# Variation Formulas for $L_{1}$-Principal Functions and Application to the Simultaneous Uniformization Problem 

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## 1. Introduction

Let $\mathcal{R}=\bigcup_{t \in B}(t, R(t))$ and $\widetilde{\mathcal{R}}=\bigcup_{t \in B}(t, \tilde{R}(t))$ be unramified sheeted domains over $B \times \mathbb{C}_{z}$, where $B=\left\{|t|_{\widetilde{R}}<\rho\right\}$ is a disk in $\mathbb{C}_{t}$ and $R(t) \Subset \tilde{R}(t)$ for $t \in B$. We set $\partial \mathcal{R}=\bigcup_{t \in B}(t, \partial R(t))$ in $\widetilde{\mathcal{R}}$. In this paper, we assume that

$$
\mathcal{R}: t \in B \rightarrow R(t)
$$

is a $C^{\omega}$ smooth variation of domains $R(t)$ with $C^{\omega}$ smooth boundaries in $\tilde{R}(t)$. Namely, we can choose a real-analytic defining function $\varphi(t, z)$ of $\partial \mathcal{R}$ such that $\frac{\partial \varphi}{\partial z} \neq 0$ on $\partial \mathcal{R}$. We denote by $C_{j}(t)(j=0,1, \ldots, v)$, where $v \geq 0$ is independent of $t \in B$, the boundary contours of $R(t)$ in $\tilde{R}(t)$ with the orientation $\partial R(t)=$ $\sum_{j=0}^{v} C_{j}(t)$. Assume that the total space $\mathcal{R}$ contains $B \times\{0\}$. Precisely, there exists at least one constant section $\mathbf{O}$ of $\mathcal{R}$ over $B \times\{0\}$. For each $t \in B$, we conventionally write 0 for the point $\mathbf{O} \cap R(t)$.

Let $t \in B$ be fixed. In the theory of one complex variable, it is known that there exists a unique real-valued function $u(t, z)$ on $R(t) \backslash\{0\}$ satisfying the following four conditions:
(1) $u(t, z)$ is harmonic on $R(t) \backslash\{0\}$ and is continuous on $\overline{R(t)}$;
(2) $u(t, z)-\log \frac{1}{|z|}$ is harmonic at $z=0$;
(3) $u(t, z)=0$ on $C_{0}(t)$;
(4) for each $i=1, \ldots, v$, we have
(i) $u(t, z)=a_{i}(t)$ : constant on $C_{i}(t)$ and
(ii) $\int_{C_{i}(t)} * d u(t, z)=0$.

We note that $u(t, z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \tilde{R}(t)$. By (2), we find a neighborhood $U_{0}(t)$ of $z=0$ such that

$$
\begin{equation*}
u(t, z)=\log \frac{1}{|z|}+\gamma(t)+h(t, z) \text { on } U_{0}(t) \tag{1.1}
\end{equation*}
$$

where $\gamma(t)$ is the constant term and $h(t, z)$ is harmonic for $z$ on $U_{0}(t)$ such that

$$
\begin{equation*}
h(t, 0)=0, \quad t \in B . \tag{1.2}
\end{equation*}
$$

[^0]The function $u(t, z)$ is called the $L_{1}$-principal function on $R(t)$ with logarithmic pole at 0 with respect to $C_{0}(t)$, and $\gamma(t)$ is called the $L_{1}$-constant for $(R(t), 0)$ with respect to $C_{0}(t)$ (cf. [7]). In this paper, we simply call $u(t, z)$ the $L_{1}$-principal function for $\left(R(t), 0, C_{0}(t)\right)$ and call $\gamma(t)$ the $L_{1}$-constant for $\left(R(t), 0, C_{0}(t)\right)$. We note that $u(t, z)>0$ in $R(t) \backslash\{0\}$ and that $a_{i}(t)>0(i=1, \ldots, v)$.

Then we have the following variation formula for the $L_{1}$-constant $\gamma(t)$ for $\left(R(t), 0, C_{0}(t)\right)$.

Lemma 1.1. It holds for $t \in B$ that

$$
\frac{\partial^{2} \gamma(t)}{\partial t \partial \bar{t}}=-\frac{1}{\pi} \int_{\partial R(t)} k_{2}(t, z)\left|\frac{\partial u(t, z)}{\partial z}\right|^{2} d s_{z}-\frac{4}{\pi} \iint_{R(t)}\left|\frac{\partial^{2} u(t, z)}{\partial \bar{t} \partial z}\right|^{2} d x d y
$$

where

$$
k_{2}(t, z)=\left(\frac{\partial^{2} \varphi}{\partial t \partial \bar{t}}\left|\frac{\partial \varphi}{\partial z}\right|^{2}-2 \operatorname{Re}\left\{\frac{\partial^{2} \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}}\right\}+\left|\frac{\partial \varphi}{\partial t}\right|^{2} \frac{\partial^{2} \varphi}{\partial z \partial \bar{z}}\right) /\left|\frac{\partial \varphi}{\partial z}\right|^{3}
$$

on $\partial \mathcal{R}$ and does not depend on the choice of defining functions $\varphi(t, z)$ of $\partial \mathcal{R}$ and where $d s_{z}$ is the arc length element of $\partial R(t)$ at $z$.

The function $k_{2}(t, z)$ on $\partial \mathcal{R}$ is due to Maitani and Yamaguchi in [4], which is based on [3]. This variation formula is formally the same as that for the Robin constant $\lambda(t)$ (induced by the Green function $g(t, z)$ on $R(t)$ with logarithmic pole at $z=0$ ) in [4, Thm. 3.1]. The essential difference in the proofs for $\gamma(t)$ and $\lambda(t)$ is because, unlike the Green function $g(t, z), u(t, z)$ is not a defining function of $\partial \mathcal{R}$.

Theorem 1.2. Under the same conditions as in Lemma 1.1, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_{z}$ then $\gamma(t)$ is a $C^{\omega}$ superharmonic function on $B$.

Remark 1. For Lemma 1.1, we assumed that $\mathcal{R}$ is unramified over $B \times \mathbb{C}_{z}$. However, Lemma 1.1 (and hence Theorem 1.2) holds even if each $R(t), t \in B$, has a finite number of branch points $\zeta_{k}(t)(k=1, \ldots, m)$ such that, for $t \in B, \zeta_{k}(t)$ is a holomorphic function on $B$ with $\zeta_{k}(t) \neq \zeta_{l}(t)(k \neq l)$. The reason is that this case can be reduced to Lemma 1.1 via the standard method by using Nishimura's theorem [5].

In the special case when $R(t)$ is a planar Riemann surface, the $L_{1}$-principal function $u(t, z)$ induces a circular slit mapping $f(t, z)$. That is, if we choose a branch $u^{*}(t, z)$ of a harmonic conjugate function of $u(t, z)$ on $R(t), t \in B$, such that

$$
f(t, z)=e^{\gamma(t)-\left(u(t, z)+i u^{*}(t, z)\right)}
$$

is of the form

$$
w=f(t, z)=z+\sum_{j=2}^{\infty} b_{j}(t) z^{j} \text { on } U_{0}(t)
$$

then $f(t, z)$ conformally maps $R(t)$ onto a circular slit domain $\left\{|w|<e^{\gamma(t)}\right\} \backslash$ $\left(\bigcup_{i=1}^{v} \ell_{i}\right)$, where $\ell_{i}(t)=f\left(t, C_{i}(t)\right)$ (an arc of the circle $\left\{|w|=e^{\gamma(t)-a_{i}(t)}\right.$ ). If $\mathcal{R}$
is pseudoconvex over $B \times \mathbb{C}_{z}$ then $e^{\gamma(t)}$ is logarithmic superharmonic on $B$, so the total space $\bigcup_{t \in B}\left\{|w|<e^{\gamma(t)}\right\}$ is a Hartogs domain in $B \times \mathbb{C}_{w}$.

Remark 2. We note that the same property of radial slit mapping does not hold for the radius $r_{0}(t)$. Indeed, as will be shown in the next section, there exist pseudoconvex domains $\mathcal{R}$ in $B \times \mathbb{C}_{z}$ such that the $\log r_{0}(t)$ are neither superharmonic nor subharmonic on $B$.

Under the same conditions as for the unramified domain $\mathcal{R}=\bigcup_{t \in B}(t, R(t))$ in $\widetilde{\mathcal{R}}$ over $B \times \mathbb{C}_{z}$ and for $\partial R(t)=\sum_{j=0}^{v} C_{j}(t)$, we assume that there exist two holomorphic sections,

$$
\Xi_{0}: z=0 \quad \text { and } \quad \Xi_{1}: z=\xi(t)
$$

of $\mathcal{R}$ over $B$ such that $\Xi_{0} \cap \Xi_{1}=\emptyset$. Let $t \in B$ be fixed. In the theory of one complex variable, there exists a unique real-valued function $p(t, z)$ on $R(t) \backslash\{0, \xi(t)\}$ satisfying the following four conditions:
(I) $p(t, z)$ is harmonic on $R(t) \backslash\{0, \xi(t)\}$ and continuous on $\overline{R(t)}$;
(II) $p(t, z)-\log \frac{1}{|z|}$ is harmonic at $z=0$ and

$$
\lim _{z \rightarrow 0}\left(p(t, z)-\log \frac{1}{|z|}\right)=0
$$

(III) $p(t, z)-\log |z-\xi(t)|$ is harmonic at $z=\xi(t)$;
(IV) for each $j=0,1, \ldots, v$, we have
(i) $p(t, z)=a_{j}(t)$ : constant on $C_{j}(t)$ and
(ii) $\int_{C_{j}(t)} * d p(t, z)=0$.

