Covariant Time-Frequency Distributions Based on Conjugate Operators

Franz Hlawatsch and Helmut Bölcskei

Abstract— **We propose classes of quadratic time-frequency distributions that retain the inner structure of Cohen's class. Each of these classes is based on a pair of "conjugate" unitary operators producing time-frequency displacements. The classes satisfy covariance and marginal properties corresponding to these operators. For each class, we define a "central member" generalizing the Wigner distribution and the Q-distribution, and we specify a transformation by which the class can be derived from Cohen's class.**

I. Introduction

Cohen's class with signal-independent kernels (Cohen's class hereafter) consists of all quadratic time-frequency representations (QTFR's) $T_x(t,f)$ that are *covariant* to time-frequency shifts $T_{\mathbf{S}_{\tau,\nu}x}(t,f) = T_x(t-\tau, f-\nu)$ [1]–[3]. Here, $x(t)$ is a signal with Fourier transform $X(f) = \int_t x(t)\, e^{-j2\pi ft}\, dt$, and $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ with the time-shift operator $(\mathbf{T}_\tau\, x)(t) = x(t-\tau)$ and the frequency-shift operator $(\mathbf{F}_\nu\, x)(t) = x(t)\, e^{j2\pi\nu t}$. The properties of the operators $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ entail a characteristic structure of Cohen's class. In this letter, this structure will be worked out in a generalized framework. We construct QTFR classes that are based on pairs of "conjugate" operators and that satisfy generalized covariance and marginal properties [4], [5]. Due to space limitations, we summarize our results without providing proofs. The concept of conjugate operators has been developed independently in [6] and [7].

II. Conjugate Operators

We consider two operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ indexed by parameters $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{G}$ with $\mathcal{G} \subseteq \mathbb{R}$. They are assumed to be *unitary* on a linear signal space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$, and to satisfy identical *composition laws* $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 \bullet \beta_2}$ where $(\mathcal{G}, \bullet)$ is a commutative group [4], [8], [9]. The *eigenvalues* $\lambda^A_{\alpha,\tilde{\alpha}}$ and *eigenfunctions* $u^A_{\tilde{\alpha}}(t)$ of $\mathbf{A}_\alpha$ are defined by $\big(\mathbf{A}_\alpha\, u^A_{\tilde{\alpha}}\big)(t) = \lambda^A_{\alpha,\tilde{\alpha}}\, u^A_{\tilde{\alpha}}(t)$; they are indexed by a "dual parameter" $\tilde{\alpha}$. The **A**-*Fourier transform* (**A**-FT) [8] is defined as $X_A(\tilde{\alpha}) = (\mathcal{F}_A\, x)(\tilde{\alpha}) \triangleq \langle x, u^A_{\tilde{\alpha}} \rangle = \int_t x(t)\, u^{A*}_{\tilde{\alpha}}(t)\, dt$. Analogous definitions apply to $\lambda^B_{\beta,\tilde{\beta}}$, $u^B_{\tilde{\beta}}(t)$, and $X_B(\tilde{\beta}) = (\mathcal{F}_B\, x)(\tilde{\beta})$. We now assume that applying one operator to an eigenfunction of the other operator merely shifts the eigenfunction parameter [4], [5]:

Manuscript received March 29, 1995. This work was supported by FWF grants P10012-ÖPH and P10531-ÖPH. The associate editor coordinating the review of this letter and approving it for publication was Prof. A. Tewfik.

The authors are with INTHFT, Vienna University of Technology, Vienna, Austria.

Publisher Item Identifier S 1070-9908(96)01162-5.

Definition 1. Two operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ as described above will be called *conjugate* if $\tilde{\alpha} \in \mathcal{G}$, $\tilde{\beta} \in \mathcal{G}$ and

$$\big(\mathbf{B}_\beta\, u^A_{\tilde{\alpha}}\big)(t) = u^A_{\tilde{\alpha}\bullet\beta}(t)\,, \qquad \big(\mathbf{A}_\alpha\, u^B_{\tilde{\beta}}\big)(t) = u^B_{\tilde{\beta}\bullet\alpha}(t)\,.$$

Two conjugate operators $\mathbf{A}_\alpha, \mathbf{B}_\beta$ can be shown to satisfy the following remarkable properties [4]:

1) Their eigenvalues can be written as $\lambda^A_{\alpha,\tilde{\alpha}} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ and $\lambda^B_{\beta,\tilde{\beta}} = e^{\mp j2\pi\,\mu(\beta)\,\mu(\tilde{\beta})} = \big(\lambda^A_{\beta,\tilde{\beta}}\big)^*$. Here, $\mu(g) \in \mathbb{R}$ maps $(\mathcal{G}, \bullet)$ onto $(\mathbb{R}, +)$ in the sense that $\mu(g_1 \bullet g_2) = \mu(g_1) + \mu(g_2)$, $\mu(g_0) = 0$, and $\mu(g^{-1}) = -\mu(g)$ where g_0 is the identity element in $\mathcal{G}$ and g^{-1} denotes the group-inverse of g. In the following, we shall simply write $\lambda^A_{\alpha,\beta} = \lambda_{\alpha,\beta}$ and $\lambda^B_{\alpha,\beta} = \lambda^*_{\alpha,\beta}$.
2) They commute up to a phase factor, $\mathbf{A}_\alpha \mathbf{B}_\beta = \lambda_{\alpha,\beta}\, \mathbf{B}_\beta \mathbf{A}_\alpha$.
3) Their eigenfunctions are related as $\langle u^B_{\tilde{\beta}}, u^A_{\tilde{\alpha}} \rangle = \lambda_{\tilde{\alpha},\tilde{\beta}}$, $\int_{\mathcal{G}} u^B_{\tilde{\beta}}(t)\, \lambda^*_{\tilde{\alpha},\tilde{\beta}}\, d\mu(\tilde{\beta}) = u^A_{\tilde{\alpha}}(t)$, and $\int_{\mathcal{G}} u^A_{\tilde{\alpha}}(t)\, \lambda_{\tilde{\beta},\tilde{\alpha}}\, d\mu(\tilde{\alpha}) = u^B_{\tilde{\beta}}(t)$, where $d\mu(g) \triangleq |\mu'(g)|\, dg$.
4) The inner product of their kernels is $\int_t \int_{t'} A_\alpha(t,t')\, B^*_\beta(t,t')\, dt\, dt' = \delta\big(\mu(\alpha)\big)\ \ \delta\big(\mu(\beta)\big)$ where $\delta(\cdot)$ denotes the Dirac delta function (cf. [10]).
5) The **A**-FT and **B**-FT satisfy $\big(\mathcal{F}_A \mathbf{B}_\beta\, x\big)(\tilde{\alpha}) = (\mathcal{F}_A\, x)(\tilde{\alpha} \bullet \beta^{-1})$ and $\big(\mathcal{F}_B \mathbf{A}_\alpha\, x\big)(\tilde{\beta}) = (\mathcal{F}_B\, x)(\tilde{\beta} \bullet \alpha^{-1})$, and they are related as $X_B(\tilde{\beta}) = \int_{\mathcal{G}} X_A(\tilde{\alpha})\, \lambda^*_{\tilde{\beta},\tilde{\alpha}}\, d\mu(\tilde{\alpha})$ and $X_A(\tilde{\alpha}) = \int_{\mathcal{G}} X_B(\tilde{\beta})\, \lambda_{\tilde{\alpha},\tilde{\beta}}\, d\mu(\tilde{\beta})$ (cf. [6], [7]).

We now compose two conjugate operators $\mathbf{A}_\alpha, \mathbf{B}_\beta$ as $\mathbf{D}_\theta \triangleq \mathbf{B}_\beta \mathbf{A}_\alpha$ where $\theta = (\alpha, \beta) \in \mathcal{G}^2$ with $\mathcal{G}^2 = \mathcal{G} \times \mathcal{G}$. It is readily shown that $\mathbf{D}_\theta$ is unitary on $\mathcal{X}$ and satisfies the *composition property* [4], [11] $\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = \lambda_{\alpha_2,\beta_1}\, \mathbf{D}_{\theta_1 \circ \theta_2}$ where $(\mathcal{G}^2, \circ)$ is the commutative group with group operation $\theta_1 \circ \theta_2 = (\alpha_1, \beta_1) \circ (\alpha_2, \beta_2) = (\alpha_1 \bullet \alpha_2, \beta_1 \bullet \beta_2)$, identity element $\theta_0 = (g_0, g_0)$, and inverse elements $\theta^{-1} = (\alpha^{-1}, \beta^{-1})$. Furthermore, $\mathbf{D}^{-1}_\theta = \lambda_{\alpha,\beta}\, \mathbf{D}_{\theta^{-1}}$ and $\mathbf{D}_{\theta_0} = \mathbf{I}$ where $\mathbf{I}$ is the identity operator on $\mathcal{X}$.

