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Vanishing viscosities and error estimate for a Cahn–Hilliard type phase field system related to tumor growth

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Abstract

In this paper we perform an asymptotic analysis for two different vanishing viscosity coefficients occurring in a phase field system of Cahn–Hilliard type that was recently introduced in order to approximate a tumor growth model. In particular, we extend some recent results obtained in [5], letting the two positive viscosity parameters tend to zero independently from each other and weakening the conditions on the initial data in such a way as to maintain the nonlinearities of the PDE system as general as possible. Finally, under proper growth conditions on the *interaction potential*, we prove an error estimate leading also to the uniqueness result for the limit system.

1 Introduction

In this paper we study the system of partial differential equations

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = p(\varphi)(\sigma - \gamma\mu) \tag{1.1}$$

$$\mu = \beta\partial_t\varphi - \Delta\varphi + F'(\varphi) \tag{1.2}$$

$$\partial_t\sigma - \Delta\sigma = -p(\varphi)(\sigma - \gamma\mu), \tag{1.3}$$

together with the boundary and initial conditions

$$\partial_\nu\mu = \partial_\nu\varphi = \partial_\nu\sigma = 0 \tag{1.4}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0\,. \tag{1.5}$$

Each of the partial differential equations (1.1)–(1.3) is meant to hold in a three-dimensional bounded domain Ω, endowed with a smooth boundary Γ, and for every positive time, and ∂_ν in (1.4) stands for the outward normal derivative on Γ. Moreover, α and β are nonnegative parameters, strictly positive in principle, while γ is a strictly positive constant. Furthermore, p is a nonnegative function, and F is a nonnegative potential. Finally, μ_0, φ_0 and σ_0 are given initial data defined in Ω.

The physical context of this paper is that of tumor growth dynamics. This topic has in recent years become of big interest in applied mathematics, especially after continuum models were developed (cf., e.g., [7, 18]). The fact that multiple constituents interact with each other made it necessary to consider diffuse interface models based on continuum mixture theory (cf., e.g., [4, 6, 12, 15, 20, 26]). These models consist of a Cahn–Hilliard type equation (in general with transport) containing reaction terms that depend on the nutrient concentration (e.g. oxygen) and in turn obey an advection-reaction-diffusion equation. Even though numerical simulations of these models have already been carried out in several papers (cf., e.g., [7, Chapter 8] and references therein), the rigorous mathematical analysis of the resulting PDEs systems is still very poor. To our knowledge, the first results are related to the so-called Cahn–Hilliard–Hele–Shaw system (cf., e.g., [19,25]) in which the nutrient is neglected, while two very recent contributions [5] and [13] (cf. also [16], where formal studies on the corresponding sharp interface limits are performed) deal with a model recently proposed in [14] (or approximations thereof, see also [27]), where the velocities are set to zero and the state variables are the tumor fraction φ and the nutrient-rich concentration σ. We can set $\varphi \simeq 1$ in the tumorous phase and $\varphi \simeq -1$ in the healthy cell phase, while σ typically satisfies $\sigma \simeq 1$ in a nutrient-rich extracellular water phase and $\sigma \simeq 0$ in a nutrient-poor extracellular water phase. Moreover, the third unknown μ is the related chemical potential, specified by (1.2) as in the case of the viscous Cahn–Hilliard or Cahn–Hilliard equation, depending on whether $\beta > 0$ or $\beta = 0$ (see [3, 10, 11]). In addition, in [5] the PDE system (1.1)–(1.5) was studied for the very particular case that $\alpha = \beta$, and the asymptotic analysis as the coefficient $\alpha = \beta$ tends to zero was performed, yielding the convergence of subsequences to weak solutions of the limit problem; moreover, in [13] the existence of weak solutions, as well as uniqueness and existence of attractors, was proved directly for the limit system where $\alpha = \beta = 0$ (cf. also the following comments in this Introduction).

In the case $\alpha = 0$, the sub-system (1.1)–(1.2) becomes of viscous or pure Cahn–Hilliard type, depending on whether $\beta > 0$ or $\beta = 0$. On the other hand, in the case $\alpha > 0$ the presence of the term $\alpha\partial_t\mu$ in (1.1) gives a parabolic structure to equation (1.1) with respect to μ.

We remark that the original model deals with functions F and p that are precisely related to each other. Namely, we have

$$p(u) = 2p_0\sqrt{F(u)} \quad \text{if } |u| \le 1 \quad \text{and} \quad p(u) = 0 \quad \text{otherwise,} \tag{1.6}$$

where p_0 is a positive constant and $F(u)$ is the classical Cahn–Hilliard double-well free energy density. However, this relation is useless in many aspects of the mathematical study. Moreover, one can allow F to be even a singular potential.

As mentioned above, [5] just deals with the case $\alpha = \beta$ for the mathematical study, although the constants α and β have a different meaning. In that paper, the existence of a unique solution to the system (1.1)–(1.5) was proved under very general conditions on p and F, and, in the same framework, the long-time behavior of the solution was discussed. In addition, in a more restricted setting for the double-well potential F, [5] investigated the asymptotic behavior of the problem as the coefficient $\alpha = \beta$ tends to zero, finding the convergence of subsequences to weak solutions of the limit problem. Moreover, under a smoothness condition on the initial values, uniqueness for the limit problem was proved and, consequently, also the convergence of the entire family. It must be pointed out that a uniqueness result was proved in [13] under weaker assumptions.

In the present paper, we first extend some of the results of [5]. Namely, we let the positive parameters α and β be independent from each other, and we weaken the assumptions on the initial data while keeping the potential as general as possible. At the same time, we establish a general a priori estimate that is uniform with respect to the parameters α and β. This is the starting point of possible asympotic analyses with respect to these parameters. Then, we confine ourselves to a class of regular potentials. In this framework, we state a convergence result as both α and β tend to zero independently, and we prove an error estimate in terms of α and β for the difference of the solution to (1.1)–(1.5) and the one of the limit problem. The case of just one of the parameters tending to zero is the subject of a work in progress.

Let us express our belief that the results of the present paper are general and interesting enough so that methods and estimates could be extended to other situations. In particular, in case of the trivial choice $p \equiv 0$ (admitted by our assumption (2.3)) our system (1.1)–(1.3) decouples and (1.1)–(1.2) reduces to a well-known phase field system of Caginalp tye which can be seen as a (doubly) viscous approximation of the Cahn–Hilliard system obtained at the limit as α and β go to zero. To this concern, let us quote the papers [8, 9, 22, 23], where different investigations on this kind of viscous approximations of Cahn–Hilliard system are performed, and point out that the results contained in [22] are here generalized and somehow improved.

Our paper is organized as follows. In the next section, we will state the assumptions and our results on the mathematical problem. In Section 3, we will prove the extensions mentioned above. The last section is devoted to the asymptotic analysis and the error estimate. In the remainder of the paper, we take $\gamma = 1$, without loss of generality.

2 Statement of the problem and results

In this section, we make precise assumptions and state our results. As in the Introduction, $\Omega \subset \mathbb{R}^3$ denotes the domain where the evolution takes place and Γ is its boundary. We assume Ω to be open, bounded, and connected, and Γ to be smooth. Moreover, the symbol ∂_ν denotes the outward normal derivative on Γ. Given a final time T, we set

$$Q := \Omega \times (0,T) \quad \text{and} \quad \Sigma := \Gamma \times (0,T). \tag{2.1}$$

Moreover, we set for brevity

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad \text{and} \quad W := \{v \in H^2(\Omega) :\ \partial_\nu v = 0 \text{ on } \Gamma\}, \tag{2.2}$$

and endow these spaces with their standard norms. For the norm in a generic Banach space X (or a power of it), we use the symbol $\|\cdot\|_X$ with the following exceptions: we simply write $\|\cdot\|_p$ and $\|\cdot\|_*$ if $X = L^p(\Omega)$ or $X = L^p(Q)$ for $p \in [1,+\infty]$ and $X = V^*$, the dual space of V, respectively. Finally, it is understood that H is embedded in V^* in the usual way, i.e., such that $\langle u, v\rangle = \int_\Omega u\, v$ for every $u \in H$ and $v \in V$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between V^* and V.

