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VANISHING THEOREMS FOR DOLBEAULT COHOMOLOGY OF LOG HOMOGENEOUS VARIETIES

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Abstract. We consider a complete nonsingular complex algebraic variety having a normal crossing divisor such that the associated logarithmic tangent bundle is generated by its global sections. We obtain an optimal vanishing theorem for logarithmic Dolbeault cohomology of nef line bundles in that setting. This implies a vanishing theorem for ordinary Dolbeault cohomology which generalizes results of Broer for flag varieties, and of Mavlyutov for toric varieties.

Introduction. The main motivation for this work comes from the well-developed theory of complete intersections in algebraic tori $(\boldsymbol{C}^*)^n$ and in their equivariant compactifications, toric varieties. In particular, the Hodge numbers of these complete intersections were determined by Danilov and Khovanskii, and their Hodge structure, by Batyrev, Cox and others (see [11, 2, 26]). This is made possible by the special features of toric geometry; two key ingredients are the triviality of the logarithmic tangent bundle $T_X(-\log D)$, where X is a complete nonsingular toric variety with boundary D, and the Bott-Danilov-Steenbrink vanishing theorem for Dolbeault cohomology: $H^i(X, L \otimes \Omega_X^j) = 0$ for any ample line bundle L on X and any $i \geq 1$, $j \geq 0$.

A natural problem is to generalise this theory to complete intersections in algebraic homogeneous spaces and their equivariant compactifications. As a first observation, the preceding two results also hold for abelian varieties and, more generally, for the "semi-abelic" varieties of [1], that is, equivariant compactifications of semi-abelian varieties. In fact, for a complete nonsingular variety X and a divisor D with normal crossings on X, the triviality of the logarithmic tangent bundle is equivalent to X being semi-abelic with boundary D, by a result of Winkelmann (see [32]). Moreover, it is easy to see that semi-abelic varieties satisfy Bott vanishing.

The next case to consider after these "log parallelisable varieties" should be that of flag varieties. Here counter-examples to Bott vanishing exist for grassmannians and quadrics, as shown by work of Snow (see [29]). For example, any smooth quadric hypersurface X in $\boldsymbol{P}^{2m}$ satisfies $H^{m-1}(X, \Omega_X^m(1)) \neq 0$.

On the other hand, a vanishing theorem due to Broer asserts that $H^i(X, L \otimes \Omega_X^j) = 0$ for any nef line bundle L on a flag variety X, and all $i > j$ (see [8], and [9] for a generalisation to homogeneous vector bundles); in this setting, a line bundle is nef (numerically effective) if and only if it is effective, or generated by its global sections. Moreover, the same vanishing

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theorem holds for any nef line bundle on a complete simplicial toric variety, in view of a recent result of Mavlyutov (see [27, Thm. 2.4]).

In this article, we obtain generalisations of Broer's vanishing theorem to any "log homogeneous" variety, that is, to a complete nonsingular variety X having a divisor with normal crossings D such that $T_X(-\log D)$ is generated by its global sections. Then X contains only finitely many orbits of the connected automorphism group $\mathrm{Aut}^0(X, D)$, and these are the strata defined by D. The class of log homogeneous varieties, introduced and studied in [7], contains of course the log parallelisable varieties, and also the "wonderful (symmetric) varieties" of De Concini-Procesi and Luna (see [12, 25]). Log homogeneous varieties are closely related to spherical varieties; in particular, every spherical homogeneous space has a log homogeneous equivariant compactification (see [4]).

Our final result (Theorem 3.18) asserts that $H^i(X, L \otimes \Omega^j_X) = 0$ *for any nef* (*resp. ample*) *line bundle L on a log homogeneous variety X, and for any* $i > j + q + r$ (*resp.* $i > j$). Here q denotes the irregularity of X, i.e., the dimension of the Albanese variety, and r its rank, i.e., the codimension of any closed stratum (these are all isomorphic). Thus, $q + r = 0$ if and only if X is a flag variety; then our final result gives back Broer's vanishing theorem.

We deduce our result from the vanishing of the logarithmic Dolbeault cohomology groups $H^i(X, L^{-1} \otimes \Omega^j_X(\log D))$ for L nef and $i < j - c$, where $c \le q + r$ is an explicit function of (X, D, L); see Theorem 3.16 for a complete (and optimal) statement. In particular, $H^i(X, \Omega^j_X(\log D)) = 0$ for all $i < j - q - r$; this also holds for all $i > j + q$ by a general result on varieties with finitely many orbits (Theorem 1.6). In view of a logarithmic version of the Lefschetz theorem due to Norimatsu (see [28]), this gives information on the mixed Hodge structure of complete intersections: specifically, *for any ample hypersurfaces* $Y_1, \dots, Y_m \subset X$ *such that* $D + Y_1 + \cdots + Y_m$ *has normal crossings, the complete intersection* $Y := Y_1 \cap \cdots \cap Y_m$ *satisfies* $H^i(Y, \Omega^j_Y(\log D)) = 0$ *unless* $i + j \ge \dim(Y)$ *or* $-q \le j - i \le q + r$.

Since the proof of our results is somewhat indirect, we first present it in the setting of flag varieties, and then sketch how to adapt it to log homogeneous varieties. For a flag variety $X = G/P$, the tangent bundle T_X is the quotient of the trivial bundle $X \times \mathfrak{g}$ (where $\mathfrak{g}$ denotes the Lie algebra of G) by the sub-bundle R_X of isotropy Lie subalgebras. Via a homological argument of "Koszul duality" (Lemma 3.1), Broer's vanishing theorem is equivalent to the assertion that $H^i(R_X, p^*L) = 0$ for all $i \ge 1$, where $p : R_X \to X$ denotes the structure map. But one checks that the canonical bundle of the nonsingular variety R_X is trivial, and the projection $f : R_X \to \mathfrak{g}$ is proper, surjective and generically finite. So the desired vanishing follows from the Grauert-Riemenschneider theorem.

For an arbitrary log homogeneous variety X with boundary D, we consider the connected algebraic group $G := \mathrm{Aut}^0(X, D)$, with Lie algebra $\mathfrak{g} := H^0(X, T_X(-\log D))$. We may still define the "bundle of isotropy Lie subalgebras" R_X as the kernel of the (surjective) evaluation map from the trivial bundle $X \times \mathfrak{g}$ to $T_X(-\log D)$, and the resulting map $f : R_X \to \mathfrak{g}$. If G is linear, we show that the connected components of the general fibres of f are toric varieties of dimension $\le r$ (Theorem 2.2 and Corollary 2.6, the main geometric ingredients of the proof).

Moreover, any nef line bundle L on X is generated by its global sections. By a generalisation of the Grauert-Riemenschneider theorem due to Kollár (see [16, Cor. 6.11]), it follows that $H^i(R_X, p^*L \otimes \omega_{R_X}) = 0$ for any $i > r$. Via homological duality arguments again, this is equivalent to the vanishing of $H^i(X, L^{-1} \otimes \Omega_X^j(\log D))$ for any such L, and all $i < j - r$ (Corollary 3.10). In turn, this easily yields our main result, under the assumption that G is linear.

The case of an arbitrary algebraic group G may be reduced to the preceding setting, in view of some remarkable properties of the Albanese morphism of X: this is a homogeneous fibration, which induces a splitting of the logarithmic tangent bundle (Lemma 1.4), and a decomposition of the ample cone (Lemma 3.14).

Homological arguments of "Koszul duality" already appear in the work of Broer mentioned above, and also in work of Weyman (see [31, Chap. 5]). The latter considers the more general setting of a sub-bundle of a trivial bundle, but mostly assumes that the resulting projection is birational, which very seldom holds in our setting.

The geometry of the morphism $f : R_X \to \mathfrak{g}$ bears a close analogy with that of the moment map $\phi : \Omega_X^1 \to \mathfrak{g}^*$, studied in depth by Knop for a variety X equipped with an action of a connected reductive group G. In particular, Knop considered the compactified moment map $\Phi : \Omega_X^1(\log D) \to \mathfrak{g}^*$, and he showed that the connected components of the general fibres of Φ are toric varieties, if X is log homogeneous under G (see [23, p. 265]). By applying another result of Kollár, he also showed that $H^i(\Omega_X^1(\log D), \mathcal{O}_{\Omega_X^1(\log D)}) = 0$ for all $i \geq 1$ (see [23, Thm. 4.1]).

However, vanishing results for $H^i(\Omega_X^1(\log D), q^*L)$, where L is a nef line bundle on X, and $q : \Omega_X^1(\log D) \to X$ denotes the structure map, are only known under restrictive assumptions on X (see [4]). Also, generalising Knop's vanishing theorem to all log homogeneous varieties (under possibly non-reductive groups) is an open question.

Our construction coincides with that of Knop in the case where X is a $G \times G$-equivariant compactification of a connected reductive group G: then one may identify R_X with $\Omega_X^1(\log D)$, and f with the compactified moment map (see Example 2.5). As applications, we obtain very simple descriptions of the algebra of differential operators on X which preserve D, and of the bi-graded algebra $H^\bullet(X, \Omega_X^\bullet(\log D))$ (see Example 3.7). The structure of the latter algebra also follows from Deligne's description of the mixed Hodge structure on the cohomology of G (see [14, Thm. 9.1.5]), while the former seems to be new. In that setting, one may also obtain more precise information on the numerical invariants of (possibly non-ample) hypersurfaces in G and X; this will be developed elsewhere.

NOTATION. Throughout this article, we consider algebraic schemes, varieties, and morphisms over the field $\boldsymbol{C}$ of complex numbers. We follow the conventions of the book [20]; in particular, a *variety* is an integral separated scheme of finite type over $\boldsymbol{C}$. By a point, we always mean a closed point; a *general point* of a variety is a point of some non-empty Zariski open subset.

We shall consider pairs (X, D), where X is a complete nonsingular variety of dimension n, and D is a *simple normal crossing divisor* on X, i.e., D is an effective, reduced divisor with nonsingular irreducible components $D_1, \dots, D_l$ which intersect transversally. We then set

$$X_0 := X \setminus \operatorname{Supp}(D),$$

the *open part* of X.

An *algebraic group* G is a group scheme of finite type over $\boldsymbol{C}$; then each connected component of G is a nonsingular variety. We denote by $G^0 \subset G$ the *neutral component*, i.e., the connected component through the identity element e, and by $\mathfrak{g}$ the Lie algebra of G.

We say that a pair (X, D) is a *G-pair*, if X is equipped with a faithful action of the algebraic group G that preserves D.

1. Logarithmic Dolbeault cohomology of varieties with finitely many orbits.

1.1. Differential forms with logarithmic poles. We begin by recalling some basic results on differential forms with logarithmic poles, referring to [16, Chap. 2] for details.

Given a pair (X, D) as above and an integer $j \geq 0$, let $\Omega^j_X(\log D)$ denote the sheaf of *differential forms of degree j with logarithmic poles along D*, consisting of rational j-forms ω on X such that ω and $d\omega$ have at worst simple poles along $D_1, \dots, D_l$. The sheaf $\Omega^j_X(\log D)$ is locally free and satisfies

$$\Omega^j_X(\log D) = \wedge^j \Omega^1_X(\log D), \quad \Omega^n_X(\log D) = \omega_X(D), \tag{1}$$

where $\omega_X := \Omega^n_X$ denotes the canonical sheaf. As a consequence, the dual sheaf of $\Omega^j_X(\log D)$ is given by

$$\Omega^j_X(\log D)^\vee = \Omega^{n-j}_X(\log D) \otimes \omega_X^{-1}(-D). \tag{2}$$

Moreover,

$$\Omega^1_X(\log D)^\vee =: \mathcal{T}_X(-\log D)$$

is the *logarithmic tangent sheaf*, i.e., the subsheaf of the tangent sheaf $\mathcal{T}_X$ consisting of derivations that preserve the ideal sheaf of D.