Examples. The shift operators $\mathbf{T}_\tau$, $\mathbf{F}_\nu$ underlying Cohen's class are conjugate with $(\mathcal{G}, \bullet) = (\mathbb{R}, +)$, $\mu(g) = g$, eigenvalues $\lambda^T_{\tau,f} = e^{-j2\pi\tau f}$, $\lambda^F_{\nu,t} = e^{j2\pi\nu t}$, eigenfunctions $u^T_f(t) = e^{j2\pi ft}$, $u^F_t(t') = \delta(t'-t)$, and dual parameters $\tilde{\tau} = f$, $\tilde{\nu} = t$. The operators are conjugate since $\big(\mathbf{F}_\nu\, u^T_f\big)(t) = u^T_{f+\nu}(t)$ and $\big(\mathbf{T}_\tau\, u^F_t\big)(t') = u^F_{t+\tau}(t')$. The operators underlying the *hyperbolic* QTFR's class [12] are conjugate as well, but the operators underlying the *affine class* and the *power classes* [13]-[15] are *not* conjugate.

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III. Covariance And Marginal Properties

Let $\nu_{\tilde{\alpha}}^{A}(t)$ and $\tau_{\tilde{\beta}}^{B}(f)$ denote the instantaneous frequency and group delay of the eigenfunctions $u_{\tilde{\alpha}}^{A}(t)$ and $u_{\tilde{\beta}}^{B}(t)$, respectively. For any $\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta}) \in \mathcal{G}^2$, the corresponding functions $\nu_{\tilde{\alpha}}^{A}(t)$ and $\tau_{\tilde{\beta}}^{B}(f)$ are assumed[1] to intersect in a unique time-frequency (TF) point $z = (t, f)$. Hence, $z = l(\tilde{\theta})$ where $l(\tilde{\theta})$ will be called the *localization function* (LF) of the operator $\mathbf{D}_\theta$ [4], [5]. The LF is constructed by solving the system of equations $\nu_{\tilde{\alpha}}^{A}(t) = f$, $\tau_{\tilde{\beta}}^{B}(f) = t$ for $(t, f) = z$ [4], [10], [16]. It is assumed to be invertible, i.e., $z = l(\tilde{\theta}) \Leftrightarrow \tilde{\theta} = l^{-1}(z)$.

The LF describes the *TF displacements* caused by $\mathbf{D}_\theta$. If a signal $x(t)$ is localized about a TF point $z = (t, f)$, then $(\mathbf{D}_\theta x)(t)$ will be localized about a new TF point $z' = (t', f')$. Since z is the intersection[2] of $u_{\tilde{\alpha}}^{A}(t)$ and $u_{\tilde{\beta}}^{B}(t)$ with $(\tilde{\alpha}, \tilde{\beta}) = \tilde{\theta} = l^{-1}(z)$, z' will be the intersection of $(\mathbf{D}_\theta u_{\tilde{\alpha}}^{A})(t)$ and $(\mathbf{D}_\theta u_{\tilde{\beta}}^{B})(t)$. Due to the conjugateness of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, $(\mathbf{D}_\theta u_{\tilde{\alpha}}^{A})(t) = \lambda_{\alpha,\tilde{\alpha}}\, u_{\tilde{\alpha}\bullet\beta}^{A}(t)$ and $(\mathbf{D}_\theta u_{\tilde{\beta}}^{B})(t) = \lambda_{\beta,\tilde{\beta}\bullet\alpha}^{*}\, u_{\tilde{\beta}\bullet\alpha}^{B}(t)$. Hence, $z' = l(\tilde{\alpha} \bullet \beta, \tilde{\beta} \bullet \alpha) = l(\tilde{\theta} \circ \theta^T) = l(l^{-1}(z) \circ \theta^T)$ with $\theta^T = (\beta, \alpha)$. This motivates the following definition [4], [5]:

Definition 2. A QTFR $T_x(z) = T_x(t, f)$ will be called *covariant to* $\mathbf{D}_\theta$ if

$$T_{\mathbf{D}_\theta x}(z) = T_x\Big(l\big(l^{-1}(z) \circ \theta^{-T}\big)\Big) \tag{1}$$

with $\theta^{-T} = (\theta^{-1})^T = (\beta^{-1}, \alpha^{-1})$.

