# Variance Reduction for Monte Carlo Methods to Evaluate Option Prices under Multi-factor Stochastic Volatility Models

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## Abstract

We present variance reduction methods for Monte Carlo simulations to evaluate European and Asian options in the context of multiscale stochastic volatility models. European option price approximations, obtained from singular and regular perturbation analysis [J.P. Fouque, G. Papanicolaou, R. Sircar and K. Solna: Multiscale Stochastic Volatility Asymptotics, SIAM Journal on Multiscale Modeling and Simulation **2(1)**, 2003], are used in important sampling techniques, and their efficiencies are compared. Then we investigate the problem of pricing arithmetic average Asian options (AAOs) by Monte Carlo simulations. A two-step strategy is proposed to reduce the variance where geometric average Asian options (GAOs) are used as control variates. Due to the lack of analytical formulas for GAOs, it is then necessary to consider efficient Monte Carlo methods to estimate the unbiased means of GAOs. The second step consists in deriving formulas for approximated prices based on perturbation techniques, and in computing GAOs by using importance sampling. Numerical results illustrate the efficiency of our method.

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# 1  Introduction

Monte Carlo methods are natural and essential tools in computational finance. Examples include pricing and hedging financial instruments with complex structure or high dimensionality [10]. This paper addresses the issue of variance reduction for Monte Carlo methods for a class of multi-factor stochastic volatility models.

In the first part of this paper, we investigate an application of importance sampling to variance reduction in evaluating European options by Monte Carlo methods. Under one-factor stochastic volatility model, Fouque and Tullie [9] proposed to use approximations of European option prices obtained from singular perturbation expansions for the importance sampling techniques. They demonstrated that the first order correction term added to the zeroth order (or homogenized) option price approximation dramatically reduce the variance. However, recent empirical studies document that at least two-factor stochastic volatility models with well-separated characteristic time scales are necessary to capture stylized facts like the observed kurtosis, fatter tailed return distributions, long memory effect, and the shape of term structure of implied volatilities. We refer to [1], [3] and [11] for detailed discussions. Therefore, this motivates an extension to apply importance sampling in the context of two-factor stochastic volatility models. Fouque et al. [8] used a combination of singular and regular perturbation expansions to derive price approximations of European options. We shall apply their results to important sampling.

The second part of this paper explores variance reduction methods for Asian options. Asian options are known as path dependent options whose payoff depends on the average stock price and a fixed or floating strike price during a specific period of time before maturity. Here we only consider continuous average stock prices in time either arithmetically or geometrically. An arithmetic average Asian option will be abbreviated as AAO and likewise an geometric average Asian option will be GAO. Using Monte Carlo simulations to evaluate Asian option prices has been an important approach in parallel to PDE approaches [4, 5, 14]. Under the Black-Scholes model, underlying risky assets are assumed to follow log-normal distributions. Among many variance reduction estimators for arithmetic average Asian options, Boyle et al [2] showed that the control variate estimators derived from the geometric mean perform best. It is noted that close-form solutions exist for GAOs under constant volatility such that the unbiased control variance esti-

mator can be calculated easily. When the volatility is randomly fluctuating, there is no analytic solution for GAO in general. To estimate unbiased prices of GAOs, we consider importance sampling by applying the first order price approximations obtained from the analysis of singular and regular perturbations. As a consequence, we propose a two-step strategy which combines a control variate estimator and importance sampling to reduce variance for AAOs.

The organization of the paper is as follows. A class of two-factor stochastic volatility models is introduced in Section 2. Section 3 includes a brief review of importance sampling for diffusion processes, an application of perturbation analysis to European option prices, and some numerical demonstrations. A two-stage variance reduction for Asian options is discussed in Section 4, in which a combination of control variates for AAO and importance sampling for GAO, and some numerical simulations are presented.

# 2 Multifactor Stochastic Volatility Models and Option Prices

Following [8], we consider a family of two-factor stochastic volatility models $(S_t, Y_t, Z_t)$, where $S_t$ is the underlying price, $Y_t$ evolves as an Ornstein-Uhlenbeck (OU) process, as a prototype of an ergodic diffusion, and $Z_t$ follows another diffusion process. Under the pricing risk-neutral probability measure $\mathbb{P}^\star$, our model is described by the following equations:

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dW_t^{(0)}, \qquad\qquad\qquad\qquad (1)\\
\sigma_t &= f(Y_t, Z_t),\\
dY_t &= \left(\alpha(m_f - Y_t) - \nu_f\sqrt{2\alpha}\,\Lambda_f(Y_t, Z_t)\right) dt\\
&\quad + \nu_f\sqrt{2\alpha}\left(\rho_1 dW_t^{(0)} + \sqrt{1-\rho_1^2}\,dW_t^{(1)}\right),\\
dZ_t &= \left(\delta(m_s - Z_t) - \nu_s\sqrt{2\delta}\,\Lambda_s(Y_t, Z_t)\right) dt\\
&\quad + \nu_s\sqrt{2\delta}\left(\rho_2 dW_t^{(0)} + \rho_{12} dW_t^{(1)} + \sqrt{1-\rho_2^2-\rho_{12}^2}\,dW_t^{(2)}\right),
\end{aligned}
$$

where $\left(W_t^{(0)}, W_t^{(1)}, W_t^{(2)}\right)$ are independent standard Brownian motions, and the instant correlation coefficients $\rho_1, \rho_2$, and $\rho_{12}$ satisfy $\rho_1^2 < 1$ and $\rho_2^2 + \rho_{12}^2 < 1$ respectively. The stock price $S_t$ has a constant rate of return equal to

