Variance-optimal hedging for time-changed Lévy processes

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Abstract

In this paper we solve the variance-optimal hedging problem in stochastic volatility models based on time-changed Lévy processes, i.e. in the setup of Carr et al. (2003). The solution is derived using results for general affine models in the companion paper Kallsen & Pauwels (2008).

Key words: variance-optimal hedging, stochastic volatility, time changed Lévy process, Laplace transform

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1 Introduction

Stochastic volatility (SV) models with and without jumps as e.g. in [13, 2, 4] have been introduced to account for stylized facts of stock return data such as heavy tails, volatility clustering, and negative correlation between returns and changes in volatility. These models typically lead to incomplete markets where perfect hedges do not exist for most contingent claims. In case of jumps they typically remain incomplete even if finitely many derivative assets are added. In this paper we compute the variance-optimal hedge for European-style options in time-changed Lévy models where the discounted stock follows a martingale. The hedging strategy and the hedging error are expressed in terms of integral representations that allow for straightforward numerical evaluation. Their derivation relies heavily on results for general affine stochastic volatility models in the companion paper [19].

Concrete numerical results for variance-optimal hedging have also been derived in [12, 7, 14, 5, 15, 18, 1]. The first reference uses PDE methods for specific continuous stochastic volatility models. [7] considers a SV model involving jumps. A partial integro-differential

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equation is solved using finite differences in order to obtain the value process of the option and the optimal hedge. The hedging error is computed by Monte-Carlo simulation. This approach is applied to exotic contingent claims and it allows for options as hedging instruments. References [14, 5, 1] consider only processes with stationary, independent increments without stochastic volatility but their integral transform approach provides the basis of the current study. Our goal here is to produce simple formulas for the optimal hedging strategy and the hedging error which can be evaluated without implementing involved numerical schemes or computer-intensive Monte-Carlo simulations. The approach does not apply to more difficult problems as in [7], at least at this stage. But it is fast and accurate in the standard hedging problems where it can be applied. Moreover, we provide rigorous proofs under assumptions that can be directly verified in terms of model parameters. For a partial extension of our methodology to the non-martingale case we refer the reader to [20]. However, the latter is more restrictive concerning the dependence between changes in stock price and volatility. Moreover, it argues on an informal mathematical level. For a discussion of discrete variance-optimal hedging in stochastic volatility models cf. also [18].

As a by-product we obtain option pricing formulas in the models under consideration. These are mostly known from the literature, cf. [13, 22, 4, 25], at least for call and put options. However, verifyable conditions on the parameters warranting that the pricing formulas do indeed hold are typically not stated. They are provided here together with rigorous proofs.

The paper is organized as follows. The subsequent short section introduces the general setup. In Section 3 we consider stochastic volatility models involving Lévy-driven Ornstein-Uhlenbeck processes. Activity processes of square-root or Cox-Ingersoll-Ross type are discussed in Section 4. The results are illustrated numerically and compared in Section 5. The final section contains proofs.

Unexplained notation is used as in [16] and [19]. In particular, we use the dot notation for stochastic integrals, i.e. $\vartheta \bullet S_t := \int_0^t \vartheta_s dS_s$.

2 Variance-optimal hedging of European claims

We consider a European-style contingent claim with discounted payoff $H = f(S_T)$ at time $T \in \mathbb{R}_+$. By S we denote the discounted price process of the underlying which is specified further in the following two sections. In order to derive concrete results, we suppose that the payoff function $f : (0, \infty) \to \mathbb{R}$ can be written in integral form

$$f(s) = \int s^\zeta \Pi(d\zeta) \tag{2.1}$$

with some finite complex measure Π on a strip $\mathcal{S}_f := \{\zeta \in \mathbb{C} : R' \leq \mathrm{Re}(\zeta) \leq R\}$, where R', $R \in \mathbb{R}$. The measure Π is supposed to be symmetric in the sense that $\Pi(\overline{A}) = \overline{\Pi(A)}$ for $A \in \mathscr{B}(\mathbb{C})$ and $\overline{A} := \{\overline{z} \in \mathbb{C} : z \in A\}$. For most concrete payoffs this rather abstract formulation reduces to

$$f(s) = \int_{R-i\infty}^{R+i\infty} s^\zeta \varrho(\zeta) d\zeta,$$

i.e. the measure Π is concentrated on a line $R + i\mathbb{R}$ and has a Lebesgue density $\zeta \mapsto -i\varrho(\zeta)$. E.g. we have

$$\varrho(\zeta) = \frac{1}{2\pi i} \frac{K^{1-\zeta}}{\zeta(1-\zeta)}$$

for $R > 1$ in the case of a European call $f(s) = (s-K)^+$. The same function ϱ yields the payoff of a put $f(s) = (K-s)^+$ if we choose instead $R < 0$. The kernels ϱ or integral representations (2.1) for many more payoffs can be found in [14].

The goal of variance-optimal hedging is to minimize the expected squared hedging error

$$E\Big[(v + \vartheta \bullet S_T - H)^2\Big] \tag{2.2}$$

over all initial endowments $v \in \mathbb{R}$ and all *admissible* trading strategies ϑ, i.e. all

$$\vartheta \in \Theta := \big\{\vartheta \text{ predictable process} : E[|\vartheta|^2 \bullet \langle S, S\rangle_T] < \infty\big\}.$$

The hedging error

$$J_0 := E\Big[(v^* + \vartheta^* \bullet S_T - H)^2\Big]$$

of the *variance-optimal initial capital* v^* and *strategy* ϑ^* is called *mimimal hedging error*. The aim of this paper is to determine these objects in Lévy-based stochastic volatility models.

From general theory it is known that $v^* = E[H]$, i.e. the variance-optimal initial capital coincides with the value of the claim if the underlying probability measure is the market's pricing measure, cf. e.g. [11, 26]. Therefore option pricing formulas are recovered as a by-product. If the initial endowment v is fixed rather than part of the optimization, the optimizer ϑ^* of (2.2) stays the same in the present martingale case. The expected squared hedging error, however, increases by $(v - v^*)^2$.

3 Ornstein-Uhlenbeck-type activity process

In this section we solve the hedging problem in the integrated Ornstein-Uhlenbeck (OU) time change model proposed in [4]. In order to allow for a more flexible dependence structure between changes in asset prices and volatility we consider the slightly extended version introduced in [17]. The discounted asset price process in the integrated OU time change model is given by $S = S_0 \exp(Z)$, where the *return process* Z is specified as

$$\begin{aligned} Z_t &= X_{Y_t} + \widetilde{X}_{z_t} - \delta t, \\ dY_t &= y_{t-} dt, \\ dy_t &= \kappa(\eta - y_{t-})dt + dz_t. \end{aligned}$$

Here $\kappa > 0$, $\eta \geq 0$, and δ are constant parameters. $X, \widetilde{X}, z$ denote three independent Lévy processes. In order to obtain a positive *activity process* Y we suppose that $y_0 > 0$ and z is increasing. More specifically, we assume that z equals the sum of its positive jumps. Integration by parts yields that

$$y_t = e^{-\kappa t} y_0 + (1 - e^{-\kappa t})\eta + \int_0^t e^{-\kappa(t-s)} dz_s.$$

Since X is a Lévy process, we have

$$E[e^{\zeta X_t}] = \exp(\psi^X(\zeta)t), \quad t \geq 0$$

for $\zeta \in i\mathbb{R} \subset \mathbb{C}$ and some continuous function $\psi^X : i\mathbb{R} \to \mathbb{C}$ with $\psi^X(0) = 0$. This *Lévy exponent* is of the form

$$\psi^X(\zeta) = b^X \zeta + \frac{1}{2} c^X \zeta^2 + \int \left(e^{\zeta x} - 1 - \zeta h(x)\right) F^X(dx),$$

where (b^X, c^X, F^X) denotes the *Lévy-Khintchine triplet* of X relative to some truncation function $h : \mathbb{R} \to \mathbb{R}$ as e.g. $h(x) = x 1_{\{|x| \leq 1\}}$. Analogously, we define Lévy exponents $\psi^{\widetilde{X}}, \psi^z$ and Lévy-Khintchine triplets $(b^{\widetilde{X}}, c^{\widetilde{X}}, F^{\widetilde{X}}), (b^z, c^z, F^z)$.

Remark 3.1 If one chooses $X_t = W_t + \mu t$ as standard Brownian motion with drift μ, moreover $\widetilde{X}_t = \varrho t$ and $\eta = 0$, then the bivariate process (Z, y) coincides in law with the solution to

$$\begin{aligned} dZ_t &= (\mu y_{t-} - \delta)\, dt + \sqrt{y_{t-}} dW_t + \varrho dz_t, \\ dy_t &= -\kappa y_{t-} dt + dz_t, \end{aligned}$$

cf. [17, Section 4.3]. This is the so-called BNS model suggested in [2].

In order to solve the hedging problem in this setup we need some moment conditions. We use the notation from the previous section.

Assumption 3.2 We suppose that

1. for some $\varepsilon > 0$
   (a) $\psi^X, \psi^{\widetilde{X}} : i\mathbb{R} \to \mathbb{C}$ have analytic extensions to the strip
   $$\{\zeta \in \mathbb{C} : (2R' \wedge 0) - \varepsilon < \mathrm{Re}(\zeta) < (2R \vee 2) + \varepsilon\},$$
   which we denote again by $\psi^X, \psi^{\widetilde{X}}$, respectively,
   (b) $\psi^z : i\mathbb{R} \to \mathbb{C}$ has an analytic extension, again denoted by ψ^z, to the strip
   $$\{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) < 2M_1 + M_2 + \varepsilon\},$$
   where
   $$\begin{aligned} M_1 &= \max\left\{\frac{1 - e^{-\kappa T}}{\kappa} \psi^X(2R' \wedge 0), \frac{1 - e^{-\kappa T}}{\kappa} \psi^X(2R \vee 2)\right\}, \\ M_2 &= \max\left\{\psi^{\widetilde{X}}(2R' \wedge 0), \psi^{\widetilde{X}}(2R \vee 2)\right\}, \end{aligned}$$

2. $\delta = \psi^z(\psi^{\widetilde{X}}(1))$ and $\psi^X(1) = 0$,

3. $\delta \neq \frac{1}{2}\psi^z(\psi^{\widetilde{X}}(2))$ or $\psi^X(2) \neq 0$.

The second assumption means that the discounted price process S is a martingale. The third one is made to exclude the degenerate case that S is constant. The conditions in the first part warrant that all expressions in Theorems 3.3, 3.4 are defined and that they really correspond to the optimal hedge. We can now state the main results of this section.

