$, the Lee form θ is ∇^g-parallel, the hermitian structure (g_G, J) is locally conformally Kähler flat, and the complex surface (M, J) is a Hopf surface.*

Proof: If the first Betti number $b_1(M)$ is even then the surface is of Kähler type with zero first Chern class and therefore it is a Calabi-Yau. Suppose $b_1(M)$ is odd and therefore there is no Kähler structure on the surface. Using (2.14), (2.13) and $\rho = 0$ we obtain from (3.28) that $\kappa = \frac{1}{2}(|\theta|^2 + d^\dagger\theta)\Omega$. Hence, the surface is Einstein-Hermitian and the main result of [27] implies the assertion. **Q.E.D.**

Theorem 4.9 can be derived also from the properties of Einstein-Weyl structures found in [6] applying the connection between SU(2) holonomy of the Bismut connection with the canonical Einstein-Weyl structure on a hermitian surface discovered in [5].

The description of the Hopf surfaces and Theorem 4.9 show that there exist compact hermitian surfaces with restricted holonomy of the Bismut connection contained in SU(2) which does not admit a HKT structure.

We finish this section with the following non-vanishing result which gives obstructions to the existence of non Calabi-Yau spaces with SU(n) holonomy.

Theorem 4.10 *Let (M, J) be a 2n-dimensional compact complex manifold with vanishing first Chern class and does not admit any Kähler metric. If there exist a hermitian structure (g, J) such that the restricted holonomy group of the Bismut connection is contained in SU(n) then either*

i) $h^{0,n} = h^{n,0} = 1$ if the Lee form is exact

or

ii) $h^{0,1} \geq 1$ if the Lee form is not exact.

Proof. If the Lee form is an exact 1-form then the Gauduchon metric is balanced and $h^{n,0} = 1$ by the result in [51] (see also [37]).

Suppose that the Lee form θ is not exact ie the Gauduchon metric $g_G = e^f g$ is not balanced and the corresponding Lee form $\theta_G \neq 0$. The equation (2.10) shows that the 2-form $dJ\theta$ is an (1,1)-form since $\rho = 0$ and ρ^D is an (1,1)-form. This yields $d\theta$ is an (1,1)-form ie $\bar{\partial}\theta^{0,1} = \bar{\partial}(J\theta)^{0,1} = 0$, where ^{0,1} means the (0,1)-part. Then, we get $\bar{\partial}\theta_G^{0,1} = 0$ due to the equality $\theta_G^{0,1} = \theta^{0,1} + (n-1)\bar{\partial}f$. Combining the latter with $d_G^\dagger\theta_G = 0$ we obtain that $\theta_G^{0,1}$ is a $\bar{\partial}$- harmonic with respect to the Gauduchon metric non zero form. Hence, $h^{0,1} \geq 1$. **Q.E.D.**

5 String equations and Hermitian Manifolds

We shall investigate the cohomological properties of solutions of the string equations (1.1) which are complex manifolds (M, g, J). For this the torsion T of the Bismut connection ∇ is related to the three-form field strength H of string theory as

$$H = T \; . \tag{5.39}$$

As it was already mentioned, for applications in string theory the restricted holonomy of the Bismut connection is contained in $SU(n)$. In this case the relation between string equations and complex manifolds can be made manifest.

We shall seek to describe vanishing theorems for solutions of the string equations which are KT manifolds for which the Bismut connection has (restricted) holonomy contained in $SU(n)$. Most of the statements below apply to almost strong KT manifolds. Therefore most of our results concern type II strings for which H is closed. Our results apply also to the heterotic string but the contribution from the gauge sector has been neglected.

The presence of a strong KT structure on (M, g, J) that is associated with type II strings allows the use of theorem shown in [5] which we shall state here without proof as follows:

Theorem 5.1 *Let (M, g, J) be a compact 2n-dimensional $(n > 1)$ strong KT manifold with Kähler form Ω. Suppose that the Lee form θ is co-closed, $d^\dagger\theta = 0$ and that the (1,1)-part of the Ricci form of the Bismut connection is non-negative everywhere on M.*

a) Then every $\bar\partial$-harmonic $(0,p)$-form, $p = 1, \ldots, n$, is parallel with respect to the Bismut connection.

b) If moreover the (1,1)-part of the Ricci form of the Bismut connection is strictly positive at some point, then the cohomology groups $H^p(M, \mathcal{O})$ vanish for $p = 1, \ldots, n$.