We note that $p(t, z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \tilde{R}(t),-\infty<p(t, z)<+\infty$, and $-\infty<$ $a_{j}(t)<+\infty$.

By (II), we find a neighborhood $U_{0}(t)$ of $z=0$ such that

$$
\begin{equation*}
p(t, z)=\log \frac{1}{|z|}+h_{0}(t, z) \text { on } U_{0}(t), \tag{1.3}
\end{equation*}
$$

where $h_{0}(t, z)$ is harmonic for $z$ on $U_{0}(t)$ and

$$
\begin{equation*}
h_{0}(t, 0)=0, \quad t \in B \tag{1.4}
\end{equation*}
$$

By (III), we find a neighborhood $U_{\xi}(t)$ of $z=\xi(t)$ such that

$$
\begin{equation*}
p(t, z)=\log |z-\xi(t)|+\alpha(t)+h_{\xi}(t, z) \text { on } U_{\xi}(t), \tag{1.5}
\end{equation*}
$$

where $\alpha(t)$ is a real constant, $h_{\xi}(t, z)$ is harmonic for $z$ on $U_{\xi}(t)$, and

$$
\begin{equation*}
h_{\xi}(t, \xi(t))=0, \quad t \in B \tag{1.6}
\end{equation*}
$$

In this paper, we simply call $p(t, z)$ the $L_{1}$-principal function for $(R(t), 0, \xi(t))$ and call $\alpha(t)$ the $L_{1}$-constant for $(R(t), 0, \xi(t))$.

Under this situation, we have the following.

Lemma 1.3. It holds for $t \in B$ that

$$
\frac{\partial^{2} \alpha(t)}{\partial t \partial \bar{t}}=\frac{1}{\pi} \int_{\partial R(t)} k_{2}(t, z)\left|\frac{\partial p(t, z)}{\partial z}\right|^{2} d s_{z}+\frac{4}{\pi} \iint_{R(t)}\left|\frac{\partial^{2} p(t, z)}{\partial \bar{t} \partial z}\right|^{2} d x d y
$$

Theorem 1.4. Under the same conditions as in Lemma 1.3, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_{z}$ then $\alpha(t)$ is a $C^{\omega}$ subharmonic function on $B$. This is also true under the same condition for $\mathcal{R}$ as in Remark 1.

As an application of Theorem 1.4, we demonstrate the following fact. Let $B$ be a simply connected domain in $\mathbb{C}_{t}$. Let $\pi: \mathcal{S} \rightarrow B$ be a holomorphic family of compact Riemann surfaces $S(t)=\pi^{-1}(t)$ over $B$ such that each fiber $S(t)$ is of genus $\geq 2$ and nonsingular in $\mathcal{S}$. For a fixed $t \in B$, we consider the Schottky covering $\tilde{S}(t)$ of each $S(t)$ (see [1, 19F, p. 241; 2, Sec. 101, p. 266]). We denote by $\tilde{\mathcal{S}}$ the total space of the variation $t \in B \rightarrow \tilde{S}(t)$; namely, $\tilde{\mathcal{S}}=\bigcup_{t \in B}(t, \tilde{S}(t))$. Then we have our final result as follows.

Theorem 1.5. The total space $\tilde{\mathcal{S}}$ consisting of the Schottky covering $\tilde{S}(t)$ of compact Riemann surfaces $S(t)$ with one complex parameter $t \in B$ is holomorphically uniformized to a univalent domain on $B \times \mathbb{P}^{1}$.

In [4], Maitani and Yamaguchi proved that, if $\mathcal{R}=\bigcup_{t \in B}(t, R(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_{z}$ such that each $R(t), t \in B$, is planar and parabolic, then $\mathcal{R}$ is holomorphically uniformizable to a domain in $B \times \mathbb{P}^{1}$. Since the Schottky covering $\tilde{S}(t)$ of a compact Riemann surface $S(t)$ of genus $g \geq 2$ is planar but not parabolic, their theorem and method cannot be applied to our case. In [8], Yamaguchi discussed Theorem 1.5 and offered a rough sketch of the proof. However, his sketch had a "gap". This paper bridges that gap by establishing the variation formula for $L_{1}$-principal functions (Lemma 1.3), leading to Theorem 1.5.

Acknowledgments. The author would like to offer thanks to Professor Hiroshi Yamaguchi for invaluable discussions and comments. The author would also like to express her gratitude to the referee, who read the manuscript with care.

## 2. Proof of Lemma 1.1

In the proof we put $a_{0}(t) \equiv 0$ on $C_{0}(t)$, so that we can simply write

$$
u(t, z)=a_{i}(t) \text { on } C_{i}(t), \quad i=0,1, \ldots, v
$$

We divide the proof into two steps.
Step 1: For each $t \in B$ and $i=0,1, \ldots, v$, it holds, along $C_{i}(t)$, that

$$
\begin{align*}
\frac{\partial^{2} u}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_{z}} d s_{z}= & 2 k_{2}(t, z)\left|\frac{\partial u}{\partial z}\right|^{2} d s_{z}+\frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_{z}} d s_{z} \\
& +4 \operatorname{Im}\left\{\frac{\partial u}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\}-4 \operatorname{Im}\left\{\frac{\partial a_{i}}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\} \tag{2.1}
\end{align*}
$$

Proof. Since $\mathcal{R}: t \in B \rightarrow R(t)$ is a $C^{\omega}$ smooth variation of unramified domains $R(t)$ with $C^{\omega}$ smooth boundary over $\mathbb{C}_{z}$, we remark from the standard argument that $u(t, z)$ is real-analytically extended for $(t, z)$ beyond $\partial \mathcal{R}$ to a neighborhood $\mathcal{V}=\bigcup_{t \in B}(t, V(t))$ of $\partial \mathcal{R}$ in $\widetilde{\mathcal{R}}$ such that, for each fixed $t \in B$, we have $V(t) \supset$ $\partial R(t)$ and $u(t, z)$ is harmonic for $z \in R(t) \cup V(t) \backslash\{0\}$. (This immediately implies that $\gamma(t)$ is also real-analytic on $B$, since $\gamma(t)=\frac{1}{2 \pi} \int_{0}^{2 \pi} h\left(t, \varepsilon e^{i \theta}\right) d \theta$.)

It suffices to prove (2.1) for $t=0$ and at an arbitrary fixed point $\zeta$ on $C_{i}(0)$. We first prove (2.1) at $(0, \zeta)$ when

$$
\begin{equation*}
\frac{\partial u}{\partial n_{\zeta}}(0, \zeta)>0 \tag{2.2}
\end{equation*}
$$

In this case we have a neighborhood $B_{0} \times V_{0}$ of $(0, \zeta)$ such that $B_{0} \Subset B$ and $V_{0} \Subset$ $V(t)$ for $t \in B_{0}$ and such that

$$
\begin{array}{lll}
u(t, z)<a_{i}(t) & \text { on } R(t) \cap V_{0} & \text { for } t \in B_{0}, \\
u(t, z)=a_{i}(t) & \text { on } \partial R(t) \cap V_{0} & \text { for } t \in B_{0}, \\
u(t, z)>a_{i}(t) & \text { on } \overline{R(t)} & \\
& \text { for } t \in V_{0}
\end{array}
$$

It follows that $u(t, z)-a_{i}(t)$ is a $C^{\omega}$ defining function of $\partial \mathcal{R} \cap\left[B_{0} \times V_{0}\right]$. We simply set

$$
u_{i}(t, z)=u(t, z)-a_{i}(t) \text { in } B_{0} \times V_{0} .
$$

Since $k_{2}(t, z)$ on $\partial \mathcal{R}$ does not depend on the choice of defining functions, it follows that on $\partial \mathcal{R} \cap\left[B_{0} \times V_{0}\right]$ we have

$$
k_{2}(t, z)=\left(\frac{\partial^{2} u_{i}}{\partial t \partial \bar{t}}\left|\frac{\partial u_{i}}{\partial z}\right|^{2}-2 \operatorname{Re}\left\{\frac{\partial^{2} u_{i}}{\partial \bar{t} \partial z} \frac{\partial u_{i}}{\partial t} \frac{\partial u_{i}}{\partial \bar{z}}\right\}+\left|\frac{\partial u_{i}}{\partial t}\right|^{2} \frac{\partial^{2} u_{i}}{\partial z \partial \bar{z}}\right)\left|\frac{\partial u_{i}}{\partial z}\right|^{-3} .
$$

