

Variance-Constrained Control for Uncertain Stochastic Systems With Missing Measurements

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Abstract—In this paper, we are concerned with a new control problem for uncertain discrete-time stochastic systems with missing measurements. The parameter uncertainties are allowed to be norm-bounded and enter into the state matrix. The system measurements may be unavailable (i.e., missing data) at any sample time, and the probability of the occurrence of missing data is assumed to be known. The purpose of this problem is to design an output feedback controller such that, for all admissible parameter uncertainties and all possible incomplete observations, the system state of the closed-loop system is mean square bounded, and the steady-state variance of each state is not more than the individual prescribed upper bound. We show that the addressed problem can be solved by means of algebraic matrix inequalities. The explicit expression of the desired robust controllers is derived in terms of some free parameters, which may be exploited to achieve further performance requirements. An illustrative numerical example is provided to demonstrate the usefulness and flexibility of the proposed design approach.

Index Terms—Algebraic matrix inequality, incomplete observation, missing signal, robust control, stochastic control, variance constraints.

I. INTRODUCTION

IT IS often the case in many practical stochastic control problems that, the performance requirements are naturally expressed in terms of the *upper bounds* on the steady-state variances [19]. For example, in an aiming control problem, the residence time in a given pointing direction (within circle of radius R) is inversely proportional to the maximum eigenvalue of the covariance matrix [10], and therefore the variance upper bounds are directly related to the aiming control performances. Traditional control design techniques, such as LQG and H_∞ control theories, cannot be directly applied in this kind of design problems. For instance, the LQG controllers minimize a linear quadratic performance index, which lacks guaranteed variance constraints with respect to individual system states.

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On the other hand, the covariance control theory has provided a more direct methodology for achieving the individual variance constraints than the LQG control theory, see [19] and references therein. The main idea of the covariance control theory is to choose a state covariance according to different requirements on the system performance and robustness, and then to design a controller so that the specified state covariance is assigned to the closed-loop system. The covariance control design follows these three steps [19]; 1) formulate the control design objectives as constraints in the space of covariance matrices, 2) use numerical techniques to obtain an assignable covariance that satisfies the design objectives and, 3) parameterize the set of all controllers that assign the desired covariance, and obtain a satisfactory controller from this set, according to some given criteria.

Covariance control theory has been successfully applied in dealing with the pointing control problem for NASA's ACES structure such as the Hubble Space Telescope [29]. In the presence of model parameter uncertainties, the robust variance-constrained control problem has received considerable research attention, where the upper bounds are stated as design objectives imposed on the variance values, see e.g., [4], [8], [22], [26]. Also, the dual robust filtering problems with error variance constraints have recently gained initial research interest [24], [25], [27].

It should be pointed out that, in all the literature mentioned above, it is implicitly assumed that the measurements always contain the signal, which can be used for the feedback control. However, in practical applications such as target tracking control, there may be a nonzero probability that any observation consists of noise alone if the target is absent, i.e., the measurements are not consecutive but contain missing observations. In such a case, the missing data phenomena must be taken into account when designing the control law. Unfortunately, if some of the measurements are unavailable, the standard definition of covariance in the data statistical analysis does not directly apply. The conventional optimal (robust, H_∞) control theories, which are dependent on the system output covariance, do not suit the case when there are missing measurements. Hence, it is important to develop a new stochastic control approach to tackling the missing measurement problem.

The missing measurements are caused by a variety of reasons, e.g., the high maneuverability of the tracked target, a certain failure in the measurement, intermittent sensor failures, accidental loss of some collected data, or some of the data may be jammed or coming from a high noise environment, etc., see [13]. Missing data analysis problem is well understood and has been studied extensively in the context of time series analysis and system identification (see [3], [17] and the references therein),

where the research attention has been focused on the statistical inference and modeling of missing data. In particular, there exists a great deal of literature on the study of the statistical law of the missing data. Recently, the robust filtering problem with missing data has been investigated in [15], [16], and [23].