If (X, D) is a G-pair for some algebraic group G, then the logarithmic (co)tangent sheaves are G-linearised, and we have a morphism of linearised sheaves

$$\mathrm{op}_{X,D} : \mathcal{O}_X \otimes_{\boldsymbol{C}} \mathfrak{g} \to \mathcal{T}_X(-\log D), \tag{3}$$

the *action map*.

Each divisor $D^k := D - D_k$ induces a simple normal crossing divisor on D_k, that we denote by $D^k|_{D_k}$ or just by D^k for simplicity. Moreover, taking the residue along D_k yields an exact sequence

$$0 \to \Omega^j_X(\log D^k) \to \Omega^j_X(\log D) \to \Omega^{j-1}_{D_k}(\log D^k) \to 0 \tag{4}$$

that provides an inductive way to relate differential forms with logarithmic poles to ordinary differential forms. Also, note the exact sequence

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \to \bigoplus_{k=1}^{l} \mathcal{O}_{D_k} \to 0\,. \tag{5}$$

Given an invertible sheaf $\mathcal{L}$ on X, we shall consider the logarithmic Dolbeault cohomology groups, $H^i(X, \mathcal{L} \otimes \Omega_X^j(\log D))$. Note the isomorphism

$$H^i(X, \mathcal{L} \otimes \Omega_X^j(\log D))^* \cong H^{n-i}(X, \mathcal{L}^{-1}(-D) \otimes \Omega_X^{n-j}(\log D))\,, \tag{6}$$

a consequence of Serre duality and (2).

Also, recall a vanishing result of Norimatsu (see [28]): *if $\mathcal{L}$ is ample, then $H^i(X, \mathcal{L}^{-1} \otimes \Omega_X^j(\log D)) = 0$ for all $i+j < n$.* When $D = 0$, this is the classical Kodaira-Akizuki-Nakano vanishing theorem, and the case of an arbitrary D follows by induction on the number of irreducible components of D in view of the exact sequence (4); see [16, Cor. 6.4] for details.

This vanishing result implies a logarithmic version of the Lefschetz theorem, also due to Norimatsu (see [28]). Let $Y_1, \dots, Y_m$ be ample hypersurfaces in X such that the divisor $D + Y_1 + \cdots + Y_m$ has simple normal crossings; in particular, $Y := Y_1 \cap \cdots \cap Y_m$ is a nonsingular complete intersection, equipped with a simple normal crossing divisor $D|_Y$. Then *the pull-back map*

$$H^i(X, \Omega_X^j(\log D)) \to H^i(Y, \Omega_Y^j(\log D))$$

is an isomorphism if $i + j < n - m$, and is injective if $i + j = n - m$.

By Hodge theory (see [13, Sec. 3.2]), it follows that *the pull-back map in cohomology,*

$$H^k(X_0, \boldsymbol{C}) \to H^k(Y_0, \boldsymbol{C})\,,$$

is an isomorphism for $k < n - m$, and is injective for $k = n - m$.

1.2. Varieties with finitely many orbits: the linear case. In this subsection, we consider a G-pair (X, D), where G is a connected algebraic group; we assume that X contains only finitely many G-orbits and that G is linear or, equivalently, affine.

We shall obtain a vanishing theorem for the groups $H^i(X, \Omega_X^j(\log D))$. In the case where $D = 0$, we have the following result, which is well-known if G is reductive:

LEMMA 1.1. (i) *With the assumptions of this subsection, X admits a cellular decomposition* (*in the sense of* [19, *Ex.* 1.9.1]).

(ii) *There are natural isomorphisms*

$$A^i(X) \cong H^i(X, \Omega_X^i)$$

for all i, where $A^i(X)$ denotes the Chow group of rational equivalence classes of cycles of codimension i, with complex coefficients. Moreover,

$$H^i(X, \Omega_X^j) = 0 \quad (i \neq j)\,.$$

PROOF. By [19, Ex. 19.1.11] and Hodge theory, it suffices to show (i). We shall deduce that assertion from the Białynicki-Birula decomposition (see [3]).

We may choose a maximal torus $T \subset G$ and a one-parameter subgroup $\lambda : \boldsymbol{G}_m \to T$ such that the fixed point subscheme X^λ equals X^T. Also, recall that X^T is nonsingular. For any component X_i of X^T, let

$$X_i^+ := \{x \in X \, ; \, \lim_{t \to 0} \lambda(t)x \in X_i\}$$

and

$$r_i : X_i^+ \to X_i \, , \quad x \mapsto \lim_{t \to 0} \lambda(t)x \, .$$

Then X is the disjoint union of the X_i^+, where X_i runs over the components of X^T; moreover, each X_i^+ is a locally closed nonsingular subvariety of X, and each retraction r_i is a locally trivial fibration into affine spaces.

Next, consider the centraliser $G^T \subset G$. Since G^T is connected and the quotient G^T/T admits no non-trivial subtorus, it follows that

$$G^T \cong T \times U,$$

where U is a unipotent group. Clearly, U acts on X^T.

We claim that X^T contains only finitely many orbits of U or, equivalently, of G^T. To check this, it suffices to show that Z^T contains only finitely many G^T-orbits, for any G-orbit Z. Given a point $x \in Z^T$, the differential at e of the orbit map $G \to Z$, $g \mapsto g \cdot x$ yields a surjective map $\mathfrak{g} \to T_x Z$ (where $\mathfrak{g}$ denotes the Lie algebra of G), and hence a surjective map $\mathfrak{g}^T \to (T_x Z)^T = T_x(Z^T)$. It follows that the orbit $G^T \cdot x$ is open in Z^T, which implies our claim.

Note that U preserves each component X_i, and r_i is U-equivariant. Thus, given an orbit $Z = U \cdot x$ in X_i, the preimage $r_i^{-1}(Z)$ is isomorphic to the homogeneous bundle $U \times^{U_x} F$ over $Z \cong U/U_x$, where U_x denotes the isotropy group of x in U, and F the fibre of r_i at x (so that F is preserved by U_x and isomorphic to an affine space). Since U is unipotent, it contains a closed subvariety V, isomorphic to an affine space, such that the multiplication of the group U induces an isomorphism $V \times U_x \cong U$. Then $r_i^{-1}(Z) \cong V \times F$ is an affine space, which yields the desired cellular decomposition. □

From that result and inductive arguments, we shall deduce:

THEOREM 1.2. *With the assumptions of this subsection, there are natural isomorphisms*

$$A^i(X_0) \cong H^i(X, \Omega_X^i(\log D)) \tag{7}$$

for all i. Moreover,

$$H^i(X, \Omega_X^j(\log D)) = 0 \quad (i > j) \, . \tag{8}$$

PROOF. We may assume that $D \neq 0$ in view of Lemma 1.1. We first prove the vanishing assertion (8), by induction on the number of irreducible components of D and the dimension of X. Write $D = D_1 + D^1$, where D_1 is irreducible. Then (4) yields an exact sequence

$$H^i(X, \Omega_X^j(\log D^1)) \to H^i(X, \Omega_X^j(\log D)) \to H^i(D_1, \Omega_{D_1}^{j-1}(\log D^1))$$

which implies our assertion.

Next, we construct the isomorphism (7). If D is irreducible, then the exact sequence $0 \to \Omega_X^i \to \Omega_X^i(\log D) \to \Omega_D^{i-1} \to 0$ (a special case of (4)) yields a diagram

$$\begin{array}{ccccccc} A^{i-1}(D) & \longrightarrow & A^i(X) & \longrightarrow & A^i(X \setminus D) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ H^{i-1}(D, \Omega_D^{i-1}) & \longrightarrow & H^i(X, \Omega_X^i) & \longrightarrow & H^i(X, \Omega_X^i(\log D)) & \longrightarrow & 0, \end{array}$$

where the top row is a standard exact sequence of Chow groups (see [19, Prop. 1.8]), the bottom row is exact since $H^i(D, \Omega_D^{i-1}) = 0$, the vertical arrows are isomorphisms, and the square commutes by functoriality of the cycle map (see [19, Sec. 19.1]). This yields the desired isomorphism.

In the general case, we argue again by induction on the number of irreducible components of D and the dimension of X. The exact sequences

$$H^{i-1}(D_k, \Omega_{D_k}^{i-1}) \to H^i(X, \Omega_X^i) \to H^i(X, \Omega_X^i(\log D_k)) \to 0$$

for $k = 1, \ldots, l$ and the natural maps

$$H^i(X, \Omega_X^i(\log D_k)) \to H^i(X, \Omega_X^i(\log D))$$

yield a complex

$$\bigoplus_{k=1}^{l} H^{i-1}(D_k, \Omega_{D_k}^{i-1}) \to H^i(X, \Omega_X^i) \to H^i(X, \Omega_X^i(\log D)) \to 0. \tag{9}$$

We claim that this complex is exact. Consider indeed the commutative diagram

$$\begin{array}{ccccccc} H^{i-1}(D_1, \Omega_{D_1}^{i-1}) & \longrightarrow & H^i(X, \Omega_X^i) & \longrightarrow & H^i(X, \Omega_X^i(\log D_1)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{i-1}(D_1, \Omega_{D_1}^{i-1}(\log D^1)) & \longrightarrow & H^i(X, \Omega_X^i(\log D^1)) & \longrightarrow & H^i(X, \Omega_X^i(\log D)) & \longrightarrow & 0. \end{array}$$

Then the rows are exact, by the vanishing of $H^i(D_1, \Omega_{D_1}^{i-1}(\log D^1))$ and the preceding argument. Moreover, the left and middle vertical maps are surjective by the induction assumption; thus, so is the right vertical map. This implies our claim.

That claim implies in turn the isomorphism (7), by comparing (9) with the complex

$$\bigoplus_{k=1}^{l} A^{i-1}(D_k) \to A^i(X) \to A^i(X_0) \to 0$$

which is exact in view of the standard exact sequence

$$A^{i-1}(\mathrm{Supp}(D)) \to A^i(X) \to A^i(X_0) \to 0$$

and the surjectivity of the natural map

$$\bigoplus_{k=1}^{l} A^{i-1}(D_k) \to A^{i-1}(\mathrm{Supp}(D)) . \qquad \square$$

REMARK 1.3. One can show that X_0 is a linear variety, as defined by Totaro in [30]. By Theorem 3 of that article, it follows that the Chow group of X_0 is isomorphic to the smallest subspace of Borel-Moore homology (with complex coefficients) with respect to the weight filtration. In turn, this yields another proof of Theorem 1.2, admittedly less direct than the proof presented here.

1.3. Varieties with finitely many orbits: the general case. We still consider a G-pair (X, D), where G is a connected algebraic group and X contains only finitely many G-orbits, but we no longer assume that G is linear.

We shall obtain a generalisation of Theorem 1.2 to that setting; for this, we recall (after [7, Prop. 2.4.1]) a reduction to the linear case, via the Albanese morphism

$$\alpha : X \to A \tag{10}$$

(the universal morphism to an abelian variety).

By Chevalley's structure theorem, G admits a largest connected affine subgroup G_{aff}; this subgroup is normal in G, and the quotient G/G_{aff} is an abelian variety. Moreover, X is equivariantly isomorphic to the total space of a homogeneous bundle $G \times^I Y$, where $I \subset G$ is a closed subgroup with neutral component G_{aff}, and $Y \subset X$ is a closed nonsingular subvariety, preserved by I; both I and Y are unique.

As a consequence, I is a normal affine subgroup of G, and the quotient G/I is an abelian variety equipped with an isogeny $G/G_{\mathrm{aff}} \to G/I$. Moreover, I acts on Y with finitely many orbits, and D induces a simple normal crossing divisor E on Y, preserved by I. Finally, the Albanese morphism (10) may be identified to the homogeneous fibration $G \times^I Y \to G/I$ with fibre Y.