Since $u_{i}(t, z)$ is harmonic for $z$ and $a_{i}(t)$ is independent of $z$, we have

$$
\frac{\partial^{2} u_{i}}{\partial z \partial \bar{z}}=0, \quad \frac{\partial u_{i}}{\partial z}=\frac{\partial u}{\partial z} \quad \text { and } \quad \frac{\partial u_{i}}{\partial \bar{z}}=\frac{\partial u}{\partial \bar{z}}, \quad \frac{\partial^{2} u_{i}}{\partial \bar{t} \partial z}=\frac{\partial^{2} u}{\partial \bar{t} \partial z} .
$$

Since $u_{i}(t, z)=0$ on $C_{i}(t)$ and $u_{i}(t, z)<0$ on $R(t) \cap V_{0}$, we have

$$
\left|\frac{\partial u}{\partial z}\right|=\left|\frac{\partial u_{i}}{\partial z}\right|=\frac{1}{2} \frac{\partial u_{i}}{\partial n_{z}}=\frac{1}{2} \frac{\partial u}{\partial n_{z}}>0 \text { along } C_{i}(t) \cap V_{0} .
$$

It follows that, along $C_{i}(t) \cap V_{0}$,

$$
\begin{aligned}
\frac{\partial^{2} u_{i}}{\partial t \partial \bar{t}} & =\left(k_{2}(t, z)\left|\frac{\partial u}{\partial z}\right|^{3}+2 \operatorname{Re}\left\{\frac{\partial u_{i}}{\partial t} \frac{\partial u}{\partial \bar{z}} \frac{\partial^{2} u}{\partial \bar{t} \partial z}\right\}\right)\left|\frac{\partial u}{\partial z}\right|^{-2} \\
& =\frac{1}{2} k_{2}(t, z) \frac{\partial u}{\partial n_{z}}+2 \operatorname{Re}\left\{\frac{\partial u_{i}}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} / \frac{\partial u}{\partial z}\right\}
\end{aligned}
$$

so that

$$
\frac{\partial^{2} u}{\partial t \partial \bar{t}}=\frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}}+\frac{1}{2} k_{2}(t, z) \frac{\partial u}{\partial n_{z}}+2 \operatorname{Re}\left\{\frac{\partial\left(u-a_{i}\right)}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} / \frac{\partial u}{\partial z}\right\}
$$

Consequently, along $C_{i}(t) \cap V_{0}$ for $t \in B_{0}$, we have

$$
\begin{aligned}
\frac{\partial^{2} u}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_{z}} & d s_{z} \\
& =\left[\frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}}+\frac{1}{2} k_{2}(t, z) \frac{\partial u}{\partial n_{z}}+2 \operatorname{Re}\left\{\frac{\partial\left(u-a_{i}\right)}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} / \frac{\partial u}{\partial z}\right\}\right] \frac{\partial u}{\partial n_{z}} d s_{z}
\end{aligned}
$$

Since $u(t, z)=a_{i}(t):$ constant on $C_{i}(t)$ for each $t \in B_{0}$, we have

$$
\frac{\partial u(t, z)}{\partial n_{z}} d s_{z}=\frac{2}{i} \frac{\partial u(t, z)}{\partial z} d z \text { along } C_{i}(t)
$$

Since $\frac{\partial u(t, z)}{\partial n_{z}} d s_{z}$ is real it follows that, along $C_{i}(t) \cap V_{0}$ for each $t \in B_{0}$,

$$
\begin{aligned}
\frac{\partial^{2} u}{\partial t \partial \bar{t}} & \frac{\partial u}{\partial n_{z}} d s_{z} \\
& =\frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_{z}} d s_{z}+\frac{1}{2} k_{2}(t, z)\left(\frac{\partial u}{\partial n_{z}}\right)^{2} d s_{z}+2 \operatorname{Re}\left\{\frac{2}{i} \frac{\partial\left(u-a_{i}\right)}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\} \\
& =\frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_{z}} d s_{z}+2 k_{2}(t, z)\left|\frac{\partial u}{\partial z}\right|^{2} d s_{z}+4 \operatorname{Im}\left\{\frac{\partial\left(u-a_{i}\right)}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\}
\end{aligned}
$$

In particular, putting $(t, z)=(0, \zeta)$ in this equality yields formula (2.1) at the point $\zeta \in C_{i}(0)$ in the case (2.2).

We next prove formula (2.1) at the point $\zeta \in C_{i}(0)$ when $\frac{\partial u}{\partial n_{\zeta}}(0, \zeta)<0$. In fact, we put $q(t, z):=-u(t, z)$ in $\mathcal{R}$ so that $\frac{\partial q}{\partial n_{\zeta}}(0, \zeta)>0$. Since $q(t, z)$ is a constant $-a_{i}(t)$ on $\partial C_{i}(t)$, we can apply our reasoning from the first case for $q(t, z)$ and obtain equation (2.1) for $q(t, z)$-that is, for $-u(t, z)$ (with $-a_{i}(t)$ instead of $a_{i}(t)$ ). On the other hand, the formula (2.1) for $u(t, z)$ (with $a_{i}(t)$ ) is identical to the formula (2.1) for $-u(t, z)$ (with $-a_{i}(t)$ ), so (2.1) for $u(t, z)$ holds when $\frac{\partial u}{\partial n_{\zeta}}(0, \zeta)<0$.

Finally, we prove formula (2.1) at the point $\zeta \in C_{i}(0)$ when

$$
\frac{\partial u}{\partial n_{\zeta}}(0, \zeta)=0
$$

or, equivalently, $\frac{\partial u}{\partial z}(0, \zeta)=0$. Let $\varphi(t, z)$ be a $C^{\omega}$ defining function of $\partial \mathcal{R}$. Since $u(t, z)-a_{i}(t) \equiv 0$ on $\partial \mathcal{R} \cap\left[B_{0} \times V_{0}\right]$, we find a neighborhood $B_{1} \times V_{1} \subset B_{0} \times$ $V_{0}$ of $(0, \zeta)$ and a $C^{\omega}$ function $f(t, z)$ on $B_{1} \times V_{1}$ such that

$$
\begin{equation*}
u(t, z)-a_{i}(t)=f(t, z) \varphi(t, z) \text { on } B_{1} \times V_{1} \tag{2.3}
\end{equation*}
$$

By differentiating (2.3) with respect to $z$, we first have

$$
\frac{\partial u}{\partial z}=\frac{\partial f}{\partial z} \varphi+f \frac{\partial \varphi}{\partial z} \text { on } B_{1} \times V_{1}
$$

Since

$$
\frac{\partial u}{\partial z}(0, \zeta)=0, \quad \varphi(0, \zeta)=0, \quad \text { and } \quad \frac{\partial \varphi}{\partial z}(0, \zeta) \neq 0
$$

we have $f(0, \zeta)=0$. Differentiating both side of (2.3) with respect to $t$, we next have

$$
\frac{\partial u}{\partial t}-\frac{\partial a_{i}}{\partial t}=\frac{\partial f}{\partial t} \varphi+f \frac{\partial \varphi}{\partial t} \text { on } B_{1} \times V_{1}
$$

so that

$$
\frac{\partial u}{\partial t}(0, \zeta)-\frac{\partial a_{i}}{\partial t}(0)=\frac{\partial f}{\partial t}(0, \zeta) \cdot 0+0 \cdot \frac{\partial \varphi}{\partial t}(0, \zeta)=0 .
$$

Together with $\frac{\partial u}{\partial n_{\zeta}}(0, \zeta)=0$, this implies that both sides of $(2.1)$ are zero at $(0, \zeta)$ and thus completes the proof of Step 1.

Step 2: Lemma 1.1 is true.
Proof. We set

$$
\frac{\partial^{2} u}{\partial t \partial \bar{t}}(t, 0):=\frac{\partial^{2} \gamma(t)}{\partial t \partial \bar{t}} .
$$

Then we see from (1.1) and (1.2) that the function $\frac{\partial^{2} u}{\partial t \partial t}(t, z)$ for each fixed $t \in B$ is harmonic for $z$ on the whole $\overline{R(t)}$. There exists a tubular neighborhood $W_{0}$ of $\partial R(0)$ such that $u(0, z)$ is harmonic on $R(0) \cup W_{0}$. Then we can take a neighborhood $B_{0} \subset B$ of $t=0$ such that, for each $t \in B_{0}, W_{0} \supset \partial R(t)$ and $u(t, z)$ is harmonic in $\left(R(0) \cup W_{0}\right) \backslash\{0, \xi(t)\}$.