the constant risk-free interest rate $r$ (under risk-neutral), and the random volatility $\sigma_t$ depending on the two volatility factors $Y_t$ and $Z_t$. The risk neutral probability measure $I\!P^\star$ is determined by the combined market prices of volatility risk $\Lambda_f$ and $\Lambda_s$ which we assume to be bounded and independent of the stock price $S$. The joint process $(S_t, Y_t, Z_t)$ is Markovian. Without $\Lambda_f$ (resp. $\Lambda_s$) the driving volatility process $Y_t$ (resp. $Z_t$) is mean-reverting around its long run mean $m_f$ (resp. $m_s$), with a rate of mean reversion $\alpha > 0$ (resp. $\delta > 0$) or a time scale $1/\alpha$ (resp. $1/\delta$), and a "vol-vol" $\nu_f\sqrt{2\alpha}$ (resp. $\nu_s\sqrt{2\delta}$) corresponding to a long run standard deviation $\nu_f$ (resp. $\nu_s$). Here we choose to write OU processes with long run distributions $\mathcal{N}(m_f, \nu_f^2)$ and $\mathcal{N}(m_s, \nu_s^2)$ as prototypes of more general ergodic diffusions. The volatility function $f(y, z)$ in (1) is assumed to be smooth in $z$, bounded and bounded away from 0 ($0 < c_1 \leq f \leq c_2$). The two stochastic volatility factors $Y_t$ and $Z_t$ are differentiated by their intrinsic time scales. The first factor $Y_t$ is fast mean-reverting on a short time scale $1/\alpha$, and the second factor $Z_t$ is slowly varying on a long time scale $1/\delta$. In other words we assume that these time scales are separated: $\alpha^{-1} < 1 < \delta^{-1}$. In this paper we will use an asymptotic theory in the regime where $\alpha \to \infty$, $\delta \to 0$, in order to compute option prices by Monte Carlo simulations for finite values of $\alpha$ and $\delta$.

The payoff of an European option is a function $H(S_T)$ of the stock price at the expire date. Using the Markov property, the no-arbitrage price of this option is obtained as the conditional expectation of the discounted payoff given the current stock price and driving volatility levels:

$$P(t, x, y, z) = I\!E^\star \left\{ e^{-r(T-t)} H(S_T) \mid S_t = x, Y_t = y, Z_t = z \right\}.$$

Payoffs of Asian options, as mentioned before, are functions of fixed strike $K$, floating strike $S_T$, and a time average of stock prices. For example, the price at time $t$ of an Asian call option is given by

$$I\!E^\star \left\{ e^{-r(T-t)} (A_T - S_T - K)^+ \mid \mathcal{F}_t \right\}, \tag{2}$$

where $(\mathcal{F}_t)$ denotes the filtration generated by the process $(S_t, Y_t, Z_t)$. The random variable $A_T$ can be the *arithmetic average*

$$A_T = \frac{1}{T} \int_0^T S_t dt,$$

in which case the option is called an arithmetic average Asian option (AAO),

or the *geometric average*

$$A_T = \exp\left(\frac{1}{T}\int_0^T \ln S_t dt\right).$$

in which case the option is called a geometric average Asian option (GAO).

# 3 Importance Sampling for European Options

To simplify the notations, we present the stochastic volatility model in (1) in a vector form as follows

$$dV_t = b(t, V_t)dt + a(t, V_t)d\eta_t, \tag{3}$$

where we set

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad V_t = \begin{pmatrix} S_t \\ Y_t \\ Z_t \end{pmatrix}, \qquad \eta_t = \begin{pmatrix} W_t^{(0)} \\ W_t^{(1)} \\ W_t^{(2)} \end{pmatrix},$$

we define the drift

$$b(t, v) = \begin{pmatrix} rx \\ \alpha(m_f - y) - \nu_f\sqrt{2\alpha}\,\Lambda_f(y, z) \\ \delta(m_s - z) - \nu_s\sqrt{2\delta}\,\Lambda_s(y, z) \end{pmatrix},$$

and the diffusion matrix

$$a(t, v) = \begin{pmatrix} f(y, z)x & 0 & 0 \\ \nu_f\sqrt{2\alpha}\,\rho_1 & \nu_f\sqrt{2\alpha}\,\sqrt{1 - \rho_1^2} & 0 \\ \nu_s\sqrt{2\delta}\,\rho_2 & \nu_s\sqrt{2\delta}\,\rho_{12} & \nu_s\sqrt{2\delta}\,\sqrt{1 - \rho_2^2 - \rho_{12}^2} \end{pmatrix}.$$

The price $P(t, x, y, z)$ of an European option at time $t$ is given by

$$P(t, v) = I\!E^\star\left\{e^{-r(T-t)}H(S_T) \mid V_t = v\right\}. \tag{4}$$

A basic Monte Carlo approximation for the price (6) is based on calculating the sample mean

$$P(t, x, y, z) \approx \frac{1}{N}\sum_{k=1}^{N} e^{-r(T-t)}H(S_T^{(k)}), \tag{5}$$

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where $N$ is the total number of independent realizations of the process, and $S_T^{(k)}$ denotes the terminal value of the stock in the $k$-th trajectory. Importance sampling techniques consist in changing the weights of these realizations in order to reduce the variance of the estimator (10).

Under classical integrability conditions on the function $h(t, v)$, the process

$$Q_t = \exp\left(\left\{\int_0^t h(s, V_s)d\eta_s + \frac{1}{2}\int_0^t ||h(s, V_s)||^2 ds\right\}\right),$$

is a martingale, and the Radon-Nikodyn derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\star} = (Q_T)^{-1}$$

defines a new probability $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}^\star$. By Girsanov Theorem, under this new measure $\tilde{\mathbb{P}}$, the process $(\tilde{\eta}_t)$ defined by

$$\tilde{\eta}_t = \eta_t + \int_0^t h(s, V_s)ds,$$

is a standard Brownian motion. The option price $P$ can be written under $\tilde{\mathbb{P}}$ as

$$P(t, v) = \tilde{\mathbb{E}}\left\{e^{-r(T-t)}H(S_T)Q_T \mid V_t = v\right\}, \tag{6}$$

where

$$Q_T = \exp\left(\left\{\int_0^T h(s, V_s)d\tilde{\eta}_s - \frac{1}{2}\int_0^T ||h(s, V_s)||^2 ds\right\}\right), \tag{7}$$

and the dynamics of our model becomes

$$dV_t = \left(b(t, V_t) - a(t, V_t)h(t, V_t)\right)dt + a(t, V_t)d\tilde{\eta}_t. \tag{8}$$

Applying Ito's formula to $P(t, V_t)Q_t$, it is a straightforward computation to obtain

$$H(V_T)Q_T = P(t, v) + \int_t^T Q_s\left(a'\nabla P + P h\right)(s, V_s) \cdot d\tilde{\eta}_s,$$

where $a'$ denotes the transpose of $a$ and the gradient is with respect to the variable $v$. Therefore the **variance** of the payoff $H(V_T)Q_T$ in (6) is simply

$$\mathrm{Var}_{\tilde{P}}(H(V_T)Q_T) = \tilde{\mathbb{E}}\left\{\int_t^T Q_s^2||a'\nabla P + P h||^2 ds\right\}.$$

Indeed, if the quantity $P$ to be computed was known, one could obtain a zero variance by choosing

$$h = -\frac{1}{P}\left(a' \nabla P\right). \tag{9}$$

Our strategy is to use in (9) known approximations to the exact value $P$. Then the Monte Carlo simulations are done under the new measure $\tilde{\mathbb{P}}$:

$$P(t, x, y, z) \approx \frac{1}{N} \sum_{k=1}^{N} e^{-r(T-t)} H(S_T^{(k)}) Q_T^{(k)}, \tag{10}$$

where $N$ is the total number of simulations, and $S_T^{(k)}$ and $Q_T^{(k)}$ denote the final value of the $k$-th realized trajectory (8) and weight (7) respectively.