Naturally, after the system model with missing measurement is built up, the corresponding control problem should then be considered, probably with relaxed performance requirements. For example, when tracking a highly maneuverable target, it is important to control the tracking system such that the tracking error is bounded in the mean square in the presence of missing measurements. Similar problems also exist in other engineering applications, such as the network congestion control when there are signals missing during transmission, where the missing observation is also named as dropout or intermittence, see [7], [9], and [18]. So far, to the best of the authors' knowledge, the control problem for stochastic systems with or without uncertainties under missing measurements has not been fully investigated and remains to be important and challenging.

In this paper, we are concerned with a new control problem for uncertain discrete-time stochastic systems with missing measurements. The parameter uncertainties are allowed to be norm-bounded and enter into the state matrix. The system measurements may be unavailable (i.e., missing data) at any sample time, and the probability of the occurrence of missing data is assumed to be known. The purpose of this problem is to design an output feedback controller such that, for all admissible parameter uncertainties and all possible incomplete observations, the system state of the closed-loop system is mean square bounded, and the steady-state variance of each state is not more than the individual prescribed upper bound. We show that the addressed problem can be solved in terms of algebraic matrix inequalities, and derive the explicit expression of the desired robust controllers. An illustrative numerical example is provided to demonstrate the usefulness and flexibility of the proposed design approach.

Notation: The notations in this paper are quite standard. \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript “ T ” denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P -null sets and is right continuous). $\mathcal{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P . $\text{Prob}\{\cdot\}$ means the occurrence probability of the event “ \cdot ”. For a real matrix B , B^+ denotes its Moore-Penrose inverse. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. PROBLEM FORMULATION AND ASSUMPTIONS

Consider the following uncertain discrete-time stochastic system:

$$x(k+1) = (A + \Delta A)x(k) + Bu(k) + w(k) \quad (1)$$

and the measurement equation

$$y(k) = \gamma(k)Cx(k) + v(k) \quad (2)$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ is a control input vector, $y \in \mathbb{R}^p$ is a measured output vector, and A , B and C are known constant matrices. $w(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^p$ are mutually uncorrelated zero mean Gaussian white noise sequences with respective covariances $W > 0$ and $V > 0$. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $X(0)$, and is uncorrelated with both $w(k)$ and $v(k)$. ΔA is a real-valued perturbation matrix that represents parametric uncertainty being of the following form:

$$\Delta A = MFN, \quad FF^T \leq I \quad (3)$$

and M and N are known constant matrices of appropriate dimensions which specify how the elements of the nominal matrix A are affected by the uncertain parameters in F . The uncertainties in ΔA are said to be admissible if (3) holds.

The stochastic variable $\gamma(k) \in \mathbb{R}$ is assumed to be a Bernoulli distributed white sequence taking values on 0 and 1 with

$$\text{Prob}\{\gamma(k) = 1\} = \mathcal{E}\{\gamma(k)\} := \bar{\gamma} \quad (4)$$

where $\bar{\gamma}$ is a known positive constant, and $\gamma(k) \in \mathbb{R}$ is assumed to be independent of $w(k)$, $v(k)$, and $x(0)$. Therefore, we have

$$\text{Prob}\{\gamma(k) = 0\} = 1 - \bar{\gamma} \quad (5)$$

$$\sigma_{\tilde{\gamma}}^2 := \mathcal{E}\{(\gamma(k) - \bar{\gamma})^2\} = (1 - \bar{\gamma})\bar{\gamma}. \quad (6)$$

Remark 1: The system measurement mode (2) was first introduced in [11], and has then been used in many papers (see e.g., [12], [23]) to account for the probabilistic measurement missing. The corresponding probability $\bar{\gamma}$ could be estimated through statistical tests.

Remark 2: The parameter uncertainty described in (3) has been frequently employed in dealing with robust filtering and control problems for uncertain systems, see e.g., [24], since many practical systems possess parameter uncertainties which can be either exactly modeled or overbounded by (3). It is noted that the unknown matrix F in (3) can even be allowed to be time-varying and state-dependent, i.e., $F = F(t, x(t))$, as long as (3) is satisfied.