For arbitrary fixed $t \in B_{0}$, it follows from Green's formula that

$$
\int_{C_{i}(0)-C_{i}(t)} \frac{\partial u(t, z)}{\partial n_{z}} d s_{z}=0 .
$$

On the other hand, we see from the condition (4)(ii) for the $L_{1}$-principal function $u(t, z)$ that, for each fixed $t \in B_{0}$,

$$
\begin{equation*}
\int_{C_{i}(0)} \frac{\partial u(t, z)}{\partial n_{z}} d s_{z}=0 \quad(i=1, \ldots, v) \tag{2.4}
\end{equation*}
$$

Differentiating both sides of (2.4) with respect to $t$ and $\bar{t}$ then yields

$$
\int_{C_{i}(0)} \frac{\partial\left(\frac{\partial^{2} u(t, z)}{\partial t \partial \bar{t}}\right)}{\partial n_{z}} d s_{z}=0 \quad(i=1, \ldots, v)
$$

Hence,

$$
\begin{equation*}
\int_{\partial R(0)} u(0, z) \frac{\partial\left(\frac{\partial^{2} u(t, z)}{\partial t \partial \bar{t}}\right)}{\partial n_{z}} d s_{z}=\sum_{i=0}^{\nu} \int_{\partial C_{i}(0)} a_{i}(0) \frac{\partial\left(\frac{\partial^{2} u(t, z)}{\partial t \partial \bar{t}}\right)}{\partial n_{z}} d s_{z}=0 . \tag{2.5}
\end{equation*}
$$

We draw a small disk $U_{0}^{\varepsilon}=\{|z|<\varepsilon\}$ such that $u(t, z)$ is harmonic for $z$ in $R(0) \backslash$ $U_{0}^{\varepsilon}$ and $\frac{\partial^{2} u}{\partial \partial \partial \bar{t}}$ is harmonic for $z$ in $\overline{R(0)}$. It follows from Green's formula that

$$
\int_{\partial R(0) \backslash U_{0}^{\varepsilon}} \frac{\partial^{2} u}{\partial t \partial \bar{t}}(0, z) \frac{\partial u(0, z)}{\partial n_{z}} d s_{z}=\int_{\partial R(0) \backslash U_{0}^{\varepsilon}} u(0, z) \frac{\partial\left(\frac{\partial^{2} u}{\partial t \partial \bar{t}}(0, z)\right)}{\partial n_{z}} d s_{z}
$$

Letting $\varepsilon \rightarrow 0$, we see from (2.5) that

$$
\int_{\partial R(0)} \frac{\partial^{2} u}{\partial t \partial \bar{t}}(0, z) \frac{\partial u(0, z)}{\partial n_{z}} d s_{z}+2 \pi \frac{\partial^{2} u}{\partial t \partial \bar{t}}(0,0)=0
$$

Thus,

$$
\frac{\partial^{2} \gamma}{\partial t \partial \bar{t}}(0)=-\frac{1}{2 \pi} \int_{\partial R(0)} \frac{\partial^{2} u}{\partial t \partial \bar{t}}(0, z) \frac{\partial u(0, z)}{\partial n_{z}} d s_{z}
$$

By (2.1) we have

$$
\begin{aligned}
\frac{\partial^{2} \gamma}{\partial t \partial \bar{t}}(0)= & -\frac{1}{2 \pi} \int_{\partial R(0)}\left(2 k_{2}(0, z)\left|\frac{\partial u}{\partial z}\right|^{2} d s_{z}+\frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_{z}} d s_{z}\right. \\
& \left.+4 \operatorname{Im}\left\{\frac{\partial u}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\}-4 \operatorname{Im}\left\{\frac{\partial a_{i}}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\}\right) \\
= & -\frac{1}{\pi} \int_{\partial R(0)} k_{2}(0, z)\left|\frac{\partial u}{\partial z}\right|^{2} d s_{z}-\frac{1}{2 \pi} \int_{\partial R(0)} \frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_{z}} d s_{z} \\
& -\frac{2}{\pi} \operatorname{Im}\left\{\int_{\partial R(0)} \frac{\partial u}{} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\}+\frac{2}{\pi} \operatorname{Im}\left\{\int_{\partial R(0)} \frac{\partial a_{i}}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right\} \\
\equiv & I_{1}+I_{2}+I_{3}+I_{4} .
\end{aligned}
$$

Here the integrand in $I_{k}(k=1, \ldots, 4)$ is evaluated at $t=0$ and $z \in \partial R(0)$.
We note that $a_{i}(t)$ does not depend on $z$ and that $a_{0}(t) \equiv 0$ on $B$. By condition (4)(ii) for the $L_{1}$-principal function $u(t, z)$, we thus have

$$
\begin{aligned}
I_{2} & =-\frac{1}{2 \pi} \sum_{i=0}^{\nu} \frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}}(0) \int_{C_{i}(0)} \frac{\partial u(0, z)}{\partial n_{z}} d s_{z} \\
& =-\frac{1}{2 \pi}\left(0 \times \int_{C_{0}(0)} \frac{\partial u(0, z)}{\partial n_{z}} d s_{z}+\sum_{i=1}^{\nu} \frac{\partial^{2} a_{i}}{\partial t \partial \bar{t}}(0) \times 0\right)=0 .
\end{aligned}
$$

By (1.1) and (1.2) we see that, if we set $\frac{\partial u}{\partial t}(t, 0):=\frac{\partial \gamma(t)}{\partial t}$ for each fixed $t \in B$, then both $\frac{\partial u}{\partial t}(t, z)$ and $\frac{\partial u}{\partial t \partial \bar{t}}(t, z)$ are harmonic for $z$ on the whole $\overline{R(t)}$. It follows from Green's formula and the harmonicity of $u$ for $z$ that

$$
\begin{aligned}
I_{3} & =-\frac{2}{\pi} \operatorname{Im} \iint_{R(0)} d\left(\frac{\partial u}{\partial t} \frac{\partial^{2} u}{\partial \bar{t} \partial z} d z\right) \\
& =-\frac{2}{\pi} \operatorname{Im} \iint_{R(0)}\left(\frac{\partial^{2} u}{\partial t \partial \bar{z}} \frac{\partial^{2} u}{\partial \bar{t} \partial z}+\frac{\partial u}{\partial t} \frac{\partial^{3} u}{\partial \bar{t} \partial z \partial \bar{z}}\right) d \bar{z} \wedge d z \\
& =-\frac{4}{\pi} \iint_{R(0)}\left|\frac{\partial^{2} u}{\partial \bar{t} \partial z}\right|^{2} d x d y .
\end{aligned}
$$

Since $2 \frac{\partial u}{\partial z} d z=i \frac{\partial u}{\partial n_{z}}+d u$ along $C_{i}(0)$, we have

$$
2 \int_{C_{i}(0)} \frac{\partial u}{\partial z} d z=i \int_{C_{i}(0)} \frac{\partial u}{\partial n_{z}} d s_{z}+\int_{C_{i}(0)} d u=i \int_{C_{i}(0)} \frac{\partial u}{\partial n_{z}} d s_{z} .
$$

It follows from (2.4) that, for each $t \in B_{0}$,

$$
\int_{C_{i}(0)} \frac{\partial u(t, z)}{\partial z} d z=\frac{i}{2} \int_{C_{i}(0)} \frac{\partial u(t, z)}{\partial n_{z}} d s_{z}=0 \quad(i=1, \ldots, \nu)
$$

Hence, by an argument similar to that used for $I_{2}=0$, we obtain

$$
I_{4}=\frac{2}{\pi} \operatorname{Im}\left\{\sum_{i=0}^{\nu} \frac{\partial a_{i}}{\partial t}(0) \int_{C_{i}(0)} \frac{\partial^{2} u}{\partial \bar{t} \partial z}(0, z) d z\right\}=0
$$

It turns out that

$$
\frac{\partial^{2} \gamma}{\partial t \partial \bar{t}}(0)=-\frac{1}{\pi} \int_{\partial R(0)} k_{2}(0, z)\left|\frac{\partial u(0, z)}{\partial z}\right|^{2} d s_{z}-\frac{4}{\pi} \iint_{R(0)}\left|\frac{\partial u}{\partial \bar{t} \partial z}(0, z)\right|^{2} d x d y
$$

which proves Step 2—namely, Lemma 1.1.
Proof of Theorem 1.2. If $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_{z}$, then $k_{2}(t, z) \geq 0$ on $\partial \mathcal{R}$. It follows from Lemma 1.1 that $\frac{\partial^{2} \gamma}{\partial t \partial \bar{t}} \leq 0$. Thus, $\gamma(t)$ is a superharmonic function on $B$.