The class of all QTFR's covariant to $\mathbf{D}_\theta$ is characterized as follows (cf. [4], [11]):

Theorem 1. A QTFR $T_x(z) = T_x(t, f)$ is covariant to an operator $\mathbf{D}_\theta$ if and only if

$$\begin{aligned} T_x(z) &= \langle x, \mathbf{H}_z^D x \rangle \\ &= \int_{t_1}\int_{t_2} x(t_1)\, x^*(t_2)\, h_z^{D*}(t_1, t_2)\, dt_1 dt_2 \end{aligned} \tag{2}$$

with $\mathbf{H}_z^D = \mathbf{D}_{[l^{-1}(z)]^T} \mathbf{H} \mathbf{D}_{[l^{-1}(z)]^T}^{-1}$, i.e. $h_z^D(t_1, t_2) = \int_{t_1'}\int_{t_2'} D_{[l^{-1}(z)]^T}(t_1, t_1')\;\; h(t_1', t_2')\;\; D_{[l^{-1}(z)]^T}^{-1}(t_2', t_2)\;\; dt_1' dt_2'$. Here, $\mathbf{H}$ is an arbitrary linear operator with kernel $h(t_1, t_2)$, assumed independent of $x(t)$, and $D_\theta(t_1, t_2)$ and $D_\theta^{-1}(t_1, t_2)$ are the kernels of $\mathbf{D}_\theta$ and $\mathbf{D}_\theta^{-1}$, respectively.

For given operator $\mathbf{D}_\theta$, (2) defines a class of QTFR's parameterized by the 2-D kernel $h(t_1, t_2)$ of the operator $\mathbf{H}$. This class consists of *all* QTFR's satisfying the covariance (1). For $\mathbf{D}_\theta = \mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$, (1) becomes the TF shift covariance $T_{\mathbf{S}_{\tau,\nu} x}(t, f) = T_x(t - \tau, f - \nu)$, and (2) becomes Cohen's class where $h_z^D(t_1, t_2) = h_z^S(t_1, t_2) = h(t_1 - t, t_2 - t)\, e^{j2\pi f(t_1 - t_2)}$.

[1] In certain cases, this assumption holds if one uses the group delay of $u_{\tilde{\alpha}}^{A}(t)$ and the instantaneous frequency of $u_{\tilde{\beta}}^{B}(t)$; here, an analogous theory can be formulated.

[2] z is the intersection of $u_{\tilde{\alpha}}^{A}(t)$ and $u_{\tilde{\beta}}^{B}(t)$ in the sense that $u_{\tilde{\alpha}}^{A}(t)$ and $u_{\tilde{\beta}}^{B}(t)$ are concentrated, in the TF plane, along $\nu_{\tilde{\alpha}}^{A}(t)$ and $\tau_{\tilde{\beta}}^{B}(f)$, respectively, and z is the intersection of $\nu_{\tilde{\alpha}}^{A}(t)$ and $\tau_{\tilde{\beta}}^{B}(f)$.

Besides the covariance property (1), the *marginal properties* [4], [8], [17]

$$\begin{aligned} &\int_{\mathcal{G}} T_x\big(l(\tilde{\theta})\big)\, d\mu(\tilde{\beta}) = |X_A(\tilde{\alpha})|^2, \\ &\int_{\mathcal{G}} T_x\big(l(\tilde{\theta})\big)\, d\mu(\tilde{\alpha}) = |X_B(\tilde{\beta})|^2 \end{aligned} \tag{3}$$

are of importance. A class of QTFR's satisfying (3) is

$$\bar{T}_x(z) = \iint_{\mathcal{G}^2} \Psi(\theta)\, A_x^D(\theta)\, \Lambda\big(l^{-1}(z), \theta\big)\, d\mu^2(\theta) \tag{4}$$

where $\Lambda(\tilde{\theta}, \theta) \triangleq \lambda_{\alpha,\tilde{\alpha}}\, \lambda_{\beta,\tilde{\beta}}^{*}$, $A_x^D(\theta) \triangleq \langle \mathbf{D}_{\theta^{-1/2}} x, \mathbf{D}_{\theta^{1/2}} x \rangle = \lambda_{\alpha,\beta}^{-1/2} \langle x, \mathbf{D}_\theta x \rangle$ (the "characteristic function"[3]), $d\mu^2(\theta) \triangleq d\mu(\alpha)\, d\mu(\beta)$, and $\Psi(\theta) = \Psi(\alpha, \beta)$ is a kernel (assumed independent of $x(t)$) satisfying $\Psi(\alpha, g_0) = \Psi(g_0, \beta) = 1$ [4], [8], [17]. In the case of the conjugate operators $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$, the marginal properties (3) become $\int_t T_x(t, f)\, dt = |X(f)|^2$ and $\int_f T_x(t, f)\, df = |x(t)|^2$, $A_x^D(\theta) = A_x^S(\tau, \nu)$ becomes the symmetric ambiguity function [3], and the QTFR class (4) becomes Cohen's class.