For technical convenience, we define a new stochastic sequence

$$\tilde{\gamma}(k) := \gamma(k) - \bar{\gamma}. \quad (7)$$

It is not difficult to see that $\tilde{\gamma}(k)$ is a scalar zero mean stochastic sequence with variance

$$\sigma_{\tilde{\gamma}}^2 = (1 - \bar{\gamma})\bar{\gamma}. \quad (8)$$

When an output feedback control law

$$u(k) = Ky(k) \quad (9)$$

is applied to the system (1) and (2), the closed-loop system is governed by

$$x(k+1) = (A + \bar{\gamma}BKC + \Delta A)x(k) + \tilde{\gamma}(k)BKCx(k) + BKv(k) + w(k). \quad (10)$$

Remark 3: It should be pointed out that a stochastic variable $\tilde{\gamma}(k)$ is contained in (10), which reflects the characteristic of the missing measurement for the addressed control problem. Hence, the system (10) is actually a stochastic parameter system, for which the performance analysis issue would become involved.

The steady-state state covariance is defined by

$$X := \lim_{k \rightarrow \infty} X(k) := \lim_{k \rightarrow \infty} \mathcal{E}[x(k)x^T(k)]. \quad (11)$$

Using the statistics of the stochastic sequences $w(k)$, $v(k)$ and $\tilde{\gamma}(k)$, we can find that the state covariance $X(k)$ defined in (11) satisfies

$$\begin{aligned} X(k+1) &= (A + \bar{\gamma}BKC + \Delta A)X(k)(A + \bar{\gamma}BKC + \Delta A)^T \\ &\quad + \sigma_{\tilde{\gamma}}^2(BKC)X(k)(BKC)^T \\ &\quad + BKVK^T B^T + W. \end{aligned} \quad (12)$$

We know from [1] and [5] that, if the state of the system (10) is mean square bounded, then the steady-state covariance X defined in (11) exists and satisfies the following discrete-time modified Lyapunov equation:

$$\begin{aligned} X &= (A + \bar{\gamma}BKC + \Delta A)X(A + \bar{\gamma}BKC + \Delta A)^T \\ &\quad + \sigma_{\tilde{\gamma}}^2(BKC)X(k)(BKC)^T \\ &\quad + BKVK^T B^T + W. \end{aligned} \quad (13)$$

Remark 4: It follows from [1] and [5] that, there exists a unique symmetric positive semi-definite solution to (13) if and only if

$$\rho \left\{ (A + \bar{\gamma}BKC + \Delta A) \otimes (A + \bar{\gamma}BKC + \Delta A) + \sigma_{\tilde{\gamma}}^2(BKC) \otimes (BKC) \right\} < 1 \quad (14)$$

where ρ is the spectral radius and \otimes is the Kronecker product. Furthermore, we also know from [1], [5] that the condition (14) is equivalent to the mean square boundedness of the state of the system (10). To this end, we achieve the conclusion that, if there exists a positive definite solution to the (13), then (14) holds, and the state covariance $X(k)$ defined in (11) will converge to its steady-state value X .

The purpose of this paper is to design the controller gain K such that in the presence of missing measurement as well as all admissible perturbations ΔA , the following two performance requirements are simultaneously satisfied:

- 1) The state of the system (10) is mean-square bounded, i.e., (14) holds.
- 2) The steady-state covariance X exists and satisfies

$$[X]_{ii} \leq \alpha_i^2, \quad i = 1, 2, \dots, n. \quad (15)$$

where $[X]_{ii}$ means the steady-state variance of the i th state, and α_i^2 ($i = 1, 2, \dots, n$) denotes the prespecified steady-state variance constraint on the i th state.

In Section III, we will give a solution to the problem addressed above. Specifically, we will first characterize an upper bound on the steady-state covariance X satisfying (13), let this upper bound meet the prespecified variance constraints, and then we will parameterize all desired controller gains with

which the resulting steady-state state covariance is not more than the obtained upper bound.

III. MAIN RESULTS AND PROOFS

We will need the following lemmas to establish our main results.

Lemma 1: [21] Let a positive scalar $\varepsilon > 0$ and a positive definite matrix $Q > 0$ be such that

$$NQN^T < \varepsilon I.$$

Assume that $\Delta A = MFN$ where $FF^T \leq I$, and M and N are constant matrices with appropriate dimensions. Then

$$\begin{aligned} (A_n + \Delta A)Q(A_n + \Delta A)^T \\ \leq A_n(Q^{-1} - \varepsilon^{-1}N^T N)^{-1}A_n^T + \varepsilon MM^T \end{aligned} \quad (16)$$

holds for all admissible perturbations ΔA .