Theorem 3.3 (Optimal hedge) *Suppose that Assumption 3.2 holds. The variance-optimal initial capital v^* and the variance-optimal hedging strategy ϑ^* are given by*

$$v^* = \int V(\zeta)_0 \Pi(d\zeta), \tag{3.1}$$

$$\vartheta^*_t = \int \frac{V(\zeta)_{t-}}{S_{t-}} \frac{\kappa_0(t,\zeta) + \kappa_1(\zeta) y_{t-}}{\delta_0 + \delta_1 y_{t-}} \Pi(d\zeta), \tag{3.2}$$

where

$$\Psi_1(t, u_1, u_2) := e^{-\kappa t} u_1 + \frac{1 - e^{-\kappa t}}{\kappa} \psi^X(u_2), \tag{3.3}$$

$$q(t, u_1, u_2) := \Psi_1(t, u_1, u_2) + \psi^{\widetilde{X}}(u_2), \tag{3.4}$$

$$\Psi_0(t, u_1, u_2) := \eta(1 - e^{-\kappa t}) u_1 + \eta \psi^X(u_2) \left(t - \frac{1 - e^{-\kappa t}}{\kappa} \right) - \delta t u_2 + \int_0^t \psi^z(q(s, u_1, u_2)) ds, \tag{3.5}$$

$$V(\zeta)_t := S_t^\zeta \exp\big(\Psi_0(T-t, 0, \zeta) + \Psi_1(T-t, 0, \zeta) y_t\big),$$

$$\kappa_0(t,\zeta) := \psi^z\Big(\Psi_1(T-t,0,\zeta) + \psi^{\widetilde{X}}(\zeta+1)\Big) - \psi^z\Big(\Psi_1(T-t,0,\zeta) + \psi^{\widetilde{X}}(\zeta)\Big) - \psi^z\big(\psi^{\widetilde{X}}(1)\big),$$

$$\kappa_1(\zeta) := \psi^X(\zeta+1) - \psi^X(\zeta),$$

$$\delta_0 := \psi^z(\psi^{\widetilde{X}}(2)) - 2\psi^z(\psi^{\widetilde{X}}(1)),$$

$$\delta_1 := \psi^X(2).$$

A similar representation yields the hedging eror.

Theorem 3.4 (Hedging error) *Under Assumption 3.2 the minimal hedging error is given by*

$$J_0 = \begin{cases} \int\int\int_0^T J_1(t,\zeta_1,\zeta_2) dt \Pi(d\zeta_1)\Pi(d\zeta_2), & \textit{if } \delta_0 \neq 0,\ \delta_1 \neq 0, \\ \int\int\int_0^T J_2(t,\zeta_1,\zeta_2) dt \Pi(d\zeta_1)\Pi(d\zeta_2), & \textit{if } \delta_0 = 0, \\ \int\int\int_0^T J_3(t,\zeta_1,\zeta_2) dt \Pi(d\zeta_1)\Pi(d\zeta_2), & \textit{if } \delta_1 = 0. \end{cases}$$

The integrals relative to Π have to be understood in the sense of the Cauchy principal value (cf. the following remark). The integrands $J_k : [0,T] \times \mathcal{S}_f{}^2 \to \mathbb{C}$, $k = 1,2,3$ in these *expressions are defined as*

$$
\begin{aligned}
& J_1(t,\zeta_1,\zeta_2) \\
= \; & e^{\xi_0} S_0^{\zeta_1+\zeta_2} \Bigg(\exp\big(\Psi_0(t,\xi_1,\zeta_1+\zeta_2) + \Psi_1(t,\xi_1,\zeta_1+\zeta_2) y_0\big) \Big(\frac{\eta_2}{\delta_1} \big(D_2\Psi_0(t,\xi_1,\zeta_1+\zeta_2) \\
& + e^{-\kappa t} y_0 \big) + \frac{\eta_1\delta_1 - \eta_2\delta_0}{\delta_1^2} \Big) + \frac{\eta_0\delta_1^2 - \eta_1\delta_0\delta_1 + \eta_2\delta_0^2}{\delta_1^3} e^{-\frac{\delta_0}{\delta_1}\xi_1} \times \\
& \times \int_0^1 \left(\frac{\delta_1}{\delta_0} + \xi_1 s \right) e^{\frac{\delta_0}{\delta_1}\xi_1 s + \Psi_0\left(t, \frac{\delta_1}{\delta_0}\log(s) + \xi_1 s, \zeta_1+\zeta_2\right) + \Psi_1\left(t, \frac{\delta_1}{\delta_0}\log(s) + \xi_1 s, \zeta_1+\zeta_2\right) y_0} ds \Bigg), \\
& J_2(t,\zeta_1,\zeta_2) \\
= \; & \frac{e^{\xi_0} S_0^{\zeta_1+\zeta_2}}{\delta_1} \exp\Big(\Psi_0(t,\xi_1,\zeta_1+\zeta_2) + \Psi_1(t,\xi_1,\zeta_1+\zeta_2) y_0 \Big) \times \\
& \times \Big(\eta_1 + \big(D_2\Psi_0(t,\xi_1,\zeta_1+\zeta_2) + e^{-\kappa t} y_0 \big) \eta_2 \Big), \\
& J_3(t,\zeta_1,\zeta_2) \\
= \; & \frac{S_0^{\zeta_1+\zeta_2}}{\delta_0} \exp\Big((\psi_0(0,\zeta_1) + \psi_0(0,\zeta_2))(T-t) + \psi_0(0,\zeta_1+\zeta_2) t \Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_0(u_1,u_2) &:= \kappa\eta u_1 - \delta u_2 + \psi^z\left(u_1 + \psi^{\widetilde{X}}(u_2)\right), && (3.6) \\
\psi_1(u_1,u_2) &:= -\kappa u_1 + \psi^X(u_2), && (3.7) \\
\alpha_j &= \alpha_j(t,\zeta_1,\zeta_2) \\
&:= \psi_j(\xi_1(t,\zeta_1,\zeta_2), \zeta_1+\zeta_2) - \psi_j(\Psi_1(T-t,0,\zeta_1),\zeta_1) \\
&\quad - \psi_j(\Psi_1(T-t,0,\zeta_2),\zeta_2), \\
\eta_0 &= \eta_0(t,\zeta_1,\zeta_2) := \delta_0\alpha_0(t,\zeta_1,\zeta_2) - \kappa_0(t,\zeta_1)\kappa_0(t,\zeta_2), \\
\eta_1 &= \eta_1(t,\zeta_1,\zeta_2) \\
&:= \delta_0\alpha_1(t,\zeta_1,\zeta_2) + \delta_1\alpha_0(t,\zeta_1,\zeta_2) - \kappa_1(\zeta_1)\kappa_0(t,\zeta_2) \\
&\quad - \kappa_1(\zeta_2)\kappa_0(t,\zeta_1), \\
\eta_2 &= \eta_2(t,\zeta_1,\zeta_2) := \delta_1\alpha_1(t,\zeta_1,\zeta_2) - \kappa_1(\zeta_1)\kappa_1(\zeta_2), \\
\xi_j &= \xi_j(t,\zeta_1,\zeta_2) := \Psi_j(T-t,0,\zeta_1) + \Psi_j(T-t,0,\zeta_2), \quad j = 0,1, \\
D_2\Psi_0(t,u_1,u_2) &= \int_0^t (\psi^z)'\left(q(s,u_1,u_2)\right) e^{-\kappa s} ds + \eta\left(1 - e^{-\kappa t}\right). && (3.8)
\end{aligned}
$$

For ease of notation we dropped the arguments of some functions in the formulae above. The mappings $\eta_0, \eta_1, \eta_2, \alpha_0, \alpha_1, \xi_0, \xi_1$ *are defined on* $[0,T] \times {\mathcal{S}_f}^2$.

Remark 3.5 The integrals in the previous theorem are to be understood in the sense that

$$
J_0 = \lim_{c \uparrow \infty} \int_{\mathcal{S}_f^c} \int_{\mathcal{S}_f^c} \int_0^T J_k(t,\zeta_1,\zeta_2) dt \Pi(d\zeta_1) \Pi(d\zeta_2)
$$

where

$$\mathcal{S}_f^c := \{\zeta \in \mathbb{C} : R' \leq \mathrm{Re}(\zeta) \leq R, |\mathrm{Im}(\zeta)| \leq c\}.$$

The integrals in (3.1, 3.2) and the triple integrals in (3.6) cannot be avoided and must be evaluated numerically. This is done in Section 5 for a concrete model. However, the integrals in (3.5) and (3.8) can be expressed analytically for the subordinators z that are considered in the literature.

Proposition 3.6 (Gamma-OU process) *Suppose that Assumption 3.2(1a) holds. Moreover, let* $\eta = 0$ *and*

$$\psi^z(\zeta) = \frac{\kappa a \zeta}{b - \zeta}$$

for some $a, b > 0$ *such that*

$$2M_1 + M_2 < b$$

i.e. y *is a Gamma-OU process in the sense of [2, 25]. Then Assumption 3.2(1b) holds for sufficiently small* $\varepsilon > 0$*. The integral in (3.5) is of the form*

$$\int_0^t \psi^z(q(s, u_1, u_2))ds = \begin{cases} \frac{a}{b - k_2(u_2)} \left(b \log\varphi(t, u_1, u_2) + \kappa k_2(u_2) t\right) & \text{if } b \neq k_2(u_2), \\ -a \left(\frac{b}{k_1(u_1, u_2)} \left(e^{\kappa t} - 1\right) + \kappa t\right) & \text{if } b = k_2(u_2), \end{cases} \tag{3.9}$$

where

$$\begin{aligned} k_1(u_1, u_2) &:= u_1 - \frac{\psi^X(u_2)}{\kappa}, \\ k_2(u_2) &:= \frac{\psi^X(u_2)}{\kappa} + \psi^{\widetilde{X}}(u_2). \end{aligned}$$

The mapping $t \mapsto \log\varphi(t, u_1, u_2)$ *denotes the distinguished logarithm of*

$$t \mapsto \varphi(t, u_1, u_2) := \frac{b - q(t, u_1, u_2)}{b - q(0, u_1, u_2)}$$

in the sense of [24, Lemma 7.6], i.e. $\log\varphi$ *is the unique continuous function* $[0, T] \to \mathbb{C}$ *with* $\log\varphi(0, u_1, u_2) = 0$ *and* $\exp(\log\varphi(t, u_1, u_2)) = \varphi(t, u_1, u_2)$*. The derivative* $D_2\Psi_0$ *needed in Thereom 3.4 can be written as*

$$D_2\Psi_0(t, u_1, u_2) = \frac{ab\left(1 - e^{-\kappa t}\right)}{\left(b - u_1 - \psi^{\widetilde{X}}(u_2)\right)\left(b - \Psi_1(t, u_1, u_2) - \psi^{\widetilde{X}}(u_2)\right)}. \tag{3.10}$$

PROOF. Since ψ^z can be extended to an analytic function on $\{v \in \mathbb{C} : \mathrm{Re}(v) < b\}$, we obtain Assumption 3.2(1b) for sufficiently small ε. (3.9) is shown by differentiation of its right-hand side. (3.10) follows by differentiation from (3.9). □

A similar result holds for another family of subordinators.