Also, note a splitting property of the logarithmic (co)tangent sheaf:

LEMMA 1.4. *With the preceding notations, there is an isomorphism of G-linearised sheaves*

$$\Omega^1_X(\log D) \cong \Omega^1_{X/A}(\log D) \oplus (\mathcal{O}_X \otimes_{\mathbf{C}} \mathfrak{a}^*) , \tag{11}$$

where $\Omega^1_{X/A}(\log D) \otimes \mathcal{O}_Y \cong \Omega^1_Y(\log E)$, and $\mathfrak{a}$ denotes the Lie algebra of A (so that G acts trivially on $\mathfrak{a}$).

Moreover, the composite map $\mathcal{O}_X \otimes_{\mathbf{C}} \mathfrak{a}^* \to \Omega^1_X(\log D) \to \mathcal{O}_X \otimes_{\mathbf{C}} \mathfrak{g}^*$ (where the map on the right is the transpose of the action map (3)) is induced from the map $\mathfrak{a}^* \to \mathfrak{g}^*$, the transpose of the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_{\mathrm{aff}} = \mathfrak{a}$.

PROOF. By [7, Prop. 2.4.1], the Albanese fibration yields an exact sequence of G-linearised sheaves

$$0 \to \mathcal{T}_{X/A}(-\log D) \to \mathcal{T}_X(-\log D) \to \alpha^*\mathcal{T}_A \to 0 , \tag{12}$$

where $\mathcal{T}_{X/A}(-\log D) \otimes \mathcal{O}_Y \cong \mathcal{T}_Y(-\log E)$. Also, note that $\alpha^*\mathcal{T}_A \cong \mathcal{O}_X \otimes \mathfrak{a}$. Since G acts on A, and α is equivariant, the composite map

$$\mathcal{O}_X \otimes_{\boldsymbol{C}} \mathfrak{g} \xrightarrow{\mathrm{op}_{X,D}} \mathcal{T}_X(-\log D) \longrightarrow \mathcal{O}_X \otimes_{\boldsymbol{C}} \mathfrak{a}$$

is induced from the quotient map $\mathfrak{g} \to \mathfrak{a}$.

Choose a subspace $\tilde{\mathfrak{a}} \subset \mathfrak{g}$ such that the composite map $\tilde{\mathfrak{a}} \to \mathfrak{g} \to \mathfrak{a}$ is an isomorphism. Then the composite map

$$\mathcal{O}_X \otimes_{\boldsymbol{C}} \tilde{\mathfrak{a}} \to \mathcal{O}_X \otimes_{\boldsymbol{C}} \mathfrak{g} \to \mathcal{T}_X(-\log D) \to \mathcal{O}_X \otimes_{\boldsymbol{C}} \mathfrak{a}$$

is an isomorphism as well; thus, the exact sequence (12) is split. Taking duals yields our assertions. □

REMARK 1.5. With the notation of the preceding proof, we may further assume that $\tilde{\mathfrak{a}}$ is contained in the centre of $\mathfrak{g}$. Indeed, if $C(G)$ denotes the centre of the group G, then the natural map $C(G) \to A$ is surjective, as follows e.g. from [7, Lem. 1.1.1].

This yields a decomposition of the logarithmic tangent sheaf into a direct sum of the integrable subsheaves $\mathcal{T}_{X/A}(-\log D)$ and $\mathcal{O}_X \otimes_{\boldsymbol{C}} \tilde{\mathfrak{a}}$, and an analogous splitting of the tangent sheaf.

We now come to the main result of this section:

THEOREM 1.6. *With the notation and assumptions of this subsection*,

$$H^i(X, \Omega_X^j(\log D)) = 0 \quad (i > j + q(X)), \tag{13}$$

where $q(X) := \dim(A)$ *denotes the irregularity of* X. *Moreover*, *there is an isomorphism*

$$H^{j+q(X)}(X, \Omega_X^j(\log D)) \cong A^j(Y_0)^I \tag{14}$$

(*the subspace of* I-*invariants in* $A^j(Y_0)$), *and* I *acts on* $A^j(Y_0)$ *via the finite quotient* I/G_{aff}.

PROOF. Lemma 1.4 yields decompositions

$$\Omega_X^j(\log D) \cong \bigoplus_{k=0}^{j} \Omega_{X/A}^k(\log D) \otimes_{\boldsymbol{C}} \wedge^{j-k}\mathfrak{a}^* \tag{15}$$

and isomorphisms

$$\Omega_{X/A}^k(\log D) \otimes \mathcal{O}_Y \cong \Omega_Y^k(\log E).$$

Since $H^i(Y, \Omega_Y^k(\log E)) = 0$ for $i > k$ by (8), this yields in turn

$$R^i\alpha_*\Omega_{X/A}^k(\log D) = 0 \quad (i > k). \tag{16}$$

Together with (15), it follows that

$$R^i\alpha_*\Omega_X^j(\log D) = 0 \quad (i > j). \tag{17}$$

This implies the vanishing (13) in view of the Leray spectral sequence

$$H^p(A, R^q\alpha_*\Omega_X^j(\log D)) \Rightarrow H^{p+q}(X, \Omega_X^j(\log D))$$

which also yields isomorphisms

$$H^{q(X)+j}(X, \Omega_X^j(\log D)) \cong H^{q(X)}(A, R^j\alpha_*\Omega_X^j(\log D)).$$

Moreover, $R^j\alpha_*\Omega_X^j(\log D) = R^j\alpha_*\Omega_{X/A}^j(\log D)$ is the G-linearised sheaf on $A = G/I$ associated with the I-module $H^j(Y, \Omega_Y^j(\log E))$, i.e., with $A^j(Y_0)$ in view of (7). By Serre duality, it follows that

$$H^{q(X)}(A, R^j\alpha_*\Omega_X^j(\log D)) \cong H^0(A, \mathcal{F})^*,$$

where $\mathcal{F}$ denotes the G-linearised sheaf on G/I associated with the dual I-module $A^j(Y_0)^*$. Since the connected linear algebraic group G_{aff} acts trivially on the Chow group $A^j(Y_0)$, we have

$$\begin{aligned} H^0(A, \mathcal{F}) = H^0(G/I, \mathcal{F}) &\cong (\mathcal{O}(G) \otimes_{\mathbf{C}} A^j(Y_0)^*)^I \\ &\cong (\mathcal{O}(G)^{G_{\mathrm{aff}}} \otimes_{\mathbf{C}} A^j(Y_0)^*)^{I/G_{\mathrm{aff}}} \cong (A^j(Y_0)^*)^{I/G_{\mathrm{aff}}}. \end{aligned}$$

This yields the isomorphism (14). □

REMARK 1.7. (i) Let $Y_1, \ldots, Y_m \subset X$ be ample hypersurfaces such that the divisor $D + Y_1 + \cdots + Y_m$ has simple normal crossings, and consider the complete intersection $Y := Y_1 \cap \cdots \cap Y_m$. Combining Theorem 1.6 with the logarithmic Lefschetz theorem recalled in Subsection 1.1, we see that

$$H^i(Y, \Omega_Y^j(\log D)) = 0 \quad (i + j < n - m \quad \text{and} \quad i > j + q(X)).$$

If G is linear, i.e., $q(X) = 0$, then also

$$H^i(Y, \Omega_Y^i(\log D)) \cong A^i(X_0) \quad (i < (n-m)/2).$$

Moreover, if $i = (n-m)/2$, then $A^i(X_0) \hookrightarrow H^i(Y, \Omega_Y^i(\log D))$. In particular, $A^i(X_0) \hookrightarrow A^i(Y_0)$ for all $i \leq (n-m)/2$.

If, in addition, X_0 is affine, then $H^i(Y, \Omega_Y^j(\log D)) = 0$ whenever $i + j > n - m$. Thus, the only "unknown" groups $H^i(Y, \Omega_Y^j(\log D))$ are those where $i + j = n - m$. To compute their dimension, it suffices to determine the Euler characteristic $\chi(Y, \Omega_Y^j(\log D))$, which is expressed in topological terms via the Riemann-Roch theorem.

Since the topological Euler characteristic of Y_0 satisfies an adjunction formula due to Norimatsu and Kiritchenko (see [28, 22]), this yields a determination of the Betti numbers of Y_0, as already observed in [11] for toric varieties.

(ii) Taking $D = 0$ in Theorem 1.6 and using Serre duality yields the vanishing

$$H^i(X, \Omega_X^j) = 0 \quad (|i - j| > q(X)) \tag{18}$$

for a complete nonsingular variety X on which an algebraic group G acts with finitely many orbits.

This is closely related to a result of Carrell and Lieberman (see [10, Thm. 1]):

$$H^i(M, \Omega_M^j) = 0 \quad (|i - j| > \dim Z(V)), \tag{19}$$

where M is a compact Kähler manifold admitting a global vector field V with non-empty scheme of zeros $Z(V)$.

In fact, (19) implies (18) when X is projective and the maximal connected affine subgroup $G_{\mathrm{aff}} \subset G$ is reductive. Consider indeed a general one-parameter subgroup $\lambda : \boldsymbol{G}_m \to G_{\mathrm{aff}}$ and the associated vector field $V \in \mathfrak{g}_{\mathrm{aff}}$. Then $Z(V) = X^{\lambda} = X^T$, where T denotes the unique maximal torus of G_{aff} containing the image of λ. It follows that $Z(V)$ meets each G_{aff}-orbit along a finite set, non-empty if the orbit is closed; as a consequence, $\dim Z(V) = q(X)$. However, it is not clear whether (18) may be deduced from (19) when (say) G_{aff} is unipotent.

2. Geometry of log homogeneous varieties.

2.1. Basic properties. Let (X, D) be a G-pair, where G is a connected algebraic group. Following [7], we say that X is *log homogeneous under G with boundary D*, or that (X, D) is *G-homogeneous*, if the action map (3) is surjective.

Under that assumption, the open part X_0 is a unique G-orbit. Moreover, by [7, Cor. 3.2.2], the G-orbit closures in X are exactly the non-empty partial intersections of boundary divisors,

$$D_{k_1,\dots,k_m} := D_{k_1} \cap \cdots \cap D_{k_m} \, ,$$

and each $D_{k_1,\dots,k_m}$ is log homogeneous under G with boundary being the restriction of the divisor

$$D^{k_1,\dots,k_m} := \sum_{k \neq k_1,\dots,k_m} D_k \, .$$

In particular, X contains only finitely many G-orbits, and these coincide with the orbits of the connected automorphism group $\mathrm{Aut}^0(X, D)$. Moreover, the closed orbits are exactly the minimal non-empty partial intersections. By [7, Thm. 3.3.3], these closed orbits are all isomorphic. We call their common codimension the *rank* of X, and denote it by $\mathrm{rk}(X)$. Note that

$$\mathrm{rk}(D_{k_1,\dots,k_m}) = \mathrm{rk}(X) - m \, . \tag{20}$$

Log homogeneity is preserved by equivariant blowing up, in the following sense. Let X' be a complete nonsingular G-variety equipped with a G-equivariant birational morphism $u : X' \to X$. Denote by D' the reduced inverse image of D. Then (X', D') is a homogeneous G-pair, by [7, Prop. 2.3.2]. We now show that logarithmic Dolbeault cohomology is also preserved:

LEMMA 2.1. *With the preceding notation and assumptions, there are isomorphisms*

$$u^* \Omega^j_X(\log D) \cong \Omega^j_{X'}(\log D') \, . \tag{21}$$

Moreover, any invertible sheaf $\mathcal{L}$ on X satisfies

$$H^i(X', u^*\mathcal{L} \otimes \Omega^j_{X'}(\log D')) \cong H^i(X, \mathcal{L} \otimes \Omega^j_X(\log D)) \tag{22}$$

for all i and j.