As far as the structure of the system is concerned, we are given two constants α and β and three functions p, $\widehat{B}$ and $\widehat{\pi}$ satisfying the conditions listed below

$$\alpha,\ \beta \in (0,1) \tag{2.3}$$

$$p : \mathbb{R} \to [0,+\infty) \text{ is bounded and Lipschitz continuous} \tag{2.4}$$

$$\widehat{B} : \mathbb{R} \to [0,+\infty] \text{ is convex, proper, lower semicontinuous} \tag{2.5}$$

$$\widehat{\pi} \in C^1(\mathbb{R}) \text{ is nonnegative, and } \pi := \widehat{\pi}\,' \text{ is Lipschitz continuous.} \tag{2.6}$$

We also define the potential $F : \mathbb{R} \to [0,+\infty]$ and the graph B in $\mathbb{R} \times \mathbb{R}$ by

$$F := \widehat{B} + \widehat{\pi} \quad \text{and} \quad B := \partial \widehat{B}. \tag{2.7}$$

We notice that if F is a C^2 function then our assumptions imply that F'' is bounded from below. We also remark that B is maximal monotone. In the following, we write $D(\widehat{B})$ and $D(B)$ for the effective domains of $\widehat{B}$ and B, respectively, and we use the same symbol B for the maximal monotone operators induced on L^2 spaces.

Remark 2.1. We notice that, among many others, the most important and typical examples of potentials fit our assumptions. Namely, we can take as F the classical double-well potential and as the logarithmic potential, which are defined by

$$F_{cl}(r) := \tfrac{1}{4}(r^2-1)^2 = \tfrac{1}{4}((r^2-1)^+)^2 + \tfrac{1}{4}((1-r^2)^+)^2 \quad \text{for } r \in \mathbb{R} \tag{2.8}$$

$$F_{log}(r) := (1-r)\ln(1-r) + (1+r)\ln(1+r) + \kappa(1-r^2)^+ \quad \text{for } |r| < 1, \tag{2.9}$$

where the decomposition $F = \widehat{B} + \widehat{\pi}$ as in (2.7) is written explicitly. In (2.9), κ is a positive constant which, depending on its value, does or does not provide a double well, and the definition of the logarithmic part of F_{log} is extended by continuity to ± 1 and by $+\infty$ outside $[-1,1]$. Moreover, another possible choice is

$$F(r) := I(r) + ((1-r^2)^+)^2 \quad \text{for } r \in \mathbb{R}, \tag{2.10}$$

where I is the indicator function of $[-1,1]$, which takes the value 0 in $[-1,1]$ and $+\infty$ elsewhere. For such an irregular potential, the associated subdifferential is multi-valued, and the precise statement of problem (1.1)–(1.5) has to introduce a selection ξ of $B(u)$.

As far as the initial data of our problem are concerned, we assume that

$$\sqrt{\alpha}\,\mu_0\,,\ \sigma_0 \in H, \quad \varphi_0 \in V, \quad \text{and} \quad F(\varphi_0) \in L^1(\Omega), \tag{2.11}$$

while the regularity properties postulated for the solution are the following:

$$\mu, \sigma \in H^1(0,T;V^*) \cap L^2(0,T;V) \tag{2.12}$$

$$\varphi \in H^1(0,T;H) \cap L^2(0,T;W) \tag{2.13}$$

$$\xi \in L^2(0,T;H), \quad \text{and} \quad \xi \in B(u) \quad \text{a.e. in } Q. \tag{2.14}$$

We notice that (2.12)–(2.13) imply that $\mu, \sigma \in C^0([0,T];H)$ and $\varphi \in C^0([0,T];V)$. At this point, we consider the problem of finding a quadruplet $(\mu, \varphi, \sigma, \xi)$ with the above regularity in order that $(\mu, \varphi, \sigma, \xi)$ and the related function

$$R = p(\varphi)(\sigma - \mu) \tag{2.15}$$

satisfy the system

$$\alpha\langle\partial_t\mu, v\rangle + \int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\mu\cdot\nabla v = \int_\Omega Rv$$
$$\text{for every } v \in V, \text{ a.e. in } (0,T) \tag{2.16}$$

$$\mu = \beta\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) \quad \text{and} \quad \xi \in B(\varphi) \quad \text{a.e. in } Q \tag{2.17}$$

$$\langle\partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma\cdot\nabla v = -\int_\Omega Rv$$
$$\text{for every } v \in V, \text{ a.e. in } (0,T) \tag{2.18}$$

$$\mu(0) = \mu_0, \quad \varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0\,. \tag{2.19}$$

This is a weak formulation of the boundary value problem (1.1)–(1.5) described in the Introduction. The homogeneous Neumann boundary condition for φ is contained in (2.13) (see (2.2) for the definition of W), while the analogous ones for μ and σ are meant in a generalized sense through the variational equations (2.16) and (2.18). We notice once and for all that the addition of (2.16) and (2.18) yields

$$\langle\partial_t\big(\alpha\mu + \varphi + \sigma\big), v\rangle + \int_\Omega \nabla(\mu+\sigma)\cdot\nabla v = 0 \tag{2.20}$$

for every $v \in V$, a.e. in $(0,T)$. We also set for convenience

$$S = \sqrt{p(\varphi)}\,(\sigma - \mu)\,. \tag{2.21}$$

Our first results deal with the well-posedness of the above problem and general a priori estimates. Namely, we have:

Theorem 2.2. *Assume* (2.3)–(2.7) *and* (2.11). *Then, for every* $\alpha, \beta \in (0,1)$*, there exists a unique quadruplet* $(\mu, \varphi, \sigma, \xi)$ *satisfying* (2.12)–(2.14) *and solving problem* (2.15)–(2.19).

Theorem 2.3. *Assume* (2.3)–(2.7) *and* (2.11). *Then, for some constant* $\widehat{C}$ *that depends only on* Ω, T *and the shapes of* π *and* p*, the following is true: for every* $\alpha, \beta \in (0,1)$*, the solution* $(\mu, \varphi, \sigma, \xi)$ *to problem* (2.15)–(2.19) *with the regularity specified by* (2.12)–(2.14) *satisfies*

$$\begin{aligned}
&\alpha^{1/2}\|\mu\|_{L^\infty(0,T;H)} + \|\nabla\mu\|_{L^2(0,T;H)} \\
&\quad + \beta^{1/2}\|\partial_t\varphi\|_{L^2(0,T;H)} + \|\varphi\|_{L^\infty(0,T;V)} + \|F(\varphi)\|^{1/2}_{L^\infty(0,T;L^1(\Omega))} \\
&\quad + \|\sigma\|_{H^1(0,T;V^*)\cap L^\infty(0,T;H)\cap L^2(0,T;V)} + \|S\|_{L^2(0,T;H)} + \|R\|_{L^2(0,T;H)} \\
&\quad + \|\partial_t(\alpha\mu+\varphi)\|_{L^2(0,T;V^*)} \\
&\leq \widehat{C}\left(\alpha^{1/2}\|\mu_0\|_H + \|\varphi_0\|_V + \|F(\varphi_0)\|^{1/2}_{L^1(\Omega)} + \|\sigma_0\|_H\right)
\end{aligned} \tag{2.22}$$

as well as

$$\begin{aligned}
&\|\mu\|_{L^2(0,T;V)} + \|\varphi\|_{L^2(0,T;W)} + \|\xi\|_{L^2(0,T;H)} \\
&\leq \widehat{C}\left(\alpha^{1/2}\|\mu_0\|_H + \|\varphi_0\|_V + \|F(\varphi_0)\|^{1/2}_{L^1(\Omega)} + \|\sigma_0\|_H + \|\mu\|_{L^2(0,T;H)} + 1\right).
\end{aligned} \tag{2.23}$$

Thus, a uniform estimate for the left-hand side of (2.22) holds in terms of the norms of the initial data related to (2.11), while an estimate for the left-hand side of (2.23) follows whenever a bound for $\|\mu\|_{L^2(0,T;H)}$ has been proved.

Remark 2.4. We note that Theorems 2.2 and 2.3 improve the results of [5], since the stronger assumption made there,

$$\mu_0, \varphi_0, \sigma_0 \in V \quad \text{and} \quad F(\varphi_0) \in L^1(\Omega), \tag{2.24}$$

is now replaced by (2.11) (and also since just the case $\alpha = \beta$ is dealt with in [5]).

