# 44. Vanishing Theorems in Asymptotic Analysis

By Hideyuki MAJIMA

Department of Mathematics, Faculty of Science,
University of Tokyo

Let $M$ be a complex manifold and let $H$ be a divisor on $M$. Denote by $\mathcal{O}$ the sheaf over $M$ of germs of holomorphic functions and by $\mathcal{O}(*H)$ the sheaf over $M$ of germs of meromorphic functions which are holomorphic in $M-H$ and have poles on $H$. We suppose that the divisor $H$ has normal crossings. At a point $p$ on $H$, we may choose local holomorphic coordinates $(x_1, \cdots, x_n)$ in a neighborhood $U = \prod_{i=1}^{n} \{|x_i| < r_i\}$ of $p = (0, \cdots, 0)$ such that the set $H \cap U = \bigcup_{i=1}^{n''} \{x : x_i = 0\}$ is the union of coordinate hyperplanes. The complement $U - (U \cap H)$ is a punctured polycylinder $P^*(n'', n)$ given by

$$\{x : |x_i| < r_i \ (i=1, \cdots, n), \ x_1, \cdots, x_{n''} \neq 0\}$$
$$= \prod_{i=1}^{n''} (D(r_i) - \{0\}) \times \prod_{i=n''+1}^{n} D(r_i),$$

where $D(r_i) = \{|x_i| < r_i\}$. Topologically, $P^*(n'', n)$ is a product $\times^{n''} S^1$ of $n''$ circles, where $S^1 = \{z \in C : |z| = 1\}$.

Let $S$ be an open polysector at $p$ of the form

$$S = \prod_{i=1}^{n''} S(c_i, r_i) \times \prod_{i=n''+1}^{n} D(r_i),$$

with the coordinates system $x$ chosen as above, where we set

$$S(c_i, r_i) = \{x_i : 0 < |x_i| < r_i, \ \arg x_i \in c_i\}$$

for a positive number $r_i$ and an open interval $c_i$ in $R$, $i=1, \cdots, n''$, and let $f$ be a holomorphic function in $S$. We denote by $[1, n]$ the set $\{1, 2, 3, \cdots, n\}$. We say that $f$ is *strongly asymptotically developable as $x$ tends to $H$ in $S$ at $p$*, if there exists a family of functions

$$F = \{f(x_I ; q_J) : \phi \neq J \subset [1, n''], \ I = [1, n] - J, \ q_J \in N^J\}$$

satisfying the following properties:

(1.1)   $f(x_I ; q_J)$ is a holomorphic function of $x_I = (x_i)_{i \in I}$ in the open polysector

$$S_I = \prod_{i \in I \cap [1, n'']} S(r_i) \times \prod_{j=n''+1}^{n} D(r_j)$$

for any non-empty subset $J$ of $[1, n'']$ and for any $q_J \in N^J$,

(1.2)   for any $N \in N^{[1, n'']}$ and for any closed subpolysector

$$S' = \prod_{i=1}^{n''} S[c_i', r_i'] \times \prod_{j=n''+1}^{n} D[r_j']$$

of $S$, there exists a constant $K_{S', N}$ such that

$$|f(x) - \mathrm{App}_N (x ; F)| \leq K_{S', N} |x_{[1, n'']}|^N,$$

or any $x \in S'$, where we set

$$S[c_i', r_i'] = \{x_i : 0 < |x_i| \leq r_i', \ \arg x_i \in c_i'\}, \ D[r_j] = \{x_j : |x_j| \leq r_j\}$$

for a positive number $r_i' < r_i$ and a closed sub-interval $c_i'$ of $c_i$,

$i=1, \cdots, n''$, for a positive real numbers $r'_j < r_j$, $j = n''+1, \cdots, n$ and $\mathrm{App}_N(x; F)$ is defined by

(1.3)   $\mathrm{App}_N(x; F) = \sum_{\phi \neq J \subset [1, n'']} (-1)^{\#J+1} \sum_{j \in J} \sum_{p_j=0}^{N_j-1} f(x_I; p_J) x_J^{p_J}$.