Remark 3. In the theory of one complex variable, the circular slit mapping and the radial slit mapping have good correspondence. But the same result for the corresponding radius of the radial slit mapping does not hold. In fact, we have two counterexamples of pseudoconvex domains $\mathcal{R}$ in $B \times \mathbb{C}_{z}$.
(i) The radius of radial slit mapping is not superharmonic on $B$. Let

$$
\begin{aligned}
& \mathcal{R}=\left\{|t|<\frac{1}{2}\right\} \times\{|z|<1\} \backslash\left\{(t, z):\left|z-\frac{1}{2}\right| \leq|t|<\frac{1}{2}\right\}, \\
& B=\left\{|t|<\frac{1}{2}\right\}, \quad R(t)=\{|z|<1\} \backslash\left\{\left|z-\frac{1}{2}\right| \leq|t|\right\} .
\end{aligned}
$$

The total space $\mathcal{R}=\bigcup_{t \in B}(t, R(t))$ is a pseudoconvex domain in $B \times \mathbb{C}_{z}$.
Let $C_{0}=\{|z|=1\}$ and $C(t)=\left\{\left|z-\frac{1}{2}\right|=|t|\right\}$, where $0 \leq t<\frac{1}{2}$; that is, $\partial R(t)=C_{0}-C(t)$. There exists a unique harmonic function $q(t, z)$ on $\overline{R(t)} \backslash\{0\}$ such that

$$
\begin{array}{ll}
q(t, z)=\log |z|+\lambda(t)+h(t, z) & \text { at } z=0, \text { where } h(t, 0)=0 ; \\
q(t, z)=0 & \text { on } C_{0} ; \text { and } \\
\frac{\partial q(t, z)}{\partial n}=0 & \text { on } C(t) .
\end{array}
$$

Note that $q(0, z)=\log |z|$ on $R(0)$ and $\lambda(0)=0$. The radial slit mapping $f_{0}(t, z)$ for $\left(R(t), 0, C_{0}\right)$ is given by $f_{0}(t, z)=e^{\lambda(t)-\left(q(t, z)+i q^{*}(t, z)\right)}$ on $R(t)$, so the radius $r_{0}(t)$ stated in Remark 2 is in this case equal to $e^{\lambda(t)}$. Our claim is thus to show that $\lambda(t)$ is not superharmonic on $B$.

Let $z$ and $z^{*}$ be inverse points with respect to $C(t)$. We use $R^{*}(t)$ to denote the domain obtained from $R(t)$ by inversion with respect to $C(t)$. If we define $q\left(t, z^{*}\right):=q(t, z)$ on $R^{*}(t)$, then $q(t, z)$ can be continued harmonically across $C(t)$ into the domain $R^{*}(t)$ by a reflection principle due to Schwarz. More precisely, let $\hat{R}(t):=R(t) \cup C(t) \cup R^{*}(t)$ and let $C_{0}^{*}(t)($ resp. $\alpha(t))$ be the circle (resp. point) obtained from $C_{0}$ (resp. $z=0$ ) by inversion with respect to $C(t)$. Here $\partial \hat{R}(t)=C_{0}-C_{0}^{*}(t)$. Then $q(t, z)$ on $R(t)$ is harmonically extended to the harmonic function $q(t, z)$ on $\hat{R}(t) \backslash\{0, \alpha(t)\}$ such that

$$
\begin{array}{ll}
q(t, z)=\log |z|+\lambda(t)+h(t, z) & \text { at } z=0, \text { where } h(t, 0)=0 ; \\
q(t, z)-\log |z-\alpha(t)| \text { is harmonic } & \text { at } z=\alpha(t) ; \text { and } \\
q(t, z)=0 & \text { on } C_{0} \cup C^{*}(t) .
\end{array}
$$

We note that $C_{0}^{*}(t) \subset C_{0}^{*}\left(\frac{1}{2}\right)=\left\{\left|z-\frac{2}{3}\right|<\frac{1}{3}\right\}=: \delta$ for $t \in B$. We set $R_{0}:=$ $\{|z|<1\} \backslash \delta$, so that $R_{0} \subset R(t)$ for $t \in B$. If we put $s(t, z):=q(t, z)-\log |z|$ and $s(t, 0)=\lambda(t)$, then $s(t, z)$ is harmonic on $\hat{R}(t)$ with pole $\log |z-\alpha(t)|$ at
$\alpha(t)$ and with boundary values 0 on $C_{0}$ and $-\log |z|$ on $C_{0}^{*}(t)$, so that $s(t, z)<$ $\log 3$ on $C_{0}^{*}(t)$. Further, we consider the harmonic function $\tilde{s}(t, z)$ on $R_{0}$ with pole $\log |z-\alpha(t)|$ at $\alpha(t)$ and with boundary values $\log 3$ on $\partial R_{0}$. Since $\tilde{s}(t, z)-s(t, z)$ is harmonic on the whole $R_{0}$, whose boundary values $\geq 0$, it follows from the maximum principle that $\tilde{s}(t, 0) \geq s(t, 0)=\lambda(t)$. Since $\alpha(t) \rightarrow 0$ as $|t| \rightarrow \frac{1}{2}$, we have $\tilde{s}(t, 0) \rightarrow-\infty$ and hence $\lambda(t) \rightarrow-\infty$ as $|t| \rightarrow \frac{1}{2}$. Since $\lambda(0)=0$, it follows that $\lambda(t)$ is not superharmonic on $B$.
(ii) The radius of radial slit mapping is not subharmonic on $B$. Let

$$
\mathcal{R}=\bigcup_{t \in B}\{|z|<r(t)\} \backslash B \times\left(C_{1} \cup C_{2}\right),
$$

where $C_{1}=\left[\frac{1}{2}, \frac{2}{3}\right], C_{2}=\left[\frac{i}{2}, \frac{2 i}{3}\right]$ in $\mathbb{C}_{z}, r(t)>1$, and $\log r(t)$ is superharmonic on $B$. Thus $\mathcal{R}$ is a pseudoconvex domain in $B \times \mathbb{C}_{z}$. In this case the radial slit mapping $f_{0}(t, z)$ for $\left(R(t), 0, C_{0}(t)\right)$, where $C_{0}(t)=\{|z|=r(t)\}$, is identical with $z$; hence $r_{0}(t)=r(t)$ for $t \in B$. Thus $\log r_{0}(t)$ is not subharmonic on $B$.

## 3. Proof of Lemma 1.3

It suffices to prove the lemma for $t=0$. We find a neighborhood $B_{0}=\left\{|t|<r_{0}\right\} \Subset$ $B$ of $t=0$, a neighborhood $V_{0} \Subset \tilde{R}(0)$ of $\partial R(0)$, and a neighborhood $U_{\xi(0)}^{\rho_{0}}:=$ $\left\{|z-\xi(0)|<\rho_{0}\right\} \Subset R(0)$ of $\xi(0)$ such that:
(i) $\xi(t) \in U_{\xi(0)}^{\rho_{0}}, t \in B_{0}$;
(ii) $U_{\xi(0)}^{\rho_{0}} \cap\{z=0\}=\emptyset$;
(iii) $\bigcup_{t \in B_{0}}(t, \partial R(t)) \subset B_{0} \times V_{0}$; and
(iv) $p(t, z), t \in B_{0}$, is harmonically extended for $z$ beyond $\partial R(t)$ onto $V_{0}$.

We see from (1.3), (1.4), and (1.5) that if we put

$$
\begin{aligned}
\frac{\partial^{2} p}{\partial t \partial \bar{t}}(0,0) & :=\frac{\partial^{2} h_{0}}{\partial t \partial \bar{t}}(0,0)=0 \quad \text { and } \\
\frac{\partial^{2} p}{\partial t \partial \bar{t}}(0, \xi(0)) & :=\frac{\partial^{2} \alpha}{\partial t \partial \bar{t}}(0)+\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(0, \xi(0))
\end{aligned}
$$

then the function $\frac{\partial^{2} p}{\partial t \partial \partial}(0, z)$ is harmonic for $z$ in the whole $\overline{R(0)}$. We note that, for arbitrary fixed $t \in B_{0}$ and $j=0, \ldots, \nu$,

$$
\int_{C_{j}(0)} \frac{\partial p(t, z)}{\partial n_{z}} d s_{z}=\int_{C_{j}(t)} \frac{\partial p(t, z)}{\partial n_{z}} d s_{z}=0
$$

by Green's formula and the condition (IV )(ii) for the $L_{1}$-principal function $p(t, z)$. Thus,

$$
\int_{C_{j}(0)} \frac{\partial}{\partial n_{z}}\left(\frac{\partial^{2} p(t, z)}{\partial t \partial \bar{t}}\right) d s_{z}=0
$$

On each boundary component $C_{j}(0), j=0,1, \ldots, \nu$, of $R(0)$, we have $p(0, z)=$ constant $=a_{j}(0)$; therefore,

$$
\begin{equation*}
\int_{\partial R(0)} p(0, z) \frac{\partial\left(\frac{\partial^{2} p}{\partial t \partial \bar{t}}(0, z)\right)}{\partial n_{z}} d s_{z}=\sum_{j=0}^{\nu} a_{j}(0) \int_{C_{j}(0)} \frac{\partial\left(\frac{\partial^{2} p}{\partial t \partial \bar{t}}(0, z)\right)}{\partial n_{z}} d s_{z}=0 . \tag{3.1}
\end{equation*}
$$