Our next results regard the asymptotic analysis as the coefficients α and β tend to zero, independently. To this end, we restrict ourselves to a particular class of potentials. Namely, we also assume that

$$D(\widehat{B}) = \mathbb{R} \quad \text{and} \quad |B^\circ(r)| \leq C\left(\widehat{B}(r) + 1\right) \quad \text{for every } r \in \mathbb{R}, \tag{2.25}$$

where B° is the element of B with minimal norm and C is a given positive constant. Let us note that, for example, all polynomially growing potentials, as well as exponential functions, comply with our assumption (2.25). Let us point out that (2.25) implies (actually, it is equivalent to) the condition

$$D(\widehat{B}) = \mathbb{R}, \quad |s| \leq C\left(\widehat{B}(r) + 1\right) \quad \text{for all } r \in \mathbb{R},\ s \in B(r) \tag{2.26}$$

for the same constant C, as checked precisely in the next remark.

Remark 2.5. In fact, a similar equivalence holds for a more general growth condition and in the general setting of Hilbert spaces, as we show at once. If X is a Hilbert space, $\widehat{B} : X \to [0, +\infty)$ is convex and l.s.c. (thus continuous since it is everywhere defined), $B := \partial\widehat{B}$ and, for every $u \in X$, $B^\circ(u)$ is the element of $B(u)$ having minimal norm, the assumption

$$\|B^\circ(u)\|_X \leq \Psi(\widehat{B}(u)) \quad \text{for every } u \in X,$$

where $\Psi : [0, +\infty) \to [0, +\infty)$ is continuous, implies

$$\|\zeta\|_X \le \Psi(\widehat{B}(u)) \quad \text{for every } u \in X \text{ and every } \zeta \in B(u).$$

Indeed, for arbitrary $u \in X$, $\zeta \in B(u)$ and $\varepsilon > 0$, we have

$$\big(B^\circ(u+\varepsilon\zeta) - \zeta, (u+\varepsilon\zeta) - u\big) \ge 0, \quad \text{whence} \quad \|\zeta\|_X \le \|B^\circ(u+\varepsilon\zeta)\|_X. \tag{2.27}$$

By applying our assumption to $u + \varepsilon\zeta$, we deduce that

$$\|\zeta\|_X \le \Psi(\widehat{B}(u+\varepsilon\zeta)).$$

By taking $\varepsilon \to 0$ and owing to the continuity of $\Psi \circ \widehat{B}$, we conclude.

Now we are ready to state our result on asymptotics.

Theorem 2.6. *Assume* (2.3)–(2.7) *and* (2.25) *on the structure and* (2.11) *on the initial data. Moreover, let* $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ *be the unique solution to problem* (2.15)–(2.19) *given by Theorem 2.2. Then, we have that there exists a quadruplet* $(\mu, \varphi, \sigma, \xi)$ *such that*

$$\mu_{\alpha,\beta} \to \mu \qquad \textit{weakly in } L^2(0,T;V) \tag{2.28}$$

$$\varphi_{\alpha,\beta} \to \varphi \qquad \textit{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W) \tag{2.29}$$

$$\sigma_{\alpha,\beta} \to \sigma \qquad \textit{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V) \tag{2.30}$$

$$\partial_t(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}) \to \partial_t\varphi \qquad \textit{weakly in } L^2(0,T;V^*) \tag{2.31}$$

$$\xi_{\alpha,\beta} \to \xi \qquad \textit{weakly in } L^2(0,T;H) \tag{2.32}$$

at least for a subsequence. Moreover, every limiting quadruplet $(\mu, \varphi, \sigma, \xi)$ *satisfies*

$$\langle \partial_t\varphi, v\rangle + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega R\, v \qquad \forall\, v \in V, \ \textit{a.e. in } (0,T) \tag{2.33}$$

$$\mu = -\Delta\varphi + \xi + \pi(\varphi), \quad \xi \in B(\varphi) \qquad \textit{a.e. in } Q \tag{2.34}$$

$$\langle \partial_t\sigma, v\rangle + \int_\Omega \nabla\sigma \cdot \nabla v = -\int_\Omega R\, v \qquad \forall\, v \in V, \ \textit{a.e. in } (0,T) \tag{2.35}$$

$$\varphi(0) = \varphi_0 \quad \textit{and} \quad \sigma(0) = \sigma_0 \qquad \textit{in } \Omega \tag{2.36}$$

where R *is defined by* (2.15)*, accordingly.*

The above result generalizes the analogous [5, Thm. 2.6] as far as the assumptions on the initial data are concerned (and also since just the case $\alpha = \beta$ was considered there). Moreover, in [5, Thm. 2.6], even uniqueness for the solution to the limit problem was proved. However, also for this point, stronger conditions on the initial data are assumed in order that the solution to the limit problem is rather smooth. Here, we can consider the natural regularity requirements, i.e.,

$$\mu \in L^2(0,T;V) \tag{2.37}$$

$$\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{2.38}$$

$$\sigma \in H^1(0,T;V^*) \cap L^2(0,T;V) \subset C^0([0,T];H)\,. \tag{2.39}$$

For uniqueness in this framework, we can quote the even more general result [13, Thm. 2]. However, uniqueness also follows from the error estimate we present at once (see the forthcoming Remark 4.1

for details). In order to state our last result we need to reinforce the assumptions we made on the potential F; namely, we assume

$$D(\widehat{B}) = \mathbb{R} \quad \text{and} \quad F = \widehat{B} + \widehat{\pi} \quad \text{is a } C^2 \text{ function on } \mathbb{R} \tag{2.40}$$

$$|F(r)| \le C_0(|r|^6 + 1), \quad |F'(r)| \le C_1(|r|^5 + 1), \quad \text{and} \quad |F''(r)| \le C_2(|r|^4 + 1). \tag{2.41}$$

Although the third condition in (2.41) implies the other two, we have written all of them for convenience. We also remark that the classical potential (2.8) fulfils such assumptions. Furthermore, we notice that (2.41) is slightly more general than the analogous assumption made in [5, Thm. 2.6]. Finally, we can observe that the exponents in (2.41) are related to the dimension of Ω and the related Sobolev embeddings. Here is our last result.

Theorem 2.7. *Assume* (2.3)–(2.7) *and* (2.40)–(2.41) *on the structure and* (2.11) *on the initial data. Then, with the notation of Theorem 2.6, the estimate*

$$\begin{aligned} &\|\varphi_{\alpha,\beta} - \varphi\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|\mu_{\alpha,\beta} - \mu\|_{L^2(0,T;V^*)} \\ &\quad + \|\sigma_{\alpha,\beta} - \sigma\|_{L^\infty(0,T;V^*)\cap L^2(0,T;H)} \le C\left(\alpha^{1/2} + \beta^{1/2}\right) \end{aligned} \tag{2.42}$$

holds true with a constant C that depends only on Ω, T, the structure of the system, and the norms of the initial data related to assumptions (2.11)*, but not on α nor on β.*

The rest of the section is devoted to list some facts. We make repeated use of the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in [0,T] \tag{2.43}$$

and of well-known inequalities, namely, of the elementary Young inequality

$$ab \le \delta a^2 + \frac{1}{4\delta}\, b^2 \quad \text{for every } a, b \ge 0 \text{ and } \delta > 0 \tag{2.44}$$

as well as of Hölder's inequality and its consequences. Moreover, as Ω is bounded and smooth, we can owe to the Poincaré and Sobolev type inequalities, namely,

$$\|v\|_V \le C\Big(\|\nabla v\|_H + \big|\textstyle\int_\Omega v\big|\Big) \quad \text{for every } v \in V \tag{2.45}$$

$$V \subset L^q(\Omega) \quad \text{and} \quad \|v\|_q \le C\|v\|_V \quad \text{for every } v \in V \text{ and } 1 \le q \le 6 \tag{2.46}$$

$$L^q(\Omega) \subset V^* \quad \text{and} \quad \|v\|_* \le C\|v\|_q \quad \text{for every } v \in L^q(\Omega) \text{ and } q \ge 6/5\,. \tag{2.47}$$

In (2.45)–(2.47), C only depends on Ω. Finally, we recall the interpolation inequality

$$\|v\|_H^2 \le \|v\|_V\, \|v\|_* \quad \text{for every } v \in V, \tag{2.48}$$

which trivially follows from the identity $\|v\|_H^2 = \langle v, v\rangle$ for every $v \in V$.