PROOF. The natural morphism

$$du : \mathcal{T}_{X'}(-\log D') \to u^*\mathcal{T}_X(-\log D)$$

is clearly surjective, and hence is an isomorphism since its source and target are locally free sheaves of the same rank. This implies (21) and, in turn, the isomorphism (22) by using the projection formula and the equalities $u_*\mathcal{O}_{X'} = \mathcal{O}_X$, $R^i u_*\mathcal{O}_{X'} = 0$ for all $i \geq 1$. □

2.2. The bundle of isotropy Lie subalgebras. We still consider a pair (X, D), homogeneous under a connected algebraic group G. The action map (3) yields an exact sequence

$$0 \to \mathcal{R}_X \to \mathcal{O}_X \otimes_{\boldsymbol{C}} \mathfrak{g} \to \mathcal{T}_X(-\log D) \to 0\,, \tag{23}$$

where $\mathcal{R}_X$ is locally free; equivalently, we have an exact sequence

$$0 \to \Omega^1_X(\log D) \to \mathcal{O}_X \otimes_{\boldsymbol{C}} \mathfrak{g}^* \to \mathcal{R}_X^\vee \to 0\,. \tag{24}$$

We denote by R_X the vector bundle over X associated with the locally free sheaf $\mathcal{R}_X$. Specifically, the structure map

$$p : R_X \to X$$

satisfies

$$p_*\mathcal{O}_{R_X} = \bigoplus_{j\geq 0} S^j\mathcal{R}_X^\vee\,, \tag{25}$$

where S^j denotes the j-th symmetric power over $\mathcal{O}_X$.

By [7, Prop. 2.1.2], the fibre $R_{X,x}$ at an arbitrary point x is an ideal of the isotropy Lie subalgebra $\mathfrak{g}_x$: the kernel $\mathfrak{g}_{(x)}$ of the representation of $\mathfrak{g}_x$ in the normal space to the orbit $G \cdot x$ at x. In particular, $R_{X,x} = \mathfrak{g}_x$ if $x \in X_0$. We may thus call R_X the *bundle of isotropy Lie subalgebras*.

We may view R_X as a closed G-stable subvariety of $X \times \mathfrak{g}$, and denote by

$$f : R_X \to \mathfrak{g}$$

the second projection. Then f is proper, G-equivariant, and its fibres may be identified to closed subschemes of X via the first projection p.

Since X is complete, the vector bundle R_X is trivial if and only if f is constant, i.e., $\mathfrak{g}_{(x)}$ is independent of $x \in X$. By [7, Thm. 2.5.1], this is also equivalent to X being a semi-abelic variety.

Returning to an arbitrary homogeneous G-pair (X, D), choose a *base point* $x_0 \in X_0$ and denote by

$$H := G_{x_0}$$

its isotropy group, with Lie algebra

$$\mathfrak{h} := \mathfrak{g}_{x_0}\,.$$

This identifies X_0 to the homogeneous space G/H, and the pull-back R_{X_0} to the homogeneous vector bundle $G \times^H \mathfrak{h}$ associated with the adjoint representation of the (possibly non-connected) algebraic group H. The restriction

$$f_0 : R_{X_0} \to \mathfrak{g}$$

is identified to the "collapsing" morphism

$$G \times^H \mathfrak{h} \to \mathfrak{g}, \quad (g, \xi)H \mapsto g \cdot \xi \tag{26}$$

where the dot denotes the adjoint action. In particular,

$$f(R_X) = \overline{G \cdot \mathfrak{h}} \tag{27}$$

and for any $\xi \in \mathfrak{h}$, the fibre of f_0 at the point $(1, \xi)H \in R_{X,x_0}$ is identified to the fixed point subscheme $(G/H)^\xi$.

2.3. The general fixed point subschemes. We keep the notation and assumptions of Subsection 2.2, and assume in addition that the algebraic group G is linear.

THEOREM 2.2. *With the preceding assumptions, the connected components of the general fibres of f over its image $\overline{G \cdot \mathfrak{h}}$ are toric varieties under subtori of G, of dimension* $\mathrm{rk}(G) - \mathrm{rk}(H)$.

PROOF. It suffices to consider fibres at points $(1, \xi)H$, where ξ is a general point of $\mathfrak{h}$. Let $\xi = s + n$ be the Jordan decomposition, that is, $s \in \mathfrak{h}$ is semi-simple, $n \in \mathfrak{h}$ is nilpotent, and $[s, n] = 0$. Then $\mathrm{rk}(G) = \mathrm{rk}(G^s)$, where G^s denotes the centralizer of s in G; likewise, $\mathrm{rk}(H) = \mathrm{rk}(H^s)$. Also, note that $n \in \mathfrak{h}^s$ (the Lie algebra of H^s). Moreover, since ξ is general, we may assume that s is a general point of the Lie algebra of a maximal torus $T_H \subset H$. Then $G^s = G^{T_H}$ acts on the fixed point subscheme $X^s = X^{T_H}$ through the quotient group G^{T_H}/T_H; moreover, $(H^0)^{T_H}/T_H$ is unipotent. Together with Lemma 2.3, this yields a reduction to the case where $\mathrm{rk}(H) = 0$; equivalently, H^0 is unipotent.

Under that assumption, we claim that H^0 is a maximal unipotent subgroup of G. Indeed, the variety G/H is spherical under any Levi subgroup L of G, by [7, Thm. 3.2.1]. Therefore, we have $\dim(G/H) \le \dim(B_L)$, where B_L denotes a Borel subgroup of L. But $\dim(B_L) = \dim(G/U)$, where $U \subset G$ is a maximal unipotent subgroup. Thus, $\dim(H) \ge \dim(U)$ which implies our claim.

By that claim, the normalizer of H is a Borel subgroup of G, that we denote by B. Moreover, for any maximal torus $T \subset B$, the intersection $T \cap H$ is finite, and $B/H \cong T/(T \cap H)$. Since B normalizes $\mathfrak{h}$, the morphism $f_0 : G \times^H \mathfrak{h} \to \mathfrak{g}$ factors as the natural map

$$u : G \times^H \mathfrak{h} \to G \times^B \mathfrak{h}$$

followed by the collapsing morphism

$$v : G \times^B \mathfrak{h} \to \mathfrak{g}.$$

Also, v is birational onto its image, the cone of nilpotent elements in $\mathfrak{g}$ (this is well-known in the case where G is reductive, and the general case follows by using a Levi decomposition of G). On the other hand, the fibre of u at any point $(1, \xi)H$ is B/H, where both are identified to subvarieties of G/H. This identifies the general fibres of f_0 to quotients of maximal tori of G by finite subgroups. □

LEMMA 2.3. *Let s be a semi-simple element of $\mathfrak{h}$ with centralizer G^s and fixed point subscheme X^s, and let Y denote the connected component of X^s through the base point x_0. Then*:

(i) *Y is a log homogeneous variety under G^s with boundary $D|_Y$. Moreover, the fixed point subscheme R_X^s is a vector bundle over X^s, and its pull-back to Y is R_Y.*

(ii) *For any nilpotent element $n \in \mathfrak{h}^s$, the fibres of $f : R_X \to \mathfrak{g}$ and of $f_Y : R_Y \to \mathfrak{g}^s$ at $s+n \in \mathfrak{h}^s$ coincide in a neighborhood of the point $(1, s+n)H$.*

PROOF. (i) Since s is semi-simple, X^s is nonsingular and $T_x(X^s) = (T_xX)^s$ for any $x \in X^s$. Moreover, $T_x((G\cdot x)^s) = (T_x(G\cdot x))^s = \mathfrak{g}^s \cdot x$. As a consequence, $G^s \cdot x$ is a component of $(G\cdot x)^s$. It follows readily that D induces a divisor with normal crossings on X^s. Moreover, the exact sequence

$$0 \to \mathfrak{g}_{(x)} \to \mathfrak{g} \to T_xX(-\log D) \to 0$$

(see [7, (2.1.4)]) yields an exact sequence

$$0 \to \mathfrak{g}^s_{(x)} \to \mathfrak{g}^s \to T_xX^s(-\log D) \to 0\,.$$

This implies both assertions.

(ii) It suffices to show that both fibres have the same tangent space at the point $(1,\xi)H$, where $\xi := s+n$. For this, consider the map

$$\Phi : G \times \mathfrak{h} \to \mathfrak{g}\,, \quad (g,\xi) \mapsto g\cdot\xi\,,$$

invariant under the action of H via $h\cdot(g,\xi) = (gh^{-1}, h\cdot\xi)$ and which induces the map f_0 on the quotient $G \times^H \mathfrak{h}$. The differential of Φ at $(1,\xi)$ may be identified to the map

$$\varphi : \mathfrak{g}\times\mathfrak{h} \to \mathfrak{g}\,, \quad (u,v) \mapsto [u,\xi] + v$$

and the differential of the orbit map

$$H \to G \times \mathfrak{h}\,, \quad h \mapsto (h^{-1}, h\cdot\xi)$$

is identified with the map

$$\psi : \mathfrak{h} \to \mathfrak{g}\times\mathfrak{h}\,, \quad w \mapsto (-w, [w,\xi])\,.$$

Thus, the tangent space at $(1,\xi)H$ of the fibre of f through that point is the homology space $\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi)$.

Next, consider the decomposition $\mathfrak{g} = \bigoplus_{\lambda\in C} \mathfrak{g}_\lambda$ into eigenspaces of s, where $\mathfrak{g}_0 = \mathfrak{g}^s$, and the induced decomposition $\mathfrak{h} = \bigoplus_{\lambda\in C}\mathfrak{h}_\lambda$. For $(u,v)\in \mathrm{Ker}(\varphi)$, this yields with obvious notation:

$$v_\lambda - (\lambda + \mathrm{ad}(n))u_\lambda = 0 \quad \text{for all } \lambda\,.$$

If λ is not zero, then $\lambda + \mathrm{ad}(n)$ is an automorphism of $\mathfrak{g}$ preserving $\mathfrak{h}$, and hence (u_λ, v_λ) is in $\mathrm{Im}(\psi)$. Thus,

$$\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi) = \mathrm{Ker}(\varphi^s)/\mathrm{Im}(\psi^s)$$

with obvious notation again; this yields the desired equality of tangent spaces. □

REMARK 2.4. Given an arbitrary closed subgroup H of a connected linear algebraic group G, and a semi-simple element $s \in \mathfrak{h}$, the orbit $G^s \cdot x_0 \cong G^s/H^s$ is open in the fixed point subscheme $(G/H)^s$. If G and H are reductive, then any general point $s \in \mathfrak{h}$ is semi-simple, and $(H^s)^0$ is a maximal torus of H, contained in the centre of the connected reductive group G^s. If, in addition, the homogeneous space G/H is spherical, then G^s/H^s is also spherical, as follows e.g. from Lemma 2.3. This implies that G^s is a torus, and yields a very simple proof of Theorem 2.2 in that special case.

EXAMPLE 2.5. Any connected reductive group G may be viewed as the homogeneous space $(G \times G)/\mathrm{diag}(G)$ for the action of $G \times G$ by left and right multiplication. This homogeneous space is spherical, and hence admits a log homogeneous equivariant compactification X.

We claim that

$$\mathcal{R}_X \cong \Omega^1_X(\log D) \tag{28}$$

as $G \times G$-linearised sheaves; in other words, R_X is equivariantly isomorphic to the total space of the logarithmic cotangent bundle. To see this, choose a non-degenerate G-invariant quadratic form q on $\mathfrak{g}$; then the quadratic form $(q, -q)$ on $\mathfrak{g} \times \mathfrak{g}$ is non-degenerate and $G \times G$-invariant. Moreover, the fibre of R_X at the identity element, $\mathfrak{g}_{(e)} = \mathrm{diag}(\mathfrak{g})$, is a Lagrangian subspace of $\mathfrak{g} \times \mathfrak{g}$. It follows that R_X is a Lagrangian sub-bundle of the trivial bundle $X \times \mathfrak{g} \times \mathfrak{g}$ equipped with the quadratic form $(q, -q)$. Thus, the quotient bundle $(X \times \mathfrak{g} \times \mathfrak{g})/R_X$ is isomorphic to the dual of R_X. This implies our claim in view of the exact sequence (23).