## 3.1  Vanilla European Option Price Approximations

We give here a brief review of the main result in [8] from the perturbation theory for European options under multiscale stochastic volatility models presented in (1). We introduce $\varepsilon = 1/\alpha$ and assume parameters $\varepsilon$ and $\delta$ are relatively small, $0 < \varepsilon, \delta \ll 1$. Denote by $P^{\varepsilon,\delta}$ the price of a European option with payoff function $H$, and apply the Feynman-Kac formula to (4). Then $P^{\varepsilon,\delta}(t, x, y, z)$ solves the three-dimensional partial differential equation

$$\mathcal{L}^{\varepsilon,\delta} P^{\varepsilon,\delta} = 0,$$
$$P^{\varepsilon,\delta}(T, x, y, z) = H(x),$$

where we define the partial differential operator $\mathcal{L}^{\varepsilon,\delta}$ by

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_{BS} + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3,$$

with each component operator given by:

$$\mathcal{L}_0 = \nu_f^2 \frac{\partial^2}{\partial y^2} + (m_f - y)\frac{\partial}{\partial y}, \tag{11}$$

$$\mathcal{L}_1 = \nu_f \sqrt{2}\left(\rho_1 x f(y, z)\frac{\partial^2}{\partial x \partial y} - \Lambda_f(y, z)\frac{\partial}{\partial y}\right), \tag{12}$$

$$\mathcal{L}_{BS}(f(y,z)) = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y, z)x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right), \tag{13}$$

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$$\mathcal{M}_1 = \nu_s\sqrt{2}\left(\rho_2 x f(y,z)\frac{\partial^2}{\partial x \partial z} - \Lambda_s(y,z)\frac{\partial}{\partial z}\right), \qquad (14)$$

$$\mathcal{M}_2 = \nu_s^2\frac{\partial^2}{\partial z^2} + (m_s - z)\frac{\partial}{\partial z}, \qquad (15)$$

$$\mathcal{M}_3 = 2\nu_f\nu_s\left(\rho_1\rho_2 + \rho_{12}\sqrt{1-\rho_1^2}\right)\frac{\partial^2}{\partial y \partial z}. \qquad (16)$$

By using a combination of singular and regular perturbations the following pointwise price approximation is derived in [8]

$$P^{\varepsilon,\delta}(t,x,y,z) \approx \tilde{P}(t,x,z),$$

where

$$\tilde{P} = P_{BS} \qquad (17)$$
$$+ \ (T-t)\left(V_0\frac{\partial}{\partial\sigma} + V_1 x\frac{\partial^2}{\partial x \partial\sigma} + V_2 x^2\frac{\partial^2}{\partial x^2} + V_3 x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right)\right)P_{BS},$$

with an accuracy of order $(\varepsilon|\log\varepsilon| + \delta)$ for call options. The leading order price $P_{BS}(t,x;\bar{\sigma}(z))$ is independent of the $y$ variable and is the homogenized price which solves the Black-Scholes equation

$$\mathcal{L}_{BS}(\bar{\sigma}(z))P_{BS} = 0,$$
$$P_{BS}(T,x;\bar{\sigma}(z)) = H(x).$$

Here the $z$-dependent **effective volatility** $\bar{\sigma}(z)$ is defined by

$$\bar{\sigma}^2(z) = \langle f^2(\cdot,z)\rangle, \qquad (18)$$

where the brackets denote the average with respect to the invariant distribution $\mathcal{N}(m_f,\nu_f^2)$ of the fast factor $(Y_t)$. The parameters $(V_0,V_1,V_2,V_3)$ are given by

$$V_0 = -\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle\Lambda_s\rangle\bar{\sigma}', \qquad (19)$$

$$V_1 = \frac{\rho_2\nu_s\sqrt{\delta}}{\sqrt{2}}\langle f\rangle\bar{\sigma}', \qquad (20)$$

$$V_2 = \frac{\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\left\langle\Lambda_f\frac{\partial\phi}{\partial y}\right\rangle, \qquad (21)$$

$$V_3 = -\frac{\rho_1\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\left\langle f\frac{\phi}{\partial y}\right\rangle, \qquad (22)$$

where $\overline{\sigma}'$ denotes the derivative of $\overline{\sigma}$, and the function $\phi(y, z)$ is a solution of the Poisson equation

$$\mathcal{L}_0 \phi(y, z) = f^2(y, z) - \overline{\sigma}^2(z).$$

The parameters $V_0$ and $V_1$ (resp. $V_2$ and $V_3$) are small of order $\sqrt{\varepsilon}$ (resp. $\sqrt{\delta}$). The parameters $V_0$ and $V_2$ reflects the effect of the market prices of volatility risk. The parameters $V_1$ and $V_3$ are proportional to the correlation coefficients $\rho_2$ and $\rho_1$ respectively. In [8], these parameters are calibrated using the observed implied volatilities. In the present work, the model (1) will be fully specified, and these parameters are computed using the formulas above.

## 3.2 Numerical Simulations

We consider vanilla European call options as examples for Monte Carlo simulations. From (17), we use successively $P_{BS}$ and $\tilde{P}$ as prior information on the true option price $P$ in (4), and we compare the efficiency of variance reduction by importance sampling. By taking $H(x) = (x - K)^+$, the leading order term $P_{BS}$ is given by the Black-Scholes formula

$$P_{BS}(t, x; \overline{\sigma}(z)) = x\mathcal{N}(d_1(x, z)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x, z)), \qquad (23)$$

where

$$
\begin{aligned}
d_1(x, z) &= \frac{\ln(x/K) + (r + \frac{1}{2}\overline{\sigma}^2(z))(T - t)}{\overline{\sigma}(z)\sqrt{T - t}}, \\
d_2(x, z) &= d_1(x, z) - \overline{\sigma}(z)\sqrt{T - t}, \\
\mathcal{N}(d) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-u^2/2} du.
\end{aligned}
$$