3 Proofs of Theorems 2.2 and 2.3

We start proving Theorem 2.3 in the following form: (2.22)–(2.23) hold for every α and β and every solution to problem (2.15)–(2.19) satisfying the regularity specified by (2.12)–(2.14). We do not know anything about well-posedness yet, indeed.

First a priori estimate. We test (2.16) and (2.18) by μ and σ, respectively, and integrate over $(0,t)$, where $t \in (0,T)$ is arbitrary. At the same time, we multiply (2.17) by $-\partial_t\varphi$ and integrate over Q_t. Then, we add the resulting equalities to each other, obtaining

$$\begin{aligned}&\frac{\alpha}{2}\int_\Omega |\mu(t)|^2 + \int_{Q_t} \partial_t\varphi\,\mu + \int_{Q_t} |\nabla\mu|^2 \\ &\quad - \int_{Q_t} \mu\,\partial_t\varphi + \beta\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi(t)|^2 + \int_\Omega F(\varphi(t)) \\ &\quad + \frac{1}{2}\int_\Omega |\sigma(t)|^2 + \int_{Q_t} |\nabla\sigma|^2 + \int_{Q_t} R(\sigma-\mu) \\ &= \frac{\alpha}{2}\int_\Omega |\mu_0|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi_0|^2 + \int_\Omega F(\varphi_0) + \frac{1}{2}\int_\Omega |\sigma_0|^2 \,.\end{aligned}$$

Clearly, two terms cancel out. Moreover, F is nonnegative by assumptions (2.5)–(2.6). Finally, we have $R(\sigma-\mu) = |S|^2$ and $|R| \le |S| \sup\sqrt{p}$ a.e. in Q with the notation (2.21). Therefore, with the help of (2.4) we immediately deduce

$$\begin{aligned}&\alpha^{1/2}\|\mu\|_{L^\infty(0,T;H)} + \|\nabla\mu\|_{L^2(0,T;H)} \\ &\quad + \beta^{1/2}\|\partial_t\varphi\|_{L^2(0,T;H)} + \|\nabla\varphi\|_{L^\infty(0,T;H)} + \|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))}^{1/2} \\ &\quad + \|\sigma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|S\|_{L^2(0,T;H)} + \|R\|_{L^2(0,T;H)} \\ &\le C\left(\alpha^{1/2}\|\mu_0\|_H + \|\nabla\varphi_0\|_H + \|F(\varphi_0)\|_{L^1(\Omega)}^{1/2} + \|\sigma_0\|_H\right) \end{aligned} \tag{3.1}$$

for some constant C that depends only on p. Thus, in order to prove (2.22), we have to complete the full norm of φ and estimate the terms that are missing in (3.1).

Second a priori estimate. We estimate the mean value of φ by testing (2.20) by $v = 1$. We obtain, for every $t \in [0,T]$,

$$\int_\Omega \big(\alpha\mu(t) + \varphi(t) + \sigma(t)\big) = \int_\Omega \big(\alpha\mu_0 + \varphi_0 + \sigma_0\big) \le |\Omega|^{1/2}\|\alpha\mu_0 + \varphi_0 + \sigma_0\|_H$$

and deduce that (since $\alpha < 1$)

$$\Big|\int_\Omega \varphi(t)\Big| \le C_\Omega\big(\alpha^{1/2}\|\mu_0\|_H + \|\varphi_0\|_H + \|\sigma_0\|_H + \alpha^{1/2}\|\mu(t)\|_H + \|\sigma(t)\|_H\big), \tag{3.2}$$

where C_Ω depends only on Ω.

Third a priori estimate. We test (2.18), written at the time t, with $v(t)$, where v is arbitrary in $L^2(0,T;V)$. Then we integrate over $(0,T)$ with respect to t and obtain

$$\Big|\int_0^T \langle\partial_t\sigma(t), v(t)\rangle\,dt\Big| \le \big(\|\nabla\sigma\|_{L^2(0,T;H)} + \|R\|_{L^2(0,T;H)}\big)\|v\|_{L^2(0,T;V)} \,.$$

This means that

$$\|\partial_t\sigma\|_{L^2(0,T;V^*)} \le \|\nabla\sigma\|_{L^2(0,T;H)} + \|R\|_{L^2(0,T;H)} \,. \tag{3.3}$$

Fourth a priori estimate. Similarly, we test (2.20), written at the time t, by $v(t)$, where v is arbitrary in $L^2(0,T;V)$. We obtain

$$\Big|\int_0^T \langle\partial_t(\alpha\mu+\varphi)(t), v(t)\rangle\,dt\Big| \le \Big|\int_0^T \langle\partial_t\sigma(t), v(t)\rangle\,dt\Big| + \|\nabla(\mu+\sigma)\|_{L^2(0,T;H)}\|v\|_{L^2(0,T;V)} \,,$$

whence immediately

$$\|\partial_t(\alpha\mu+\varphi)\|_{L^2(0,T;V^*)} \le \|\partial_t\sigma\|_{L^2(0,T;V^*)} + \|\nabla\mu\|_{L^2(0,T;H)} + \|\nabla\sigma\|_{L^2(0,T;H)}. \qquad (3.4)$$

First conclusion. We combine (3.1)–(3.4) with the Poincaré inequality (2.45) applied to φ and immediately deduce (2.22) with a constant $\widehat{C}$ that depends only on p, Ω and T.

Fifth a priori estimate and conclusion. By estimate (2.22) and the Lipschitz continuity of π, we deduce that

$$\|\pi(\varphi)\|_{L^2(0,T;H)} \le \widehat{C}\left(\alpha^{1/2}\|\mu_0\|_H + \|\varphi_0\|_V + \|F(\varphi_0)\|_{L^1(\Omega)}^{1/2} + \|\sigma_0\|_H + 1\right), \qquad (3.5)$$

with the same $\widehat{C}$, without loss of generality, provided that we allow $\widehat{C}$ to depend on π as well. Now, we write (2.17) in the form

$$-\Delta\varphi + \xi = f := -\beta\partial_t\varphi - \pi(\varphi) + \mu$$

and observe that (2.22), (3.5) and $\beta < 1$ imply

$$\|f\|_{L^2(0,T;H)} \le \widehat{C}\left(\alpha^{1/2}\|\mu_0\|_H + \|\varphi_0\|_V + \|F(\varphi_0)\|_{L^1(\Omega)}^{1/2} + \|\sigma_0\|_H + 1 + \|\mu\|_{L^2(0,T;H)}\right),$$

with the same $\widehat{C}$ once more, without loss of generality. If M denotes the right-hand side of this inequality, a standard argument (formally multiply by $-\Delta\varphi$) shows that both $\Delta\varphi$ and ξ are bounded in $L^2(Q)$ by a multiple of M. Therefore, the same holds for $\|\varphi\|_{L^2(0,T;W)}$ by elliptic regularity. Finally, the full norm $\|\mu\|_{L^2(0,T;V)}$ is is equivalent to the sum of $\|\nabla\mu\|_{L^2(0,T;H)}$ and $\|\mu\|_{L^2(0,T;H)}$. Thus, (2.23) follows and the proof of Theorem 2.3 is complete. □

Proof of Theorem 2.2. As far as uniqueness is concerned, we can refer to the proof of the uniqueness part of [5, Thm. 2.2] since it holds under the present assumptions. In order to prove the existence of a solution, we approximate the data μ_0 and σ_0 by functions $\mu_{0,\varepsilon}$ and $\sigma_{0,\varepsilon}$ satisfying

$$\mu_{0,\varepsilon}, \sigma_{0,\varepsilon} \in V \quad \text{for } \varepsilon > 0, \qquad \mu_{0,\varepsilon} \to \mu_0 \quad \text{and} \quad \sigma_{0,\varepsilon} \to \sigma_0 \quad \text{in } H \quad \text{as } \varepsilon \searrow 0.$$