If $f$ is strongly asymptotically developable, the family of functions satisfying (1.1) and (1.2) is *uniquely* determined, which will be called the *total family of coefficients of strongly asymptotic expansion* of $f$ and denoted by $TA(f)$. For a non-empty subset $J$ of $[1, n'']$, we denote $f(x_I; q_J)$ by $TA_J(f)_{q_J}$ and define the formal series

(1.4)   $FA_J(f) = \sum_{q_J \in N^J} TA_J(f)_{q_J} x_J^{q_J}$

which will be called the *formal series of strongly asymptotic expansion* of $f(x)$ for $J \subset [1, n'']$. In the case in which $f(x_I; q_J)$ can be extended to a holomorphic function in $\prod_{i \in I} D(r_i)$, we say that $f$ is *strongly asymptotically developable to the formal series*

$$f'(x_{[1,n'']}, x_{[n''+1,n]}) = \sum_{q \in N^{n''}} f(x_{[n''+1,n]}, q)(x_{[1,n'']})^q$$
$$= \sum_{q_j \in N} f(x_{[1,n]-\{j\}}; q_j)(x_j)^{q_j} (j \in [1, n'']),$$

in $\mathcal{O}_{H \cap D(r)'} = \bigcap_{n=n''+1}^{j} \mathcal{O}(\prod_{i \neq j} D(r_i))[[x_j]]$ as $x$ tends to $H$ in $S$ at $p$, where $\mathcal{O}(\prod_{i \neq j} D(r_i))[[x_j]]$ is the *C*-algebra of formal series of $x_j$ with coefficients in the *C*-algebra $\mathcal{O}(\prod_{i \neq j} D(r_i))$ of holomorphic functions in $\prod_{i \neq j} D(r_i)$. Let $\mathcal{J}_H$ be the nullstellen ideal of $H$ and put $\mathcal{O}_{M \hat{\frown} H} = \mathrm{proj.\,lim}_{k \to \infty} \mathcal{O}/(\mathcal{J}_H)^k$. Note that $\mathcal{O}_{H \cap D(r)'}$ coincides with the *C*-algebra $\mathcal{O}_{M \hat{\frown} H}(D(r))$ of all sections of $\mathcal{O}_{M \hat{\frown} H}$ over $D(r)$.

Let $U$ be an open set included in $M - H$ and including polysectors at any point in $\mathrm{cl}\,(U) \cap H$, where $\mathrm{cl}\,(U)$ denotes the closure of $U$ in $M$. We say that a holomorphic function $f$ in $U$ is *strongly asymptotically developable as variables tend to $H$ in $U$*, if, for any point $p \in \mathrm{cl}\,(U) \cap H$ with holomorphic local coordinates system $x$ and for any open polysector $S = \prod_{i=1}^{n''} S(c_i, r_i) \times \prod_{i=n''+1}^{n} D(r_i)$ included in $U$ at $p$, $f$ is strongly asymptotically developable as $x$ tends to $H$ in $S$ at $p$. We denote by $\mathcal{A}(U)$ the set of all functions holomorphic in $U$ and strongly asymptotically developable as the variables tend to $H$ in $U$, and denote by $\mathcal{A}'(U)$ (or $\mathcal{A}_0(U)$) the subset of $\mathcal{A}(U)$ of functions strongly asymptotically developable to some formal series (or the identically zero series 0) as the variables tend to $H$ in any open polysector at any point in $\mathrm{cl}\,(U) \cap H$. The set $\mathcal{A}(U)$ has naturally a *C*-algebraic structure, and $\mathcal{A}'(U)$ and $\mathcal{A}_0(U)$ are sub-*C*-algebras of $\mathcal{A}(U)$: $\mathcal{A}_0(U)$ is also a sub *C*-algebra of $\mathcal{A}'(U)$. With these materials, we can define sheaves over the *real blow-up along $H$*, which is constructed in a similar manner as the construction of blow-up of a complex manifold along a submanifold as follows. We start by constructing the real blow-up of a polydisc along coordinate planes. Let $D(r)$ be an $n$-dimensional disc at the origin in $C^n$ with holomorphic coordinates $x_1, \cdots, x_n$, and let $V \subset D(r)$ be the locus $\bigcup_{i=1}^{n''} \{x : x_i = 0\}$. Let $D(r)^-$ be the real analytic subvariety

of $D(r)\times(S^1)^{n''}$ defined by the relations

$D(r)^- = \{(x_1, \cdots, x_n, z_1, \cdots, z_{n''})\,;\, \mathrm{Im}\,(x_i\bar{z}_i) = 0,\ \mathrm{Re}\,(x_i\bar{z}_i) \geqq 0,\ i \in [1, n'']\}$.