The correction in (17) is then obtained by computing the Greeks

$$\frac{\partial P_{BS}}{\partial \sigma}, \ x\frac{\partial^2 P_{BS}}{\partial x \partial \sigma}, \ x^2\frac{\partial^2 P_{BS}}{\partial x^2}, \ x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_{BS}}{\partial x^2}\right).$$

Our numerical experiments consist of substituting the approximations $P_{BS}$ or $\tilde{P}$ into (9), and compare their efficiency in reducing the variance of Monte

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Table 1: Parameters used in the two-factor stochastic volatility model (1).

| $r$ | $m_f$ | $m_s$ | $\nu_f$ | $\nu_s$ | $\rho_1$ | $\rho_2$ | $\rho_{12}$ | $\Lambda_f$ | $\lambda_s$ | $f(y,z)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10% | -0.8 | -0.8 | 0.5 | 0.8 | -0.2 | -0.2 | 0 | 0 | 0 | $\exp(y+z)$ |

Carlo simulations. We start with the homogenized price $P_{BS}$ which leads to

$$h(t,x,y,z) = \frac{-1}{P_{BS}(t,x;\bar{\sigma}(z))} a' \nabla P_{BS}(t,x;\bar{\sigma}(z))$$

$$= \frac{-1}{P_{BS}(t,x;\bar{\sigma}(z))} \begin{pmatrix} f(y,z)x & \frac{\rho_1 \nu_f \sqrt{2}}{\sqrt{\varepsilon}} & \rho_2 \nu_s \sqrt{2\delta} \\ 0 & \nu_f \frac{\sqrt{2}}{\sqrt{\varepsilon}}\sqrt{1-\rho_1^2} & \rho_{12}\nu_s\sqrt{2\delta} \\ 0 & 0 & \nu_s\sqrt{2\delta}\sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix} \begin{pmatrix} \frac{\partial P_{BS}}{\partial x} \\ \frac{\partial P_{BS}}{\partial y} \\ \frac{\partial P_{BS}}{\partial z} \end{pmatrix}$$

$$= \frac{-\frac{\partial P_{BS}}{\partial x}}{P_{BS}(t,x;\bar{\sigma}(z))} \begin{pmatrix} f(y,z)x \\ 0 \\ 0 \end{pmatrix} - \nu_s\sqrt{2\delta}\frac{\bar{\sigma}'(z)\frac{\partial P_{BS}}{\partial \sigma}}{P_{BS}(t,x;\bar{\sigma}(z))} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix},$$

where we have used that $P_{BS}$ does not depend on $y$. The Vega is given by

$$\frac{\partial P_{BS}}{\partial \sigma} = x\sqrt{T-t}\mathcal{N}'(d_1(x,z)).$$

Likewise we construct a function $\tilde{h}$ by using the higher order approximation $\tilde{P}$. Since we are only interested in terms of order less than or equal to $\sqrt{\varepsilon}$ or $\sqrt{\delta}$, we shall drop any higher order terms, and obtain

$$\tilde{h}(t,x,y,z) = \frac{-\frac{\partial \tilde{P}}{\partial x}}{\tilde{P}(t,x,z)} \begin{pmatrix} f(y,z)x \\ 0 \\ 0 \end{pmatrix} - \sqrt{2\delta}\frac{\bar{\sigma}'(z)\frac{\partial P_{BS}}{\partial \sigma}}{P_{BS}(t,x;\bar{\sigma}(z))} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix}$$

Relevant parameters and functions for this model are chosen as in Table 1. The price computations will be done with various values of the time scale parameters $\alpha$ and $\delta$ given in Table 3.

There is a total of $N = 5000$ sample paths in (10), simulated based on the discretization of the diffusion process $V_t$ using an Euler scheme [10] with time step $\Delta t = 0.005$.

The other values (initials conditions and option parameters) are given in Table 2.

Table 2: Initial conditions and call option parameters.

| $S_0$ | $Y_0$ | $Z_0$ | $K$ | $T$ years |
|---|---|---|---|---|
| 55 | -1 | -1 | 50 | 1 |

Table 3: Comparison of simulated option prices and their variances for various values of $\alpha$ and $\delta$; $P^{MC}$ is obtained by basic Monte Carlo simulation, $P_{BS}$ is computed by (23), $\tilde{P}$ by (17), $P^{IS}(P_{BS})$ and $P^{IS}(\tilde{P})$ are computed by Monte Carlo simulations using importance sampling with $P_{BS}$ and $\tilde{P}$ respectively (means are shown in parenthesis next to the variances).

| $\alpha$ | $\delta$ | $P^{MC}$ | $P_{BS}$ | $\tilde{P}$ | $P^{IS}(P_{BS})$ | $P^{IS}(\tilde{P})$ |
|---|---|---|---|---|---|---|
| 100 | 0.01 | 0.024114 (10.93) | 10.779 | 11.069 | 0.004006 (11.13) | 0.000986 (11.03) |
| 50 | 0.05 | 0.022995 (11.03) | 10.779 | 11.208 | 0.000703 (11.03) | 0.000698 (10.99) |
| 20 | 0.1 | 0.022596 (11.09) | 10.779 | 11.449 | 0.002161 (11.09) | 0.001284 (11.00) |
| 5 | 1 | 0.032745 (11.50) | 10.779 | 12.20 | 0.003841 (12.03) | 0.002435 (11.60) |

The results presented in Table 3 generalize those presented in [9] in the case of only the fast factor to the case of two factors, fast and slow. One can observe the significant variance reduction from the plain Monte Carlo simulation $P^{MC}$ to the important sampling simulations $P^{IS}(P_{BS})$ and $P^{IS}(\tilde{P})$. This reduction is indeed drastic in the regime ($\alpha$ large, $\delta$ small) where the approximation $\tilde{P}$ is very efficient, but it is also significant in the regime where the time scales are not so well-separated ($\alpha = 5$, $\delta = 1$ for instance).

# 4 Two-Step Variance Reduction for Asian Options

From the definition of arithmetic average Asian options in (2), it is convenient to introduce the running sum process $I_t = \int_0^t S_u du$ or, in its differential form,

$$dI_t = S_t dt, \tag{24}$$

such that the joint dynamics $(S_t, Y_t, Z_t, I_t)$ is Markovian. Under the risk-neutral probability measure $\mathbb{P}^\star$ the price of an arithmetic average Asian call

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option is given by

$$P(t, x, y, z, I) = E^* \left\{ e^{-r(T-t)} \left( \frac{I_T}{T} - S_T - K \right)^+ \mid S_t = x, Y_t = y, Z_t = z, I_t = I \right\}. \tag{25}$$

We will use this type of options as typical examples when we discuss the variance reduction of Monte Carlo simulations in Section 4.3.