Note that the above theorem has been stated in a different but equivalent way in [5]. The solutions of the string equations can be separated into two classes depending on whether the dilaton is a constant or not.

5.1 Constant Dilaton

Let the dilation ϕ be constant. In such case the string equations (1.1), after using the above identification (5.39) of H and T, become

$$\begin{aligned} Ric^g_{ij} - \frac{1}{4} T_{imn} T_j{}^{mn} &= 0 \\ \nabla^g_i T^{imn} &= 0 \; . \end{aligned} \tag{5.40}$$

or equivalently

$$Ric = 0 \; . \tag{5.41}$$

In particular the above equation implies that $d^\dagger T = 0$ (see (3.25)). At present, we take $dT \neq 0$. The restrictions imposed on T necessary for the various theorems will be stated explicitly.

Assuming that the (restricted) holonomy of the Bismut connection is in $SU(n)$, and so $\rho = 0$, we find using (3.16) that the string equations (5.41) can be rewritten as

$$(\nabla_X \theta) Y = \frac{1}{4} \lambda^\Omega(X, JY). \tag{5.42}$$

For manifolds with an almost strong KT structure $\lambda^\Omega = 0$ and so the above equation implies that the Lee form θ is parallel with respect to the Bismut connection, i.e.

$$\nabla\theta = 0 \ .$$

In particular, $\theta^\#$ is a Killing vector field. This can be easily seen using the definition of the Bismut connection and (2.7).

Remark 2. For almost strong KT manifolds, $\lambda^\Omega = 0$, the equation (3.24) shows that the Lee form is identically zero iff (M, g, J) is a Kähler manifold. Hence, on any non-Kähler almost strong KT manifold which is a solution of (5.41), there is a globally defined non-zero Killing vector field $\theta^\#$.

There is an alternative way to characterize the string equations the following:

Theorem 5.2 *Let (M, g, J, ∇) be a 2n-dimensional compact strong KT manifold and* $\mathrm{hol}(\nabla) \subseteq SU(n)$*. Then (M, g, J, ∇) is a solution of the string equations (5.41) if and only if the scalar curvature of the Bismut connection vanishes, $Scal^\nabla = 0$.*

*Proof:*From the assumptions of the theorem we have

$$\rho = dT = 0 \ .$$

Since $dT = 0$, $\lambda^\Omega = 0$. Substituting this into equation (3.24), we find

$$2|d^\dagger\theta|^2 + 2|\theta|^2 - \frac{1}{3}|T|^2 = 0 \ . \tag{5.43}$$

To show the theorem in one direction, we insert $\rho = Ric = \lambda^\Omega = 0$ into (3.16) and find that

$$\nabla\theta = 0$$

which in turn implies $d^\dagger\theta = 0$; therefore g is the Gauduchon metric. Substituting this into (5.43), we get

$$2|\theta|^2 - \frac{1}{3}|T|^2 = 0 \ . \tag{5.44}$$

Moreover since $\rho = 0$, then $b = 0$. Substituting this, (5.44) and $d^\dagger\theta = 0$ into (3.17), we find $Scal^\nabla = 0$. Which proves the theorem in one direction.