Lemma 2: [6] Let $Y \in \mathbb{R}^{m \times p}$ ($m \leq p$) and $Z \in \mathbb{R}^{m \times n}$. There exists a matrix U that satisfies simultaneously

$$Y = ZU, \quad UU^T = I$$

if and only if

$$YY^T = ZZ^T.$$

In this case, a general solution for U can be expressed as

$$U = U_Z \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} U_Y^T, \quad U \in \mathbb{R}^{(n-r_Z) \times (p-r_Z)}, \quad TT^T = I \quad (17)$$

where U_Z and U_Y come from the singular value decomposition of Z and Y , respectively

$$\begin{aligned} Z &= T_Z \begin{bmatrix} H_Z & 0 \\ 0 & 0 \end{bmatrix} U_Z^T \\ &= [T_{Z1} \quad T_{Z2}] \begin{bmatrix} H_Z & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{Z1}^T \\ U_{Z2}^T \end{bmatrix} \end{aligned} \quad (18)$$

$$\begin{aligned} Y &= T_Y \begin{bmatrix} H_Y & 0 \\ 0 & 0 \end{bmatrix} U_Y^T \\ &= [T_{Y1} \quad T_{Y2}] \begin{bmatrix} H_Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_{Y1}^T \\ U_{Y2}^T \end{bmatrix} \end{aligned} \quad (19)$$

and

$$r_Z = \text{rank}(Z), \quad T_Z = T_Y, \quad H_Z = H_Y.$$

For presentation convenience, we denote

$$\Phi := (Q^{-1} - \varepsilon^{-1}N^T N)^{-1} \quad (20)$$

$$R := \bar{\gamma}^2 C \Phi C^T + \sigma_{\tilde{\gamma}}^2 C Q C^T + V \quad (21)$$

$$\begin{aligned} \Omega &:= Q - A \Phi A^T - \varepsilon M M^T - W \\ &\quad + \bar{\gamma}^2 A \Phi C^T R^{-1} C \Phi A^T \end{aligned} \quad (22)$$

$$Y := \bar{\gamma}(I - BB^+)A\Phi C^T R^{-1/2} \quad (23)$$

$$Z := (I - BB^+)\Omega^{1/2} \quad (24)$$

where $\bar{\gamma}$ and $\sigma_{\tilde{\gamma}}$ are defined in (4) and (8), respectively.

The following theorem provides the conditions for the existence of a solution to the addressed variance-constrained control problem with missing measurements, which are derived by

using the singular value decomposition technique and the generalized inverse theory.

Theorem 1: If there exist a positive scalar $\varepsilon > 0$ and a positive definite matrix $Q > 0$ such that (25)–(27), shown at the bottom of the page hold, where Φ is defined in (20) and B^+ denotes the Moore-Penrose inverse of B , then there exists a feedback gain, K , such that the state of the system (10) is mean square bounded, and the steady-state covariance X exists and satisfies $X \leq Q$.

Proof: It follows immediately from the matrix inverse lemma that

$$(Q^{-1} - \varepsilon^{-1}N^TN)^{-1} = Q + QN^T(\varepsilon I - NQN^T)^{-1}NQ$$

and therefore the inequality in (25), shown at the bottom of the page, implies that $\Phi > 0$. Also, from the definition of Ω in (22), we know that the inequality in (26), shown at the bottom of the page, indicates $\Omega \geq 0$, and that (27), shown at the bottom of the page, can be rewritten as

$$(I - BB^+)(\Omega - \bar{\gamma}^2 A\Phi C^T R^{-1} C\Phi A^T)(I - BB^+) = 0 \quad (28)$$

or

$$YY^T = ZZ^T \quad (29)$$

where Y and Z are defined in (23) and (24), respectively.