Proposition 3.7 (IG-OU process) *Suppose that $\eta = 0$ and*

$$\psi^z(\zeta) = \frac{\kappa a \zeta}{\sqrt{b^2 - 2\zeta}}$$

for some $a, b > 0$ such that

$$2M_1 + M_2 < \frac{1}{2}b^2 \tag{3.11}$$

i.e. y is an inverse Gaussian OU process in the sense of [2, 25]. Then the integral in (3.5) is of the form

$$\int_0^t \psi^z(q(s, u_1, u_2))ds = \begin{cases} C_1(t, u_1, u_2) & \text{if } b^2 \neq 2k_2(u_2) \text{ and } k_1(u_1, u_2) \neq 0, \\ C_2(t, u_1, u_2) & \text{if } b^2 = 2k_2(u_2), \\ C_3(t, u_1, u_2) & \text{if } b^2 \neq 2k_2(u_2) \text{ and } k_1(u_1, u_2) = 0, \end{cases} \tag{3.12}$$

where k_1, k_2 are defined as in Proposition 3.6,

$$\begin{aligned} \alpha(t, u_1, u_2) &:= \sqrt{b^2 - 2q(t, u_1, u_2)}, \\ \beta(u_2) &:= \sqrt{b^2 - 2k_2(u_2)}, \end{aligned}$$

$t \mapsto \log\varphi(t, u_1, u_2)$ denotes the distinguished logarithm of

$$\begin{aligned} \varphi(t, u_1, u_2) &:= \frac{\beta(u_2)(\beta(u_2) - \alpha(0, u_1, u_2)) - k_1(u_1, u_2)}{k_1^2(u_1, u_2)} \times \\ &\quad \times \left[\beta(u_2)(\alpha(t, u_1, u_2) + \beta(u_2)) - e^{-\kappa t}k_1(u_1, u_2)\right], \end{aligned}$$

and

$$\begin{aligned} C_1(t, u_1, u_2) &:= a\left(\frac{k_2(u_2)}{\beta(u_2)}(\log\varphi(t, u_1, u_2) + \kappa t) + \alpha(t, u_1, u_2) - \alpha(0, u_1, u_2)\right), \\ C_2(t, u_1, u_2) &:= \frac{2a}{\alpha(0, u_1, u_2)}\left(k_1(u_1, u_2)\left(1 - e^{-\frac{1}{2}\kappa t}\right) + k_2(u_2)\left(e^{\frac{1}{2}\kappa t} - 1\right)\right), \\ C_3(t, u_1, u_2) &:= \frac{\kappa a k_2(u_2) t}{\beta(u_2)}. \end{aligned}$$

The derivative $D_2\Psi_0$ needed in Thereom 3.4 can be written as

$$D_2\Psi_0(t, u_1, u_2) = \begin{cases} \frac{\psi^z(q(0,u_1,u_2)) - \psi^z(q(t,u_1,u_2))}{\kappa k_1(u_1,u_2)} & \text{if } k_1(u_1, u_2) \neq 0, \\ \frac{a(b^2 - k_2(u_2))\left(1 - e^{-\kappa t}\right)}{\beta^3(u_2)} & \text{if } k_1(u_1, u_2) = 0. \end{cases}$$

PROOF. ψ^z can be analytically extended to $\{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) < \frac{1}{2}b^2\}$. This yields Assumption 3.2(1b) for sufficiently small ε. Condition (3.11) warrants that $b^2 - 2q(t, u_1, u_2) \neq 0$ for $\mathrm{Re}(u_1) < 2M_1$, which implies that the square roots are well defined. Differentiation of the right-hand side yields (3.12). In the proof of Theorem 3.3 it is shown that $(u_1, u_2) \mapsto \Psi_0(t, u_1, u_2)$ is analytic. Differentiation yields the expression for $k_1(u_1, u_2) \neq 0$.

The formula for $k_1(u_1, u_2) = 0$ is obtained by considering the limit of the first expression for $k_1(u_1, u_2) \to 0$. $\square$

In addition to the subordinator z, the Lévy processes $X, \widetilde{X}$ must be specified in order apply the results in practice. We discuss here the two examples.

Example 3.8 (NIG-Gamma-OU model) We consider the general model of this section with the subordinator z from Proposition 3.6, $\delta = 0$, $\widetilde{X} = 0$ and a normal inverse Gaussian Lévy process X, i.e. with

$$\psi^X(\zeta) = \mu\zeta + \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + \zeta)^2} \tag{3.13}$$

for some parameters $\alpha > 0, \beta \in (-\alpha, \alpha), \mu \in \mathbb{R}$. Suppose that $R = R'$ in Section 2. In order for all conditions in Assumption 3.2 to be satisfied, we require

$$\begin{aligned} -\alpha - \beta &< 2R, \\ 2R \vee 2 &< \alpha - \beta, \\ \psi^X(2) \vee \psi^X(2R) &< \frac{b\kappa}{2(1 - e^{-\kappa T})}, \qquad (3.14) \\ \mu &= \sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2}. \qquad (3.15) \end{aligned}$$

Example 3.9 (BNS model) If we choose Brownian motion instead of the NIG Lévy process for X, we end up with the model from Remark 3.1. More specifically, let $\delta = 0$,

$$\psi^X(\zeta) = \frac{\zeta^2 - \zeta}{2},$$

$\widetilde{X} = 0$ and z as in Proposition 3.6. Again we suppose that $R = R'$ in Section 2. In order for all conditions in Assumption 3.2 to be satisfied, we require (3.14).

4 Square-root type activity process

In the so-called Heston [13] model the activity process is chosen as a square-root process, which is also known as Cox-Ingersoll-Ross process in interest rate theory. Together with a time-changed Lévy process as in the previous section, we end up with another class of stochastic volatility models from [4]. In this setup the discounted stock price $S = S_0 \exp(Z)$ is of the form

$$\begin{aligned} Z_t &= X_{Y_t} + \lambda(y_t - y_0) - \delta t, \qquad (4.1) \\ dY_t &= y_t dt, \\ dy_t &= \kappa(\eta - y_t)dt + \sigma\sqrt{y_t} dW_t. \end{aligned}$$

Here $\kappa, \eta, \sigma > 0$, $\lambda \in \mathbb{R}$ are constants and W a standard Wiener process. X denotes a Lévy process which is independent of W. As in the previous section we write ψ^X for the Lévy exponent of X.

Remark 4.1 If we choose $X_t = B_t + \mu t$ as Brownian motion with drift, we recover the dynamics of the Heston model up to a rescaling of the activity process y. More specifically, (Z, y) coincides in law with the solution to

$$
\begin{aligned}
dZ_t &= (\lambda\kappa\eta - \delta + (\mu - \kappa)y_t)dt + \sqrt{y_t}d(\widetilde{W}_t + \lambda\sigma W_t), \\
dy_t &= \kappa(\eta - y_t)dt + \sigma\sqrt{y_t}dW_t,
\end{aligned}
\tag{4.2}
$$

where $W, \widetilde{W}$ denote independent standard Wiener processes, cf. e.g. [17, Section 4.2].

In order to solve the hedging problem from Section 2 in this setup, we need some regularity conditions.

Assumption 4.2 Let

$$
\begin{aligned}
l(u_1, u_2) &:= \sigma^2 u_1 + \lambda\sigma^2 u_2 - \kappa, \\
\gamma(\zeta) &:= \sqrt{\kappa^2 - 2\sigma^2\psi^X(\zeta)}, \\
g(\zeta) &:= \frac{1}{2\sigma^2}\left(\gamma(\zeta)\coth\left(\frac{1}{2}\gamma(\zeta)T\right) + \kappa\right) - \lambda\zeta,
\end{aligned}
\tag{4.3}
$$

$$
\Psi_1(t, u_1, u_2) := \frac{\gamma^2(u_2) - l^2(u_1, u_2)}{\sigma^2\left(l(u_1, u_2) - \gamma(u_2)\coth(\frac{1}{2}\gamma(u_2)t)\right)} + u_1. \tag{4.4}
$$

We assume that

1. for some $\varepsilon > 0$ we have that

   (a) $\psi^X : i\mathbb{R} \to \mathbb{C}$ has an analytic extension to the strip

   $$\{u \in \mathbb{C} : (2R' \wedge 0) - \varepsilon < \mathrm{Re}(u) < (2R \vee 2) + \varepsilon\},$$

   which we denote again by ψ^X,

   (b) $M_2 < \kappa^2/(2\sigma^2)$ and $M_1 < M_3$ for

   $$
   \begin{aligned}
   M_1 &= \max\left\{\Psi_1(T, 0, 2R' \vee 0), \Psi_1(T, 0, 2R \wedge 2)\right\}, \\
   M_2 &= \max\left\{\psi^X(2R' \wedge 0), \psi^X(2R \vee 2)\right\}, \\
   M_3 &= \min\left\{g(2R' \wedge 0, 0, T), g(2R \vee 2, 0, T)\right\},
   \end{aligned}
   $$

2. $$T < \frac{1}{\gamma(2R \vee 2)}\log\left(\frac{l(0, 2R \vee 2) + \gamma(2R \vee 2)}{l(0, 2R \vee 2) - \gamma(2R \vee 2)}\right) \text{ if } l(0, 2R \vee 2) - \gamma(2R \vee 2) > 0,$$

3. $\delta = \lambda\kappa\eta$ and $\psi^X(1) = \lambda\left(\kappa - \frac{1}{2}\lambda\sigma^2\right)$,

4. $\psi^X(2) \neq 2\lambda\kappa - 2\lambda^2\sigma^2$.

Assumption 3 means that the discounted price process S is a martingale. The last one excludes the degenerate case that S is constant. The remaining conditions warrant that all expressions in Theorems 4.3, 4.4 are defined and that they really correspond to the optimal hedge. We can now state the main results of this section.

Theorem 4.3 *Suppose that Assumption 4.2 holds. The variance-optimal initial capital v^* and the variance-optimal hedging strategy ϑ^* are given by*

$$\begin{aligned} v^* &= \int V(\zeta)_0 \Pi(d\zeta), & (4.5)\\ \vartheta_t^* &= \int \frac{V(\zeta)_{t-}}{S_{t-}} \frac{\kappa_1(t,\zeta)}{\delta_1} \Pi(d\zeta), & (4.6) \end{aligned}$$

where

$$\varphi(t, u_1, u_2) := \frac{2\gamma(u_2)}{l(u_1, u_2)(1 - e^{\gamma(u_2)t}) + \gamma(u_2)(1 + e^{\gamma(u_2)t})}, \tag{4.7}$$

$\log\varphi$ *denotes the distinguished logarithm of* $t \mapsto \varphi(t, u_1, u_2)$*, i.e.*

$$\begin{aligned} \log\varphi(t, u_1, u_2) &= \int_0^t \frac{D_1\varphi(s, u_1, u_2)}{\varphi(s, u_1, u_2)} ds \\ &= \int_0^t \frac{\gamma(u_2)\left(l(u_1, u_2) - \gamma(u_2)\right) e^{\gamma(u_2)t}}{l(u_1, u_2)(1 - e^{\gamma(u_2)t}) + \gamma(u_2)(1 + e^{\gamma(u_2)t})} ds, & (4.8)\\ \Psi_0(t, u_1, u_2) &:= \frac{\kappa\eta}{\sigma^2}\left(2\log\varphi(t, u_1, u_2) + (\gamma(u_2) - l(u_1, u_2))\, t\right) + \kappa\eta u_1 t, & (4.9)\\ V(\zeta)_t &:= S_t^\zeta \exp(\Psi_0(T - t, 0, \zeta) + \Psi_1(T - t, 0, \zeta) y_t), \\ \kappa_1(t, \zeta) &:= \lambda\sigma^2\left(\Psi_1(T - t, 0, \zeta) + \lambda\zeta\right) + \psi^X(\zeta + 1) - \psi^X(\zeta) - \psi^X(1), & (4.10)\\ \delta_1 &:= \psi^X(2) - 2\psi^X(1) + \lambda^2\sigma^2. \end{aligned}$$

A similar representation yields the hedging eror.