For the converse, suppose that $Scal^\nabla = 0$. Then, substituting $b = 0$ and (5.43) into (3.17) yields

$$d^\dagger\theta = 0 \ . \tag{5.45}$$

Next taking the trace of (3.16) using $\rho = Scal^\nabla = \lambda^\Omega = 0$, we find that $\sum_{i=1}^{2n}(\nabla_{e_i}\theta)Je_i = 0$. In turn using (2.9) this implies that

$$\sum(\nabla^g_{e_i}\theta)Je_i = 0 \ . \tag{5.46}$$

From (5.45) and (5.46), we conclude that $\theta^{0,1}$ is $\bar\partial$ co-closed, ie

$$\bar\partial^\dagger\theta^{0,1} = 0 \ ,$$

where the superscript ^{0,1} means the (0,1)-part of the form. Note that the Lee form θ is not identically zero because of Remark 2. Since $\rho = 0$, equation (2.10) implies that $\rho^D = d(J\theta)$ and therefore $d(J\theta)$ is an (1,1)-form. This in turn implies that $d\theta$ is a (1,1) from, ie

$$\bar{\partial}\theta^{0,1} = 0 \ .$$

Thus $\theta^{0,1}$ is both $\bar{\partial}$ closed and co-closed and therefore $\bar{\partial}$ harmonic. Applying the vanishing theorem of [5] stated in 5.1, we conclude that

$$\nabla\theta = 0 \ .$$

Substituting this together with $\rho = \lambda^{\Omega} = 0$ into (3.16), we get $Ric = 0$. Hence, we recover the string equations for constant dilation ϕ.

Q.E.D.

We remark that the solutions of the string equations that we investigate are *not* required to satisfy the second Killing spinor equation in (1.2). If they did, then the second in (1.2) implies that $\theta = 2d\phi$ (see [51, 37]). Since the dilaton ϕ is constant, this in turn implies that $\theta = 0$. If (M, g, J) satisfies the assumptions of the theorem above, substituting $\theta = 0$ in (5.43), we find that the torsion T vanishes and the manifold M is Calabi-Yau.

There are some restrictions on the cohomology of manifolds that are solutions of the string equations (5.41). More precisely the following result holds:

Theorem 5.3 *Let (M, g, J, ∇) be a 2n-dimensional compact strong KT manifold and be a solution of the string equations (5.41). Moreover let* $\mathrm{hol}(\nabla) \subseteq SU(n)$*. Then (M, g, J, ∇) has the following properties:*

i) Every $\bar{\partial}$-harmonic (0,p)-form, $p = 1, 2, ..., n$ is parallel with respect to the Bismut connection ∇. Therefore $h^{0,p} = dimH^p(M, \mathcal{O}) \leq \binom{n}{p}$, $h^{0,1} \geq 1$ and the dimension of the space of Killing vector fields is at least $2h^{0,1} \geq 2$;

ii) every holomorphic (p,0)-form , $p = 1, 2, \ldots, 2n$ is parallel with respect to the Chern connection and therefore $h^{p,0} = \leq \binom{n}{p}$;

iii) the plurigenera $p_m(J) = 0, \quad m > 0$ provided (g, J) is not Kähler.

Proof: To show (i), we apply the vanishing theorem of [5] stated in 5.1 using the assumptions that $\rho = dT = 0$.

Moreover using the assumptions $\rho = \lambda^{\Omega} = b = 0$, (ii) and (iii) are direct consequence of Theorems 4.3 and 4.1 , respectively. **Q.E.D.**

Corollary 5.4 *Let (M, J, g) be as in Theorem 5.3. If in addition the Lee form θ is closed, then θ is ∇^g-parallel and (M, g) is locally isometric to $N^{2n-1} \times \mathbb{R}$, where N^{2n-1} is a (2n-1)-dimensional Riemannian manifold with non-negative Riemannian Ricci curvature.*

*Proof:*Using the assumptions of the theorem, we have shown that θ is parallel with respect to the Bismut connection. This and the additional assumption that θ is closed imply that

$$i_{\theta^\#}T = 0 \ .$$

Substituting this back into $\nabla\theta = 0$, we find that θ is parallel with respect to the Levi-Civita connection. The non-negativity of the Riemannian Ricci curvature follows from (3.15) and the assumptions of the corollary. **Q.E.D.**

There are various applications to the above theorems. One such application is in the context of HKT manifolds. Since the holonomy of the Bismut connection for HKT manifolds is contained in $Sp(k)$, $n = 2k$, it is also contained in $SU(n)$. In particular as a direct application of Theorem 5.2, we have the following:

Theorem 5.5 *A compact strong HKT manifold is a solution of the string equations with constant dilaton (5.41), iff* $Scal^{\nabla} = 0$*. In addition the statements (i), (ii) and (iii) of Theorem 5.3 taken with respect to any of the complex structure* $J_a, a = 1, 2, 3$ *also hold.*

As another application, we can relate KT manifolds and HKT manifolds as follows:

Corollary 5.6 *Let* (M, g, J, ∇) *be a 4k-dimensional compact strong KT manifold, the (restricted) holonomy of* ∇ *be in SU(2k) with* $Scal^{\nabla} = 0$*. If there exists a non-degenerate* $\bar{\partial}$*-harmonic (0,2)-form* ϕ*, then* (M, g, J, ∇) *is a strong HKT manifold.*

Proof: If there is such a (0,2)-form ϕ, then applying Theorem 5.3 we conclude that it is parallel with respect to the Bismut connection. Hence, the (restricted) holonomy of ∇ is contained in Sp(k). **Q.E.D.**

We remark that the existence of $\bar{\partial}$-harmonic (0,2)-form is not necessary for the existence of a ∇-parallel one. For example, $SU(3)$ with any left invariant complex structure has $h^{0,2} = 0$ but there exist a left invariant HKT structure on this group. The same example can be used to demonstrate that $h^{4,0} = p_1(J) = \dim H^0(M, \mathcal{O}(K)) = 0$ despite the fact that there is a ∇-parallel $(4, 0)$-form because the holonomy of ∇ is trivial in accordance with Theorem 5.3.

Examples. Compact strong solutions are even dimensional Lie groups endowed with an left invariant complex structure compatible with a bi-invariant Riemannian metric [50, 45] The induced Bismut connection is flat and the torsion is parallel and so both closed and co-closed. The Lee form is not always closed in these examples. Some of these group manifold examples admit in fact a HKT structure [50, 45, 30].

In four dimensions, (2.14) implies $\nabla^g\theta = \nabla\theta$. The conditions $dT = 0, d^{\dagger}T = 0$ are equivalent to $d^{\dagger}\theta = 0, d\theta = 0$, respectively. The standard hermitian structure on the Hopf surfaces HS_o has ∇^g-parallel Lee form and $\rho = 0$. Then $Ric = 0$ by (3.16) and therefore it is a solution of the string equations (5.41). In fact, the Hopf surfaces HS_o with the standard hermitian structure are the unique compact non Calabi-Yau solution of the string equation with constant dilation and (restricted) holonomy of the Bismut connection contained in SU(2) by Theorem 4.9. In particular, every compact strong HKT surface solves the string equation (5.41) with constant dilation.

5.2 Non-constant dilaton

Let (M, g, J) be a KT manifold equipped with a Bismut connection with (restricted) holonomy contained in $SU(n)$. In addition, we take the dilation ϕ not to be a constant, $\phi \neq$ const. The string equations (1.1) can be written using (5.39) and (3.15) as

$$Ric + \frac{1}{2}d^{\dagger}T + 2\nabla^g d\phi = 0, \qquad d^{\dagger}T + 2i_{d\phi^{\#}}T = 0, \tag{5.47}$$

where i_X denotes the interior multiplication by a vector X.

Using (3.16), (3.27) and the assumption that the (restricted) holonomy of the Bismut connection is contained in $SU(n)$, so we have $\rho = 0$, we find that the string equations (5.47) can be expressed in terms of the Lee form θ as

$$
\begin{aligned}
-(\nabla_X\theta)Y - (\nabla_Y\theta)X + \frac{1}{2}\lambda^{\Omega}(X,JY) + 4(\nabla^g_X d\phi)Y &= 0, \\
(\nabla_X\theta)Y - (\nabla_Y\theta)X + 2T(d\phi^{\#},X,Y) &= 0.
\end{aligned} \tag{5.48}
$$