Then, for every $\varepsilon > 0$, the condition (2.24) holds for the approximating data so that the assumptions of [5, Thm. 2.2] are fulfilled. Thus, the problem (2.15)–(2.19) has a unique solution $(\mu_\varepsilon, \varphi_\varepsilon, \sigma_\varepsilon, \xi_\varepsilon)$ with R_ε defined by (2.15) accordingly. Moreover, such a solution must satisfy (2.22)–(2.23) due to the above proof. As α and β are fixed, such estimates provide uniform boundedness with respect to ε even for μ_ε in $L^\infty(0,T;H)$ and $\partial_t\varphi_\varepsilon$ in $L^2(0,T;H)$. Therefore, (2.23) implies that μ_ε, φ_ε and ξ_ε are bounded in $L^2(0,T;V)$, $H^1(0,T;H) \cap L^2(0,T;W)$ and $L^2(0,T;H)$, respectively. Finally, the estimate for the time derivative of $\alpha\mu_\varepsilon + \varphi_\varepsilon$ derived from (2.22) and the estimate for $\partial_t\varphi_\varepsilon$ mentioned before imply that $\partial_t\mu_\varepsilon$ is bounded in $L^2(0,T;V^*)$. Hence, we have

$$\begin{aligned}
&\mu_\varepsilon \to \mu \quad \text{weakly star in } H^1(0,T;V^*) \cap L^2(0,T;V) \\
&\varphi_\varepsilon \to \varphi \quad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;W) \\
&\sigma_\varepsilon \to \sigma \quad \text{weakly star in } H^1(0,T;V^*) \cap L^2(0,T;V) \\
&\xi_\varepsilon \to \xi \ \text{ and } R_\varepsilon \to R \quad \text{weakly in } L^2(0,T;H)
\end{aligned}$$

as $\varepsilon \searrow 0$, at least for a subsequence. This implies, in particular, that the initial conditions for (μ, φ, σ) are satisfied. Moreover, the above convergence for φ_ε and the Aubin-Lions lemma (see, e.g., [17, Thm. 5.1, p. 58]) imply that

$$\mu_\varepsilon \to \mu, \quad \varphi_\varepsilon \to \varphi, \quad \sigma_\varepsilon \to \sigma \quad \text{strongly in } L^2(0,T;H).$$

Then, $\pi(\varphi_\varepsilon)$ and $p(\varphi_\varepsilon)$ converge to $\pi(\varphi)$ and $p(\varphi)$, respectively, strongly in $L^2(0,T;H)$. Therefore, we can identify the limits of the nonlinear terms ξ_ε and R_ε. For the former, we can apply, e.g., [1, Cor. 2.4, p. 41] and conclude that $\xi \in B(\varphi)$ a.e. in Q. For the latter we note that R_ε converges to $p(\varphi)(\sigma-\mu)$ strongly in $L^1(Q)$, whence (2.15) follows. At this point, we can write the integrated–in–time version of problem (2.16)–(2.18) for the approximating solution with time dependent test functions and take the limit as ε tends to zero. We obtain the analogous systems for (μ,φ,σ,ξ), and this implies (2.16)–(2.18) for such a quadruplet. This completes the proof of Theorem 2.2. □

4 Asymptotics

This section is devoted to the proof of Theorems 2.6 and 2.7. In order to simplify the notation, we follow a general rule in performing our a priori estimates. The small-case italic c without any subscript stands for different constants, that may only depend on Ω, T, the shape of the nonlinearities and the norms of the initial data related to assumption (2.11). A notation like c_δ signals a constant that depends also on the parameter δ. We point out that c and c_δ do not depend on α and β and that their meaning might change from line to line and even in the same chain of inequalities. On the contrary, those constants we need to refer to are always denoted by different symbols, e.g., by a capital letter.

Proof of Theorem 2.6. We follow the argument done for [5, Thm. 2.6] rather closely, but we have to modify the types of convergence since our assumptions are different and more general. We start from (2.22)–(2.23), written for the solution $(\mu_{\alpha,\beta},\varphi_{\alpha,\beta},\sigma_{\alpha,\beta},\xi_{\alpha,\beta})$, and improve the latter by estimating the norm of $\mu_{\alpha,\beta}$ on its right-hand side. However, we omit the subscripts α and β for a while. Thanks to (2.17) and (2.26), we have that $|\xi| \le C\bigl(\widehat{B}(\varphi)+1\bigr)$ a.e. in Q. Then, by integrating over Ω we obtain

$$\int_\Omega |\xi(t)| \le C\int_\Omega \bigl(\widehat{B}(\varphi(t))+1\bigr) \quad \text{for a.a. } t\in(0,T). \tag{4.1}$$

At this point, we can estimate the mean value of μ on account of (2.6) and (2.7). Indeed, by just integrating (2.17) over Ω, we deduce that

$$\begin{aligned} \Bigl|\int_\Omega \mu(t)\Bigr| &= \Bigl|\int_\Omega \bigl(\beta\partial_t\varphi+\xi+\pi(\varphi)\bigr)(t)\Bigr| \\ &\le \beta\|\partial_t\varphi(t)\|_1 + c\bigl(\|\widehat{B}(\varphi(t))\|_1+\|\varphi(t)\|_1+1\bigr) \\ &\le c\,\beta^{1/2}\|\partial_t\varphi(t)\|_H + c\bigl(\|F(\varphi(t))\|_1+\|\varphi(t)\|_H+1\bigr) \end{aligned} \tag{4.2}$$

for a.a. $t\in(0,T)$, beacause of the Lipschitz continuity of π and the nonnegativity of $\widehat{\pi}$. Then, (2.22) implies that the function $t\mapsto \bigl|\int_\Omega \mu(t)\,dt\bigr|$ is bounded in $L^2(0,T)$. By combining this with (2.22) and the Poincaré inequality (2.45), we derive that μ is bounded in $L^2(0,T;V)$. Hence, recalling estimates (2.22)–(2.23) it turns out that the convergences (2.28)–(2.32) and a convergence for $R_{\alpha,\beta}$ hold, at least for a subsequence. For the reader's convenience, we write this conclusion explicitly, as well as the consequences we are interested in. These are obtained by means of strong compactness results (see, e.g., [24, Sect. 8, Cor. 4]), the Sobolev inequality (2.46) and the Lipschitz continuity of π and p.

We have

$$\mu_{\alpha,\beta} \to \mu \quad \text{weakly in } L^2(0,T;V) \cap L^2(0,T;L^6(\Omega))$$
$$\varphi_{\alpha,\beta} \to \varphi \quad \text{weakly star in } L^\infty(0,T;V) \cap L^2(0,T;W)$$
$$\sigma_{\alpha,\beta} \to \sigma \quad \text{weakly in } H^1(0,T;V^*) \cap L^2(0,T;V) \cap L^2(0,T;L^6(\Omega))$$
$$\xi_{\alpha,\beta} \to \xi \text{ and } R_\varepsilon \to R \quad \text{weakly in } L^2(0,T;H)$$
$$\alpha\mu_{\alpha,\beta} \to 0 \quad \text{strongly in } L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^2(0,T;L^6(\Omega))$$
$$\beta\partial_t\varphi_{\alpha,\beta} \to 0 \quad \text{strongly in } L^2(0,T;H)$$
$$\partial_t(\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta}) \to \partial_t\varphi \quad \text{weakly in } L^2(0,T;V^*)$$
$$\alpha\mu_{\alpha,\beta} + \varphi_{\alpha,\beta} \to \varphi \quad \text{strongly in } C^0([0,T];V^*) \cap L^2(0,T;H)$$
$$\pi(\varphi_{\alpha,\beta}) \to \pi(\varphi) \text{ and } p(\varphi_{\alpha,\beta}) \to p(\varphi) \quad \text{strongly in } L^2(0,T;H).$$

Hence, we infer that φ and σ satisfy the initial conditions (2.36). Moreover, we deduce that $\xi \in B(\varphi)$ (apply, e.g., [2, Prop. 2.5, p. 27]) and that $R_{\alpha,\beta}$ also converges to $p(\varphi)(\sigma-\mu)$ weakly in $L^1(0,T;L^p(\Omega))$ for some $p \in (1,2)$: consequently, we have $R = p(\varphi)(\sigma - \mu)$.