So far, we have formulated the QTFR class $\mathcal{T} = \{T_x(z)\}$ in (2) comprising all QTFR's satisfying the covariance property (1), and the QTFR class $\bar{\mathcal{T}} = \{\bar{T}_x(z)\}$ in (4) related to the marginal properties (3). These classes are equivalent in the conjugate case [4], [5]:

Theorem 2. For conjugate operators $\mathbf{A}_\alpha, \mathbf{B}_\beta$, there is $\mathcal{T} = \bar{\mathcal{T}}$ or equivalently $T_x(z) \equiv \bar{T}_x(z)$ where the kernel $h(t_1, t_2)$ of $T_x(z)$ and the kernel $\Psi(\theta)$ of $\bar{T}_x(z)$ are related as $h(t_1, t_2) = \iint_{\mathcal{G}^2} \Psi^*(\theta)\, D_\theta(t_1, t_2)\, \lambda_{\alpha,\beta}^{1/2}\, d\mu^2(\theta)$.

Hence, in the conjugate case considered, the "covariance approach" and the "characteristic function approach" to the construction of QTFR classes are fully equivalent.

With $\Psi(\theta) \equiv 1$, the "central member" $W_x^D(z) \triangleq \iint_{\mathcal{G}^2} A_x^D(\theta)\; \Lambda\big(l^{-1}(z), \theta\big)\; d\mu^2(\theta)$ of the QTFR class $\mathcal{T} = \bar{\mathcal{T}}$ is obtained [5], [18]. It can be expressed as

$$\begin{aligned} W_x^D(z) &= \int_{\mathcal{G}} X_A(\tilde{\alpha} \bullet \beta^{1/2})\, X_A^*(\tilde{\alpha} \bullet \beta^{-1/2})\, \lambda_{\beta,\tilde{\beta}}^{*}\, d\mu(\beta) \\ &= \int_{\mathcal{G}} X_B(\tilde{\beta} \bullet \alpha^{1/2})\, X_B^*(\tilde{\beta} \bullet \alpha^{-1/2})\, \lambda_{\alpha,\tilde{\alpha}}\, d\mu(\alpha) \end{aligned}$$

where$(\tilde{\alpha}, \tilde{\beta}) = l^{-1}(z)$. Any QTFR $T_x(z)$ of $\mathcal{T} = \bar{\mathcal{T}}$ can be derived from $W_x^D(z)$ as

$$T_x(z) = \iint_{\mathcal{G}^2} W_x^D\big(l(\tilde{\theta})\big)\, \psi\Big(l^{-1}(z) \circ \tilde{\theta}^{-1}\Big)\, d\mu^2(\tilde{\theta})$$

where $\psi(\tilde{\theta}) = \iint_{\mathcal{G}^2} \Psi(\theta)\, \Lambda(\tilde{\theta}, \theta)\, d\mu^2(\theta)$ [5]. In the special cases of Cohen's class and the hyperbolic class, the central member becomes the Wigner distribution and the Q-distribution, respectively [3], [12].

[3] We note that $\theta^{1/2}$ is defined by $\theta^{1/2} \circ \theta^{1/2} = \theta$, and that $\lambda_{\alpha,\beta}^{-1/2} = (e^{\pm j2\pi\, \mu(\alpha)\, \mu(\beta)})^{-1/2} = e^{\mp j\pi\, \mu(\alpha)\, \mu(\beta)}$.