Theorem 4.4 *Using the assumption and notation above the minimal hedging error is given by*

$$J_0 = \int\int\int_0^T J(t, \zeta_1, \zeta_2) dt \Pi(d\zeta_1)\Pi(d\zeta_2) \tag{4.11}$$

with

$$\begin{aligned} a(t, \zeta_1, \zeta_2) &:= \sigma^2\left(\Psi_1(T - t, 0, \zeta_1) + \lambda\zeta_1\right)\left(\Psi_1(T - t, 0, \zeta_2) + \lambda\zeta_2\right) \\ &\quad + \psi^X(\zeta_1 + \zeta_2) - \psi^X(\zeta_1) - \psi^X(\zeta_2), \\ b = b(t, \zeta_1, \zeta_2) &:= \delta_1 a(t, \zeta_1, \zeta_2) - \kappa_1(t, \zeta_1)\kappa_1(t, \zeta_2), \\ \xi_j = \xi_j(t, \zeta_1, \zeta_2) &:= \Psi_j(T - t, 0, \zeta_1) + \Psi_j(T - t, 0, \zeta_2), \quad j = 0, 1, \end{aligned}$$

$$
\begin{aligned}
D_2\Psi_0(t,u_1,u_2) &= \frac{2\kappa\eta\sinh\left(\frac{1}{2}\gamma(u_2)t\right)}{\gamma(u_2)\cosh\left(\frac{1}{2}\gamma(u_2)t\right) - l(u_1,u_2)\sinh\left(\frac{1}{2}\gamma(u_2)t\right)},\\
D_2\Psi_1(t,u_1,u_2) &= \frac{\gamma^2(u_2)}{\left(\gamma(u_2)\cosh(\frac{1}{2}\gamma(u_2)t) - l(u_1,u_2)\sinh(\frac{1}{2}\gamma(u_2)t)\right)^2},\\
J(t,\zeta_1,\zeta_2) &:= \frac{b}{\delta_1}e^{\xi_0}S_0^{\zeta_1+\zeta_2}\exp(\Psi_0(t,\xi_1,\zeta_1+\zeta_2)+\Psi_1(t,\xi_1,\zeta_1+\zeta_2)y_0)\times\\
&\quad\times\left(D_2\Psi_0(t,\xi_1,\zeta_1+\zeta_2)+D_2\Psi_1(t,\xi_1,\zeta_1+\zeta_2)y_0\right). \qquad (4.12)
\end{aligned}
$$

For ease of notation we have dropped the arguments of b,ξ_1,ξ_2 in formula (4.12). As in Theorem 3.4 the integrals relative to Π are to be understood in the sense of Cauchy principal values (cf. Remark 3.5).

Similarly to the previous section we obtain explicit results up to single integrals in (4.5, 4.6) and triple integrals in (4.11). A numerical example is provided in Section 5. The integral in 4.8 is just an alternative representation of the distinguished logarithm of φ, which coincides with the main branch up to a multiple of $2\pi i$.

In the following examples we consider particular Lévy processes X.

Example 4.5 (NIG-CIR model) Let X above denote a normal inverse Gaussian Lévy process with Lévy exponent

$$\psi^X(\zeta) = \mu\zeta + \sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+\zeta)^2}$$

for some parameters $\alpha > 0, \beta \in (-\alpha,\alpha), \mu \in \mathbb{R}$. Moreover, let $\lambda = 0$ and $\delta = 0$. Suppose that $R = R'$ in Section 2. In order for all conditions in Assumption 4.2 to be satisfied, we require

$$
\begin{aligned}
-\alpha-\beta &< 2R,\\
2R\vee 2 &< \alpha-\beta,\\
\psi^X(2)\vee\psi^X(2R) &< \frac{\kappa^2}{2\sigma^2},\\
\mu &= \sqrt{\alpha^2-(\beta+1)^2} - \sqrt{\alpha^2-\beta^2}.
\end{aligned}
$$

Example 4.6 (Heston model) We obtain the dynamics of the Heston model in Remark 4.1 if X is Brownian motion with drift or, more precisely,

$$\psi^X(\zeta) = \frac{1}{2}\left(\zeta^2 - (1+2\lambda\kappa-\lambda^2\sigma^2)\zeta\right).$$

Again we suppose that $R = R'$ in Section 2. Assumption 4.2(1a,3,4) hold automatically. Conditions 1b and 2 depend on the choice of parameters. $M_1 < M_3$ and Assumption 4.2(2) hold generally if $\lambda = 0$.

A slight extension of the above results allows to incorporate a popular class of stochastic volatility models.

Remark 4.7 (Bates model) The setup in this section can be modified to include the Bates [3] model of the form

$$\begin{aligned} dZ_t &= (\lambda\kappa\eta + (\mu-\kappa)y_t)dt + \sqrt{y_t}d(\widetilde{W}_t + \lambda\sigma W_t) + dL_t, \\ dy_t &= \kappa(\eta - y_t)dt + \sigma\sqrt{y_t}dW_t, \end{aligned}$$

where $W, \widetilde{W}$ denote independent standard Wiener processes, L a Lévy process independent of $W, \widetilde{W}$, and $\kappa, \eta, \sigma, \lambda, \mu$ constants as in Remark 4.1. To this end, we just have to allow for an arbitrary Lévy process L in (4.1) instead of the $-\delta t$-term, i.e. we consider

$$\begin{aligned} Z_t &= X_{Y_t} + \lambda(y_t - y_0) + L_t, \\ dY_t &= y_t dt, \\ dy_t &= \kappa(\eta - y_t)dt + \sigma\sqrt{y_t}dW_t \end{aligned}$$

with some Brownian motion with drift X as in Remark 4.1, more specifically with characteristic exponent $\psi^X(\zeta) = \mu\zeta + \frac{\zeta^2}{2}$. We assume that 1(a) in Assumption 4.2 holds for the Lévy exponent ψ^L of L as well. Moreover, $\delta = \lambda\kappa\eta$ in Assumption 4.2(3) must be replaced by

$$\psi^L(1) = -\lambda\kappa\eta.$$

Under these conditions Theorems 4.3, 4.4 hold with the following modifications:

- Equation (4.6) must be replaced by (6.6),
- Equation (4.9) must be replaced by

$$\begin{aligned} \Psi_0(t, u_1, u_2) := & \frac{\kappa\eta}{\sigma^2}\left(2\log\varphi(t, u_1, u_2) + (\gamma(u_2) - l(u_1, u_2))\, t\right) + \kappa\eta u_1 t \\ & + \big(\lambda\kappa\eta u_2 + \psi^L(u_2)\big)t, \end{aligned}$$

- $\kappa_0(t, \zeta) := \psi^L(\zeta + 1) - \psi^L(\zeta) + \lambda\kappa\eta$,
- $\delta_0 := \psi^L(2) + 2\lambda\kappa\eta$,
- unless L is deterministic, Equation (4.12) must be replaced by the more involved expression in (6.8) with

$$\begin{aligned} \alpha_0 &= \alpha_0(\zeta_1, \zeta_2) = \psi^L(\zeta_1 + \zeta_2) - \psi^L(\zeta_1) - \psi^L(\zeta_2), \\ \alpha_1 &= \alpha_1(t, \zeta_1, \zeta_2) \\ &= \sigma^2\left(\Psi_1(T - t, 0, \zeta_1) + \lambda\zeta_1\right)\left(\Psi_1(T - t, 0, \zeta_2) + \lambda\zeta_2\right) \\ &\quad + \psi^X(\zeta_1 + \zeta_2) - \psi^X(\zeta_1) - \psi^X(\zeta_2), \\ \eta_0 &= \eta_0(t, \zeta_1, \zeta_2) = \delta_0\alpha_0(\zeta_1, \zeta_2) - \kappa_0(t, \zeta_1)\kappa_0(t, \zeta_2), \\ \eta_1 &= \eta_1(t, \zeta_1, \zeta_2) \\ &:= \delta_0\alpha_1(t, \zeta_1, \zeta_2) + \delta_1\alpha_0(\zeta_1, \zeta_2) - \kappa_1(t, \zeta_1)\kappa_0(t, \zeta_2) - \kappa_1(t, \zeta_2)\kappa_0(t, \zeta_1), \\ \eta_2 &= \eta_2(t, \zeta_1, \zeta_2) = \delta_1\alpha_1(t, \zeta_1, \zeta_2) - \kappa_1(t, \zeta_1)\kappa_1(t, \zeta_2). \end{aligned}$$

The proofs remain practically unchanged.

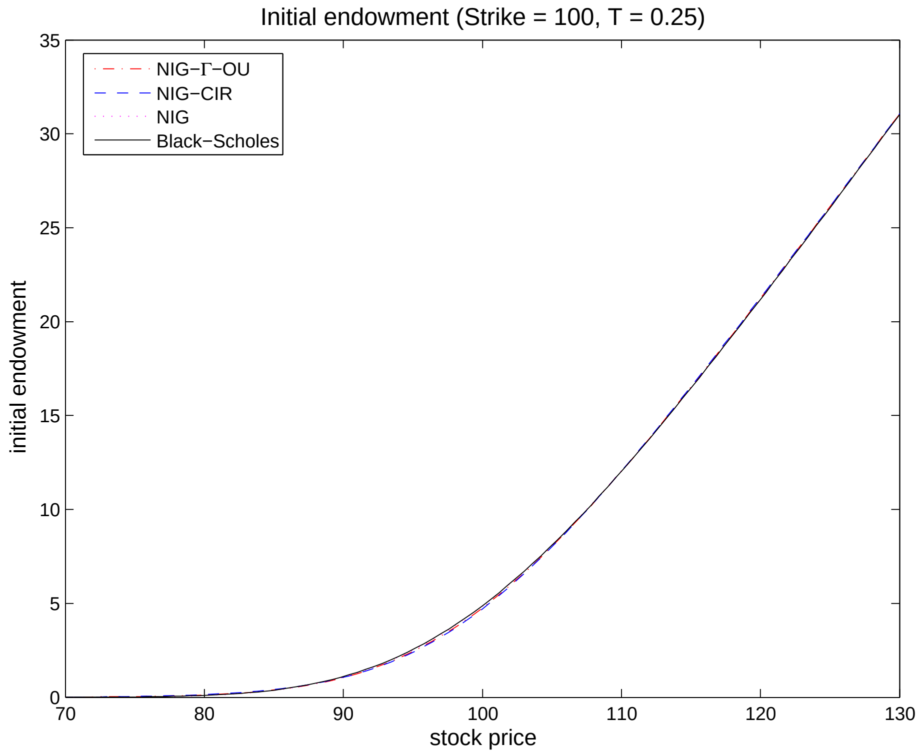

Figure 1: Variance-optimal initial capital

5 Numerical illustration

In this section we illustrate the results of the previous sections in a numerical study. In order to make strategies and hedging errors comparable, we use the same data set for parameter estimation in the different models, namely daily German stock index (DAX) data from June 14, 1988 to April 10, 2008. For simplicity we use a constant interest rate of 4% for discounting. We compare the NIG-Gamma-OU model from Example 3.8, the NIG-CIR model from Example 4.5, a NIG-Lévy process model without stochastic volatility, and geometric Brownian motion as a benchmark. The pure NIG-Lévy process model is obtained within the setup of Section 3 if we choose ψ^X as in (3.13, 3.15), $\psi^z = 0$, and $y_0 = \eta$. The estimated parameters are $\kappa = 2.54, a = 0.847, b = 0.204, \alpha = 90.1, \beta = -16.0$ for the NIG-Gamma-OU model, $\kappa = 2.54, \eta = 4.16, \sigma = 4.99, \alpha = 90.1, \beta = -16.0$ for the NIG-CIR model, $\eta = 2.53, \alpha = 53.0, \beta = -5.1$ for the NIG-Lévy model, and a variance $\sigma^2 = 0.0484$ for the Black-Scholes model. They have been estimated by Johannes Muhle-Karbe using the approach put forward in [21].

In each case a European call with strike 100 and maturity $T = 0.25$ years is hedged by trading in the underlying. The initial activity y_0 in the stochastic volatility models is chosen to equal its expectation, i.e. $y_0 = 4.16$ in the stochastic volatility models. The numerical results for the models with jumps are calculated from the formulas in the previous two sections setting $R = 1.1$ in (2.1). The geometric Brownian motion case could be derived from Theorems 3.3, 3.4 as well but of course it is easier to use the Black-Scholes formulas.