A basic Monte Carlo simulation consists in generating $N$ independent trajectories governed by equations (1) and (24), and averaging the discounted payoffs to obtain the approximation

$$P \approx P^{MC} = \frac{e^{-r(T-t)}}{N} \sum_{k=1}^{N} \left( \frac{I_T^{(k)}}{T} - S_T^{(k)} - K \right)^+. \tag{26}$$

Since the dynamics of $(S_t, Y_t, Z_t, I_t)$ is simply a special case of (3), one would apply importance sampling to reduce variance of $P^{MC}$ in (26) by approximated price of $P$ in (25). Unlike the case of European options considered in Section 3, the approximated prices of AAOs obtained in [5] do not have close-form solutions. Consequently, one has to rely on numerical PDE solutions to evaluate price approximations along each trajectory of Monte Carlo simulations. We remark that this strategy implies tremendous computational efforts so that it is not proper to apply directly the importance sampling to evaluate AAOs. This drawback therefore motivates our investigation of a two-step variance reduction strategy by combining control variates and importance sampling.

## 4.1 Control Variates for Arithmetic Average Asian Options

In the case of constant volatility, Boyle *et al.* [2] proposed a variance reduction method for arithmetic average Asian option prices (AAOs) based on using geometric average Asian options (GAOs) as control variates. The control variate estimator $P^{CV}$ is defined by

$$P^{CV} \triangleq P^{MC} + \lambda(\hat{P}_G - P_G), \tag{27}$$

where $\hat{P}_G$ is an unbiased Monte Carlo estimator of the GAO price denoted by $P_G$, computed using the same run as for $P^{MC}$. The company price $P_G$, i.e. the

counterpart geometric average Asian option, has an analytic solution. The parameter $\lambda$ is chosen to minimize the sample variance. For Asian options, $\lambda$ is often chosen equal to -1. The methodology described above performs very well among other variance reduction methods [2].

Within the context of stochastic volatility, for example our two-factor model (1), there no longer exist close-form solutions for GAOs. In order to proceed with the control variates method described above, we propose to evaluate GAOs by Monte Carlo simulations using the variance reduction technique presented and tested in the previous section for European options.

The price of a geometric average Asian call option $P_G$ is defined by

$$P_G(t, v, L) = I\!E^{\star} \left\{ e^{-r(T-t)} \left( \exp\left( \frac{L_T}{T} \right) - S_T - K \right)^+ \mid V_t = v, L_t = L \right\}, \quad (28)$$

where the dynamics $V_t = (S_t, Y_t, Z_t)$ follows (1), and the additional running sum process $(L_t)$ is given by

$$dL_t = \ln S_t \, dt. \quad (29)$$

In Section 3, we have shown an application of importance sampling for pricing European options under two-factor stochastic volatility, in which the existence of explicit formulas for approximated European option prices are crucial. Recently, Wong and Cheung [13] derived first-order approximated GAO prices under one fast mean-reverting stochastic volatility model. In the Appendix we generalize their results to two-factor models including an additional slowly time varying mean-reverting process. We derive first-order price approximations for GAOs which admit close-form solutions. We will use those price approximations as prior knowledge of the true GAO prices such that the importance sampling technique can be applied efficiently.

## 4.2 Importance Sampling for Geometric Average Asian Options

We consider the pricing problem of GAOs given in (28). The dynamics of our model consist of $(S_t, Y_t, Z_t, L_t)$ whose transpose is denoted by $\tilde{V}_t$. The vector form of the dynamics can be represented as

$$d\tilde{V}_t = \left( b(t, \tilde{V}_t) - a(t, \tilde{V}_t) h(t, \tilde{V}_t) \right) dt + a(t, \tilde{V}_t) d\tilde{\eta}_t. \quad (30)$$

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where we set

$$
\tilde{v} = \begin{pmatrix} x \\ y \\ z \\ L \end{pmatrix}, \ b(t, \tilde{v}) = \begin{pmatrix} rx \\ \alpha(m_f - y) - \nu_f \sqrt{2\alpha}\, \Lambda_f \\ \delta(m_s - z) - \nu_s \sqrt{2\delta}\Lambda_s \\ \ln x \end{pmatrix}, \ \eta_t = \begin{pmatrix} W_t^{(0)} \\ W_t^{(1)} \\ W_t^{(2)} \\ 0 \end{pmatrix},
$$

$$
\tilde{\eta}_t \ = \ \eta_t + \int_0^t h(s, \tilde{V}_s) ds
$$

and the (degenerated) diffusion matrix is

$$
a(t, v) = \begin{pmatrix} f(y, z)x & 0 & 0 & 0 \\ \nu_f \sqrt{2\alpha}\, \rho_1 & \nu_f \sqrt{2\alpha}\, \sqrt{1 - \rho_1^2} & 0 & 0 \\ \nu_s \sqrt{2\delta} \rho_2 & \nu_s \sqrt{2\delta} \rho_{12} & \nu_s \sqrt{2\delta}\, \sqrt{1 - \rho_2^2 - \rho_{12}^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The importance sampling argument follows the same lines as in Section 3, except for the construction of a deterministic function

$$
h(t, \tilde{v}) = -\frac{1}{P_G(t, \tilde{v})} a' \tilde{\nabla} P_G(t, \tilde{v}), \tag{31}
$$

where the gradient $\tilde{\nabla}$ is taken with respect to $(x, y, z, L)$. Again, the GAO price $P_G$ in (31) is unknown, and we will use asymptotic price approximations given in the following section.