We draw small disks $U_{0}^{\varepsilon}=\{|z|<\varepsilon\}$ and $U_{\xi(0)}^{\varepsilon}=\{|z-\xi(0)|<\varepsilon\}$, where $0<\varepsilon<\rho_{0}$ and $U_{0}^{\varepsilon} \cap U_{\xi(0)}^{\varepsilon}=\emptyset$, so that $p(t, z), t \in B_{0}$, is harmonic for $z$ in $\overline{R(0)} \backslash\left(U_{0}^{\varepsilon} \cup U_{\xi(0)}^{\varepsilon}\right)$. By Green's formula we have

$$
\begin{aligned}
& \int_{\partial R(0)-\partial U_{0}^{\varepsilon}-\partial U_{\xi(0)}^{\varepsilon}} \frac{\partial^{2} p}{\partial t \partial \bar{t}}(0, z) \frac{\partial p(0, z)}{\partial n_{z}} d s_{z} \\
&=\int_{\partial R(0)-\partial U_{0}^{\varepsilon}-\partial U_{\xi(0)}^{\varepsilon}} p(0, z) \frac{\partial\left(\frac{\partial^{2} p}{\partial t \partial \bar{t}}(0, z)\right)}{\partial n_{z}} d s_{z}
\end{aligned}
$$

Since $p(0, z)$ has singularities (II) and (III) at $z=0$ and $z=\xi(0)$, respectively, and since $\frac{\partial^{2} p}{\partial t \partial \partial t}(0, z)$ is harmonic on $\overline{R(0)}$, by (3.1) we have

$$
\int_{\partial R(0)} \frac{\partial^{2} p}{\partial t \partial \bar{t}}(0, z) \frac{\partial p(0, z)}{\partial n_{z}} d s_{z}-2 \pi\left(\frac{\partial^{2} \alpha}{\partial t \partial \bar{t}}(0)+\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(0, \xi(0))\right)=0
$$

as $\varepsilon$ tends to 0 . Therefore,

$$
\frac{\partial^{2} \alpha}{\partial t \partial \bar{t}}(0)+\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(0, \xi(0))=\frac{1}{2 \pi} \int_{\partial R(0)} \frac{\partial^{2} p}{\partial t \partial \bar{t}}(0, z) \frac{\partial p(0, z)}{\partial n_{z}} d s_{z}
$$

Reasoning much as in the proof of (2.1),

$$
\begin{aligned}
\frac{\partial^{2} \alpha}{\partial t \partial \bar{t}}(0) & +\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(0, \xi(0)) \\
= & \frac{1}{2 \pi} \sum_{j=0}^{v} \int_{C_{j}(0)}\left(2 k_{2}(0, z)\left|\frac{\partial p}{\partial z}\right|^{2} d s_{z}+\frac{\partial^{2} a_{j}}{\partial t \partial \bar{t}} \frac{\partial p}{\partial n_{z}} d s_{z}\right. \\
& \left.+4 \operatorname{Im}\left\{\frac{\partial p}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z\right\}-4 \operatorname{Im}\left\{\frac{\partial a_{j}}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z\right\}\right) \\
= & \frac{1}{\pi} \int_{\partial R(0)} k_{2}(0, z)\left|\frac{\partial p}{\partial z}\right|^{2} d s_{z}+\frac{1}{2 \pi} \sum_{j=0}^{\nu} \int_{C_{j}(0)} \frac{\partial^{2} a_{j}}{\partial t \partial \bar{t}} \frac{\partial p}{\partial n_{z}} d s_{z} \\
& +\frac{2}{\pi} \operatorname{Im}\left\{\int_{\partial R(0)} \frac{\partial p}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z\right\}-\frac{2}{\pi} \operatorname{Im}\left\{\sum_{j=0}^{\nu} \int_{C_{j}(0)} \frac{\partial a_{j}}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z\right\} \\
\equiv & J_{1}+J_{2}+J_{3}+J_{4} .
\end{aligned}
$$

By reasoning similar to the case $I_{2}=I_{4}=0$ for $i=1, \ldots$, $v$, we have $J_{2}=J_{4}=$ 0 for $j=0,1, \ldots, \nu$. Hence for proving the lemma it suffices to show that

$$
\begin{equation*}
J_{3}=\frac{4}{\pi} \iint_{R(0)}\left|\frac{\partial^{2} p}{\partial \bar{t} \partial z}\right|^{2} d x d y+\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(0, \xi(0)) \tag{3.2}
\end{equation*}
$$

We remark that $\frac{\partial^{2} p}{\partial \bar{t} \partial z}(0, z)$ is regular on the whole $\overline{R(0)}$ and that $\frac{\partial p}{\partial t}(0, z)$ is harmonic on $\overline{R(0)}$ except at $z=\xi(0)$. By Green's formula we have

$$
\begin{aligned}
\int_{\partial R(0)-\partial U_{\xi(0)}^{\varepsilon}} \frac{\partial p}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z & =\iint_{R(0)-U_{\xi(0)}^{\varepsilon}} d\left(\frac{\partial p}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z\right) \\
& =\iint_{R(0)-U_{\xi}^{\varepsilon}(0)}\left|\frac{\partial^{2} p}{\partial t \partial \bar{z}}\right|^{2} d \bar{z} d z
\end{aligned}
$$

Letting $\varepsilon \rightarrow 0$, we note that

$$
\lim _{\varepsilon \rightarrow 0} \iint_{U_{\xi(0)}^{\varepsilon}}\left|\frac{\partial^{2} p}{\partial t \partial \bar{z}}\right|^{2} d \bar{z} d z=0
$$

and

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \int_{\partial U_{\xi(0)}^{\varepsilon}} \frac{\partial p}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z & =-\frac{1}{2} \lim _{\varepsilon \rightarrow 0} \int_{\partial U_{\xi(0)}^{\varepsilon}} \frac{\xi^{\prime}(0)}{z-\xi(0)} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z \\
& =-\pi i \xi^{\prime}(0) \frac{\partial^{2} p}{\partial \bar{t} \partial z}(0, \xi(0)) .
\end{aligned}
$$

Thus,

$$
\int_{\partial R(0)} \frac{\partial p}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z=2 i \iint_{R(0)}\left|\frac{\partial^{2} p}{\partial t \partial \bar{z}}(0, z)\right|^{2} d x d y-\pi i \xi^{\prime}(0) \frac{\partial^{2} p}{\partial \bar{t} \partial z}(0, \xi(0))
$$

On the other hand, it follows from (1.6) that

$$
\frac{\partial h_{\xi}}{\partial t}(t, \xi(t))+\frac{\partial h_{\xi}}{\partial z}(t, \xi(t)) \xi^{\prime}(t) \equiv 0
$$

and then

$$
\begin{aligned}
\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(t, \xi(t)) & +\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{z}}(t, \xi(t)) \overline{\xi^{\prime}(t)} \\
& +\frac{\partial^{2} h_{\xi}}{\partial z \partial \bar{t}}(t, \xi(t)) \xi^{\prime}(t)+\frac{\partial^{2} h_{\xi}}{\partial z \partial \bar{z}}(t, \xi(t))\left|\overline{\xi^{\prime}(t)}\right|^{2} \equiv 0
\end{aligned}
$$

on $B$. Since $h_{\xi}$ is harmonic for $z$, we have $\frac{\partial^{2} h_{\xi}}{\partial z \partial \bar{z}}(t, z)=0$. Hence,

$$
\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(0, \xi(0))+2 \operatorname{Re}\left\{\frac{\partial^{2} h_{\xi}}{\partial \bar{t} \partial z}(0, \xi(0)) \xi^{\prime}(0)\right\}=0
$$

Thus,

$$
\begin{aligned}
J_{3} & =\frac{2}{\pi} \operatorname{Im}\left\{\int_{\partial R(0)} \frac{\partial p}{\partial t} \frac{\partial^{2} p}{\partial \bar{t} \partial z} d z\right\} \\
& =\frac{2}{\pi} \operatorname{Im}\left\{\left(2 i \iint_{R(0)}\left|\frac{\partial^{2} p}{\partial t \partial \bar{z}}(0, z)\right|^{2} d x d y\right)-\pi i \xi^{\prime}(0) \frac{\partial^{2} p}{\partial \bar{t} \partial z}(0, \xi(0))\right\} \\
& =\frac{4}{\pi} \iint_{R(0)}\left|\frac{\partial^{2} p}{\partial t \partial \bar{z}}(0, z)\right|^{2} d x d y-2 \operatorname{Re}\left\{\xi^{\prime}(0) \frac{\partial^{2} p}{\partial \bar{t} \partial z}(0, \xi(0))\right\} \\
& =\frac{4}{\pi} \iint_{R(0)}\left|\frac{\partial^{2} p}{\partial t \partial \bar{z}}(0, z)\right|^{2} d x d y+\frac{\partial^{2} h_{\xi}}{\partial t \partial \bar{t}}(0, \xi(0)) .
\end{aligned}
$$

Therefore, we conclude (3.2), which implies Lemma 1.3.
Note that the same proof of Theorem 1.2 and Remark 1 under Lemma 1.1 implies Theorem 1.4.