It follows from Lemma 2 that, there exists an orthogonal matrix U (i.e., $UU^T = I$) such that

$$Y = ZU \quad (30)$$

which can be easily rearranged as

$$(I - BB^+)(\Omega^{1/2}UR^{-1/2} - \bar{\gamma}A\Phi C^T R^{-1}) = 0. \quad (31)$$

It is noticed from [2] that, the relation (31) is the *necessary and sufficient* condition for the existence of a solution K satisfying

$$BK = \Omega^{1/2}UR^{-1/2} - \bar{\gamma}A\Phi C^T R^{-1} \quad (32)$$

which is equivalent to

$$BKR^{1/2} + \bar{\gamma}A\Phi C^T R^{-1/2} = \Omega^{1/2}U. \quad (33)$$

Since U is an orthogonal matrix, it follows again from Lemma 2 that, (33) leads to

$$\begin{aligned} \Omega &= (\Omega^{1/2}U)(\Omega^{1/2}U)^T \\ &= (BKR^{1/2} + \bar{\gamma}A\Phi C^T R^{-1/2}) \\ &\quad \cdot (BKR^{1/2} + \bar{\gamma}A\Phi C^T R^{-1/2})^T \\ &= (BK)R(BK)^T + \bar{\gamma}BKC\Phi A^T \\ &\quad + \bar{\gamma}A\Phi C^T (BK)^T + \bar{\gamma}^2 A\Phi C^T R^{-1} C\Phi A^T. \end{aligned} \quad (34)$$

Substituting the definition (22) of Ω into (34), we obtain that

$$\begin{aligned} Q &= A\Phi A^T + \varepsilon MM^T + W + \bar{\gamma}A\Phi C^T (BK)^T \\ &\quad + \bar{\gamma}(BK)C\Phi A^T + (BK)R(BK)^T. \end{aligned} \quad (35)$$

Moreover, substituting (20) and (21) into (35) and performing straightforward manipulations result in

$$\begin{aligned} Q &= (A + \bar{\gamma}BKC)(Q^{-1} - \varepsilon^{-1}N^TN)^{-1}(A + \bar{\gamma}BKC)^T \\ &\quad + \varepsilon MM^T + \sigma_{\bar{\gamma}}^2 (BKC)Q(BKC)^T \\ &\quad + BKVK^T B^T + W. \end{aligned} \quad (36)$$

Next, we define

$$\begin{aligned} \Xi &:= (A + \bar{\gamma}BKC)(Q^{-1} - \varepsilon^{-1}N^TN)^{-1}(A + \bar{\gamma}BKC)^T \\ &\quad + \varepsilon MM^T - (A + \bar{\gamma}BKC + \Delta A)Q \\ &\quad \cdot (A + \bar{\gamma}BKC + \Delta A)^T \end{aligned} \quad (37)$$

and know from Lemma 1 that $\Xi \geq 0$. To this end, it follows from (36) and (37) that, there exist a positive definite matrix $Q > 0$ and a feedback gain K such that

$$\begin{aligned} Q &= (A + \bar{\gamma}BKC + \Delta A)Q(A + \bar{\gamma}BKC + \Delta A)^T \\ &\quad + \sigma_{\bar{\gamma}}^2 (BKC)Q(BKC)^T + BKVK^T B^T \\ &\quad + W + \Xi. \end{aligned} \quad (38)$$

Subsequently, we have

$$\begin{aligned} (A + \bar{\gamma}BKC + \Delta A)Q(A + \bar{\gamma}BKC + \Delta A)^T \\ - Q + \sigma_{\bar{\gamma}}^2 (BKC)Q(BKC)^T < 0 \end{aligned} \quad (39)$$

which leads to (14). As discussed in Remark 4, we know that the state of the system (10) is mean square bounded, and the steady-state covariance X exists and satisfies (13).