The first string equation in (5.48) can be also written in the following equivalent way

$$
(\nabla_X\eta)Y + (\nabla_Y\eta)X = \frac{1}{2}\lambda^{\Omega}(X,JY), \tag{5.49}
$$

where the 1-form η is given by

$$
\eta = \theta - 2d\phi. \tag{5.50}
$$

Remark 3. For compact non Kähler solutions of (5.49) the Lee form θ cannot be identically zero. Indeed, combining (5.49) together with (3.24), one finds

$$
6d^{\dagger}\theta - \frac{2}{3}|T|^2 + 4|\theta|^2 - 4d^{\dagger}d\phi = 0\ .
$$

Integrating this equality over a compact manifold without boundary, we obtain $T = 0$ if and only if $\theta = 0$.

We have

Theorem 5.7 *A 2n-dimensional KT manifold (M,g,J) is a (local) solution to the first string equation of (1.1) and to the Killing spinor equations (1.2) with non-constant dilation if and only if (M,g,J) is an almost strong KT manifold with closed Lee form and* $\mathrm{hol}(\nabla) \subseteq SU(n)$. *If in addition M is an eight-dimensional compact space, then it admits a strong KT structure.*

Proof: If the Killing spinor equations (1.2) are satisfied, then the second in (1.2) implies that $\theta = 2d\phi$ (see [51, 37]). From (5.50), we find that $\eta = 0$ and the conditions of the theorem imply (5.49).

The last statement in the theorem follows because in eight dimensions the condition $\lambda^{\Omega} = 0$ and the fact that dT is a (2,2) form imply that dT is self-dual 4-form with respect to the Hodge * operator. Now if M is compact, we get $\int_M |dT|^2\, dV = -\int_M < *d*dT, T > dV = -\int_M < *d^2T, T > dV = 0$. So M is a strong KT manifold. **Q.E.D.**

We remark that any manifold that satisfies the assumptions of the above theorem and equation (5.49) admits an almost strong KT structure. This appears to be in conflict with the heterotic five-brane solution of [14]. However this is not the case because the contribution of the gauge sector in the string equations (1.1) have been neglected; although the contribution of the gauge sector due to the sigma model anomaly in the non-closure of T (4.38) has been taken into account. In fact it has been shown in [35] that for the heterotic five-brane to solve the string equations, the two loop contribution should be taken into account. Such corrections to the string equations involve quadratic terms in the curvatures and they have not be taken into account in this paper.

Next using (3.27) together with $\rho = 0$, we find that

$$
d^{\dagger}T = d^{\nabla}\theta = d\theta - i_{\theta^{\#}}T. \tag{5.51}
$$

Combining (5.51) with the second string equation in (5.47), we get

$$d\theta = i_{(\theta^\# - 2d\phi^\#)}T.$$

Finally, we observe that if (M, g, J) is a KT manifold and the (restricted) holonomy of ∇ is contained in SU(n), then the second string equation in (1.1) is equivalent to the equation

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0. \tag{5.52}$$

Combining (5.49) with (5.52) we get that the string equations (1.1) are equivalent to the following one

$$\nabla_X \eta(Y) = \frac{1}{4}\lambda^\Omega(X, JY), \tag{5.53}$$

Thus we have shown the following theorem:

Theorem 5.8 *Let (M, g, J) be a 2n-dimensional KT manifold and* $\mathrm{hol}(\nabla) \subseteq SU(n)$*. (M, g, J) is a solution to the string equations (1.1) if and only if the equation (5.53) holds.*

If in addition (M, g, J) is an almost strong KT manifold again with $\rho = 0$, as it will be in the case of type II strings, then the string equations are equivalent to

$$\nabla \eta = 0 \tag{5.54}$$

and so η is ∇-parallel. There are two special cases that one can consider the following:

Case 1. Let us suppose that the Lee form is closed i.e.

$$d\theta = 0\ .$$

Then locally $\theta = df$ for a smooth function f. Taking

$$\phi = \frac{1}{2}f$$

we obtain a local solution of the string equations (5.54) for which $\eta = 0$. Thus, every almost strong KT manifold with (restricted) holonomy in SU(n) and closed Lee form gives a local solution of the string equations (1.1) and (1.2) with non-constant dilation ϕ.