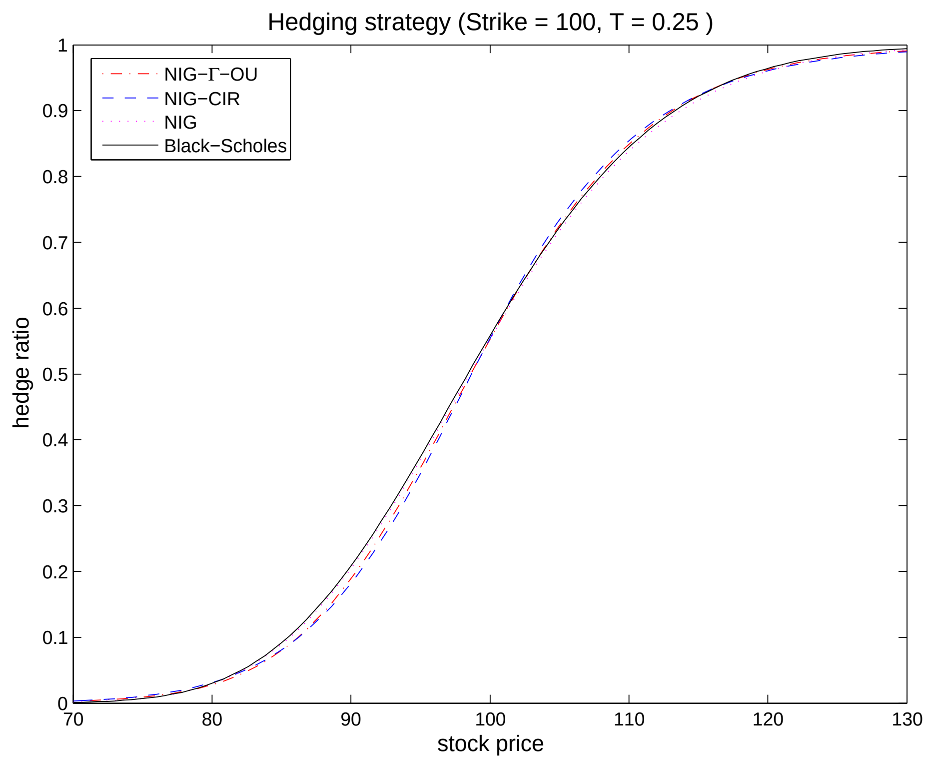

Figure 2: Variance-optimal initial hedge

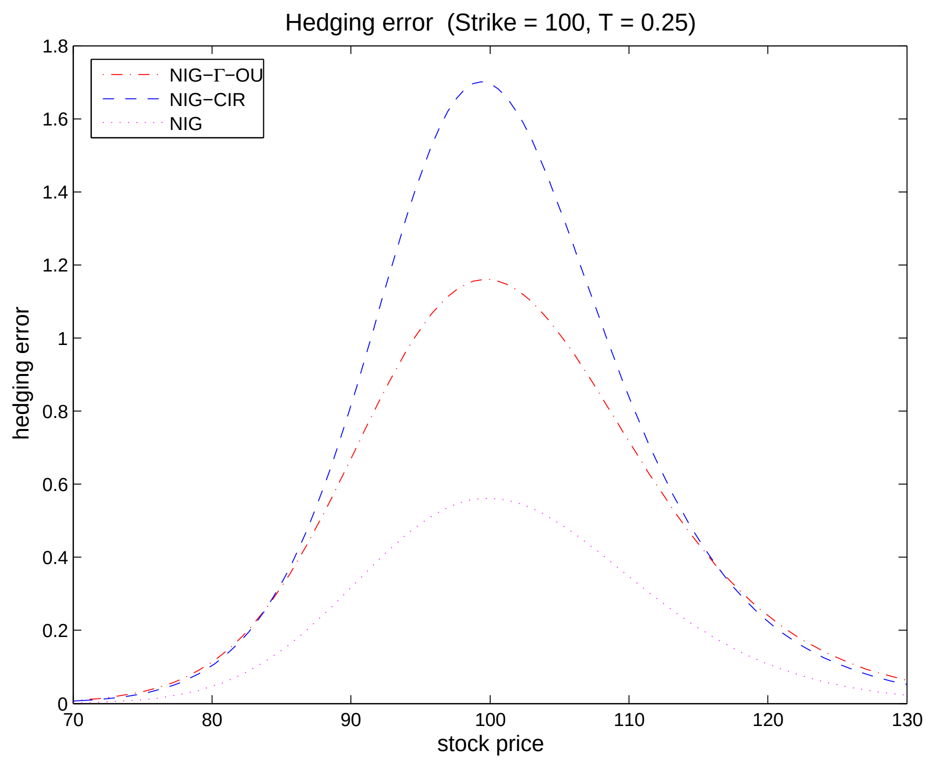

Figure 3: Minimal hedging error

The variance-optimal initial capital and hedge obviously equal the Black-Scholes price and delta of the option, respectively. Moreover, market completeness implies that the minimal hedging error is 0 for geometric Brownian motion.

The results are shown in Figures 1–3. The first one represents the variance-optimal initial capital as a function of the stock price at time $t = 0$. One observes that it hardly differs among the four models. The same holds for the optimal hedging strategy at time $t = 0$, which is shown as a function of the stock price in Figure 2. This suggests that the variance-optimal hedge is quite robust against model misspecification as long as plain vanilla options are considered. Of course, the situation changes dramatically for the hedging error, which vanishes in the Black-Scholes case. Its variance J_0 is represented in Figure 3 for the NIG-Gamma-OU, the NIG-CIR and the NIG-Lévy model, respectively. The at-the-money hedging error in the NIG-Gamma-OU case equals approximately the hedging error in the Black-Scholes model with weekly rebalancing, cf. [18].

6 Proofs

This section is devoted to the proofs of Theorems 3.3, 3.4 and 4.3, 4.4. They rely on results for general affine stochastic volatility models in [19], which we summarize here for the convenience of the reader. For details and unexplained notation we refer to [19]. In particular, $h : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a truncation function on $\mathbb{R}^2$, which is needed for the specification of Lévy-Khintchine triplets and semimartingale characteristics. Its components are denoted by h_1, h_2. A more detailed account of some proofs can be found in [23].

Assumption 6.1 As before, the discounted price process of a univariate stock is denoted by $S = S_0 \exp(Z)$. Moreover we consider a positive activity process y leading to randomly changing volatility. We make the following assumptions.

1. The characteristics $(B^{y,Z}, C^{y,Z}, \nu^{y,Z})$ of the $\mathbb{R}_+ \times \mathbb{R}$-valued semimartingale (y, Z) are supposed to be of the form

$$\begin{aligned} B_t^{y,Z} &= \int_0^t \left(\beta_{(0)} + \beta_{(1)} y_{s-}\right) ds, \\ C_t^{y,Z} &= \int_0^t \left(\gamma_{(0)} + \gamma_{(1)} y_{s-}\right) ds, \\ \nu^{y,Z}([0,t] \times G) &= \int_0^t \left(\varphi_{(0)}(G) + \varphi_{(1)}(G) y_{s-}\right) ds \end{aligned}$$

   for all $G \in \mathscr{B}^2$ and $t \in [0, T]$, where $(\beta_{(j)}, \gamma_{(j)}, \varphi_{(j)})$, $j = 0, 1$, are Lévy-Khintchine triplets on $\mathbb{R}^2$ which are admissible in the sense of [9, Definition 2.6] or [17, Definition 3.1]. More precisely, we require that

   - $\beta_{(j)} \in \mathbb{R}^2$, $\gamma_{(j)}$ is a symmetric, non-negative matrix in $\mathbb{R}^{2\times 2}$, and the measure $\varphi_{(j)}$ on $\mathbb{R}^2 \setminus \{0\}$ satisfies $\int (1 \wedge |x|^2) \varphi_{(j)}(dx) < \infty$,

- $\gamma^{1,1}_{(0)} = \gamma^{1,2}_{(0)} = \gamma^{2,1}_{(0)} = 0$,
- $\varphi_{(0)}((\mathbb{R}_+ \times \mathbb{R})^C) = \varphi_{(1)}((\mathbb{R}_+ \times \mathbb{R})^C) = 0$,
- $\int h_1(x)\varphi_{(0)}(dx) < \infty$ and $\beta^1_{(0)} - \int h_1(x)\varphi_{(0)}(dx) \geq 0$,
- $\int x_1 \varphi_{(1)}(dx) < \infty$.

In view of results from [9] this implies that the conditional characteristic function of (y, Z) is of exponentially affine form

$$E\left[e^{u_1 y_{t+s} + u_2 Z_{t+s}} \middle| \mathscr{F}_t\right] = \exp\left(\Psi_0(s, u_1, u_2) + \Psi_1(s, u_1, u_2) y_t + u_2 Z_t\right) \tag{6.1}$$

for any $s, t \geq 0$ with $s + t \leq T$, any $u \in \mathbb{C}_- \times i\mathbb{R}$ and some functions $\Psi_0, \Psi_1 : [0, T] \times \mathbb{C}_- \times i\mathbb{R} \to \mathbb{C}$, where $\mathbb{C}_- := \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) \leq 0\}$. These functions Ψ_0, Ψ_1 are obtained as solutions to

$$\frac{\partial}{\partial t}\Psi_1(t, u_1, u_2) = \psi_1(\Psi_1(t, u_1, u_2), u_2), \quad \Psi_1(0, u_1, u_2) = u_1 \tag{6.2}$$

and

$$\Psi_0(t, u_1, u_2) = \int_0^t \psi_0(\Psi_1(s, u_1, u_2), u_2) ds \tag{6.3}$$

with Lévy exponents $\psi_0 : U_0 \to \mathbb{C}$, $\psi_1 : U_1 \to \mathbb{C}$ defined as

$$\psi_j(u) := u^\top \beta_{(j)} + \frac{1}{2} u^\top \gamma_{(j)} u + \int \left(e^{u^\top x} - 1 - u^\top h(x)\right) \varphi_{(j)}(dx) \tag{6.4}$$

for

$$U_j := \left\{ u \in \mathbb{C}^2 : \int_{\{|x| \geq 1\}} \exp(\mathrm{Re}(u)^\top x) \varphi_{(j)}(dx) < \infty \right\}.$$

2. For some $\widetilde{\varepsilon} > 0$, the mappings $(u_1, u_2) \mapsto \Psi_0(t, u_1, u_2), \Psi_1(t, u_1, u_2)$ are assumed to have an analytic extension on

$$\mathcal{S} := \{u \in \mathbb{C}^2 : (\mathrm{Re}(u_1), \mathrm{Re}(u_2)) \in \mathcal{V}_{\widetilde{\varepsilon}}(0)\}$$

for all $t \in [0, T]$, where

$$M_0 := \sup\left\{2\Psi_1(t, 0, r) : r \in [R' \wedge 0, R \vee 0], t \in [0, T]\right\} \tag{6.5}$$

and

$$\mathcal{V}_{\widetilde{\varepsilon}}(a) := \left(-\infty, (M_0 \vee 0) + \widetilde{\varepsilon}\right) \times \left((2R' \wedge 0) - \widetilde{\varepsilon}, (2R \vee a) + \widetilde{\varepsilon}\right)$$

for $a \in \mathbb{R}_+$. These extensions are again denoted Ψ_0 resp. Ψ_1.

3. The mappings $t \mapsto \Psi_0(t, u_1, u_2)$, $t \mapsto \Psi_1(t, u_1, u_2)$ are continuous on $[0, T]$ for any $(u_1, u_2) \in \mathcal{S}$.

4. $\mathcal{V}_{\widetilde{\varepsilon}}(2) \subset U_0 \cap U_1$. This is satisfied if the mappings $i\mathbb{R} \to \mathbb{C}$, $u \mapsto \psi_j(u)$ for $j = 0, 1$ have analytic extensions to $\mathcal{V}_{\widetilde{\varepsilon}}(2)$, in which case representation (6.4) holds for this extension.