Finally, subtract (13) from (38) to give

$$\begin{aligned} (A + \bar{\gamma}BKC + \Delta A)(Q - X)(A + \bar{\gamma}BKC + \Delta A)^T \\ - (Q - X) + \sigma_{\bar{\gamma}}^2 (BKC)(Q - X)(BKC)^T = -\Xi \leq 0 \end{aligned} \quad (40)$$

which indicates again from Remark 4 that $Q - X \geq 0$. This completes the proof of this theorem. \blacksquare

Remark 5: It is evident from Theorem 1 that, if there exist $\varepsilon > 0$ and $Q > 0$ satisfying (25)–(27) as well as

$$[Q]_{ii} \leq \alpha_i^2, \quad i = 1, 2, \dots, n \quad (41)$$

then there exists an output feedback gain K such that: 1) the state of the closed-loop system (10) is mean square bounded, and 2) the steady-state covariance X exists and satisfies

$$[X]_{ii} \leq [Q]_{ii} \leq \alpha_i^2, \quad i = 1, 2, \dots, n.$$

Hence, the design problem of variance-constrained output feedback control with missing measurements will be feasible.

$$NQN^T < \varepsilon I \quad (25)$$

$$Q - A\Phi A^T - \varepsilon MM^T - W + \bar{\gamma}^2 A\Phi C^T R^{-1} C\Phi A^T \geq 0 \quad (26)$$

$$(I - BB^+)(Q - A\Phi A^T - \varepsilon MM^T - W)(I - BB^+) = 0 \quad (27)$$

Remark 6: The results of Theorem 1 may be conservative due to the use of the inequalities (16). However, such conservatism can be significantly reduced by appropriate choices of the parameter $\varepsilon > 0$ in a matrix norm sense. The relevant discussion and corresponding numerical algorithm can be found in [28] and references therein.

Having established the existence conditions of the desired feedback gains, our next goal is obviously to characterize the set of these gains. First, we introduce the singular value decompositions (18) and (19), where Y, Z are defined in (23) and (24), respectively.

Suppose now that the conditions of Theorem 1 are satisfied. It follows from [2] that a general solution to (32) is given by

$$K = B^+ \left(\Omega^{1/2} U R^{-1/2} - \bar{\gamma} A \Phi C^T \right) + (I - B^+ B) \Psi \quad (42)$$

where $\Psi \in \mathbb{R}^{m \times p}$ is arbitrary, U is any orthogonal matrix satisfying $Y = ZU$ and can be expressed, by Lemma 2, as

$$U = U_Z \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} U_Y^T \quad (43)$$

where

$$U \in \mathbb{R}^{(n-r_Z) \times (p-r_Z)}, \quad T T^T = I$$

and the orthogonal matrix T is arbitrary, $r_Z = \text{rank}(Z)$.

Summing up the above results, we are now in a position to characterize a set of desired controller gains satisfying (32). The following theorem can be easily proven by substituting (43) into (42).

Theorem 2: Assume that the conditions of Theorem 1 are satisfied. Then the set of all controller gains satisfying (32) is parameterized by

$$\mathcal{K} = \left\{ K : K = B^+ \left(\Omega^{1/2} U_Z \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} U_Y^T R^{-1/2} - \bar{\gamma} A \Phi C^T \right) + (I - B^+ B) \Psi \right\} \quad (44)$$

where $\varepsilon > 0$ and $Q > 0$ satisfy (25)–(27), Ω is defined in (22), U_Y and U_Z come from the singular value decomposition of Y and Z in (18) and (19), $\Psi \in \mathbb{R}^{m \times p}$ is arbitrary, and $T \in \mathbb{R}^{(n-r_Z) \times (p-r_Z)}$ is arbitrary orthogonal, $r_Z = \text{rank}(Z)$.

Remark 7: It should be mentioned that, in the design of the controller gains, if the solution set is not empty, there exists much *explicit* freedom, such as the choices of the free parameter $\Psi \in \mathbb{R}^{m \times p}$ and the orthogonal matrix $T \in \mathbb{R}^{(n-r_Z) \times (p-r_Z)}$. This freedom does not affect the mean square boundedness of the closed-loop system, but could be potentially utilized to improve other system performances such as reliability behavior. The main results of this paper can also be extended to the case when there are some deterministic parameter uncertainties on the system outputs. The reason why we discuss the case when the parameter uncertainties only enter into the state matrix is just to make our theory more understandable and to avoid unnecessarily complicated notations.

The following corollary, which can be easily obtained from Theorem 1, Theorem 2, and Remark 5, solves the variance-constrained robust output feedback control problem with missing measurements addressed in Section II.