In fact, via the isomorphism $\mathfrak{g} \times \mathfrak{g} \cong \mathfrak{g}^* \times \mathfrak{g}^*$ defined by $(q, -q)$, the map $f : R_X \to \mathfrak{g} \times \mathfrak{g}$ is identified to the compactified moment map of the logarithmic cotangent bundle, considered in [23]. Also, note that $R_{X_0} \cong G \times \mathfrak{g}$ over $X_0 \cong G$, and the restriction $f_0 : R_{X_0} \to \mathfrak{g} \times \mathfrak{g}$ is identified with the map

$$G \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}, \quad (g, \xi) \mapsto (\xi, g \cdot \xi).$$

Thus, if $\xi \in \mathfrak{g}$ is regular and semi-simple, then the fibre of f_0 at (g, ξ) is isomorphic to the maximal torus $G^{g \cdot \xi}$. As a consequence, the general fibres of f are exactly the closures in X of maximal tori of G.

Another natural map associates with any $x \in X$ the Lagrangian subalgebra $\mathfrak{g}_{(x)} \subset \mathfrak{g} \times \mathfrak{g}$. In fact, this yields a morphism from X onto an irreducible component of the variety of Lagrangian subalgebras, isomorphic to the wonderful compactification of the adjoint semi-simple group $G/C(G)$ (see [15, Sec. 2]).

Returning to the general setting, we denote by $r(X)$ the dimension of the general fibres of f; this is also the codimension of $\overline{G \cdot \mathfrak{h}}$ in $\mathfrak{g}$, since $\dim R_X = \dim \mathfrak{g}$.

COROLLARY 2.6. *With the preceding notation and assumptions, we have the inequality* $r(X) \leq \mathrm{rk}(X)$.

PROOF. We argue by induction on the rank of X. If $\mathrm{rk}(X) = 0$ then $X \cong G/H$, where H is a parabolic subgroup of G; thus, $\mathrm{rk}(H) = \mathrm{rk}(G)$. For an arbitrary rank r, consider a boundary divisor D_1 and its open orbit $G \cdot x_1 \cong G/H_1$. Since $\mathrm{rk}(D_1) = \mathrm{rk}(X) - 1$, it suffices to show that

$$\mathrm{rk}(H_1) \le \mathrm{rk}(H) + 1\,. \tag{29}$$

Choose a maximal torus $T_1 \subset H_1$ and consider its action on the normal space to D_1 at x_1. This one-dimensional representation defines a non-trivial character χ of H_1 (see [7, Prop. 2.1.2]) and hence of T_1. Let $S := \mathrm{Ker}(\chi|_{T_1})^0$. Then S has fixed points in G/H, as follows from the existence of an étale linearisation of the action of T_1 at x_1 (that is, a T_1-stable affine open subset $X_1 \subset X$ containing x_1, and a T_1-equivariant étale morphism $X_1 \to T_{x_1}X$ that maps x_1 to 0). Thus, $\dim(S) \le \mathrm{rk}(H)$, which yields (29). □

3. Vanishing theorems.

3.1. The linear case. We still consider a homogeneous pair (X, D) under a connected linear algebraic group G.

Since X contains only finitely many orbits of G, Theorem 1.2 yields the vanishing of $H^i(X, \Omega^j_X(\log D))$ for all $i > j$. From that vanishing theorem for exterior powers of $\Omega^1_X(\log D)$, we shall deduce a vanishing theorem for symmetric powers of $\mathcal{R}^\vee_X$, via the following homological trick (a generalisation of [9, Prop. 1]):

LEMMA 3.1. *Let Z be a variety, and*

$$0 \to \mathcal{E} \to \mathcal{O}_Z \otimes_{\mathbf{C}} V \to \mathcal{F} \to 0 \tag{30}$$

an exact sequence of locally free sheaves, where V is a finite-dimensional complex vector space. Then the following assertions are equivalent for an invertible sheaf $\mathcal{L}$ on Z and an integer m:

(i) $H^i(Z, \mathcal{L} \otimes S^j\mathcal{F}) = 0$ *for all $i > m$ and all j.*

(ii) $H^i(Z, \mathcal{L} \otimes \wedge^j\mathcal{E}) = 0$ *for all $i > j + m$.*

PROOF. Taking the Koszul complex associated with (30) and tensoring with $\mathcal{L}$ yields an exact sequence

$$\begin{aligned} 0 \to \mathcal{L} \otimes \wedge^j\mathcal{E} \to \mathcal{L} \otimes \wedge^{j-1}\mathcal{E} \otimes_{\mathbf{C}} V \to \cdots \to \mathcal{L} \otimes \wedge^{j-k}\mathcal{E} \otimes_{\mathbf{C}} S^kV \to \cdots \\ \cdots \to \mathcal{L} \otimes \mathcal{E} \otimes_{\mathbf{C}} S^{j-1}V \to \mathcal{L} \otimes S^jV \to \mathcal{L} \otimes S^j\mathcal{F} \to 0\,. \end{aligned}$$

We break this long exact sequence into short exact sequences:

$$\begin{gathered} 0 \to \mathcal{L} \otimes \wedge^j\mathcal{E} \to \mathcal{L} \otimes \wedge^{j-1}\mathcal{E} \otimes_{\mathbf{C}} V \to \mathcal{F}_1 \to 0\,, \\ 0 \to \mathcal{F}_1 \to \mathcal{L} \otimes \wedge^{j-2}\mathcal{E} \otimes_{\mathbf{C}} S^2V \to \mathcal{F}_2 \to 0\,, \ldots \\ 0 \to \mathcal{F}_{k-1} \to \mathcal{L} \otimes \wedge^{j-k}\mathcal{E} \otimes_{\mathbf{C}} S^kV \to \mathcal{F}_k \to 0\,, \ldots \\ 0 \to \mathcal{F}_{j-2} \to \mathcal{L} \otimes \mathcal{E} \otimes_{\mathbf{C}} S^{j-1}V \to \mathcal{F}_{j-1} \to 0\,, \\ 0 \to \mathcal{F}_{j-1} \to \mathcal{L} \otimes S^jV \to \mathcal{L} \otimes S^j\mathcal{F} \to 0\,. \end{gathered}$$

If (i) holds, then $H^i(Z, \mathcal{L} \otimes_{\mathbf{C}} S^jV) = 0$ for all $i > m$, and hence $H^i(Z, \mathcal{F}_{j-1}) = 0$ for all $i > m + 1$. We now prove (ii) by induction on j. If $j = 1$, then $\mathcal{F}_{j-1} = \mathcal{L} \otimes \mathcal{E}$ which yields the assertion. For an arbitrary j, the induction assumption implies that $H^i(Z, \mathcal{L} \otimes \wedge^{j-k}\mathcal{E} \otimes_{\mathbf{C}} S^k V) = 0$ for all $j \geq k \geq 0$ and $i > j-k+m$. By a decreasing induction on k, it follows that $H^i(Z, \mathcal{F}_k) = 0$ for all $i > j-k+m$. In particular, $H^i(Z, \mathcal{F}_1) = 0 = H^i(Z, \mathcal{L} \otimes \wedge^{j-1}\mathcal{E} \otimes_{\mathbf{C}} V)$ for all $i > j+m-1$, which implies the desired vanishing.

The converse implication is obtained by reversing these arguments. □

We apply Lemma 3.1 to the exact sequence (24) and to $\mathcal{L} = \mathcal{O}_X$. Then the assertion (ii) holds for $m = 0$; this yields:

THEOREM 3.2. *With the assumptions of this subsection, we have*

$$H^i(X, S^j \mathcal{R}_X^\vee) = 0 \quad (i \geq 1,\ j \geq 0)\,. \tag{31}$$

We now derive several geometric consequences of this vanishing theorem. We shall need the following observation:

LEMMA 3.3. *The canonical sheaf of the nonsingular variety R_X equals $p^*\mathcal{O}_X(-D)$.*

PROOF. For any locally free sheaf $\mathcal{E}$ of rank r on X, the associated vector bundle E satisfies

$$\omega_E = p^*(\wedge^r \mathcal{E}^\vee \otimes \omega_X)\,.$$

Here, the rank of $\mathcal{R}_X$ equals $\dim(G) - n$. Moreover,

$$\wedge^{\dim(G)-n}\mathcal{R}_X^\vee \cong \wedge^n \mathcal{T}_X(-\log D) \cong \omega_X^{-1}(-D)\,,$$

as follows from (24) and (1). □

PROPOSITION 3.4. *Denote by*

$$R_X \xrightarrow{g} I_X \xrightarrow{h} \mathfrak{g}$$

the Stein factorisation of the proper morphism f. Then I_X is an affine variety with rational singularities, and its canonical sheaf satisfies

$$\omega_{I_X} \cong R^{r(X)} g_*(p^*\mathcal{O}_X(-D))\,.$$

PROOF. Since the morphism h is finite, I_X is affine; it is also normal, since R_X is nonsingular and the natural map $\mathcal{O}_{I_X} \to g_*\mathcal{O}_{R_X}$ is an isomorphism.

We claim that $R^i g_*\mathcal{O}_{R_X} = 0$ for each $i \geq 1$. For this, it suffices to show the vanishing of $h_*(R^i g_*\mathcal{O}_{R_X}) = R^i f_*\mathcal{O}_{R_X}$. Since the image of f is affine, it suffices in turn to show that $H^i(R_X, \mathcal{O}_{R_X}) = 0$. But this follows from (31), since

$$H^i(R_X, \mathcal{O}_{R_X}) = H^i(X, p_*\mathcal{O}_{R_X}) = \bigoplus_{j \geq 0} H^i(X, S^j\mathcal{R}_X^\vee)\,.$$

We now deduce the rationality of singularities of I_X from a result of Kollár (see [24, Thm. 7.1]): let $\pi : Y \to Z$ be a morphism of projective varieties, where Y is nonsingular. If $\pi_*\mathcal{O}_Y = \mathcal{O}_Z$ and $R^i\pi_*\mathcal{O}_Y = 0$ for all $i \geq 1$, then Z has rational singularities.

To reduce to that setting, we compactify the morphism g as follows. Consider the vector bundle $R_X \oplus O_X$ over X, where O_X denotes the trivial line bundle. This yields a subvariety of $X \times (\mathfrak{g} \oplus \boldsymbol{C})$, and hence a proper morphism

$$\phi : R_X \oplus O_X \to \mathfrak{g} \oplus \boldsymbol{C} .$$

By the preceding argument,

$$R^i \phi_* \mathcal{O}_{R_X \oplus O_X} = 0 \quad (i \geq 1) . \tag{32}$$

Moreover, the set-theoretic fibre of ϕ at the origin of $\mathfrak{g} \oplus \boldsymbol{C}$ is the zero section. Thus, ϕ yields a morphism between projectivisations

$$\bar{f} : \boldsymbol{P}(R_X \oplus O_X) \to \boldsymbol{P}(\mathfrak{g} \oplus \boldsymbol{C})$$

which extends $f : R_X \to \mathfrak{g}$. Furthermore, (32) easily implies that

$$R^i \bar{f}_* \mathcal{O}_{\boldsymbol{P}(R_X \oplus O_X)} = 0 \quad (i \geq 1) .$$

It follows that the Stein factorisation of $\bar{f}$,

$$\bar{g} : \boldsymbol{P}(R_X \oplus O_X) \to \bar{I}_X$$

extends g and satisfies the same vanishing properties.