5. Finally, we assume

$$\psi_0(0,1) = \psi_1(0,1) = 0$$

and

$$\psi_0(0,2) \neq 0 \quad \text{or} \quad \psi_1(0,2) \neq 0.$$

Theorem 6.2 *Under Assumption 6.1 the variance-optimal initial capital v^* and the variance-optimal hedging strategy ϑ^* are given by*

$$\begin{aligned} v^* &= \int V(\zeta)_0 \Pi(d\zeta), \\ \vartheta^*_t &= \int \frac{V(\zeta)_{t-}}{S_{t-}} \frac{\kappa_0(t,\zeta) + \kappa_1(t,\zeta) y_{t-}}{\delta_0 + \delta_1 y_{t-}} \Pi(d\zeta), \end{aligned} \tag{6.6}$$

where the process $V(\zeta)$ is determined by

$$V(\zeta)_t = S_t^\zeta \exp(\Psi_0(T-t,0,\zeta) + \Psi_1(T-t,0,\zeta) y_t\}, \quad \zeta \in \mathcal{S}_f.$$

Here, $\delta_0, \delta_1 \in \mathbb{R}$ and functions $\kappa_0, \kappa_1 : [0,T] \times \mathcal{S}_f \to \mathbb{C}$ are defined by

$$\begin{aligned} \kappa_j(t,\zeta) &:= \psi_j(\Psi_1(T-t,0,\zeta), \zeta+1) - \psi_j(\Psi_1(T-t,0,\zeta),\zeta), \\ \delta_j &:= \psi_j(0,2), \quad j = 0,1. \end{aligned} \tag{6.7}$$

The minimal hedging error is given by

$$J_0 = \begin{cases} \int\int\int_0^T J_1(t,\zeta_1,\zeta_2) dt \Pi(d\zeta_1)\Pi(d\zeta_2), & \text{if } \delta_0 \neq 0,\ \delta_1 \neq 0, \\ \int\int\int_0^T J_2(t,\zeta_1,\zeta_2) dt \Pi(d\zeta_1)\Pi(d\zeta_2), & \text{if } \delta_0 = 0, \\ \int\int\int_0^T J_3(t,\zeta_1,\zeta_2) dt \Pi(d\zeta_1)\Pi(d\zeta_2), & \text{if } \delta_1 = 0. \end{cases}$$

The integrals over $\mathcal{S}_f$ have to be understood in the sense of the Cauchy principal value (cf. Remark 3.5). The integrands $J_k : [0,T] \times {\mathcal{S}_f}^2 \to \mathbb{C}$, $k = 1,2,3$ in these expressions are defined as

$$\begin{aligned} &J_1(t,\zeta_1,\zeta_2) \\ &= S_0^{\zeta_1+\zeta_2} e^{\xi_0} \Bigg(\exp\big(\Psi_0(t,\xi_1,\zeta_1+\zeta_2) + \Psi_1(t,\xi_1,\zeta_1+\zeta_2) y_0\big) \Big(\frac{\eta_2}{\delta_1} \big(D_2\Psi_0(t,\xi_1,\zeta_1+\zeta_2) \\ &\quad + D_2\Psi_1(t,\xi_1,\zeta_1+\zeta_2) y_0\big) + \frac{\eta_1\delta_1 - \eta_2\delta_0}{\delta_1^2} \Big) + \frac{\eta_0\delta_1^2 - \eta_1\delta_0\delta_1 + \eta_2\delta_0^2}{\delta_1^3} e^{-\frac{\delta_0}{\delta_1}\xi_1} \times \\ &\quad \times \int_0^1 \left(\frac{\delta_1}{\delta_0} + \xi_1 s \right) e^{\frac{\delta_0}{\delta_1}\xi_1 s + \Psi_0\left(t, \frac{\delta_1}{\delta_0}\log(s) + \xi_1 s, \zeta_1+\zeta_2\right) + \Psi_1\left(t,\frac{\delta_1}{\delta_0}\log(s)+\xi_1 s, \zeta_1+\zeta_2\right) y_0} ds \Bigg), \end{aligned} \tag{6.8}$$

$$\begin{aligned} &J_2(t,\zeta_1,\zeta_2) \\ &= \frac{S_0^{\zeta_1+\zeta_2} e^{\xi_0}}{\delta_1} \exp\Big(\Psi_0(t,\xi_1,\zeta_1+\zeta_2) + \Psi_1(t,\xi_1,\zeta_1+\zeta_2) y_0\Big) \times \\ &\quad \times \Big(\eta_1 + \big(D_2\Psi_0(t,\xi_1,\zeta_1+\zeta_2) + D_2\Psi_1(t,\xi_1,\zeta_1+\zeta_2) y_0 \big) \eta_2 \Big), \end{aligned}$$

$$\begin{aligned} &J_3(t,\zeta_1,\zeta_2) \\ &= \frac{S_0^{\zeta_1+\zeta_2}\eta_0}{\delta_0} \exp\Big((\psi_0(0,\zeta_1) + \psi_0(0,\zeta_2))(T-t) + \psi_0(0,\zeta_1+\zeta_2)t \Big), \end{aligned}$$

where

$$\begin{aligned}
\alpha_j &= \alpha_j(t,\zeta_1,\zeta_2) \\
&:= \psi_j(\xi_1(t,\zeta_1,\zeta_2),\zeta_1+\zeta_2) - \psi_j(\Psi_1(T-t,0,\zeta_1),\zeta_1) - \psi_j(\Psi_1(T-t,0,\zeta_2),\zeta_2), \\
\eta_0 &= \eta_0(t,\zeta_1,\zeta_2) := \delta_0\alpha_0(t,\zeta_1,\zeta_2) - \kappa_0(t,\zeta_1)\kappa_0(t,\zeta_2), \\
\eta_1 &= \eta_1(t,\zeta_1,\zeta_2) \\
&:= \delta_0\alpha_1(t,\zeta_1,\zeta_2) + \delta_1\alpha_0(t,\zeta_1,\zeta_2) - \kappa_1(t,\zeta_1)\kappa_0(t,\zeta_2) - \kappa_1(t,\zeta_2)\kappa_0(t,\zeta_1), \\
\eta_2 &= \eta_2(t,\zeta_1,\zeta_2) := \delta_1\alpha_1(t,\zeta_1,\zeta_2) - \kappa_1(t,\zeta_1)\kappa_1(t,\zeta_2), \\
\xi_j &= \xi_j(t,\zeta_1,\zeta_2) := \Psi_j(T-t,0,\zeta_1) + \Psi_j(T-t,0,\zeta_2), \quad j=0,1,
\end{aligned}$$

with κ_0, κ_1 from (6.7). For ease of notation we dropped the arguments of these functions in the formulae above. The mappings $\eta_0,\eta_1,\eta_2,\alpha_0,\alpha_1,\xi_0,\xi_1$ are defined on $[0,T]\times\mathcal{S}_f{}^2$.

PROOF. [19, Theorems 4.1, 4.2] □

We also need the following simple lemma, which is used repeatedly in this section.

Lemma 6.3 *Suppose that $\psi : i\mathbb{R}\to\mathbb{C}$ denotes a Lévy exponent, i.e.*

$$\psi(\zeta) = \zeta\beta + \frac{1}{2}\zeta^2\gamma + \int(e^{\zeta x} - 1 - \zeta h(x))\varphi(dx)$$

for some $\beta\in\mathbb{R}$, $\gamma\geq 0$, some measure φ on $\mathbb{R}\setminus\{0\}$ satisfying $\int(1\wedge x^2)\varphi(dx)<\infty$, and some truncation function h on $\mathbb{R}$. If ψ has an analytic extension $\widetilde{\psi}$ to $\mathcal{S}_{a,b} := (a,b)+i\mathbb{R}\subset\mathbb{C}$ for some $a<0<b$, we have $\mathrm{Re}(\widetilde{\psi}(\zeta))\leq\widetilde{\psi}(\mathrm{Re}(\zeta))$, $\zeta\in\mathcal{S}_{a,b}$ and $(a,b)\to\mathbb{R}$, $\zeta\mapsto\widetilde{\psi}(\zeta)$ is a convex function. Moreover, ψ has such an analytic extension if

$$\int_{|x|>1} e^{\zeta x}\varphi(dx) < \infty \tag{6.9}$$

for any $\zeta\in(a,b)$.

PROOF. We have $E[e^{\zeta L_1}] = e^{\psi(\zeta)}$, $\zeta\in i\mathbb{R}$ for some Lévy process L. If ψ has an analytic extension $\widetilde{\psi}$ to $\mathcal{S}_{a,b}$, the equality $E[e^{\zeta L_1}] = e^{\widetilde{\psi}(\zeta)}$ holds for any $\zeta\in\mathcal{S}_{a,b}$, cf. [9, Lemmas A.2 and A.4]. Hence

$$e^{\mathrm{Re}(\widetilde{\psi}(\zeta))} = |e^{\widetilde{\psi}(\zeta)}| = |E(e^{\zeta L_1})| \leq E(|e^{\zeta L_1}|) = E(e^{\mathrm{Re}(\zeta)L_1}) = e^{\widetilde{\psi}(\mathrm{Re}(\zeta))},$$

which yields the first assertion. The second follows from the explicit representation of $\widetilde{\psi}$, which holds also on (a,b), cf. [24, Theorem 25.17].

Condition (6.9) implies that the mapping $\mathcal{S}_{a,b}\to\mathbb{C}$, $\zeta\mapsto m_{L_1}(\zeta) := E[e^{\zeta L_1}]$ is well defined, cf. [24, Theorem 25.17]. From [9, Lemma A.2] it follows that m_{L_1} is analytic. Since $\mathcal{S}_{a,b}$ is open and convex and $m_{L_1}(\zeta)\neq 0$ for all $\zeta\in\mathcal{S}_{a,b}$, there exists an analytic function $g:\mathcal{S}_{a,b}\to\mathbb{C}$ with $m_{L_1}(\zeta) = e^{g(\zeta)}$ for all $\zeta\in\mathcal{S}_{a,b}$, cf. [10, Satz V.1.4]. By $e^{g(\zeta)} = m_{L_1}(\zeta) = e^{\psi(\zeta)}$ we have $\psi(\zeta) - g(\zeta)\in 2\pi i\mathbb{Z}$ for $\zeta\in i\mathbb{R}$. Since both ψ and g are continuous, $\psi - g$ is constant on $i\mathbb{R}$. This yields the assertion. $\square$

Theorems 3.3, 3.4 are now reduced to Theorem 6.2 above.

PROOF OF THEOREMS 3.3 AND 3.4. From [17, Section 4.7] it follows that the Lévy exponents ψ_0, ψ_1 corresponding as in (6.4) to the affine process (y, Z) are given by (3.6, 3.7). By assumption, we have that ψ_1 can be analytically extended to

$$\widetilde{U}_1 := \{u \in \mathbb{C}^2 : (2R' \wedge 0) - \widetilde{\varepsilon} < \mathrm{Re}(u_2) < (2R \vee 2) + \widetilde{\varepsilon}\}$$

and ψ_0 to

$$\widetilde{U}_0 := \widetilde{U}_1 \cap \{u \in \mathbb{C}^2 : \mathrm{Re}(u_1) < 2M_1 + \widetilde{\varepsilon}\}$$

for sufficiently small $\widetilde{\varepsilon} > 0$. The functions $\Psi_0, \Psi_1 : [0,T] \times \mathbb{C}_- \times i\mathbb{R} \to \mathbb{C}_-$ in (6.1) are given by (3.3, 3.5) because the latter solve the generalized Riccati equations (6.2, 6.3).