## 4.3  Two-Step Strategy and Numerical Simulations

To limit the length of this paper, we only choose geometric average Asian call options with fixed strikes as examples to demonstrate the efficiency on importance sampling variance reduction methods. We derive the first order price approximation of GAO in the Appendix A based on a combination of singular and regular perturbation analysis and the result is as follows:

$$
P_G(t, x, y, z, L) \approx \tilde{P}_G(t, x, z, L),
$$

where

$$
\begin{aligned}
\tilde{P}_G \ = \ & P_0^{fix} - (T - t)\sqrt{2} V_0 \frac{\partial P_0^{fix}}{\partial \sigma} + (T + t) V_1 x \frac{\partial^2 P_0^{fix}}{\partial x \partial \sigma} \\
& - \frac{(T - t)^2}{2} V_2 \frac{\partial P_0^{fix}}{\partial x} + \frac{(T - t)^3}{3} (V_2 - V_3) \frac{\partial^2 P_0^{fix}}{\partial x^2} + \frac{(T - t)^4}{4} V_3 \frac{\partial^3 P_0^{fix}}{\partial x^3}.
\end{aligned} \tag{32}
$$

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The zero order term $P_0^{fix}(t, x, L; \bar{\sigma})$ satisfies the homogenized Black-Schols type formula:

$$P_0^{fix}(t, x, L; \bar{\sigma}) \tag{33}$$
$$= \exp\left(\frac{L - t \ln x}{T} + \ln x + R(t, T, z)\right) \mathcal{N}(d_1(x, z, L)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x, z, L)),$$

where

$$R(t, T, z) = \left(r - \frac{\bar{\sigma}^2}{2}\right) \frac{(T - t)^2}{2T} + \bar{\sigma}^2 \frac{(T - t)^3}{6T^2} - r(T - t),$$

$$d_1(x, z, L) = \frac{T \ln(x/K) + L - t \ln x + (r - \bar{\sigma}^2/2)(T - t)^2/2 + \bar{\sigma}^2 \frac{(T-t)^3}{3T}}{\bar{\sigma}\sqrt{\frac{(T-t)^3}{3}}}$$

$$d_2(x, z, L) = d_1(x, z, L) - \bar{\sigma}\sqrt{\frac{(T - t)^3}{3T^2}}$$

The Vega of GAO is equal to

$$\frac{\partial P_0^{fix}}{\partial \sigma} = \frac{T - t}{3}\sigma x^2 \frac{\partial^2 P_0^{fix}}{\partial x^2} - \frac{T - t}{6}\sigma x \frac{\partial P_0^{fix}}{\partial x}.$$

Substituting the approximation (32) into (31), we get

$$\tilde{h}_G(t, x, z, L)$$
$$= \frac{-\frac{\partial \tilde{P}_G}{\partial x}}{\tilde{P}_G(t, x, z, L)} \begin{pmatrix} f(y, z)x \\ 0 \\ 0 \\ 0 \end{pmatrix} - \nu_s\sqrt{2\delta} \frac{\bar{\sigma}(z)\frac{\partial P_0^{fix}}{\partial \sigma}}{P_0^{fix}(t, x, z, L)} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1 - \rho_2^2 - \rho_{12}^2} \\ 0 \end{pmatrix}.$$

We present numerical results from Monte Carlo simulations to evaluate fixed-strike GAO prices in this section. Parameters in our model are shown in Table 4. The other values (initials conditions and option parameters) are given in Table 5. The sample paths in (30) are simulated based on the discretization of the diffusion process $V_t$ using an Euler scheme with time step $\Delta t = 0.005$ and the number of total paths are 5000. The price computations will be done with various values of the time scale parameters $\alpha$ and $\delta$ given in Table 6.

We now consider the control variates for AAOs with the same parameters given in Tables 4 and 5. Fixing the time scale parameters $\alpha = 75$ and

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Table 4: Parameters used in the two-factor stochastic volatility model (1).

| $r$ | $m_f$ | $m_s$ | $\nu_f$ | $\nu_s$ | $\rho_1$ | $\rho_2$ | $\rho_{12}$ | $\Lambda_f$ | $\lambda_s$ | $f(y,z)$ |
|-----|-------|-------|---------|---------|----------|----------|-------------|-------------|-------------|----------|
| 10% | -0.8 | -0.6 | 0.7 | 1 | -0.2 | -0.2 | 0 | 0 | 0 | $\exp(y+z)$ |

Table 5: Initial conditions and Asian call option parameters.

| $\$S_0$ | $Y_0$ | $Z_0$ | $L_0$ | $\$K$ | $T$ years |
|---------|-------|-------|-------|-------|-----------|
| 100 | -1 | -0.5 | 110 | 0 | 1 |

Table 6: Comparison of simulated option prices and their variances for various values of $\alpha$ and $\delta$; $P^{MC}$ is obtained by basic Monte Carlo simulation and $P_G^{IS}(\tilde{P}_G)$ are computed by Monte Carlo simulations using importance sampling with $\tilde{P}_G$ (means are shown in parenthesis next to the variances).

| $\alpha$ | $\delta$ | $P_G^{MC}$ | $P_G^{IS}(\tilde{P}_G)$ |
|----------|----------|------------|-------------------------|
| 100 | 0.05 | 0.048341 (7.97) | 0.006334 (7.76) |
| 75 | 0.1 | 0.043363 (7.57) | 0.007707 (7.46) |
| 50 | 0.5 | 0.051290 (7.45) | 0.009676 (7.17) |
| 25 | 1 | 0.058433 (7.31) | 0.014814 (6.96) |

$\delta = 0.1$, we first compute the unbiased price, $P_G$, of the counterpart GAO, then use it in (27) as a control variate. Figures 1 presents the result of Monte Carlo simulations as a function of realizations. The dash (or blue) line indicates sample means of the basic Monte Carlo with respect to the number of simulations. The solid (or green) line indicates the Monte Carlo using control variates with an unbiased estimator $P_G$, which is computed separately from the importance sampling. In Figure 1 we illustrate with an AAO that the combination of control variates using GAOs computed with importance sampling provides a great improvement on variance reduction compare to the basic Monte Carlo. The variance is reduced from $(1.5411)10^{-4}$ to $(1.6201)10^{-6}$ with sample means 8.4604 and 8.4965, respectively. These Monte Carlo simulations are done by choosing the time step equal to 0.005, and with 5000 realizations.
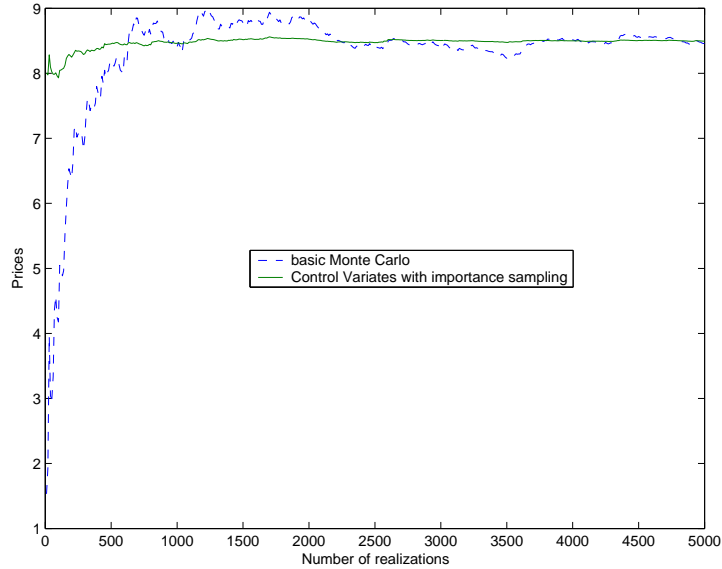


Figure 1: Mone Carlo simulations for the price of an arithmetic average Asian option. Rates of mean-reversion are chosen as $\alpha = 75$ and $\delta = 0.1$.