Finally, we take the limit in the integrated–in–time version of problem (2.16)–(2.18) for $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta}, \xi_{\alpha,\beta})$ with time-dependent test functions. We obtain the analogue for the system (2.33)–(2.35). Finally, as the solution of the limit problem is unique by Theorem 4.1, the convergences we have obtained for a subsequence hold for the whole family. This completes the proof of Theorem 2.6. □

Proof of Theorem 2.7. As we use some ideas of [13], it is convenient to rewrite the equations (2.16) and (2.18) as abstract equations in the framework of the Hilbert triplet (V, H, V^*) related to an invertible operator. To this end, we introduce the Riesz isomorphism $\mathcal{A} : V \to V^*$ associated to the standard scalar product of V, that is

$$\langle \mathcal{A}u, v\rangle := (u,v)_V = \int_\Omega \big(\nabla u \cdot \nabla v + uv\big) \quad \text{for } u, v \in V. \tag{4.3}$$

We notice that $\mathcal{A}u = -\Delta u + u$ if $u \in W$ and that the restriction of $\mathcal{A}$ to W is an isomorphism from W onto H. We also remark that

$$\langle \mathcal{A}u, \mathcal{A}^{-1}v^*\rangle = \langle v^*, u\rangle \quad \text{for every } u \in V \text{ and } v^* \in V^* \tag{4.4}$$
$$\langle u^*, \mathcal{A}^{-1}v^*\rangle = (u^*, v^*)_* \quad \text{for every } u^*, v^* \in V^*, \tag{4.5}$$

where $(\,\cdot\,,\,\cdot\,)_*$ is the dual scalar product in V^* associated with the standard one in V, and recall that $\langle v^*, u\rangle = \int_\Omega v^* u$ if $v^* \in H$. As a consequence of (4.5), we have

$$\frac{d}{dt}\,\|v^*\|_*^2 = 2\langle \partial_t v^*, \mathcal{A}^{-1}v^*\rangle \quad \text{for every } v^* \in H^1(0,T;V^*). \tag{4.6}$$

In view of the regularity conditions (2.12)–(2.14) and (2.37)–(2.39), we rewrite (2.16)–(2.18) and (2.33)–(2.35) for the solution $(\mu_{\alpha,\beta}, \varphi_{\alpha,\beta}, \sigma_{\alpha,\beta})$ to (2.15)–(2.19) and the one of the limit problem, respectively. If we term the latter $(\overline{\mu}, \overline{\varphi}, \overline{\sigma})$, we have

$$\alpha\partial_t\mu_{\alpha,\beta} + \partial_t\varphi_{\alpha,\beta} + \mathcal{A}\mu_{\alpha,\beta} = R_{\alpha,\beta} + \mu_{\alpha,\beta} \tag{4.7}$$
$$\mu_{\alpha,\beta} = \beta\partial_t\varphi_{\alpha,\beta} + \mathcal{A}\varphi_{\alpha,\beta} + F'(\varphi_{\alpha,\beta}) - \varphi_{\alpha,\beta} \tag{4.8}$$
$$\partial_t\sigma_{\alpha,\beta} + \mathcal{A}\sigma_{\alpha,\beta} = -R_{\alpha,\beta} + \sigma_{\alpha,\beta} \tag{4.9}$$
$$\partial_t\overline{\varphi} + \mathcal{A}\overline{\mu} = \overline{R} + \overline{\mu} \tag{4.10}$$
$$\overline{\mu} = \mathcal{A}\overline{\varphi} + F'(\overline{\varphi}) - \overline{\varphi} \tag{4.11}$$
$$\partial_t\overline{\sigma} + \mathcal{A}\overline{\sigma} = -\overline{R} + \overline{\sigma}, \tag{4.12}$$

where $R_{\alpha,\beta}$ and $\overline{R}$ are defined by (2.15) according to the equations we are considering. All these equations are meant in V^* a.e. in $(0,T)$. However, (4.8) and (4.11) also hold a.e. in Q. Moreover, the solutions have to satisfy the initial conditions (2.19) and (2.36), respectively. Now, we take the differences between (4.7)–(4.9) and (4.10)–(4.12) at time $s \in (0,T)$ and test them by

$$\mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)(s), \quad -(\alpha\mu_{\alpha,\beta} + \varphi)(s), \quad \text{and} \quad \mathcal{A}^{-1}\sigma(s),$$

respectively, where we have set for convenience

$$\mu := \mu_{\alpha,\beta} - \overline{\mu}, \quad \varphi := \varphi_{\alpha,\beta} - \overline{\varphi}, \quad \sigma := \sigma_{\alpha,\beta} - \overline{\sigma}, \quad \text{and} \quad R := R_{\alpha,\beta} - \overline{R}.$$

Next, we sum up and integrate over $(0,t)$ with respect to s, for an arbitrary $t \in (0,T)$. We obtain (by omitting the evaluation at s inside integrals, for brevity)

$$\begin{aligned}
&\int_0^t \langle \partial_t(\alpha\mu_{\alpha,\beta} + \varphi), \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)\rangle\, ds + \int_0^t \langle \mathcal{A}\mu, \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)\rangle\, ds \\
&\quad - \int_0^t \langle \mu, \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds + \int_0^t \langle \beta\partial_t\varphi_{\alpha,\beta}, \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds + \int_0^t \langle \mathcal{A}\varphi, \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds \\
&\quad + \int_0^t \langle F'(\varphi_{\alpha,\beta}) - F'(\overline{\varphi}), \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds \\
&\quad + \int_0^t \langle \partial_t\sigma, \mathcal{A}^{-1}\sigma\rangle\, ds + \int_0^t \langle \mathcal{A}\sigma, \mathcal{A}^{-1}\sigma\rangle\, ds \\
&= \int_0^t \langle R, \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)\rangle\, ds + \int_0^t \langle \mu, \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)\rangle\, ds + \int_0^t \langle \varphi, \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds \\
&\quad - \int_0^t \langle R, \mathcal{A}^{-1}\sigma\rangle\, ds + \int_0^t \langle \sigma, \mathcal{A}^{-1}\sigma\rangle\, ds\,.
\end{aligned}$$

For the reader's convenience, we just rearrange and use the decomposition $F' = B + \pi$. We have

$$\begin{aligned}
&\int_0^t \langle \partial_t\big(\alpha\mu_{\alpha,\beta} + \varphi\big), \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)\rangle\, ds \\
&\quad + \int_0^t \langle \mathcal{A}\mu, \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)\rangle\, ds - \int_0^t \langle \mu, \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds \\
&\quad + \int_0^t \langle \mathcal{A}\varphi, \varphi\rangle\, ds + \int_0^t \langle B(\varphi_{\alpha,\beta}) - B(\overline{\varphi}), \varphi\rangle\, ds \\
&\quad + \int_0^t \langle \partial_t\sigma, \mathcal{A}^{-1}\sigma\rangle\, ds + \int_0^t \langle \mathcal{A}\sigma, \mathcal{A}^{-1}\sigma\rangle\, ds \\
&= \int_0^t \langle R, \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi - \sigma)\rangle\, ds - \int_0^t \langle \beta\partial_t\varphi_{\alpha,\beta}, \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds - \int_0^t \langle \mathcal{A}\varphi, \alpha\mu_{\alpha,\beta}\rangle\, ds \\
&\quad + \int_0^t \langle \mu, \mathcal{A}^{-1}(\alpha\mu_{\alpha,\beta} + \varphi)\rangle\, ds - \int_0^t \langle \pi(\varphi_{\alpha,\beta}) - \pi(\overline{\varphi}), \varphi\rangle\, ds \\
&\quad - \int_0^t \langle F'(\varphi_{\alpha,\beta}) - F'(\overline{\varphi}), \alpha\mu_{\alpha,\beta}\rangle\, ds + \int_0^t \langle \varphi, \alpha\mu_{\alpha,\beta} + \varphi\rangle\, ds + \int_0^t \langle \sigma, \mathcal{A}^{-1}\sigma\rangle\, ds.
\end{aligned}$$