Corollary 1: If there exist a positive scalar $\varepsilon > 0$ and a positive definite matrix $Q > 0$ such that (25)–(27) hold, and $Q > 0$ satisfies $[Q]_{ii} \leq \alpha_i^2$ ($i = 1, 2, \dots, n$), then the controller (9) with the gain matrix K determined by (44) will achieve the desired robust control performance for uncertain systems with missing measurements.

Remark 8: In practical applications, the designers would wish to construct $\varepsilon > 0$ and matrix $Q > 0$, which solve (26), subject to the constraints (25), (27) and (41), and then obtain the desired feedback gains from (44) readily. The condition (26) is actually quadratic matrix inequality (QMI), which could be dealt with by using the algorithms proposed in [14]. Also, for relatively lower-order models, the parameterization approach can be used to deal with the addressed matrix inequalities/equation, which will be demonstrated in Section IV. One of the future research topics would be to utilize the linear matrix inequality (LMI) approach for accommodating various desired performance requirements within the same LMI framework, since LMIs can be readily solved using LMI toolbox in Matlab.

IV. AN EXTENSION

The aim of this section is to show that it is straightforward to extend the main results of the previous section to the dynamic output feedback case, which would facilitate the engineering applications.

Consider the following uncertain discrete-time stochastic system

$$x_p(k+1) = (A_p + \Delta A_p)x_p(k) + B_p u(k) + w_p(k) \quad (45)$$

and the measurement equation

$$z(k) = \gamma(k) C_p x_p(k) + v_p(k) \quad (46)$$

where $x \in \mathbb{R}^{n_p}$ is a state vector, $u \in \mathbb{R}^{m_p}$ is a control input vector, $z \in \mathbb{R}^{p_p}$ is a measured output vector, and A_p, B_p and C_p are known constant matrices. $w_p(k) \in \mathbb{R}^{n_p}$ and $v_p(k) \in \mathbb{R}^{p_p}$ are mutually uncorrelated zero-mean Gaussian white noise sequences with respective covariances $W_p > 0$ and $V_p > 0$. The initial state $x_p(0)$ has the mean $\bar{x}_p(0)$ and covariance $X_p(0)$, and is uncorrelated with both $w_p(k)$ and $v_p(k)$. ΔA_p is an uncertain matrix satisfying

$$\Delta A_p = M_p F_p N_p, \quad F_p F_p^T \leq I \quad (47)$$

and M_p and N_p are known constant matrices of appropriate dimensions. Again, the stochastic variable $\gamma(k) \in \mathbb{R}$ is a Bernoulli distributed white sequence taking values on 0 and 1 with

$$\text{Prob}\{\gamma(k) = 1\} = \mathcal{E}\{\gamma(k)\} := \bar{\gamma} \quad (48)$$

where $\bar{\gamma}$ is a known positive constant, and $\gamma(k) \in \mathbb{R}$ is independent of $w_p(k)$, $v_p(k)$, and $x_p(0)$.

We consider a dynamic controller of order n_c with state-space representation

$$x_c(k+1) = A_c x_c(k) + B_c z(k), \quad (49)$$

$$u(k) = C_c x_c(k) + D_c z(k). \quad (50)$$

It is expected to design a dynamic controller (49) and (50) such that, for all parameter uncertainties and all possible incomplete observations, the system state of the closed-loop system is mean square bounded, and the steady-state variance of each state is not more than the individual prescribed upper bound.

Combining (45), (46), (49), and (50), the closed-loop system can be expressed as the following augmented system:

$$x(k+1) = (A + \gamma(k)BKC + \Delta A)x(k) + BKv(k) + w(k) \quad (51)$$

where

$$\begin{aligned} x(k) &:= \begin{bmatrix} x_p(k) \\ x_c(k) \end{bmatrix}, & y(k) &:= \begin{bmatrix} z(k) \\ x_c(k) \end{bmatrix} \\ A &:= \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix}, & B &:= \begin{bmatrix} B_p & 0 \\ 0 & I \end{bmatrix} \\ C &:= \begin{bmatrix} C_p & 0 \\ 0 & I \end{bmatrix}, & K &:= \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \\ M &:= \begin{bmatrix} M_p \\ 0 \end{bmatrix}, & F &:= F_p \\ N &:= \begin{bmatrix} N_p & 0 \end{bmatrix}, & \Delta A &:= MFN, \\ w(k) &:= \begin{bmatrix} w_p(k) \\ 0 \end{bmatrix}, & v(k) &:= \begin{bmatrix} v_p(k) \\ 0 \end{bmatrix}. \end{aligned}$$