If the variety X is projective, then so is $\boldsymbol{P}(R_X \oplus O_X)$; thus, Kollár's result may be applied to $\bar{g}$ and hence to g. For an arbitrary (complete nonsingular) variety X, there exists a nonsingular projective variety X' together with a birational morphism

$$u : X' \to X .$$

Then the pull-back to X' of the projective bundle $\boldsymbol{P}(R_X \oplus O_X)$ is a projective variety Y equipped with a birational morphism

$$v : Y \to \boldsymbol{P}(R_X \oplus O_X) .$$

Since $\boldsymbol{P}(R_X \oplus O_X)$ is nonsingular, we have $v_* \mathcal{O}_Y = \mathcal{O}_{\boldsymbol{P}(R_X \oplus O_X)}$ and $R^i v_* \mathcal{O}_Y = 0$ for all $i \geq 1$. Thus, Kollár's result applies to the composite morphism $\bar{g} \circ v$.

This completes the proof of the rationality of singularities of I_X. The formula for its canonical sheaf follows from [21, Theorem 5] in view of Lemma 3.3. □

Next, we determine the algebra

$$H^\bullet(X, \Omega_X^\bullet(\log D)) := \bigoplus_{i,j} H^i(X, \Omega_X^j(\log D))$$

in terms of the coordinate ring of the affine variety I_X,

$$\boldsymbol{C}[I_X] = H^0(R_X, \mathcal{O}_{R_X}) = \bigoplus_j H^0(X, S^j \mathcal{R}_X^\vee) , \tag{33}$$

which is viewed as a graded module over the algebra $\boldsymbol{C}[\mathfrak{g}]$ (the symmetric algebra of $\mathfrak{g}^*$) via the natural map $\mathfrak{g}^* \to H^0(X, \mathcal{R}_X^\vee)$. For this, consider the Koszul complex associated with the exact sequence (24):

$$\cdots \to S^\bullet \mathcal{R}_X^\vee \otimes_{\boldsymbol{C}} \wedge^2 \mathfrak{g}^* \to S^\bullet \mathcal{R}_X^\vee \otimes_{\boldsymbol{C}} \mathfrak{g}^* \to S^\bullet \mathcal{R}_X^\vee .$$

This complex of graded sheaves decomposes as a direct sum of complexes

$$\mathcal{O}_X \otimes_{\boldsymbol{C}} \wedge^j \mathfrak{g}^* \to \mathcal{R}_X^\vee \otimes_{\boldsymbol{C}} \wedge^{j-1} \mathfrak{g}^* \to \cdots \to S^{j-1}\mathcal{R}_X^\vee \otimes_{\boldsymbol{C}} \mathfrak{g}^* \to S^j \mathcal{R}_X^\vee\,,$$

with homology sheaves $\Omega_X^j(\log D)$ in degree $-j$, and 0 in all other degrees (where $S^j\mathcal{R}_X^\vee$ is of degree 0). Moreover, each sheaf $S^{j-k}\mathcal{R}_X^\vee \otimes_{\boldsymbol{C}} \wedge^k \mathfrak{g}^*$ is acyclic by Theorem 3.2. This yields:

PROPOSITION 3.5. *With the assumptions of this subsection, each group $H^i(\Omega_X^j(\log D))$ is the i-th homology group of the complex*

$$\wedge^j \mathfrak{g}^* \to H^0(X, \mathcal{R}_X^\vee) \otimes \wedge^{j-1}\mathfrak{g}^* \to \cdots \to H^0(X, S^{j-1}\mathcal{R}_X^\vee) \otimes \mathfrak{g}^* \to H^0(X, S^j\mathcal{R}_X^\vee)\,.$$

Moreover, we have isomorphisms

$$H^i(X, \Omega_X^j(\log D)) \cong \operatorname{Tor}_{j-i}^{j,\boldsymbol{C}[\mathfrak{g}]}(\boldsymbol{C}, \boldsymbol{C}[I_X])\,, \tag{34}$$

where $\boldsymbol{C}$ is the quotient of $\boldsymbol{C}[\mathfrak{g}]$ by its maximal graded ideal, and the exponent j denotes the subspace of degree j. These isomorphisms are compatible with the multiplicative structures of $H^\bullet(X, \Omega_X^\bullet(\log D))$ and of $\operatorname{Tor}_\bullet^{\bullet,\boldsymbol{C}[\mathfrak{g}]}(\boldsymbol{C}, \boldsymbol{C}[I_X])$.

In turn, this will imply a description of the graded subalgebra $H^0(X, \Omega_X^\bullet(\log D))$. To state it, consider the group $\mathcal{X}(G)$ of multiplicative characters of G, and its subgroup $\mathcal{X}(G)^H$ of characters which restrict trivially to H. Then $\mathcal{X}(G)^H$ is a free abelian group of finite rank, and every $f \in \mathcal{X}(G)^H$ may be regarded as an invertible regular function on $X_0 = G/H$; this yields an isomorphism

$$\mathcal{X}(G)^H \cong \mathcal{O}(X_0)^\times/\boldsymbol{C}^\times\,.$$

Also, $\mathcal{O}(X_0)^\times/\boldsymbol{C}^\times$ may be identified to a subgroup of $H^0(X, \Omega_X^1(\log D))$ via the map $f \mapsto d\log(f) := df/f$.

COROLLARY 3.6. *With the preceding notation, $H^0(X, \Omega_X^\bullet(\log D))$ is a free exterior algebra on $df_1/f_1, \dots, df_r/f_r$, where $f_1, \dots, f_r$ is any basis of the abelian group $\mathcal{X}(G)^H$.*

PROOF. Denote by K the kernel of the map $\mathfrak{g}^* \to H^0(X, \mathcal{R}_X^\vee)$. Then $\wedge^j K$ is the kernel of the induced map $\wedge^j \mathfrak{g}^* \to H^0(X, \mathcal{R}_X^\vee) \otimes \wedge^{j-1}\mathfrak{g}^*$. By Proposition 3.5, it follows that

$$H^0(X, \Omega_X^\bullet(\log D)) \cong \wedge^\bullet K$$

as graded algebras. In particular, $H^0(X, \Omega_X^1(\log D)) \cong K$.

On the other hand, the exact sequence (5) yields an exact sequence

$$0 \to H^0(X, \Omega_X^1(\log D)) \to \boldsymbol{C}^l \to H^1(X, \Omega_X^1) \to H^1(\Omega_X^1(\log D)) \to 0$$

in view of Theorem 1.2. Moreover, the map $\boldsymbol{C}^l \to H^1(\Omega_X^1)$ may be identified with the natural map $\bigoplus_{k=1}^l A^0(D_k) \to A^1(X)$ as in the proof of that theorem. The kernel of the latter map is the complexification of the abelian group $\{\operatorname{div}(f),\ f \in \mathcal{O}(X_0)^\times\}$, and each $\operatorname{div}(f)$ is mapped to df/f under the preceding identification. □

EXAMPLE 3.7. Consider a $G \times G$-equivariant compactification X of a connected reductive group G, as in Example 2.5. Then the image of $f : R_X \to \mathfrak{g} \times \mathfrak{g}$ is the closure of the set $\{(\xi, g \cdot \xi)\,;\ \xi \in \mathfrak{g},\ g \in G\}$, the graph of the adjoint action of G on $\mathfrak{g}$.

We claim that

$$I_X = f(R_X) = \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}\,. \tag{35}$$

Indeed, $f(R_X)$ is a variety of dimension $2\dim(G) - \mathrm{rk}(G)$, contained in $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$. Moreover, since the quotient morphism $\pi : \mathfrak{g} \to \mathfrak{g}//G$ is flat and its scheme-theoretic fibres are varieties of dimension $\dim(G) - \mathrm{rk}(G)$, the same holds for the first projection $p_1 : \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g} \to \mathfrak{g}$. It follows that $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ is a variety of dimension $2\dim(G) - \mathrm{rk}(G)$, which implies the second equality in (35). To prove the first equality, it suffices to show that $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ is normal, since R_X is smooth and the general fibres of f are connected. But $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ is a complete intersection in the affine space $\mathfrak{g} \times \mathfrak{g}$, defined by the equations

$$P_1(x) = P_1(y), \ldots, P_r(x) = P_r(y)$$

where $P_1, \ldots, P_r$ are homogeneous generators of the graded polynomial ring $\boldsymbol{C}[\mathfrak{g}]^G$, and $r = \mathrm{rk}(G)$. In particular, $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ is Cohen-Macaulay. Moreover, the differentials of $P_1, \ldots, P_r$ are linearly independent at any regular element of $\mathfrak{g}$, and these form an open subset with complement of codimension 3. It follows that $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$ is regular in codimension 1, and hence normal by Serre's criterion.

Together with (28) and (33), the equalities (35) imply that

$$H^0(X, S^\bullet \mathcal{T}_X(-\log D)) \cong S^\bullet(\mathfrak{g}) \otimes_{S^\bullet(\mathfrak{g})^G} S^\bullet(\mathfrak{g})\,.$$

Moreover, Theorem 3.2 yields the vanishing of $H^i(X, S^\bullet \mathcal{T}_X(-\log D))$ for all $i \geq 1$, which is also a special case of [23, Thm. 4.1]. Combining both results and arguing as in [loc. cit.], we obtain an isomorphism

$$H^0(X, \mathfrak{U}_X) \cong U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g})\,,$$

where $\mathfrak{U}_X$ denotes the sheaf of differential operators on X generated by $\mathcal{O}_X$ and $\mathfrak{g} \times \mathfrak{g}$ (the "completely regular" differential operators of [23]), and $U(\mathfrak{g})$ stands for the enveloping algebra of $\mathfrak{g}$, with centre $Z(\mathfrak{g})$.

On the other hand, (34) yields isomorphisms

$$H^i(X, \Omega^j_X(\log D)) \cong \mathrm{Tor}^{j, \boldsymbol{C}[\mathfrak{g}] \otimes \boldsymbol{C}[\mathfrak{g}]}_{j-i}(\boldsymbol{C}, \boldsymbol{C}[\mathfrak{g}] \otimes_{\boldsymbol{C}[\mathfrak{g}]^G} \boldsymbol{C}[\mathfrak{g}])\,.$$

Since the algebra $\boldsymbol{C}[\mathfrak{g}] \otimes_{\boldsymbol{C}[\mathfrak{g}]^G} \boldsymbol{C}[\mathfrak{g}]$ is the quotient of the polynomial algebra $\boldsymbol{C}[\mathfrak{g} \times \mathfrak{g}]$ by the ideal generated by the regular sequence $(P_1(x) - P_1(y), \ldots, P_r(x) - P_r(y))$, this yields in turn

$$H^i(X, \Omega^j_X(\log D)) \cong \mathrm{Tor}^{j, \boldsymbol{C}[\mathfrak{g}]^G}_{j-i}(\boldsymbol{C}, \boldsymbol{C})\,.$$

As a consequence, the bi-graded algebra $H^\bullet(X, \Omega^\bullet_X(\log D))$ is a free exterior algebra on generators of bi-degrees $(d_1 - 1, d_1), \ldots, (d_r - 1, d_r)$, where $d_1, \ldots, d_r$ denote the degrees of $P_1, \ldots, P_r$.