At this point, we account for (4.3)–(4.6) and observe that the second and third terms on the left-hand side cancel out. Finally, owing to the initial conditions $(\alpha\mu_{\alpha,\beta} + \varphi)(0) = \alpha\mu_0$ and $\sigma(0) = 0$, we deduce

$$
\begin{aligned}
&\frac{1}{2}\,\|(\alpha\mu_{\alpha,\beta}+\varphi)(t)\|_*^2+\int_0^t\|\varphi\|_V^2\,ds+\int_{Q_t}\bigl(B(\varphi_{\alpha,\beta})-B(\overline{\varphi})\bigr)\varphi\\
&\quad+\frac{1}{2}\,\|\sigma(t)\|_*^2+\int_{Q_t}|\sigma|^2\\
&=\frac{1}{2}\,\|\alpha\mu_0\|_*^2+\int_0^t\bigl(R,\alpha\mu_{\alpha,\beta}+\varphi-\sigma\bigr)_*\,ds-\int_0^t\langle\beta\partial_t\varphi_{\alpha,\beta},\alpha\mu_{\alpha,\beta}+\varphi\rangle\,ds\\
&\quad-\int_0^t\bigl(\varphi,\alpha\mu_{\alpha,\beta}\bigr)_V\,ds+\int_0^t\bigl(\mu,\alpha\mu_{\alpha,\beta}+\varphi\bigr)_*\,ds-\int_{Q_t}\bigl(\pi(\varphi_{\alpha,\beta})-\pi(\overline{\varphi})\bigr)\,\varphi\\
&\quad-\int_{Q_t}\bigl(F'(\varphi_{\alpha,\beta})-F'(\overline{\varphi})\bigr)\,\alpha\mu_{\alpha,\beta}+\int_{Q_t}\varphi(\alpha\mu_{\alpha,\beta}+\varphi)+\int_0^t\|\sigma\|_*^2\,ds\,. \qquad (4.13)
\end{aligned}
$$

All of the terms on the left-hand side are nonnegative, the third one by monotonicity. Now, we treat each integral on the right-hand side, separately. In the sequel, δ is a positive parameter whose value will be chosen at the end of the procedure. We first observe that (2.22) holds for the solution $(\mu_{\alpha,\beta},\varphi_{\alpha,\beta},\sigma_{\alpha,\beta})$ and that Theorem 2.6 improves (2.23) for such a solution. Indeed, the restricted setting of regular potentials satisfying (2.41) led to (4.2). So, as we have seen in the previous proof, (2.22) and (2.23) imply

$$
\|\mu_{\alpha,\beta}\|_{L^2(0,T;V)}+\|\varphi_{\alpha,\beta}\|_{L^2(0,T;W)}\le c. \qquad (4.14)
$$

Now, we prepare estimates for $\|\mu\|_*$ and $\|R\|_*$ a.e. in $(0,T)$. Again for simplicity, in performing them, we omit writing the evaluation point. From the mean value theorem and the third assumption in (2.41) we easily derive that

$$
|F'(\varphi_{\alpha,\beta})-F'(\overline{\varphi})|\le c|\varphi|(|\varphi_{\alpha,\beta}|^4+|\overline{\varphi}|^4+1)\quad\text{a.e. in } Q.
$$

Therefore, by the Hölder and Sobolev inequalities, we infer that

$$
\begin{aligned}
&\|F'(\varphi_{\alpha,\beta})-F'(\overline{\varphi})\|_{6/5}\le c\,\|\varphi\|_6\bigl(\|\varphi_{\alpha,\beta}^4\|_{3/2}+\|\overline{\varphi}^{\,4}\|_{3/2}+1\bigr)\\
&\quad=c\,\|\varphi\|_6\bigl(\|\varphi_{\alpha,\beta}\|_6^4+\|\overline{\varphi}\|_6^4+1\bigr)\le c\|\varphi\|_V\bigl(\|\varphi_{\alpha,\beta}\|_V^4+\|\overline{\varphi}\|_V^4+1\bigr)\le c\|\varphi\|_V\,, \qquad (4.15)
\end{aligned}
$$

the last inequality following from estimate (2.22) for $\varphi_{\alpha,\beta}$ and the regularity (2.38) of $\overline{\varphi}$. Taking the difference between (4.8) and (4.11) and using the dual Sobolev inequality (2.47), we deduce that

$$
\begin{aligned}
&\|\mu\|_*=\|\beta\partial_t\varphi_{\alpha,\beta}+\mathcal{A}\varphi+F'(\varphi_{\alpha,\beta})-F'(\overline{\varphi})-\varphi\|_*\\
&\le\beta\|\partial_t\varphi_{\alpha,\beta}\|_*+\|\varphi\|_V+c\|F'(\varphi_{\alpha,\beta})-F'(\overline{\varphi})\|_{6/5}+\|\varphi\|_*\\
&\le c\bigl(\beta\|\partial_t\varphi_{\alpha,\beta}\|_*+\|\varphi\|_V\bigr)\le c\,\beta^{1/2}+c\,\|\varphi\|_V\,, \qquad (4.16)
\end{aligned}
$$

where the last inequality follows from (2.22). In order to estimate $\|R\|_*$, we first observe that the boundedness and the Lipschitz continuity of p and the Sobolev inequality (applied to $\nabla\overline{\varphi}$ and the test function $v\in V$) imply that, for every $v\in V$,

$$
\begin{aligned}
&\|p(\overline{\varphi})v\|_V\le\|p(\overline{\varphi})v\|_H+\|\nabla p(\overline{\varphi})\,v\|_H+\|p(\overline{\varphi})\nabla v\|_H\\
&\le c\|v\|_H+c\|\nabla\overline{\varphi}\|_4\,\|v\|_4+c\|\nabla v\|_H\le c(\|\overline{\varphi}\|_W+1)\|v\|_V\,.
\end{aligned}
$$

Hence, we have for every $v \in V$ the estimate

$$\begin{aligned}
&\int_\Omega Rv = \int_\Omega \big(p(\varphi_{\alpha,\beta})(\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}) - p(\overline{\varphi})(\overline{\sigma} - \overline{\mu})\big)v \\
&\le \int_\Omega |p(\varphi_{\alpha,\beta}) - p(\overline{\varphi})|\, |\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}|\, |v| + \Big| \int_\Omega p(\overline{\varphi})(\sigma - \mu)\, v \Big| \\
&\le c\|\varphi\|_3 \, \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_3 \|v\|_3 + \|\sigma - \mu\|_* \, \|p(\overline{\varphi})\, v\|_V \\
&\le c\|\varphi\|_V \, \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_V \|v\|_V + c\|\sigma - \mu\|_* \, (\|\overline{\varphi}\|_W + 1)\|v\|_V \, .
\end{aligned}$$

Therefore, we can estimate $\|R\|_*$ a.e. in $(0,T)$, also owing to (4.16), in this way:

$$\begin{aligned}
&\|R\|_* \le c\|\varphi\|_V \, \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_V + c\|\sigma - \mu\|_* \, (\|\overline{\varphi}\|_W + 1) \\
&\le c\|\varphi\|_V \, \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_V + c\big(\|\sigma\|_* + \beta^{1/2} + \|\varphi\|_V\big)(\|\overline{\varphi}\|_W + 1) \\
&\le c\, \psi_{\alpha,\beta}(\|\varphi\|_V + \|\sigma\|_*) + c\, \beta^{1/2}\, \overline{\psi} \, ,
\end{aligned}$$

where $\psi_{\alpha,\beta}, \overline{\psi} : (0,T) \to \mathbb{R}$ are defined by

$$\psi_{\alpha,\beta} := \|\sigma_{\alpha,\beta} - \mu_{\alpha,\beta}\|_V + \|\overline{\varphi}\|_W + 1 \quad \text{and} \quad \overline{\psi} := \|\overline{\varphi}\|_W + 1, \quad \text{a.e. in } (0,T) \, .$$