Note that (51) has the same form as (10). Therefore, our main results can be readily extended to the dynamical output feedback controller design case for uncertain stochastic systems with missing measurements and variance constraints.

V. A NUMERICAL EXAMPLE

In this section, we demonstrate the theory developed in this note by means of a simple example.

We consider an uncertain discrete-time stochastic system (1) and (2) with parameters given by

$$\begin{aligned} A &= \begin{bmatrix} 1.5 & 0.1 \\ 0.1 & -0.5 \end{bmatrix}, & B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & M &= \begin{bmatrix} 0.1 & 0.05 \\ -0.02 & 0.8 \end{bmatrix} \\ N &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, & W &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \\ V &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \end{aligned}$$

and the probability for complete observation is assumed to be 0.9, i.e., the missing probability is 0.1.

The purpose of this example is to design the output feedback gain, K , such that for the missing measurements with given

missing law as well as all admissible perturbations ΔA , the state of the system (10) is mean square bounded, and the steady-state covariance X exists and satisfies

$$[X]_{11} \leq 2, \quad [X]_{22} \leq 1. \quad (52)$$

Now, it is supposed that the positive definite matrix Q has the form $Q = \text{diag}\{q_{11}, q_{22}\}$. Note that (27) reduces to the following:

$$q_{22} - [0.01(q_{11}^{-1} - 0.01\varepsilon^{-1})^{-1} + 0.25(q_{22}^{-1} - 0.01\varepsilon^{-1})^{-1}] - 0.6404\varepsilon - 0.1 = 0. \quad (53)$$

Thus, subject to the constraints (25), (26) and (41), we can choose

$$q_{11} = 1.9106, \quad \varepsilon = 0.8286.$$

Hence, (53) implies that

$$q_{22} = 0.8700.$$

Using the results provided in the previous section, we can obtain the important matrices as follows:

$$\begin{aligned} R &= \begin{bmatrix} 2.2560 & 0 \\ 0 & 1.2905 \end{bmatrix}, & \Omega &= \begin{bmatrix} 0.4857 & -0.0992 \\ -0.0992 & 0.1350 \end{bmatrix} \\ Y &= \begin{bmatrix} 0 & 0 \\ 0.1172 & -0.3483 \end{bmatrix}, & Z &= \begin{bmatrix} 0 & 0 \\ -0.0948 & 0.3550 \end{bmatrix}. \end{aligned}$$

Let us now consider the analytical expression (44). In this case, since

$$I - B^+B = 0$$

the matrix Ψ does not affect the solution. Hence, substituting $T = 1$ and $T = -1$ into this expression leads to the following two desired controller gains:

$$\begin{aligned} K_1 &= [-2.1774 \quad -0.1239] \\ K_2 &= [-2.9883 \quad -0.4846]. \end{aligned}$$

It is not difficult to verify that the specified mean square boundedness as well as the steady-state variance constraint are achieved.

VI. CONCLUSION

In this paper, the robust control problem has been considered for parameter uncertain discrete-time stochastic systems where there is a nonzero probability of signal being absent in the measurement. This problem has been investigated by assigning an upper bound to the steady-state covariance, and by parameterizing a set of controller gains that could achieve such an upper bound. It has been shown that, the problem is solvable if several matrix inequalities/equations have positive definite solutions. Specifically, the characterization of the desired controller

gains has been given, where the design flexibility could be utilized to achieve more expected performance requirements. An numerical example has been provided to illustrate the effectiveness of the proposed design approach.

There are still plenty of topics for future research, for example, the maximization of the missing data probability while the controlled process is still guaranteed to be stable, and handling the signal transmission process that is the output of a Markov chain with known transition probabilities. The results will appear in the near future.

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