## 4. Approximation Condition

Let $R$ be a noncompact Riemann surface. Fix points $a$ and $b$ in $R$ and let $|z|<1$ (resp. $|w|<1$ ) be a local coordinate of $a$ (resp. b) such that $a$ (resp. $b$ ) corresponds to $z=0$ (resp. $w=0$ ). Let $\left\{R_{n}\right\}_{n}$ be a canonical exhaustion of $R$. In particular, $R_{n} \Subset R_{n+1} \Subset R, R=\bigcup_{n=1}^{\infty} R_{n}$, and $\partial R_{n}$ consists of a finite number of $C^{\omega}$ smooth dividing closed curves $\left\{C_{n}^{(i)}\right\}_{i=1, \ldots, v_{n}}$ such that each $C_{n}^{(i)}$ is homologous to a finite number of dividing curves $\left\{C_{n+1}^{\left(i_{k}\right)}\right\}_{k=1, \ldots, i_{\mu}}$ of $\partial R_{n+1}$; that is,

$$
\begin{equation*}
\partial R_{n}=\sum_{i=1}^{v_{n}} C_{n}^{(i)}, \quad C_{n}^{(i)} \sim \sum_{k=1}^{i_{\mu}} C_{n+1}^{\left(i_{k}\right)} \tag{4.1}
\end{equation*}
$$

We assume that $a, b \in R_{1}$ and $\{|z|<1\} \cup\{|w|<1\} \Subset R_{1}$. On each $R_{n}$, we uniquely have the principal function $p_{n}(z)$ such that:
(1) $p_{n}(z)$ is harmonic on $R_{n} \backslash\{a, b\}$ and is harmonically extended beyond $\partial R_{n}$ in a neighborhood $V_{n}$ of $R_{n+1}$ such that $\partial R_{n} \Subset V_{n} \Subset R_{n+1}$;
(2) $p_{n}(z)-\log \frac{1}{|z|}$ is harmonic at $z=0$;
(3) $p_{n}(w)-\log |w|$ is harmonic at $w=0$ and

$$
\lim _{w \rightarrow b}\left(p_{n}(w)-\log |w|\right)=0
$$

(4) for each $i=1, \ldots, v_{n}$, we have
(i) $p_{n}(z)=$ constant $a_{i}$ on $C_{n}^{(i)}$ and
(ii) $\int_{C_{n}^{(i)}(t)} * d p_{n}(z)=0$.

Thus we have

$$
\begin{align*}
& p_{n}(z)=\log \frac{1}{|z|}+\operatorname{Re}\left\{\sum_{k=1}^{\infty} a_{k}^{(n)} z^{k}\right\} \text { and }  \tag{4.2}\\
& p_{n}(z)=\log |w|+\alpha_{n}+\operatorname{Re}\left\{\sum_{k=1}^{\infty} b_{k}^{(n)} w^{k}\right\} \tag{4.3}
\end{align*}
$$

where $\alpha_{n}$ is a real constant.
It is known that, for $m \geq n$,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \| d\left(p_{n}(z)\right. & \left.-p_{m}(z)\right) \|_{R_{n}}^{2} \\
& :=\lim _{n \rightarrow \infty} \iint_{R_{n}}\left[\left(\frac{\partial\left(p_{n}-q_{m}\right)}{\partial x}\right)^{2}+\left(\frac{\partial\left(p_{n}-q_{m}\right)}{\partial y}\right)^{2}\right] d x d y=0
\end{aligned}
$$

This implies that $\left\{p_{n}(z)\right\}_{n}$ uniformly converges in any compact set $K$ in $R \backslash\{a, b\}$ to a harmonic function $q(z)$ on $R \backslash\{a, b\}$ such that $\left\{\alpha_{n}\right\}_{n}$ converges to a real constant $\alpha$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|p_{n}(z)-p(z)\right\|_{R_{n}}=0 \tag{4.4}
\end{equation*}
$$

Therefore we have

$$
p(z)=\log \frac{1}{|z|}+\operatorname{Re}\left\{\sum_{k=1}^{\infty} a_{k} z^{k}\right\} \text { near } z=0
$$

and

$$
p(w)=\log |w|+\alpha+\operatorname{Re}\left\{\sum_{k=1}^{\infty} b_{k} w^{k}\right\} \text { near } w=0
$$

Let $(\mathcal{R}, B, \pi)$ be a two-dimensional holomorphic triple: $\mathcal{R}$ is a two-dimensional complex manifold, $B=\{|t|<\rho\}$ is a disk, and $\pi$ is a holomorphic mapping from $\mathcal{R}$ onto $B$ such that the fiber $\pi^{-1}(t):=R(t)$ over each $t \in B$ is connected and nonsingular in $\mathcal{R}$. Thus $R(t)$ is a Riemann surface. We can conventionally write

$$
\mathcal{R}=\bigcup_{t \in B}(t, R(t)) .
$$

We assume that $\mathcal{R}$ is a $C^{\omega}$ topologically trivial triple in the sense that there exist a noncompact Riemann surface $R$ and a $C^{\omega}$ topological mapping

$$
T:(t, w) \in B \times R \mapsto(t, z)=(t, \varphi(t, w)) \in \mathcal{R}
$$

here, for each $t \in B$, the mapping $w \in R \mapsto z=\varphi(t, w)$ is a homeomorphism from $R$ onto $R(t)$.

Let $\left\{R_{n}\right\}_{n}$ with $\mathcal{R}=\bigcup_{n=1}^{\infty} \mathcal{R}_{n}$ be a canonical exhaustion of $R$. We put

$$
\mathcal{R}_{n}=T\left(B \times R_{n}\right)=: \bigcup_{t \in B}\left(t, R_{n}(t)\right)
$$

so that, for each $t \in B,\left\{R_{n}\right\}_{n}$ with $R(t)=\bigcup_{n=1}^{\infty} R_{n}(t)$ is a canonical exhaustion such that each connected component $C_{n}^{i}, i=1, \ldots, v_{n}$ (which is $C^{\omega}$ smooth closed dividing curve with $v_{n}$ independent of $\left.t \in B\right)$ of $\partial R(t)$ moves real-analytically smooth for $t \in B$.

Assume that there exist two disjoint holomorphic sections $\gamma$ and $\alpha$ of $\mathcal{R}$ over $B$ via $\pi$. In particular, let $\gamma: t \in B \rightarrow \gamma(t) \in R(t)$ and $\zeta: t \in B \rightarrow \zeta(t)$ be holomorphic from $B$ into $\mathcal{R}$. Let $B \times\left\{|z|<\rho_{0}\right\}$ and $B \times\left\{|w|<\rho_{1}\right\}$ be local coordinates of neighborhoods of $\gamma$ and $\zeta$ in $\mathcal{R}$ such that $\gamma$ and $\zeta$ correspond to $B \times\{z=0\}$ and $B \times\{w=0\}$, respectively. Then, by the previous argument, for each fixed $t \in B$ we uniquely have the principal function $p(t, z)$ on $R(t)$ such that $p(t, z)$ is harmonic on $R(t) \backslash\{z=0\} \cup\{w=0\}$ and such that

$$
p(t, z)=\log \frac{1}{|z|}+\operatorname{Re}\left\{\sum_{k=1}^{\infty} a_{k}(t) z^{k}\right\} \text { near } z=0
$$

and

$$
p(t, w)=\log |w|+\alpha(t)+\operatorname{Re}\left\{\sum_{k=1}^{\infty} b_{k}(t) w^{k}\right\} \text { near } w=0
$$

where $\alpha(t)$ is a constant.
We shall prove the following statement.
Lemma 4.1. Assume that each $\mathcal{R}_{n}(n=1,2, \ldots)$ is pseudoconvex in $\mathcal{R}_{n+1}$. Then $-\alpha(t)$ is a subharmonic function on $B$.

Proof. Let $\mathcal{R}_{n}$ be fixed. We consider the principal function $p_{n}(t, z)$ on $R_{n}(t)$ with respect to $z=0$ and $w=0$. We denote by $\alpha_{n}(t)$ the constant term of $p_{n}(t, z)$ at $z=0$ :

$$
p_{n}(t, z)=\log \frac{1}{|z|}+\alpha_{n}(t)+\operatorname{Re}\left\{\sum_{n=1}^{\infty} a_{k}^{(n)} z^{n}\right\} \text { near } z=0 .
$$

By Nishimura's theorem, we may assume that $\mathcal{R}_{n}$ is holomorphically equivalent to a unramified domain $\mathcal{R}_{n}$ over $B \times \mathbb{C}_{\zeta}$ such that $B \times\{a\}$ corresponds to $B \times\{a\}$ and $B \times\{b\}$ corresponds to a holomorphic section $\xi: z=\xi(t), t \in B$, where $\xi(t)$ is a holomorphic function on $B$ such that $\xi(t) \neq 0$. Let's say that

$$
T_{n}:(t, z) \in \mathcal{R}_{n} \mapsto(t, \zeta)=\left(t, f_{n}(t, z)\right) \in \mathcal{R}_{n}
$$

Here $\mathcal{R}_{n}=\bigcup_{t \in B}\left(t, R_{n}(t)\right.$ ) (where $R_{n}(t)$ is equivalent to $D_{n}(t)$ as Riemann surfaces) is a pseudoconvex domain with $C^{\omega}$ boundary in $\mathcal{R}_{n+1}$ and each $\partial D_{n}(t)$ is of class $C^{\omega}$ in $D_{n+1}(t)$. In Section 3, for each $t \in B$ we constructed the principal function $\tilde{p}_{n}(t, \zeta)$ for $D_{n}(t)$ with respect to $\zeta=0$ and $\zeta=\xi(t)$. As usual, we denote by $\tilde{\alpha}(t)$ the constant term of $p_{n}(t, \zeta)$ at $\zeta=0$ :

$$
\tilde{p}_{n}(t, \zeta)=\log \frac{1}{|\zeta|}+\tilde{\alpha}(t)+\operatorname{Re}\left\{\sum_{k=1}^{\infty} a_{k}^{(n)} \zeta^{k}\right\}
$$

It is not difficult to show that

$$
\alpha(t)=\tilde{\alpha}(t)+\log \left|\frac{\partial f_{n}}{\partial z}\right|_{z=0}
$$

Since

$$
\left|\frac{\partial f_{n}}{\partial z}\right|_{z=0} \neq 0
$$

we have

$$
\frac{\partial \alpha(t)}{\partial t \partial \bar{t}}=\frac{\partial \tilde{\alpha}(t)}{\partial t \partial \bar{t}}
$$

We showed in Lemma 4.1 that, since $\mathcal{R}_{n}$ is pseudoconvex, $\tilde{\alpha}_{n}(t)$ is subharmonic on $B$; it follows that $\alpha_{n}(t)$ is subharmonic on $B$.