Coming back to the right-hand side of (4.13), we can treat the first term as follows:

$$\begin{aligned}
&\int_0^t \big(R, \alpha\mu_{\alpha,\beta} + \varphi - \sigma\big)_* \, ds \le \int_0^t \|R\|_* \, \|\alpha\mu_{\alpha,\beta} + \varphi - \sigma\|_* \, ds \\
&\le \int_0^t \big(c\psi_{\alpha,\beta}(\|\varphi\|_V + \|\sigma\|_*) + c\beta^{1/2}\, \overline{\psi}\big)\big(\|\alpha\mu_{\alpha,\beta} + \varphi\|_* + \|\sigma\|_*\big)\, ds \\
&\le \delta \int_0^t \|\varphi\|_V^2 \, ds + \beta \int_0^T |\overline{\psi}|^2 \, ds + c_\delta \int_0^t \psi_{\alpha,\beta}^2 \big(\|\alpha\mu_{\alpha,\beta} + \varphi\|_*^2 + \|\sigma\|_*^2\big)\, ds \, .
\end{aligned} \tag{4.17}$$

We observe at once that the regularity (2.38) for $\overline{\varphi}$ and estimates (2.22) for $\sigma_{\alpha,\beta}$ and (4.14) for $\mu_{\alpha,\beta}$ imply that $\overline{\psi} \in L^2(0,T)$ and that $\psi_{\alpha,\beta}$ is bounded in $L^2(0,T)$, so that $\psi_{\alpha,\beta}^2$ is bounded in $L^1(0,T)$. This will allow us to apply the Gronwall lemma. Now, we estimate the next term on the right-hand side of (4.13). Using (2.22), we see that

$$\begin{aligned}
&- \int_0^t \langle \beta\partial_t\varphi_{\alpha,\beta}, \alpha\mu_{\alpha,\beta} + \varphi \rangle \, ds \\
&\le \alpha^2 \, \|\mu_{\alpha,\beta}\|_{L^2(0,T;H)}^2 + \delta \int_0^t \|\varphi\|_H^2 \, ds + c_\delta \, \beta^2 \, \|\partial_t\varphi_{\alpha,\beta}\|_{L^2(0,T;H)}^2 \\
&\le c\alpha + \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \, \beta \, .
\end{aligned} \tag{4.18}$$

Next, we have

$$- \int_0^t \big(\varphi, \alpha\mu_{\alpha,\beta}\big)_V \, ds \le \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \, \alpha^2 \tag{4.19}$$

thanks to (4.14) for $\mu_{\alpha,\beta}$, as well as, by (4.16),

$$\begin{aligned}
&\int_0^t \big(\mu, \alpha\mu_{\alpha,\beta} + \varphi\big)_* \, ds \le c \int_0^t \big(\beta^{1/2} + \|\varphi\|_V\big) \|\alpha\mu_{\alpha,\beta} + \varphi\|_* \, ds \\
&\le \delta \int_0^t \|\varphi\|_V^2 + \beta + c_\delta \int_0^t \|\alpha\mu_{\alpha,\beta} + \varphi\|_*^2 \, ds \, .
\end{aligned} \tag{4.20}$$

Moreover, by using the Lipschitz continuity of π, the interpolation inequality (2.48) and (4.14) for $\mu_{\alpha,\beta}$ once more, we can write

$$- \int_{Q_t} \big(\pi(\varphi_{\alpha,\beta}) - \pi(\overline{\varphi})\big)\, \varphi \le c \int_{Q_t} |\varphi|^2 \le c \int_0^t \|\varphi\|_V \, \|\varphi\|_* \, ds$$

$$\le \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \int_0^t \|\varphi\|_*^2 \, ds$$

$$\le \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \int_0^t \|\alpha\mu_{\alpha,\beta} + \varphi\|_*^2 \, ds + c_\delta \alpha^2 \,. \tag{4.21}$$

The next term to deal with is the one involving F'. We use (4.15), the Hölder, Sobolev and Young inequalities, and the estimate (4.14) for $\mu_{\alpha,\beta}$. Thus, we have

$$- \int_{Q_t} \big(F'(\varphi_{\alpha,\beta}) - F'(\overline{\varphi})\big)\, \alpha\mu_{\alpha,\beta} \le \int_0^t \|F'(\varphi_{\alpha,\beta}) - F'(\overline{\varphi})\|_{6/5} \, \|\alpha\mu_{\alpha,\beta}\|_6 \, ds$$

$$\le c \int_0^t \|\varphi\|_V \, \alpha \|\mu_{\alpha,\beta}\|_V \, ds \le \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \, \alpha^2 \,. \tag{4.22}$$

Finally, the last integral on the right-hand side of (4.13) does not need any treatment and the preceding term can be estimated in this way:

$$\int_0^t \langle \varphi, \alpha\mu_{\alpha,\beta} + \varphi \rangle \, ds \le \delta \int_0^t \|\varphi\|_V^2 \, ds + c_\delta \int_0^t \|\alpha\mu_{\alpha,\beta} + \varphi\|_*^2 \, ds. \tag{4.23}$$

At this point, we combine (4.13) and the list (4.17)–(4.23) of estimates we have obtained. Then, we choose δ small enough, recall that $\overline{\psi} \in L^2(0,T)$ and that $\psi_{\alpha,\beta}^2$ is bounded in $L^1(0,T)$, and apply the Gronwall lemma in the form [2, Lemma A.4, p. 156]. We obtain

$$\frac{1}{2} \, \|(\alpha\mu_{\alpha,\beta} + \varphi)(t)\|_*^2 + \int_0^t \|\varphi\|_V^2 \, ds + \frac{1}{2} \, \|\sigma(t)\|_*^2 + \int_{Q_t} |\sigma|^2 \le c(\alpha + \beta)$$

for every $t \in [0,T]$. As $\|\alpha\mu_{\alpha,\beta}(t)\|_*^2 \le c\alpha$ for every $t \in [0,T]$ by (2.22), the above inequality implies

$$\|\varphi\|_{L^\infty(0,T;V^*) \cap L^2(0,T;V)} + \|\sigma\|_{L^\infty(0,T;V^*) \cap L^2(0,T;H)} \le C \left(\alpha^{1/2} + \beta^{1/2}\right). \tag{4.24}$$

Now, we take the differences of equations (4.8) and (4.11) and estimate the $L^2(0,T;V^*)$ norm of it. With the help of (2.22) and (4.15) it is straightforward to infer that

$$\|\mu\|_{L^2(0,T;V^*)} \le c\,\beta \, \|\partial_t \varphi_{\alpha,\beta}\|_{L^2(0,T;H)} + c \, \|\varphi\|_{L^2(0,T;V)}$$

$$+ c \, \|F'(\varphi_{\alpha,\beta}) - F'(\overline{\varphi})\|_{L^2(0,T;L^{6/5}(\Omega))} + \|\varphi\|_{L^2(0,T;V^*)}$$

$$\le c\,\beta^{1/2} + c \, \|\varphi\|_{L^2(0,T;V)}. \tag{4.25}$$

Hence, in view of (4.24) and (4.25) we finally obtain the estimate (2.42), where one has to read $\overline{\varphi}$, $\overline{\mu}$ and $\overline{\sigma}$ in place of φ, μ and σ, respectively, due to the change of notations within this proof. □

Remark 4.1. By going through the above proof, one sees that uniqueness for the limit problem (2.33)–(2.36) has been never used, that is, the following formulation of Theorem 2.7 has been proved: the error estimate (2.42) holds for every solution (μ, φ, σ) of the limit problem satisfying the regularity requirements (2.37)–(2.39). This implies the uniqueness for such a solution. Indeed, if $(\mu_i, \varphi_i, \sigma_i)$,

$i = 1, 2$, are two solutions of the limit problem, by writing (2.42) for both of them and using uniqueness for the solution $(\varphi_{\alpha,\beta}, \mu_{\alpha,\beta}, \sigma_{\alpha,\beta})$, one immediately derives

$$\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|\mu_1 - \mu_2\|_{L^2(0,T;V^*)} \\ + \|\sigma_1 - \sigma_2\|_{L^\infty(0,T;V^*)\cap L^2(0,T;H)} \leq C\left(\alpha^{1/2} + \beta^{1/2}\right)$$

for every $\alpha, \beta \in (0,1)$, whence $\varphi_1 = \varphi_2$, $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$. Then, by comparison in (2.34), it follows that $\xi_1 = \xi_2$, as well.

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