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# 5 Conclusion

Under the context of multi-factor stochastic volatility model, two types of derivative pricing problems, namely European options and Asian options, are dealt by Monte Carlo simulations. The first set of numerical experiments demonstrates that importance sampling methods significantly reduce variances of Monte Carlo European option estimators. In particular, the price approximations used in importance sampling are obtained from a combination of singular and regular perturbation analysis detailed in [8]. The analysis is done under the assumption of the appearance of large and small time scales in the stochastic volatility models. However, even Monte Carlo simulations are done in the regime where time scales are not well separated, we still observe gains on the variance reduction. This illustrates the robustness of these price approximations. The second set of numerical experiments deals with Asian options. We propose a two-step variance reduction strategy which combines the control variates and importance sampling. Both methods can be applied seperately and hence increase the flexibility to implement the algorithm. Moreover we derive the first order price approximations for geometric average Asian call options with fixed strikes, which play an essential role in obtaining the unbiased estimator used for arithemetic average call Asian option control variates. Other type of geometric Asian option price approximations are considered in our following work [6]. In the end, we remark that, although numerical simulations are done in two-factor stochastic volatility models, it is perceived that the same approach can be used in high dimensional problems such as taking stochastic interest rate into account.

# A First-Order Price Approximations of GAOs with Fixed Srtikes

We perform an asymptotic analysis for the pricing problems of geometric average Asian options under multiscale stochastic volatility model defined in (1). The derivation for price approximations of the prices of GAOs with floating strikes, their accuracy results, and calibration are detailed in [6]. Denote by $P^{\varepsilon,\delta}$ the price of GAO and apply the Feynman-Kac formula to (28), then $P^{\varepsilon,\delta}(t, x, y, z, L)$ solves a four-dimensional partial differential equation

$$\mathcal{L}_L^{\varepsilon,\delta} P^{\varepsilon,\delta} = 0,$$

$$P^{\varepsilon,\delta}(T, x, y, z, L) = (\exp(L/T) - K)^+,$$

where we denote the partial differential operator $\mathcal{L}_L^{\varepsilon,\delta}$ by

$$\mathcal{L}_L^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_L + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3,$$

with each component as given in (11 - 16) except

$$\mathcal{L}_L(f(y,z)) = \mathcal{L}_{BS}(f(y,z)) + \ln x\frac{\partial}{\partial L}$$

By the change of variables

$$\hat{x} = L - t\,\ln x \qquad \text{and} \qquad \hat{z} = \ln x,$$

a modified PDE is obtained

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2 + \sqrt{\delta}\hat{\mathcal{M}}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3\right)P^{\varepsilon,\delta} = 0, \quad (34)$$

$$P^{\varepsilon,\delta}(T, \hat{x}, y, z, \hat{z}) = (\exp((\hat{x} + T\hat{z})/T) - K)^+,$$

where

$$\hat{\mathcal{L}}_1 = \nu_f\sqrt{2}\left[\rho_1 f(y,z)\left(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{x}}\right)\frac{\partial}{\partial y} - \Lambda_f(y,z)\frac{\partial}{\partial y}\right],$$

$$\mathcal{L}_2(f(y,z)) = \frac{\partial}{\partial t} + \frac{f^2(y,z)}{2}\left(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{x}}\right)^2 + \left(r - \frac{f^2(y,z)}{2}\right)\left(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{x}}\right) - r\cdot,$$

$$\hat{\mathcal{M}}_1 = \nu_s\sqrt{2}\left(\rho_2 f(y,z)\left(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{x}}\right)\frac{\partial}{\partial z} - \Lambda_s(y,z)\frac{\partial}{\partial z}\right),$$

We consider an asymptotic expansion in powers of $\sqrt{\delta}$

$$P^{\varepsilon,\delta}(t, \hat{x}, y, z, \hat{z}) = P_0^\varepsilon(t, \hat{x}, y, z, \hat{z}) + \sqrt{\delta}P_1^\varepsilon(t, \hat{x}, y, z, \hat{z}) + \delta P_2(t, \hat{x}, y, z, \hat{z}) + \cdots$$

and substitute this into (34) such that

$$0 = \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2\right)P_0^\varepsilon$$

$$+ \sqrt{\delta}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2\right)P_1^\varepsilon + \hat{\mathcal{M}}_1 P_0^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3 P_0^\varepsilon\right) + \cdots$$

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is deduced. We find that the leading order term $P_0^\varepsilon$ solves the singular perturbation problem with an additional $z$-dependent variable,

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2\right)P_0^\varepsilon = 0$$

with the terminal condition $P_0^\varepsilon(T, \hat{x}, y, z, \hat{z}) = (\exp((\hat{x} + T\hat{z})/T) - K)^+$. Performing the singular perturbation detailed in [13], the following approximation is obtained

$$P_0^\varepsilon \approx P_0(t, \hat{x}, z, \hat{z}) + \tilde{P}_{1,0}(t, \hat{x}, z, \hat{z}), \tag{35}$$

where the leading order term $P_0(t, \hat{x}, z, \hat{z})$ solves

$$\begin{aligned}
\langle\mathcal{L}_2\rangle P_0 &= 0, \tag{36}\\
P_0(T, x, z, \hat{z}) &= (\exp((\hat{x} + T\hat{z})/T) - K)^+,
\end{aligned}$$

and $\tilde{P}_{1,0}(t, \hat{x}, z, \hat{z}) \equiv \sqrt{\varepsilon}P_{1,0}(t, \hat{x}, z, \hat{z})$ solves

$$\begin{aligned}
\langle\mathcal{L}_2\rangle\tilde{P}_{1,0}(t, x, z, L) = & \tag{37}\\
-V_2\left((T-t)^2\frac{\partial^2}{\partial\hat{x}^2} - (T-t)\frac{\partial}{\partial\hat{x}}\right)P_0 &- V_3\left((T-t)^3\frac{\partial^3}{\partial\hat{x}^3} - (T-t)^2\frac{\partial^2}{\partial\hat{x}^2}\right)P_0,\\
\tilde{P}_{1,0}(T, x, z, \hat{z}) = 0.
\end{aligned}$$

The small parameters $V_2$ and $V_3$ are given as in (21) and (22). In fact, there exist explicit solutions in terms of $(x, z, L)$ for these two PDEs:

1. $P_0$, given in (33), is the price of GAO with fixed strike under the effective volatility $\bar{\sigma}(z)$.