REMARK 3.8. More generally, consider a log homogeneous variety X under a connected reductive group G, and assume that H is connected and reductive as well. Then the invariant rings $\boldsymbol{C}[\mathfrak{g}]^G$ and $\boldsymbol{C}[\mathfrak{h}]^H$ are graded polynomial rings, and $\boldsymbol{C}[\mathfrak{h}]^H$ is a finite module over $\boldsymbol{C}[\mathfrak{g}]^G$ via the restriction map. Moreover, the quotient morphism $\mathfrak{h} \to \mathfrak{h}//H := \mathrm{Spec}\, \boldsymbol{C}[\mathfrak{h}]^H$

yields a morphism $G \times^H \mathfrak{h} \to \mathfrak{h}//H$, the quotient of the affine variety $G \times^H \mathfrak{h}$ by the action of G. One may check that the latter morphism extends to R_X, and that the product map $R_X \to \mathfrak{g} \times \mathfrak{h}//H$ factors through an isomorphism

$$I_X \cong \mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{h}//H\,.$$

In other words,

$$\boldsymbol{C}[I_X] \cong \boldsymbol{C}[\mathfrak{g}] \otimes_{\boldsymbol{C}[\mathfrak{g}]^G} \boldsymbol{C}[\mathfrak{h}]^H\,.$$

Together with the isomorphism (34) and the freeness of the $\boldsymbol{C}[\mathfrak{g}]^G$-module $\boldsymbol{C}[\mathfrak{g}]$, it follows that

$$H^i(X, \Omega^j_X(\log D)) \cong \mathrm{Tor}^{j,\boldsymbol{C}[\mathfrak{g}]^G}_{j-i}(\boldsymbol{C}, \boldsymbol{C}[\mathfrak{h}]^H)\,.$$

Also, note that $\boldsymbol{C}[\mathfrak{g}]^G$ (resp. $\boldsymbol{C}[\mathfrak{h}]^H$) is the cohomology ring of the classifying space BG (resp. BH) with complex coefficients. This yields a description of the algebra $H^\bullet(X, \Omega^\bullet_X(\log D))$ in topological terms, which can be extended to any homogeneous space—not necessarily having a log homogeneous compactification—in view of a result of Franz and Weber (see [17, Thm. 1.6]).

3.2. The linear case (continued). We still consider a homogeneous G-pair (X, D), where G is a connected linear algebraic group. Let $\mathcal{L}$ denote an invertible sheaf on X, and assume that $\mathcal{L}$ is nef; since X is a spherical variety under a Levi subgroup of G, this is equivalent to $\mathcal{L}$ being generated by its global sections (see e.g. [6, Lem. 3.1]).

Recall from Subsection 2.2 that any component F of a general fibre of $f : R_X \to \overline{G \cdot \mathfrak{h}}$ may be identified to a toric subvariety of X. We denote by $\kappa_f(\mathcal{L})$ the Kodaira-Iitaka dimension of the pull-back $\mathcal{L}|_F$. Since $\mathcal{L}$ is globally generated, $\kappa_f(\mathcal{L})$ is the dimension of the image of the natural map $\varphi : F \to \boldsymbol{P}(H^0(F, \mathcal{L})^*)$. Note that $\kappa_f(\mathcal{L}) \le \dim(F) = r(X)$, and equality holds e.g. if $\mathcal{L}$ is big.

THEOREM 3.9. *With the notation and assumptions of this subsection,*

$$H^i(X, \mathcal{L}(-D) \otimes S^j\mathcal{R}^\vee_X) = 0 \quad (i \ne r(X) - \kappa_f(\mathcal{L}),\ \ j \ge 0)\,.$$

PROOF. Since R_X is nonsingular, f is proper and the invertible sheaf $p^*\mathcal{L}$ is f-semi-ample, it follows that the sheaf $R^i f_*(p^*\mathcal{L} \otimes \omega_{R_X})$ is torsion-free on the image of f, for any $i \ge 0$ (see [16, Cor. 6.12]). But

$$H^i(F, \mathcal{L} \otimes \omega_{R_X}) = H^i(F, \mathcal{L} \otimes \omega_F)$$

vanishes for all $i \ne r(X) - \kappa_f(\mathcal{L})$, by a result of Fujino (see [18, Cor. 1.7]). Thus, the same vanishing holds for $R^i f_*(p^*\mathcal{L} \otimes \omega_{R_X})$, i.e., for $R^i f_*(p^*\mathcal{L}(-D))$ in view of Lemma 3.3. This implies our statement, by arguing as in the beginning of the proof of Proposition 3.4. □

In view of Lemma 3.1, Theorem 3.9 yields the vanishing of the groups $H^i(X, \mathcal{L}(-D) \otimes \Omega^j_X(\log D))$ for all $i > j + r(X) - \kappa_f(\mathcal{L})$. By Serre duality (6), this implies the following:

COROLLARY 3.10. *With the notation and assumptions of this subsection,*

$$H^i(X, \mathcal{L}^{-1} \otimes \Omega^j_X(\log D)) = 0 \quad (i < j - r(X) + \kappa_f(\mathcal{L}))\,.$$

In particular, this vanishing holds for all $i < j$ if $\mathcal{L}$ is big.

REMARK 3.11. (i) Assume that $H^k(F, \mathcal{L}^{-1}) \neq 0$ for some $k \geq 0$ and some nef invertible sheaf $\mathcal{L}$ on X. Then $k = \kappa_f(\mathcal{L})$, and

$$H^{r(X)-k}(X, \mathcal{L}(-D) \otimes S^j \mathcal{R}_X^\vee) \neq 0$$

for some $j \geq 0$, by the proof of Theorem 3.9. Equivalently,

$$H^i(X, \mathcal{L}^{-1} \otimes \Omega_X^j(\log D)) \neq 0$$

for some $i = j - r(X) + k$.

The preceding assumption is fulfilled if $\mathcal{L} = \mathcal{O}_X$ and $k = 0$. Thus, Theorem 3.9 and Corollary 3.10 are optimal for the trivial invertible sheaf.

The non-vanishing of $H^{r(X)}(X, (S^j \mathcal{R}_X^\vee)(-D))$ for some j also follows from Proposition 3.4 (since I_X is affine, the space of global sections of ω_{I_X} is non-zero).

Likewise, Theorem 3.9 and Corollary 3.10 are optimal for big invertible sheaves, since $H^0(F, \mathcal{L}(-D)) \neq 0$ for sufficiently big $\mathcal{L}$; equivalently, $H^{r(X)}(F, \mathcal{L}^{-1}) \neq 0$.

(ii) If $\mathcal{L}$ is big, then we also have a refinement of Norimatsu's vanishing theorem mentioned in Subsection 1.1; namely,

$$H^i(X, \mathcal{L}^{-1} \otimes \Omega_X^j(\log D)) = 0 \quad (i + j < n),$$

as follows from [16, Cor. 6.7].

3.3. The case where the open orbit is proper over an affine. We consider again a homogeneous G-pair (X, D), where G is a connected linear algebraic group, and a nef invertible sheaf $\mathcal{L}$ on X. We assume in addition that the open orbit X_0 is proper over an affine; this holds e.g. if H is reductive (and hence X_0 is affine), or if X is a flag variety. We now obtain a stronger vanishing theorem than Theorem 3.2:

THEOREM 3.12. *With the assumptions of this subsection,*

$$H^i(X, \mathcal{L} \otimes S^j \mathcal{R}_X^\vee) = 0 \quad (i \geq 1).$$

Equivalently,

$$H^i(X, \mathcal{L} \otimes \Omega_X^j(\log D)) = 0 \quad (i > j).$$

PROOF. By Lemma 2.1, we may replace X with X' and $\mathcal{L}$ with $\mathcal{L}' := u^*\mathcal{L}$, where X' is a complete nonsingular G-variety and $u : X' \to X$ is a G-equivariant birational morphism. We claim that we may choose X' so that D' (the reduced inverse image of D) is the support of an effective base-point-free divisor.

By assumption, we have a proper morphism $\varphi : X_0 \to Y_0$, where Y_0 is an affine variety. We may assume in addition that $\varphi_* \mathcal{O}_{X_0} = \mathcal{O}_{Y_0}$. Then φ is a surjective G-equivariant morphism with connected fibres, and hence a fibration in flag varieties. The affine G-variety Y_0 admits a closed G-equivariant immersion into some G-module V. Consider the associated rational map $X - \to \boldsymbol{P}(V \oplus \boldsymbol{C})$, and its graph X'. Then X' is a complete G-variety, and the projection $u : X' \to X$ is an isomorphism above X_0. Moreover, the complement $X' \setminus u^{-1}(X_0) = \mathrm{Supp}(u^{-1}(D))$ is the set-theoretic preimage of the hyperplane section $\boldsymbol{P}(V \oplus 0)$ under the projection $X' \to \boldsymbol{P}(V \oplus \boldsymbol{C})$. Now replace X' with an equivariant desingularisation to obtain the desired setting.

Thus, we may assume that there exists a base-point-free divisor $\sum_{k=1}^{l} a_k D_k$, where the a_k are positive integers. Choose an integer $N > a_1, \ldots, a_l$. We now apply [16, Cor. 6.12] to the morphism $f : R_X \to \overline{G \cdot \mathfrak{h}}$, the invertible sheaf $\mathcal{M} := p^*\mathcal{L}(D)$, and the divisor $E := p^*(\sum_{k=1}^{l}(N - a_k) D_k)$. Then E has normal crossings, and the invertible sheaf

$$\mathcal{M}^N(-E) = p^*\mathcal{L}^N\left(\sum_{k=1}^{l} a_k D_k\right)$$

is f-semi-ample, so that the assumptions of [loc. cit.] are satisfied. Hence each sheaf $R^i f_*(p^*\mathcal{M} \otimes \omega_{R_X})$ is torsion-free on the image of f. By Lemma 3.3, this means that $R^i f_*(p^*\mathcal{L})$ is torsion-free. But $H^i(F, \mathcal{L}) = 0$ for any $i \geq 1$ and any component F of a general fibre of f, in view of Theorem 2.2. Thus, $R^i f_*(p^*\mathcal{L}) = 0$ for all $i \geq 1$. This implies our statements by the arguments of the preceding subsection. □

REMARK 3.13. The preceding argument also yields a simpler proof of [4, Thm. 3.2], the main result of that paper. It asserts that $H^i(X, \mathcal{L} \otimes S^j \mathcal{T}_X(-\log D)) = 0$ for all $i \geq 1$ and $j \geq 0$, where (X, D) is a homogeneous pair under a connected reductive group G, the open orbit X_0 is proper over an affine, and $\mathcal{L}$ is a nef invertible sheaf on X. Here the morphism $f : R_X \to \mathfrak{g}$ is replaced with the compactified moment map of [23].

3.4. The general case. We now consider a G-pair (X, D), where G is a connected algebraic group, not necessarily linear.

We shall obtain a generalisation of Corollary 3.10 to this setting. Note that the arguments of Subsection 3.2 need to be substantially modified, since the assumption of semi-ampleness in Kollár's result is not satisfied (e.g., for algebraically trivial invertible sheaves on abelian varieties). Thus, we begin with a closer study of the Albanese fibration.

With the notation of Subsection 1.3, we identify X to $G \times^I Y$, and the Albanese morphism (10) to the natural map $G \times^I Y \to G/I$. Let $E := D|_Y$, then (Y, E) is a homogeneous pair under $G_{\mathrm{aff}} = I^0$, and I preserves each G_{aff}-orbit in Y (see [7, Thm. 3.2.1]). As a consequence, $\mathrm{rk}(X) = \mathrm{rk}(Y)$. Also, recall that Y is a spherical variety under any Levi subgroup of G_{aff}.

We denote by $C(G)$ the centre of G, so that

$$G = C(G)^0 G_{\mathrm{aff}}$$

(see e.g. [7, Lem. 1.1.1]). Thus, $C(G)^0$ acts transitively on A, and $I = (I \cap C(G)^0)G_{\mathrm{aff}}$. Also, $C(G)^0$ is a semi-abelian variety by [7, Prop. 3.4.2]. Moreover, the isotropy subgroup G_x is affine for any $x \in X$ (see e.g. [7, Lem. 1.2.1]); as a consequence, $\mathfrak{g}_{(x)} \subset \mathfrak{g}_x \subset \mathfrak{g}_{\mathrm{aff}}$. It follows that

$$R_X \cong G \times^I R_Y \quad \text{and} \quad \overline{G \cdot \mathfrak{h}} = \overline{G_{\mathrm{aff}} \cdot \mathfrak{h}} \subset \mathfrak{g}_{\mathrm{aff}}\,.$$

Moreover, denoting by $X^{(\xi)}$ (resp. $Y^{(\xi)}$) the fibre at $\xi \in \mathfrak{g}_{\mathrm{aff}}$ of the map f (resp. $f_Y : R_Y \to \mathfrak{g}_{\mathrm{aff}}$), we see that $Y^{(\xi)}$ is preserved by I, and

$$X^{(\xi)} = G \times^I Y^{(\xi)} .$$

In particular, *the connected components of the general fibres of f are semi-abelic varieties of dimension $q(X) + r(Y)$*. In view of Corollary 2.6, this yields

$$r(X) = q(X) + r(Y) \le q(X) + \mathrm{rk}(X) . \tag{36}$$

Next, we describe the invertible sheaves on X. For this, choose a Borel subgroup $B \subset G_{\mathrm{aff}}$. Then

$$G_1 := C(G)^0 B$$

is a maximal connected solvable subgroup of G, and $(G_1)_{\mathrm{aff}} = B$.