The projection $pr: D(r)^- \to D(r)$ on the first factor is clearly isomorphism away from $V$, while, for a point $x \in V$ such that $x_i = 0$, $i \in I$, and $x_j \neq 0$, $j \in [1, n''] - I$, where $I \subset [1, n'']$, the inverse image of the point $x$ is $(S^1)^{\sharp I}$. The real manifold $D(r)^-$, together with the map $pr: D(r)^- \to D(r)$, will be called the real blow-up of $D(r)$ along $V$; the inverse image $pr^{-1}(V)$ is called the set of directions of the real blow-up. Note that the real blow-up $D(r)^- \xrightarrow{pr} D(r)$ is independent of the coordinates chosen in $D(r)$; this fact allows us to globalize our construction. Let $\{U_a\}$ be an open covering of $M$ such that in each $U_a$ with holomorphic coordinates $x_{a,i}$, $i \in [1, n]$, the subset $H \cap U_a$ may be given as the locus $\bigcup_{i=1}^{n''} \{x : x_{a,i} = 0\}$, and let $U_a^- \xrightarrow{pr} U_a$ be the real blow-up of $U_a$ along $U_a \cap H$. We have then isomorphisms

$$pr_{ab}: pr_a^{-1}(U_a \cap U_b) \longrightarrow pr_b^{-1}(U_a \cap U_b)$$

and using them, we can patch together the local real blow-ups $U_a^-$ to form a real analytic manifold $M^- = \bigcup_{pr_{ab}} U_a^-$ with a map $M^- \to M$. The manifold $M^-$, together with the map $pr: M^- \to M$, is called the real blow-up of $M$ along $H$. By the construction, $pr$ is an isomorphism away from $H \subset M$.

Let $M^-$ be the real blow-up of $M$ along $H$. For an open set $U^-$ in $M^-$, we define $\mathcal{A}^-(U^-)$, $\mathcal{A}'^-(U^-)$ and $\mathcal{A}_0^-(U^-)$ as follows; if $pr(U^-) \cap H = \phi$,

$$\mathcal{A}^-(U^-) = \mathcal{A}'^-(U^-) = \mathcal{A}_0^-(U^-) = \mathcal{O}(pr(U^-)),$$

if $pr(U^-) \cap H \neq \phi$,

$$\mathcal{A}^-(U^-) = \mathcal{A}(pr(U^-) - H),\quad \mathcal{A}'^-(U^-) = \mathcal{A}'(pr(U^-) - H),$$
$$\mathcal{A}_0^-(U^-) = \mathcal{A}_0(pr(U^-) - H).$$

Then, with the natural restriction mapping

$$i_{U^- U^{-\prime}}: \mathcal{A}^-(U^{-\prime}) \longrightarrow \mathcal{A}^-(U^-)$$

for any open sets $U^-$, $U^{-\prime}$ in $M^-$, $U^- \subset U^{-\prime}$, $\{\mathcal{A}^-(U^-), i_{U^- U^{-\prime}}\}$ becomes a presheaf over $M^-$ which satisfies the sheaf conditions. We denote by $\mathcal{A}^-$ the associated sheaf on $M^-$ and call $\mathcal{A}^-$ the sheaf of germs of functions strongly asymptotically developable (as the variables tend to the normal crossing divisor $H$). In the same way we obtain the sheaf $\mathcal{A}'^-$ and $\mathcal{A}_0^-$ over $M^-$ of germs of functions strongly asymptotically developable to $\mathcal{O}_{M\uparrow H}$ and to $0$, respectively. By the definition, there exists a natural inclusion $i$ of $\mathcal{A}_0^-$ to $\mathcal{A}'^-$ and there exists a natural homomorphism $FA$ of $\mathcal{A}'^-$ to the inverse image $pr^*(\mathcal{O}_{M\uparrow H})$ of $\mathcal{O}_{M\uparrow H}$ which is obtained by taking the strongly asymptotic expansions.