Obviously $(u_1, u_2) \mapsto \Psi_1(t, u_1, u_2)$ has an analytic extension, again denoted Ψ_1, to $\widetilde{U}_1$. Moreover, $(t, u_1, u_2) \mapsto \Psi_1(t, u_1, u_2)$ is continuous on $[0,T] \times \widetilde{U}_1$. Convexity of ψ^X implies that $M_0 < M_1$ for M_0 from (6.5). Together, we obtain Assumption 6.1(2, 4). Assumption 6.1(5) follows from Assumption 3.2(2, 3).

The mapping $(u_1, u_2) \mapsto q(t, u_1, u_2)$ from (3.4) is analytic on $\widetilde{U}_1$. Using Lemma 6.3 we conclude that

$$q([0,T] \times U_q) \subset \{\zeta \in \mathbb{C} : \mathrm{Re}(\zeta) < M_1 + M_2 + \varepsilon\}$$

for

$$U_q := \{u \in \mathbb{C}^2 : \mathrm{Re}(u_1) < M_1 + \widetilde{\varepsilon}, (2R' \wedge 0) - \widetilde{\varepsilon} < \mathrm{Re}(u_2) < (2R \vee 2) + \widetilde{\varepsilon}\}.$$

By [8, (9.3.2)] the mapping

$$(u_1, u_2) \mapsto g(t, u_1, u_2) := \psi^z(q(t, u_1, u_2))$$

is analytic on U_q for any fixed $t \in [0,T]$. Note that

$$\begin{aligned}
(t, u_1, u_2) &\mapsto g(t, u_1, u_2) := \psi^z(q(t, u_1, u_2)), \\
(t, u_1, u_2) &\mapsto D_2 g(t, u_1, u_2) = (\psi^z)'(q(t, u_1, u_2)) e^{-\kappa t}, \\
(t, u_1, u_2) &\mapsto D_3 g(t, u_1, u_2) = (\psi^z)'(q(t, u_1, u_2)) \left(\frac{1 - e^{-\kappa t}}{\kappa} (\psi^X)'(u_2) + (\psi^{\widetilde{X}})'(u_2) \right)
\end{aligned}$$

are continuous on $[0,T] \times U_q$. From [8, Problem 9.10.2] we conclude that

$$(u_1, u_2) \mapsto \int_0^t \psi^z(q(s, u_1, u_2)) ds$$

is analytic on U_q for any $t \in [0,T]$. Consequently, $(u_1, u_2) \mapsto \Psi_0(t, u_1, u_2)$ from (3.5) is analytic on U_q for any $t \in [0,T]$. Standard arguments show that its derivative relative to u_1 is given by (3.8). The continuity of ψ^X, g, q, ψ^z implies that Ψ_0 is continuous on $[0,T] \times U_q$. This yields Assumption 6.1(3).

Theorems 3.3, 3.4 follow now from Theorem 6.2. □

We turn now to the proof of Theorems 4.3, 4.4, which is broken down into several steps. We start with two purely analytical lemmas whose proofs can be found in [23].

Lemma 6.4 *Let $I \subset \mathbb{R}$ be a compact interval and $f : \mathbb{R} \times I \to \mathbb{R}$ a twice continuously differentiable convex function. If $x(t,u)$ denotes the solution to the initial value problem*

$$\frac{\partial}{\partial t} x(t,u) = f(x(t,u),u), \quad x(0,u) = 0,$$

the function $u \mapsto x(t,u)$ is convex for all $t \in [0,T]$.

PROOF. [23, Lemma 4.16] □

Lemma 6.5 *Let $a, b, c > 0$ with $0 < c^2 \leq b \leq a$. For*

$$x_+ := \sqrt{\frac{1}{2}(a+b)} \quad \textit{and} \quad x_- := \sqrt{\frac{1}{2}(a-b)}$$

and any $t \geq 0$ we have

$$c \sinh(ct) \left(\cosh(x_+ t) - \cos(x_- t)\right) \leq \left(\cosh(ct) - 1\right) \left(x_+ \sinh(x_+ t) + x_- \sin(x_- t)\right).$$

PROOF. [23, Lemma 4.20] □

PROOF OF THEOREMS 4.3 AND 4.4. *Step 1:* By Assumption 4.2(1a) the function $\zeta \mapsto \kappa^2 - 2\sigma^2 \psi^X(\zeta)$ is analytic on

$$U_\gamma := \{\zeta \in \mathbb{C} : (2R' \wedge 0) - \varepsilon < \mathrm{Re}(\zeta) < (2R \vee 2) + \varepsilon\}.$$

Since ψ^X is convex on the interval $((2R' \wedge 0) - \varepsilon, (2R \vee 2) + \varepsilon)$, Assumption 4.2(1b) implies that $\kappa^2 - 2\sigma^2 \mathrm{Re}(\psi^X(\zeta)) > 0$ for all $\zeta \in U_\gamma$. Together, it follows that γ as defined in (4.3) is well defined and analytic on U_γ.

Step 2: Let

$$\begin{aligned}
\Delta^+(\zeta) &:= l(0,\zeta) + \gamma(\zeta), \\
\Delta^-(\zeta) &:= l(0,\zeta) - \gamma(\zeta), \\
\tau(\zeta) &:= \begin{cases} \infty & \text{if } \Delta^-(\zeta) \leq 0, \\ \frac{1}{\gamma(\zeta)} \log\left(\frac{\Delta^+(\zeta)}{\Delta^-(\zeta)}\right) & \text{otherwise.} \end{cases}
\end{aligned}$$

We show that $\Psi_1(t, 0, \zeta) \in \mathbb{R}$ is defined for all $t < \tau(2R)$ and all $\zeta \in [2R' \wedge 0, 2R \vee 0]$ and that it solves the Riccati equation

$$\frac{\partial}{\partial t}\Psi_1(t,0,\zeta) = \frac{1}{2}\sigma^2\Psi_1^2(t,0,\zeta) + (\lambda\sigma^2\zeta - \kappa)\Psi_1(t,0,\zeta) - \lambda\kappa\zeta + \frac{1}{2}\lambda^2\sigma^2\zeta^2 + \psi^X(\zeta). \quad (6.10)$$

In [6, Lemmas A.1 and A.2] it is shown that $\Psi_1(t,0,\varphi)$ is well defined and solves (6.10) for $t < \tau(\zeta)$. It remains to prove that $\tau(\zeta) \leq \tau(2R)$ for all $\zeta \in [2R' \wedge 0, 2R \vee 0]$. By convexity the function $A(\zeta) := \psi^X(\zeta) + \frac{1}{2}\lambda^2\sigma^2\zeta^2 - \kappa\lambda\zeta$ has exactly two zeros on the interval $((2R' \wedge 0) - \widetilde{\varepsilon}, (2R \vee 2) + \widetilde{\varepsilon})$, namely $\zeta = 0$ and $\zeta = 1$. Moreover, $\Delta^-(\zeta) = 0$ happens only if $A(\zeta) = 0$. Together we conclude that Δ^- has at most one root on $I = [2R' \wedge 0, 2R \vee 2]$, namely 1. More specifically,

1. if $\kappa > \lambda\sigma^2$, then $\Delta^-(\zeta) \leq 0$ for all $\zeta \in I$,
2. if $\kappa \leq \lambda\sigma^2$, then $\Delta^-(1) = 0$, $\Delta^-(\zeta) < 0$ for $\zeta \in [2R' \wedge 0, 1)$, and $\Delta^-(\zeta) > 0$ for $\zeta \in (1, 2R \vee 2]$.

If $\lambda \leq 0$, the equality $\tau(\zeta) = \infty$ holds for all $\zeta \in I$. In the following we suppose that $\lambda > 0$ and $\lambda\sigma^2 - \kappa \geq 0$. We know that

$$\tau(\zeta) = \begin{cases} \infty & \text{if } 2R' \wedge 0 \leq \zeta \leq 1, \\ \frac{1}{\gamma(\zeta)} \log\left(\frac{\Delta^+(\zeta)}{\Delta^-(\zeta)}\right) & \text{if } 1 < \zeta \leq 2R \vee 2. \end{cases}$$

We show that τ is decreasing on $(1, 2R \vee 2]$. For $\zeta \in (1, 2R \vee 2]$ we have

$$\tau(\zeta) = G(A(\zeta), B(\zeta))$$

with

$$\begin{aligned} G(x_1, x_2) &:= \frac{1}{\sqrt{x_2^2 - 2\sigma^2 x_1}} \log\left(\frac{x_2 + \sqrt{x_2^2 - 2\sigma^2 x_1}}{x_2 - \sqrt{x_2^2 - 2\sigma^2 x_1}}\right), \\ B(\zeta) &:= \lambda\sigma^2\zeta - \kappa. \end{aligned}$$

The partial derivatives of G are given by

$$\begin{aligned} D_1G(x_1, x_2) &= -\frac{\sigma^2}{x_3^3 k(x_1, x_2)^3} \int_0^{k(x_1,x_2)} \left(\frac{2r}{1 - r^2}\right)^2 dr, \\ D_2G(x_1, x_2) &= -\frac{2}{x_2^2 k(x_1, x_2)^3} \int_0^{k(x_1,x_2)} \frac{1}{r^{-2} - 1} dr \end{aligned}$$

with

$$k(x_1, x_2) := \frac{\sqrt{x_2^2 - 2\sigma^2 x_1}}{x_2}.$$

Note that $0 < k(x_1, x_2) < 1$ if $0 < \sqrt{x_2^2 - 2\sigma^2 x_1} < x_2$. By $B(\zeta) > \lambda\sigma^2 - \kappa \geq 0$ and $B^2(\zeta) - 2\sigma^2 A(\zeta) = \gamma(\zeta)^2$ we have $D_1G(A(\zeta), B(\zeta)) \leq 0$ and $D_2G(A(\zeta), B(\zeta)) \leq 0$ for $\zeta \in (1, 2R \vee 2]$. Convexity of A and $A(0) = A(1) = 0$ imply $A'(\zeta) \geq 0$ for all $\zeta \in (1, 2R \vee 2]$. Hence

$$\tau'(\zeta) = D_1G(A(\zeta), B(\zeta))A'(\zeta) + D_2G(A(\zeta), B(\zeta))B'(\zeta) \leq 0$$

for all $\zeta \in (1, 2R \vee 2]$. Consequently, $\tau(\zeta) \geq \tau(2R \vee 2)$ holds for all $\zeta \in I$.