LEMMA 3.14. (i) *Any invertible sheaf $\mathcal{L}$ on X admits a decomposition*

$$\mathcal{L} = (\alpha^* \mathcal{M})(\Delta) , \tag{37}$$

where $\mathcal{M}$ is an invertible sheaf on A, and Δ is a G_1-stable divisor on X. Moreover, $\mathcal{M}$ (*resp. $\Delta|_Y$*) *is uniquely determined by $\mathcal{L}$ up to algebraic* (*resp. rational*) *equivalence.*

(ii) *$\mathcal{L}$ is nef* (*resp. ample*) *if and only if both $\mathcal{M}$ and $\Delta|_Y$ are nef* (*resp. ample*).

PROOF. (i) Note that B has an open orbit Y_1 in Y, and hence G_1 has an open orbit X_1 in X; the map

$$\alpha_1 := \alpha|_{X_1} : X_1 \to A$$

is a G_1-equivariant fibration with fiber Y_1.

We claim that the pull-back map $\alpha_1^* : \mathrm{Pic}(A) \to \mathrm{Pic}(X_1)$ is surjective. To see this, identify X_1 to the homogeneous space G_1/H_1; then $A = X/B = G_1/H_1B$ and hence α_1 is the composite morphism

$$G_1/H_1 \xrightarrow{\alpha_U} G_1/H_1U \xrightarrow{\alpha_T} G_1/H_1B ,$$

where U denotes the unipotent part of B, and T denotes the torus B/U. Note that G_1/U is a semi-abelian variety with maximal torus T. Thus, the quotient G_1/H_1U is a semi-abelian variety as well, and α_T is the quotient map by its maximal torus. It follows that

$$\alpha_T^* : \mathrm{Pic}(A) \to \mathrm{Pic}(G_1/H_1U)$$

is surjective. On the other hand, since U is unipotent, α_U may be factored into quotients by free actions of the additive group, and hence

$$\alpha_U^* : \mathrm{Pic}(G_1/H_1U) \to \mathrm{Pic}(G_1/H_1)$$

is an isomorphism.

By the claim, there exists an invertible sheaf $\mathcal{M}$ on A such that $\mathcal{L}|_{X_1} \cong \alpha_1^*\mathcal{M}$. Then $\mathcal{L} \cong (\alpha^*\mathcal{M})(\Delta)$ for some divisor Δ supported in $X \setminus X_1$. In particular, Δ is preserved by G_1. This proves the existence of the decomposition (37).

For the uniqueness properties, we may assume that $\mathcal{L}$ is trivial. Then $\alpha^*(\mathcal{M}) = \mathcal{O}_X(-\Delta)$, and hence $g^*\alpha^*(\mathcal{M}) \cong \alpha^*(\mathcal{M})$ for any $g \in G_1$. It follows that $a^*(\mathcal{M}) \cong \mathcal{M}$

for any $a \in A$; thus, $\mathcal{M}$ is algebraically trivial. Moreover, $\mathcal{O}_Y(\Delta|_Y) = \mathcal{L} \otimes \mathcal{O}_Y$ is trivial as well.

(ii) Recall that any effective 1-cycle on X is rationally equivalent to an effective 1-cycle preserved by B (see e.g. [6, Sec. 1.3]). Thus, $\mathcal{L}$ is nef if and only if $\mathcal{L} \cdot C \geq 0$ for any irreducible curve $C \subset X$, preserved by B.

If B acts non-trivially on C, then C is rational and hence contained in a fibre of α. Thus, $\mathcal{L} \cdot C = \Delta \cdot C$. Since Δ is preserved by the group B_1 which permutes transitively the fibres of α, we may assume that $C \subset Y$; then $\mathcal{L} \cdot C = \Delta|_Y \cdot C$.

On the other hand, if B acts trivially on C, then the orbit $G \cdot x$ is closed in X for any $x \in C$. Since X contains only finitely many G-orbits, it follows that $C \subset G \cdot x$ for any such x. By [7, Thm. 3.3.3], there is a G-equivariant isomorphism $G \cdot x \cong A \times (G_{\mathrm{aff}} \cdot x)$. This identifies the fixed point subscheme $(G \cdot x)^B$ to $A \times \{x\}$, and hence C to a curve in $A \times \{x\}$; in particular, α restricts to an isomorphism $C \cong \alpha(C)$. Since $A \times \{x\}$ is an orbit of G_1, the restriction $\Delta|_{A \times \{x\}}$ is algebraically trivial by the argument of (ii). It follows that $\mathcal{L} \cdot C = (\alpha^* \mathcal{M}) \cdot C = \mathcal{M} \cdot \alpha(C)$.

Thus, $\mathcal{L}$ is nef iff so are $\mathcal{M}$ and $\Delta|_Y$. This implies the corresponding statement for ampleness, in view of Kleiman's criterion: $\mathcal{L}$ is ample iff for any invertible sheaf $\mathcal{L}'$ on X, there exists a positive integer $n = n(\mathcal{L}')$ such that $\mathcal{L}^n \otimes \mathcal{L}'$ is nef. □

REMARK 3.15. (i) Lemma 3.14 implies readily a decomposition of the Néron-Severi group:

$$\mathrm{NS}(X) \cong \mathrm{NS}(A) \times \mathrm{NS}(Y) .$$

Also, $\mathrm{NS}(A)$ is a free abelian group; moreover, $\mathrm{NS}(Y)$ is also free and isomorphic to $\mathrm{Pic}(Y)$, since Y admits a cellular decomposition (Lemma 1.1). It follows that $\mathrm{NS}(X)$ is a free abelian group as well; in other words, algebraic and numerical equivalence coincide for invertible sheaves on X. Moreover, the cone of numerical equivalence classes of nef invertible sheaves decomposes accordingly:

$$\mathrm{Nef}(X) \cong \mathrm{Nef}(A) \times \mathrm{Nef}(Y) .$$

The nef cones of abelian varieties are well understood (see e.g. [5]). Those of spherical varieties (like Y) are studied in [6]; in particular, these cones are polyhedral.

(ii) If X is a semi-abelic variety, then G_{aff} is a torus T, and Y is a toric variety under that torus. Thus, $G_1 = G$, and G_1-stable divisors on X correspond bijectively to T-stable divisors on Y. In that case, Lemma 3.14 gives back a description of the ample invertible sheaves on semi-abelic varieties, due to Alexeev (see [1, Sec. 5.2]).

Consider a nef invertible sheaf $\mathcal{L}$ on X, and its decomposition (37); then $\mathcal{L}|_Y \cong \mathcal{O}_Y(\Delta|_Y)$. Let

$$K(\mathcal{M}) := \{a \in A \, ; \, a^* \mathcal{M} \cong \mathcal{M}\} ;$$

this is a closed subgroup of A. Since $\mathcal{L}$ determines $\mathcal{M}$ up to multiplication by an algebraically trivial invertible sheaf, we see that $K(\mathcal{M})$ depends only on $\mathcal{L}$; we shall denote that group by $K(\mathcal{L})$. We now are in a position to state:

THEOREM 3.16. *With the preceding notation and assumptions*,

$$H^i(X, \mathcal{L}^{-1} \otimes \Omega_X^j(\log D)) = 0 \quad (i < j - r(Y) + \kappa_f(\mathcal{L}|_Y) - \dim K(\mathcal{L}))\,.$$

In particular, this vanishing holds for all $i < j$ if $\mathcal{L}$ is ample.

PROOF. By (6), it suffices to show that $H^i(X, \mathcal{L}(-D) \otimes \Omega_X^j(\log D)) = 0$ for all $i > j + r(Y) - \kappa_f(\mathcal{L}|_Y) + \dim K(\mathcal{L})$. In view of the decomposition (15), this reduces to showing that

$$H^i(X, \mathcal{L}(-D) \otimes \Omega_{X/A}^j(\log D)) = 0 \tag{38}$$

for all such i and j. Consider the Leray spectral sequence

$$E_2^{p,q} := H^p(A, R^q\alpha_*(\mathcal{L}(-D) \otimes \Omega_{X/A}^j(\log D))) \Rightarrow H^{p+q}(X, \mathcal{L}(-D) \otimes \Omega_{X/A}^j(\log D))\,.$$

By the projection formula,

$$R^q\alpha_*(\mathcal{L}(-D) \otimes \Omega_{X/A}^j(\log D)) \cong \mathcal{M} \otimes R^q\alpha_*(\Omega_{X/A}^j(\log D)(-\Delta - D))\,.$$

Note that the sheaf $R^q\alpha_*(\Omega_{X/A}^j(\log D)(-\Delta - D))$ is locally free and G_1-linearised. Hence this sheaf has a filtration with subquotients being algebraically trivial invertible sheaves. Since $\mathcal{M}$ is nef, it follows that $E_2^{p,q} = 0$ for any $p > \dim K(\mathcal{L}) = \dim K(\mathcal{M})$, by a classical vanishing theorem for abelian varieties (see [5, Lem. 3.3.1, Thm. 3.4.5]). On the other hand, since $\Delta|_Y$ is nef, Corollary 3.10 yields that $H^q(Y, \Omega_Y^j(\log E)(-\Delta|_Y - E)) = 0$ for any $q > i + r(Y) - \kappa_f(\mathcal{L}|_Y)$, and hence $E_2^{p,q} = 0$ for all such q. This implies (38). □

Taking for $\mathcal{L}$ the trivial invertible sheaf and combining Theorems 1.6 and 3.16 with the equality (36), we obtain:

COROLLARY 3.17. *With the notation and assumptions of this subsection*,

$$H^i(X, \Omega_X^j(\log D)) = 0 \quad \text{unless} \quad -q(X) \le j - i \le r(X)\,.$$

Another consequence of Theorem 3.16 is a vanishing result for ordinary Dolbeault cohomology:

THEOREM 3.18. *Let $\mathcal{L}$ be an invertible sheaf on a log homogeneous variety X of irregularity q and rank r.*

If $\mathcal{L}$ is nef (resp. ample), then $H^i(X, \mathcal{L}^{-1} \otimes \Omega_X^j(\log D')) = 0$ for any effective subdivisor D' of D, and for all $i < j - q - r$ (resp. $i < j$).

In particular, $H^i(X, \mathcal{L} \otimes \Omega_X^j) = 0$ for all $i > j + q + r$ if $\mathcal{L}$ is nef, and for all $i > j$ if $\mathcal{L}$ is ample.

PROOF. If $D' = D$, then the first assertion is a consequence of Theorem 3.16 and Lemma 3.14. Indeed, if $\mathcal{L}$ is nef, then $r(Y) - \kappa_f(\mathcal{L}|_Y) \le \mathrm{rk}(Y) = r$ by Corollary 2.6, and $\dim K(\mathcal{L}) \le q$. If $\mathcal{L}$ is ample, then we have the equalities $r(Y) - \kappa_f(\mathcal{L}|_Y) = 0 = \dim K(\mathcal{L})$.

The case of an arbitrary divisor D' follows by decreasing induction on the number of irreducible components of D' and the dimension of X, using (4).

Taking $D' = 0$ yields the second assertion, by Serre duality. □

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