Now suppose that in addition the class $[\theta]$ of the Lee form in $H^1(M)$ is trivial, ie θ is exact. As it has already been mentioned this is precisely the case that the second Killing spinor equation in (1.2) admits a solution. Therefore only backgrounds with θ an exact form are supersymmetric. Moreover let (M, g, J) be a 2n-dimensional compact almost strong KT manifold with $\mathrm{hol}(\nabla) \subseteq SU(n)$ Then one can apply the corollary 4.7 shown in the context of balanced hermitian manifolds to show that (M, J) is Calabi-Yau.

Case 2. Alternatively, the Lee form θ may not be closed, $d\theta \neq 0$. If (M, g, J) is an almost strong KT manifold for which the associated Bismut connection has (restricted) holonomy contained in $SU(n)$, then as we have already mentioned in (5.54) the string equations are equivalent to the condition that the form η is ∇-parallel . In particular $\eta^\#$ is a non-zero Killing vector field and

$$d\eta = d\theta\ .$$

Conversely, if an almost strong KT manifold (M, g, J) for which the associated Bismut connection has (restricted) holonomy contained in $SU(n)$ admits a ∇-parallel one-form η such that $d\eta = d\theta$, then (M, g, J) is a local solution of the string equations with dilaton $\phi = -\frac{1}{2}f$, where f is determined by $df = \eta - \theta$. Thus we have shown the following corollary:

Corollary 5.9 *Let (M, g, J) be an almost strong 2n-dimensional KT manifold for which the associated Bismut connection has (restricted) holonomy contained in $SU(n)$ and the Lee form is not closed. Then (M, g, J) is a solution of the string equations with non-constant dilaton $\phi = (1/2)f$ if and only if it admits a ∇-parallel one-form η such that $d\eta = d\theta$ and $\eta - \theta = df$ is an exact one-form.*

Next suppose that (M, g, J) is a four-dimensional hermitian manifold. Using (2.13), we get that string equations (5.53) take the form

$$\nabla\eta = \frac{1}{2}d^\dagger\theta \otimes g, \tag{5.55}$$

(M, g, J) where η is given by (5.50). In particular, $\eta^\#$ is a non-zero conformal Killing vector field provided $d^\dagger\theta \neq 0$.

6 Concluding Remarks

We have found conditions for the existence of Bismut connections on hermitian manifolds (KT) for which their (restricted) holonomy contained in $SU(n)$. These conditions can be expressed in terms of the vanishing of certain cohomology groups. For example under certain additional assumptions the plurigenera vanish. We also consider various applications of our results in the context of string theory and in the context of balanced hermitian manifolds.

Despite the various developments the last few years in the context of KT and HKT manifolds, the existence of an HKT structure on a manifold has not been expressed in terms of conditions on its cohomology groups in parallel with similar developments in the case of hyper-Kähler manifolds. For this an analogue of the Calabi-Yau theorem is needed for the case of KT manifolds which will involve hermitian manifolds with topologically trivial canonical bundle. It is not clear for example under which conditions a KT manifold with trivial canonical bundle admits a KT structure associated with Bismut connection for which its (restricted) holonomy is contained in $SU(n)$. We have found in Theorem 4.10 obstructions to the existence of KT structure with holonomy in SU(n) on a compact complex manifold with vanishing first Chern class of non-Kähler type. There seem though to be counter examples for the existence of strong KT structures of this type.

An alternative but essentially equivalent way to state the Calabi-Yau type of conjecture mentioned above which seems to be of importance for the further development of the theory is the following: Given a 2n-dimensional $(n > 2)$ compact complex manifold with zero first Chern class and $h^{n,0} = 1$ or $h^{0,1} \geq 1$, does there exist a hermitian metric with vanishing Ricci form of the Bismut connection? The dimension $(n > 2)$ in the above question is essential since the Inoe surface has vanishing first Chern class and $h^{0,1} = 1$ but it does not admit Hermitian structure with SU(2) holonomy of the Bismut connection by Theorem 4.9.

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