Step 3: Differentiation yields

$$D_1\Psi_1(t,0,r) = \frac{\gamma(r)^2\left(l^2(0,r)-\gamma^2(r)\right)}{2\sigma^2\left(\gamma(u_2)\cosh(\frac{1}{2}\gamma(u_2)t) - l(u_1,u_2)\sinh(\frac{1}{2}\gamma(u_2)t)\right)^2}$$

for $(t,r)\in[0,T]\times[2R'\wedge 0, 2R\vee 0]$. Since

$$\operatorname{sgn}\left(D_1\Psi_1(t,0,r)\right) = \operatorname{sgn}\left(l^2(0,r)-\gamma^2(r)\right)$$

does not depend on t, we have

$$\Psi_1(t,0,r) \leq \Psi_1(0,0,r)\vee\Psi_1(T,0,r) = \Psi_1(T,0,r)\vee 0 \tag{6.11}$$

for any $(t,r)\in[0,T]\times[2R'\wedge 0,2R\vee 0]$. Convexity of ψ^X, (6.10) and Lemma 6.4 yield that $\Psi_1(T,0,r)$ is convex on $[2R'\wedge 0,2R\vee 0]$. In view of (6.11) we have

$$2\Psi_1(t,0,r)\leq\Psi_1(t,0,2r)\leq\Psi_1(T,0,2R'\wedge 0)\vee\Psi_1(T,0,2R\vee 0)$$

for all $(t,r)\in[0,T]\times[R'\wedge 0,R\vee 0]$, and hence

$$\begin{aligned} M_0 &:= \sup\{2\Psi_1(t,0,r) : r\in[R'\wedge 0,R\vee 0], t\in[0,T]\} \\ &\leq \max\{\Psi_1(T,0,2R'\wedge 0),\Psi_1(T,0,2R\vee 0)\} = M_1. \end{aligned}$$

Step 4: Define $\mathcal{S}$ and $\mathcal{V}_{\widetilde{\varepsilon}}(a)$ as in Assumption 6.1 for sufficiently small $\widetilde{\varepsilon}>0$. We show that the denominator in (4.4) does not vanish on $[0,T]\times\mathcal{S}$. This would happen only if

$$u_1 = \frac{1}{\sigma^2}\gamma(u_2)\coth\left(\frac{1}{2}\gamma(u_2)t\right)+\frac{\kappa}{\sigma^2}-\lambda u_2.$$

Suppose that $(t,u_1,u_2)\in[0,T]\times\mathcal{S}$ with

$$\operatorname{Im}(u_1) = \operatorname{Im}\left(\frac{1}{\sigma^2}\gamma(u_2)\coth\left(\frac{1}{2}\gamma(u_2)t\right)+\frac{\kappa}{\sigma^2}-\lambda u_2\right).$$

Since $M_3>M_1\geq M_0$ and $\widetilde{\varepsilon}$ can be chosen arbitrarily small, it suffices to show that

$$\widetilde{g}(r,s,t) := \frac{1}{\sigma^2}\operatorname{Re}\left(\gamma(r+is)\coth\left(\frac{1}{2}\gamma(r+is)t\right)\right)+\frac{\kappa}{\sigma^2}-\lambda r\geq M_3.$$

for $(r,s,t)\in(2R'\wedge 0,2R\vee 2)\times\mathbb{R}\times[0,T]$. The identity

$$\operatorname{Re}\left(\zeta\coth\left(\frac{\zeta}{2}\right)\right) = \frac{\operatorname{Re}(\zeta)\sinh(\operatorname{Re}(\zeta))+\operatorname{Im}(\zeta)\sin(\operatorname{Im}(\zeta))}{2(\cosh(\operatorname{Re}(\zeta))-\cos(\operatorname{Im}(\zeta)))}$$

implies

$$\widetilde{g}(r,s,t) = \frac{x_+\sinh(x_+t)+x_-\sin(x_-t)}{2\sigma^2\left(\cosh(x_+t)-\cos(x_-t)\right)}+\frac{\kappa}{\sigma^2}-\lambda r$$

with

$$\begin{aligned}
x_+ &= \mathrm{Re}(\gamma(r+is)) \\
&= \left(\frac{1}{2}\left(\left|\kappa^2 - 2\sigma^2\psi^X(r+is)\right| \pm \left(\kappa^2 - 2\sigma^2\mathrm{Re}(\psi^X(r+is))\right)\right)\right)^{\frac{1}{2}}, \\
x_- &= \mathrm{Im}(\gamma(r+is)) \\
&= \left(\frac{1}{2}\left(\left|\kappa^2 - 2\sigma^2\psi^X(r+is)\right| - \left(\kappa^2 - 2\sigma^2\mathrm{Re}(\psi^X(r+is))\right)\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

For $s = 0$ this equals

$$\widetilde{g}(r,0,t) = \frac{\gamma(r)\sinh(\gamma(r)t)}{2\sigma^2\left(\cosh(\gamma(r)t) - 1\right)} + \frac{\kappa}{\sigma^2} - \lambda r.$$

Lemma 6.5 yields $\widetilde{g}(r,s,t) \geq \widetilde{g}(r,0,t)$ for all $(r,s,t) \in (2R' \wedge 0, 2R \vee 2) \times \mathbb{R} \times [0,T]$. Since

$$D_3\widetilde{g}(r,0,t) = -\frac{\gamma^2(r)}{4\sigma^2\sinh^2\left(\frac{1}{2}\gamma(r)t\right)} < 0,$$

it follows that $\widetilde{g}(r,s,t) \geq \widetilde{g}(r,0,T) = g(r)$ for all $(r,s,t) \in (2R' \wedge 0, 2R \vee 2) \times \mathbb{R} \times [0,T]$. Note that

$$g''(r) = \frac{1}{2\sigma^2}\left(\gamma''(r)\coth(\gamma(r)T) + \frac{\gamma'(r)^2T}{2}\left(1 + \coth(\gamma(r)T)^2\right)\left(2 + \gamma T\coth(\gamma(r)T)\right)\right).$$

Since

$$\gamma''(\zeta) = -\sigma^2\frac{(\psi^X)''(\zeta)}{\gamma(\zeta)} - \sigma^4\frac{(\psi_X)'(\zeta)^2}{\gamma(\zeta)^2} < 0,$$

we have that $g''(r) < 0$, which means that g is concave on $(2R' \wedge 0, 2R \vee 2)$. Hence

$$\widetilde{g}(r,s,t) \geq g(r) \geq g(2R' \wedge 0) \wedge g(2R \vee 2) =: M_3$$

for all $(r,s,t) \in (2R' \wedge 0, 2R \vee 2) \times \mathbb{R} \times [0,T]$ as desired.

Consequently, $\Psi_1(t,u_1,u_2)$ is analytic on $\mathcal{S}$ for every $t \in [0,T]$. Moreover, Ψ_1 is continuous on $[0,T] \times \mathcal{S}$. From [17, Section 4.4] it follows that the Lévy exponents ψ_0, ψ_1 corresponding as in (6.4) to the affine process (y,Z) are given by

$$\begin{aligned}
\psi_0(u_1,u_2) &= \kappa\eta u_1 + (\lambda\kappa\eta - \delta)u_2 = \kappa\eta u_1, \\
\psi_1(u_1,u_2) &= \frac{1}{2}\sigma^2u_1^2 + (\lambda\sigma^2u_2 - \kappa)u_1 - \lambda\kappa u_2 + \frac{1}{2}\lambda^2\sigma^2u_2^2 + \psi^X(u_2).
\end{aligned}$$

ψ_0 can obviously be analytically extended to all of $\mathbb{C}^2$. By assumption ψ_1 can be analytically extended to $\{u \in \mathbb{C}^2 : (2R' \wedge 0) - \varepsilon < \mathrm{Re}(u_2) < (2R \vee 2) + \varepsilon\}$ for sufficiently small $\varepsilon > 0$. The functions $\Psi_0, \Psi_1 : [0,T] \times \mathbb{C}_- \times i\mathbb{R} \to \mathbb{C}_-$ in (6.1) are given by (4.4, 4.9) because the latter solve (6.2, 6.3), cf. also [6, Lemma A.1].

Step 5: Note that the denominator in (4.7) vanishes only if the denominator in (4.4) does, which does not happen on $[0,T] \times \mathcal{S}$. From [8, Problem 9.10.2] we conclude that $(u_1,u_2) \mapsto \log\varphi(t,u_1,u_2)$ and hence also $(u_1,u_2) \mapsto \Psi_0(t,u_1,u_2)$ are analytic on $\mathcal{S}$ for all $t \in [0,T]$. Moreover, Ψ_0 is continuous on $[0,T] \times \mathcal{S}$. The assertion follows now from Theorem 6.2. Indeed, Assumption 6.1(1–4) are shown in Step 4. Assumption 6.1(5) follows from Assumption 4.2(3, 4). □

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References

[1] F. Angelini and S. Herzel. Measuring the error of dynamic hedging: a Laplace transform approach. *Journal of Computational Finance*, 2009. To appear.

[2] O. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society, Series B*, 63:167–241, 2001.

[3] D. Bates. Jumps and stochastic volatility: the exchange rate process implicit in deutsche mark options. *Review of Financial Studies*, 9:69–107, 1996.

[4] P. Carr, H. Geman, D. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13:345–382, 2003.

[5] A. Černý. Dynamic programming and mean-variance hedging in discrete time. *Applied Mathematical Finance*, 11:1–25, 2004.

[6] A. Černý and J. Kallsen. Mean-variance hedging and optimal investment in Heston's model with correlation. *Mathematical Finance*, 18:473–492, 2008.

[7] R. Cont, P. Tankov, and E. Voltchkova. Hedging with options in models with jumps. In F. Benth et al., editor, *Stochastic analysis and applications. The Abel symposium 2005*, pages 197–217, Berlin, 2007. Springer.

[8] J. Dieudonné. *Foundations of modern analysis*. Academic Press, New York, 1960.

[9] D. Duffie, D. Filipovic, and W. Schachermayer. Affine processes and applications in finance. *The Annals of Applied Probability*, 13:984–1053, 2003.

[10] W. Fischer and I. Lieb. *Funktionentheorie*. Vieweg, Braunschweig, 1994.

[11] H. Föllmer and D. Sondermann. Hedging of nonredundant contingent claims. In W. Hildenbrand and A. Mas-Colell, editors, *Contributions to Mathematical Economics*, pages 205–223. North-Holland, Amsterdam, 1986.

[12] D. Heath, E. Platen, and M. Schweizer. A comparison of two quadratic approaches to hedging in incomplete markets. *Mathematical Finance*, 11(4):385–413, 2001.

[13] S. Heston. A closed-form solution for options with stochastic volatilities with applications to bond and currency options. *The Review of Financial Studies*, 6:327–343, 1993.

[14] F. Hubalek, L. Krawczyk, and J. Kallsen. Variance-optimal hedging for processes with stationary independent increments. *The Annals of Applied Probability*, 16:853–885, 2006.

[15] F. Hubalek and C. Sgarra. Quadratic hedging for the Bates model. *International Journal of Theoretical and Applied Finance*, 10:873–885, 2007.

[16] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Berlin, second edition, 2003.

[17] J. Kallsen. A didactic note on affine stochastic volatility models. In Yu. Kabanov, R. Liptser, and J. Stoyanov, editors, *From Stochastic Calculus to Mathematical Finance*, pages 343–368. Springer, Berlin, 2006.

[18] J. Kallsen, J. Muhle-Karbe, N. Shenkman, and R. Vierthauer. Discrete-time variance-optimal hedging in affine stochastic volatility models. In R. Kiesel, M. Scherer, and R. Zagst, editors, *Alternative Investments and Strategies*. World Scientific, Singapore, 2009. To appear.

[19] J. Kallsen and A. Pauwels. Variance-optimal hedging in general affine stochastic volatility models. *Advances in Applied Probability*, 2009. To appear.

[20] J. Kallsen and R. Vierthauer. Quadratic hedging in affine stochastic volatility models. *Review of Derivatives Research*, 12:3–27, 2009.

[21] J. Muhle-Karbe. *On Utility-Based Investment, Pricing and Hedging in Incomplete Markets*. Ph.D. dissertation (TU München), München, 2009.

[22] E. Nicolato and E. Venardos. Option pricing in stochastic volatility models of the Ornstein-. Uhlenbeck type. *Mathematical Finance*, 13:445–466, 2003.

[23] A. Pauwels. *Varianz-optimales Hedging in affinen Volatilitätsmodellen*. Ph.D. dissertation (TU München), München, 2007.

[24] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.

[25] W. Schoutens. *Lévy Processes in Finance*. Wiley, New York, 2003.

[26] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, Cambridge, 2001.