2. $\tilde{P}_{1,0}(t, x, z, L) = -\frac{(T-t)^2}{2}V_2\frac{\partial P_0^{fix}}{\partial x} + \frac{(T-t)^3}{3}(V_2 - V_3)\frac{\partial^2 P_0^{fix}}{\partial x^2} + \frac{(T-t)^4}{4}V_3\frac{\partial^3 P_0^{fix}}{\partial x^3}$.

Next, we consider the expansion of $P_1^\varepsilon(t, \hat{x}, y, z, \hat{z})$, which solves

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2\right)P_1^\varepsilon = -\left(\hat{\mathcal{M}}_1 + \frac{\mathcal{M}_3}{\sqrt{\varepsilon}}\right)P_0^\varepsilon \tag{38}$$

with a zero terminal condition. Similarly, we look for an expansion of the following form

$$P_1^\varepsilon(t, \hat{x}, y, z, \hat{z}) = P_{0,1}(t, \hat{x}, y, z, \hat{z}) + \sqrt{\varepsilon}P_{1,1}(t, \hat{x}, y, z, \hat{z}) + \varepsilon P_{2,1}(t, \hat{x}, y, z, \hat{z}) + \cdots.$$

Substituting this expansion into the PDE (38) and using the expansion (35), it follows that $P_{0,1}$, $P_{1,1}$, and $P_{2,1}$ solve the following PDEs

$$\mathcal{L}_0 P_{0,1} = 0,$$
$$\hat{\mathcal{L}}_1 P_{0,1} + \mathcal{L}_0 P_{1,1} = -\mathcal{M}_3 P_0 = 0,$$
$$\mathcal{L}_2 P_{0,1} + \hat{\mathcal{L}}_1 P_{1,1} + \mathcal{L}_0 P_{2,1} = -\hat{\mathcal{M}}_1 P_0.$$

Following a similar argument, we conclude that $P_{0,1}$ and $P_{1,1}$ are independent of the variable $y$, and $P_{0,1}$ solves

$$\langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \hat{\mathcal{M}}_1 \rangle P_0,$$

where the homogenized partial differential operator $\langle \hat{\mathcal{M}}_1 \rangle$ is written as

$$\langle \hat{\mathcal{M}}_1 \rangle = \nu_s \sqrt{2} \left( \rho_2 \langle f(y,z) \rangle \left( \frac{\partial}{\partial \hat{z}} - t \frac{\partial}{\partial \hat{x}} \right) \frac{\partial}{\partial z} - \langle \Lambda_s(y,z) \rangle \frac{\partial}{\partial z} \right).$$

Using the homogeneous property of the solution $P_0$

$$\frac{\partial^n P_0}{\partial \hat{z}^n} = T^n \frac{\partial^n P_0}{\partial \hat{x}^n},$$

we simplify

$$\langle \hat{\mathcal{M}}_1 \rangle P_0 = (T-t)\nu_s \sqrt{2} \rho_2 \langle f(y,z) \rangle \bar{\sigma}'(z) \frac{\partial^2 P_0}{\partial \hat{x} \partial \sigma} - \nu_s \sqrt{2} \langle \Lambda_s(y,z) \rangle \bar{\sigma}'(z) \frac{\partial P_0}{\partial \sigma},$$

where the Vega of $P_0$ in terms of $\hat{x}, y, z$, and $\hat{z}$ is

$$\frac{\partial P_0^{fix}}{\partial \sigma} = \frac{(T-t)^3}{3} \sigma \frac{\partial^2 P_0^{fix}}{\partial \hat{x}^2} - \frac{(T-t)^2}{2} \sigma \frac{\partial P_0^{fix}}{\partial \sigma}.$$

Since the differential operators with respect to $\hat{x}$ commute with the operator $\langle \mathcal{L}_2 \rangle$, and $P_0^{fix}$ itself is an homogeneous solution to (36), by Theorem 3.2 in [13], it is easy to obtain the following explicit solution

$$P_{0,1} = \frac{T^2 - t^2}{2} \nu_s \sqrt{2} \rho_2 \langle f \rangle \bar{\sigma}'(z) \frac{\partial^2 P_0}{\partial \hat{x} \partial \sigma} + (T-t)\nu_s \sqrt{2} \langle \Lambda_s \rangle \bar{\sigma}'(z) \frac{\partial P_0}{\partial \sigma},$$

or, in terms of $(t, x, z, L)$ with the definition $\tilde{P}_{0,1} = \sqrt{\delta} P_{0,1}$,

$$\tilde{P}_{0,1} = (T+t)x V_1 \frac{\partial^2 P_0}{\partial x \partial \sigma} - (T-t)\sqrt{2} V_0 \frac{\partial P_0}{\partial \sigma}, \tag{39}$$

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where $V_0$ and $V_1$ are the same as in (19) and (20).

**Remark:** To obtain an accuracy result of the approximation

$$P_G^{\varepsilon,\delta}(t, x, y, z, L) \approx \tilde{P}_G(t, x, y, z, L) = P_0^{fix} + \tilde{P}_{1,0} + \tilde{P}_{0,1},$$

one needs to regularize the payoff, and consider the corresponding residuals by calculating higher order derivatives of $P_0^{fix}$ with respect to $\hat{x}$ and $\hat{z}$; then one estimates the upper bound of the residuals. We refer to our ongoing work [6] for details, and we present the main result here.

For any given point $t < T$, $x \in \mathcal{R}^+$, and $(y, z, L) \in \mathcal{R}^3$, the accuracy of the approximation for fixed strike Asian call options is given by

$$\left| P_G^{\varepsilon,\delta}(t, x, y, z, L) - \tilde{P}_G(t, x, z, L) \right| \leq C \max\{\varepsilon, \delta, \sqrt{\varepsilon\delta}\}.$$

for all $0 < \delta < \bar{\delta}$ and $0 < \varepsilon < \bar{\varepsilon}$. Other types of GAO such as floating strike and the issue of calibration are also discussed in [6].

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