IV. Transformation of Operators and QTFR Classes

The QTFR class $\mathcal{T} = \tilde{\mathcal{T}}$ can be constructed using a transformation approach, a fact linking our theory to the "warping" theory in [10], [16]. Let $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ be conjugate operators on a signal space $\mathcal{X}$, with group $(\mathcal{G}, \bullet)$, and consider the operators $\mathbf{C}_\gamma \triangleq \mathbf{V}\,\mathbf{A}_{s(\gamma)}\mathbf{V}^{-1}$ and $\mathbf{D}_\delta \triangleq \mathbf{V}\,\mathbf{B}_{s(\delta)}\mathbf{V}^{-1}$. Here, $\mathbf{V}$ is an isometric isomorphism mapping $\mathcal{X}$ onto some other space $\mathcal{Y}$, and $s(\cdot)$ is a one-to-one function mapping some commutative group $(\mathcal{H}, *)$ onto $(\mathcal{G}, \bullet)$, such that $s(h_1 * h_2) = s(h_1) \bullet s(h_2)$ for all $h_1, h_2 \in \mathcal{H}$. Assuming suitable choice of the dual parameters $\tilde{\gamma}$ and $\tilde{\delta}$, the eigenvalues/functions of $\mathbf{C}_\gamma$ and $\mathbf{D}_\delta$ are $\lambda^C_{\gamma,\tilde{\gamma}} = \lambda^A_{s(\gamma),s(\tilde{\gamma})}$, $u^C_{\tilde{\gamma}}(t) = \left(\mathbf{V} u^A_{s(\tilde{\gamma})}\right)(t)$ and $\lambda^D_{\delta,\tilde{\delta}} = \lambda^B_{s(\delta),s(\tilde{\delta})}$, $u^D_{\tilde{\delta}}(t) = \left(\mathbf{V} u^B_{s(\tilde{\delta})}\right)(t)$, respectively, and $\mathbf{C}_\gamma$ and $\mathbf{D}_\delta$ are *conjugate* operators on $\mathcal{Y}$, with group $(\mathcal{H}, *)$. Thus, isometric isomorphisms $\mathbf{V}$ and one-to-one group transformations $s(\cdot)$ preserve the conjugateness property of two operators. The following theorem [5] states that *any* QTFR class $\mathcal{T} = \tilde{\mathcal{T}}$ corresponding to conjugate operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ can be derived from Cohen's class using a transformation. Similar results have been derived independently in [6], [7].

Theorem 3: Let $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ be conjugate with group $(\mathcal{G}, \bullet)$ corresponding to function $\mu(\cdot)$, so that $\lambda^A_{\alpha,\tilde{\alpha}} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$. If $\lambda^A_{\alpha,\tilde{\alpha}} = e^{-j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ ($-$ sign), then $\mathbf{A}_\alpha = \mathbf{V}\,\mathbf{T}_{t_r\mu(\alpha)}\mathbf{V}^{-1}$ and $\mathbf{B}_\beta = \mathbf{V}\,\mathbf{F}_{\mu(\beta)/t_r}\mathbf{V}^{-1}$, where $t_r > 0$ is an arbitrary reference time constant, and $(\mathbf{V}_x^{-1})(t) = \frac{1}{\sqrt{t_r}} X_B(\mu^{-1}(\frac{t}{t_r}))$ with $\mu^{-1}(\cdot)$ denoting the function inverse to $\mu(\cdot)$. Furthermore, any QTFR $T_x(z) = T_x(t,f)$ of the QTFR class $\mathcal{T} = \tilde{\mathcal{T}}$ associated to $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ can be derived from a corresponding QTFR $C_x(t,f)$ of Cohen's class as

$$T_x(z) = C_{\mathbf{V}^{-1}x}\left(t_r\mu(\tilde{\beta}), \frac{\mu(\tilde{\alpha})}{t_r}\right)\Bigg|_{\tilde{\theta}=l^{-1}(z)}$$

where $l^{-1}(\cdot)$ is the inverse LF of $\mathbf{D}_\theta = \mathbf{B}_\beta\mathbf{A}_\alpha$. If $\lambda^A_{\alpha,\tilde{\alpha}} = e^{j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ (+ sign), then the above relations have to be replaced by $\mathbf{A}_\alpha = \mathbf{V}\,\mathbf{F}_{\mu(\alpha)/t_r}\mathbf{V}^{-1}$ and $\mathbf{B}_\beta = \mathbf{V}\,\mathbf{T}_{t_r\mu(\beta)}\mathbf{V}^{-1}$, $(\mathbf{V}_x^{-1})(t) = \frac{1}{\sqrt{t_r}} X_A(\mu^{-1}(\frac{t}{t_r}))$, and $T_x(z) = C_{\mathbf{V}^{-1}x}\left(t_r\mu(\tilde{\alpha}), \frac{\mu(\tilde{\beta})}{t_r}\right)\Big|_{\tilde{\theta}=l^{-1}(z)}$.

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