**Theorem 1.** *The following short sequence of sheaves over $M^-$ is exact:*

$$0 \longrightarrow \mathcal{A}_0^- \xrightarrow{\ i\ } \mathcal{A}'^- \xrightarrow{\ FA\ } pr^*(\mathcal{O}_{M\uparrow H}) \longrightarrow 0.$$

From the exact sequence, we obtain the long exact sequences

$$0 \longrightarrow H^0(p^-, \mathcal{A}_0^-|_{p-}) \longrightarrow H^0(p^-, \mathcal{A}'^-|_{p-}) \longrightarrow H^0(p^-, pr^*(\mathcal{O}_{M\widehat{\upharpoonright}H})|_{p-})$$
$$\longrightarrow H^1(p^-, \mathcal{A}_0^-|_{p-}) \longrightarrow H^1(p^-, \mathcal{A}'^-|_{p-}) \longrightarrow H^1(p^-, pr^*(\mathcal{O}_{M\widehat{\upharpoonright}H})|_{p-}) \cdots$$

for any point $p \in H$, where $p^- = pr^{-1}(p)$, and

$$0 \longrightarrow H^0(M^-, \mathcal{A}_0^-) \longrightarrow H^0(M^-, \mathcal{A}'^-) \longrightarrow H^0(M^-, pr^*(\mathcal{O}_{M\widehat{\upharpoonright}H}))$$
$$\longrightarrow H^1(M^-, \mathcal{A}_0^-) \longrightarrow H^1(M^-, \mathcal{A}'^-) \longrightarrow H^1(M^-, pr^*(\mathcal{O}_{M\widehat{\upharpoonright}H})) \cdots.$$

**Theorem 2.** *For any $p \in H$, the image of the mapping of $H^1(p^-, \mathcal{A}_0^-|_{p-})$ to $H^1(p^-, \mathcal{A}'^-|_{p-})$ is zero and so $H^1(p^-, \mathcal{A}_0^-|_{p-})$ is isomorphic to $(\mathcal{O}_{M\widehat{\upharpoonright}H})_p/(\mathcal{O})_p$. Moreover, if $H^1(M, \mathcal{O}) = 0$ then the image of mapping of $H^1(M^-, \mathcal{A}_0^-)$ to $H^1(M^-, \mathcal{A}'^-)$ is zero and so $H^1(M^-, \mathcal{A}_0^-)$ is isomorphic to*

$$H^0(M^-, pr^*(\mathcal{O}_{M\widehat{\upharpoonright}H}))/H^0(M^-, \mathcal{A}'^-) = H^0(M, \mathcal{O}_{M\widehat{\upharpoonright}H})/H^0(M, \mathcal{O}).$$

**Remark 1.** We prove first that the mapping of $H^1(p^-, \mathcal{A}_0^-|_{p-})$ to $H^1(p^-, \mathcal{A}'^-|_{p-})$ is a zero mapping, from which we can deduce Theorem 2.

Put $\mathcal{A}'^-(*H) = \mathcal{A}'^- \otimes_{pr^*\mathcal{O}} pr^*(\mathcal{O}(*H))$, and $\mathcal{O}_{M\widehat{\upharpoonright}H}(*H) = \mathcal{O}_{M\widehat{\upharpoonright}H} \otimes_{\mathcal{O}} \mathcal{O}(*H)$. Then, the analogous results are valid for the following sequence of sheaves over $M^-$:

$$0 \longrightarrow \mathcal{A}_0^- \longrightarrow \mathcal{A}'^-(*H) \longrightarrow pr^*(\mathcal{O}_{M\widehat{\upharpoonright}H}(*H)) \longrightarrow 0.$$

For $E = \mathcal{O}_{M\widehat{\upharpoonright}H}, \mathcal{O}_{M\widehat{\upharpoonright}H}(*H), \mathcal{A}'^-$ and $\mathcal{A}'^-(*H)$, denote by $GL(m, E)$ the *sheaf of germs of invertible $m$-by-$m$ matrices of the entries in $E$.* Denote by $I_m$ the $m$-by-$m$ unit matrix and denote by $GL(m, \mathcal{A}^-)_{I_m}$ the *sheaf over $M^-$ of germs of $m$-by-$m$ invertible matricial functions strongly asymptotically developable to $I_m$.* Then, for the sequences

$$I_m \longrightarrow GL(m, \mathcal{A}^-)_{I_m} \longrightarrow GL(m, \mathcal{A}'^-) \longrightarrow pr^*(GL(m, \mathcal{O}_{M\widehat{\upharpoonright}H})) \longrightarrow I_m,$$
$$I_m \longrightarrow GL(m, \mathcal{A}^-)_{I_m} \longrightarrow GL(m, \mathcal{A}'^-(*H))$$
$$\longrightarrow pr^*(GL(m, \mathcal{O}_{M\widehat{\upharpoonright}H}(*H))) \longrightarrow I_m,$$

the similar results are valid. These are extensions of the results due to Y. Sibuya [4], [5] and B. Malgrange [3] in one variable case. The detail and the applications are published elsewhere (see Majima [1], [2]).

## References

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