Furthermore, we showed that for each fixed $t \in B, \alpha_{n}(t)$ converges $\alpha(t)$. By its proof we can also show that

$$
\lim _{n \rightarrow \infty} \alpha_{n}(t)=\alpha(t) \text { is uniform on } B
$$

Hence $\alpha(t)$ is subharmonic on $B$.

## 5. An Example of Theorem 1.4

We begin with a simple example of our general result shown in this paper. Let $B=\{|t|<\rho\}$ be a disk in $\mathbb{C}_{t}$. For each $t \in B$, let $R(t)$ be a disk $\{|z|<r(t)\}$ in $\mathbb{C}_{z}$, where $\log r(t)$ is a superharmonic function on $B$. If we set the Hartogs domain of disks $\mathcal{R}=\bigcup_{t \in B}(t, R(t))$, then $\mathcal{R}$ is a pseudoconvex domain. Assume that there exists a holomorphic section $\xi: t \in B \mapsto \xi(t) \in R(t)$, where $\xi(t) \neq 0$. We consider the following function:

$$
f(t, z)=-\frac{1}{\xi(t)} \cdot \frac{r(t)^{2}(z-\xi(t))}{z\left(r(t)^{2}-\bar{\xi}(t) z\right)} \text { on } R(t)
$$

Then $f$ is a circular slit mapping on $R(t)$ with a zero at $z=\xi(t)$ and a pole at $z=$ 0 . The $L_{1}$-constant $\lambda(t)$ on $B$ is written into

$$
\begin{aligned}
\lambda(t)=\log \left|\frac{\partial f}{\partial z}(t, \xi(t))\right| & =\log \left|-\frac{1}{\xi(t)^{2}} \cdot \frac{r(t)^{2}}{r(t)^{2}-|\xi(t)|^{2}}\right| \\
& =-2 \log |\xi(t)|+\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{|\xi(t)|}{r(t)}\right)^{2 n}
\end{aligned}
$$

Since $\xi(t)$ is holomorphic on $B$ and since $\log r(t)$ is superharmonic on $B$, it follows that $\log \frac{|\xi(t)|}{r(t)}$ is subharmonic on $B$ and so is the second term on the right-hand side. Hence, $\lambda(t)$ is a subharmonic function on $B$.

## 6. Proof of Theorem 1.5

First, we take the holomorphic sections $\xi_{0}$ and $\xi_{1}$ on $B$ of $\mathcal{S}$, and we may assume that they are constant: $z=0$ and $z=1$ on $B$. We can make a canonical exhaustion $\left\{\tilde{S}_{n}(t)\right\}_{n}$ of the Schottky covering $\tilde{S}(t)$ of $S(t)$ for each $t$; namely, $\tilde{S}_{j}(t) \Subset \tilde{S}_{j+1}(t)$ $(j=1,2, \ldots)$ and $\bigcup_{n=1}^{\infty} \tilde{S}_{n}(t)=\tilde{S}(t)$ such that each $\bigcup_{t \in B}\left(t, \tilde{S}_{n}(t)\right)=\tilde{\mathcal{S}}_{n}$ is a smooth pseudoconvex domain and the projection of $\partial \tilde{\mathcal{S}}_{n}$ to $S(t)$ does not contain 0,1 in $S(t)$. We thus have $\tilde{\mathcal{S}}=\bigcup_{n=1}^{\infty} \tilde{\mathcal{S}}_{n}$. Let $\hat{\mathcal{S}}=\tilde{\mathcal{S}} \backslash(\underset{\sim}{B} \times\{0\} \cup B \times\{1\})$. We fix two points 0,1 in $\tilde{S}_{n}(t)$ over 0,1 in $S(t)$. We note that $\tilde{S}_{n}(t)$ is a planar Riemann surface. Then, there exists an $L_{1}$-principal function $u_{1 n}(t, z)$ on $\tilde{S}_{n}(t)$ such that

$$
u_{1 n}(t, z)= \begin{cases}\log \frac{1}{|z|}+h_{0 n}(t, z) & \text { on } U_{0} \\ \log |z-1|+\lambda_{n}(t)+h_{1 n}(t, z) & \text { on } U_{1}\end{cases}
$$

where $h_{0 n}(t, 0)=0$ and $h_{1 n}(t, 1)=0$. Then $u_{1 n}(t, z)$ induces the circular slit mapping $f_{1 n}(t, z)=e^{u_{1 n}(t, z)+i u_{1 n}^{*}(t, z)}$ on $\tilde{S}_{n}(t)$ such that

$$
f_{1 n}(t, z)= \begin{cases}\frac{1}{z}+A_{0 n}(t)+A_{1 n}(t) z+\cdots & \text { on } U_{0} \\ B_{1 n}(t)(z-1)+B_{2 n}(t)(z-1)^{2}+\cdots & \text { on } U_{1}\end{cases}
$$

where $f_{1 n}(t, 1)=0$ and $f_{1 n}(t, 0)=\infty$. Since $\tilde{\mathcal{S}}_{n}$ is pseudoconvex, Theorem 1.4 implies that $\lambda_{1 n}(t)=\log \left|B_{1 n}(t)\right|=\log \left|\frac{\partial f_{1 n}}{\partial z}\right|(t, 1)$ is a subharmonic function on $B$. As we have already shown, $u_{1 n}(t, z)$ and hence $f_{1 n}(t, z)$ uniformly converge (respectively) to a harmonic function $u_{1}(t, z)$ and to a univalent holomorphic function $f_{1}(t, z)$ on every compact set in $\hat{S}$ such that

$$
u_{1}(t, z)= \begin{cases}\log \frac{1}{|z|}+h_{0}(t, z) & \text { on } U_{0} \\ \log |z-1|+\lambda(t)+h_{1}(t, z) & \text { on } U_{1}\end{cases}
$$

where $h_{0}(t, 0)=0$ and $h_{1}(t, 1)=0$, and

$$
f_{1}(t, z)= \begin{cases}\frac{1}{z}+A_{0}(t)+A_{1}(t) z+\cdots & \text { on } U_{0} \\ B_{1}(t)(z-1)+B_{2}(t)(z-1)^{2}+\cdots & \text { on } U_{1}\end{cases}
$$

We note that $\lim _{n \rightarrow \infty} \lambda_{n}(t)=\lambda(t)$ uniformly converges on $B$. It follows that $\lambda(t)=\log \left|\frac{\partial f}{\partial z}\right|(t, 1)$ is a subharmonic function on $B$.

Next we shall show that $\log \left|\frac{\partial f_{1}(t, z)}{\partial z}\right|$ is a plurisubharmonic function on $\hat{\mathcal{S}}$. It is enough to show that $\log \left|\frac{\partial f_{1}}{\partial z}(t, \xi(t))\right|$ restricted to every holomorphic section $\xi \neq$ 0,1 of $\tilde{\mathcal{S}}$ is subharmonic on $B$. In fact, we consider another slit mapping $f_{\xi}(t, z)$ on $R(t)$ such that

$$
f_{\xi}(t, z)= \begin{cases}\frac{1}{z}+C_{0}(t)+C_{1}(t) z+\cdots & \text { on } U_{0} \\ D_{1}(t)(z-\xi(t))+D_{2}(t)(z-\xi(t))^{2}+\cdots & \text { on } U_{\xi(t)}\end{cases}
$$

We consider the translation $\phi(t, z)=f_{1}(t, z)-f_{1}(t, \xi(t))$ on $R(t)$; then $\phi(t, z)$ is a holomorphic mapping on $\tilde{R}(t)$ such that

$$
\phi(t, z)= \begin{cases}\frac{1}{z}+\tilde{A}_{0}(t)+\tilde{A}_{1}(t) z+\cdots & \text { on } U_{0} \\ \tilde{B}_{1}(t)(z-\xi(t))+\tilde{B}_{2}(t)(z-\xi(t))^{2}+\cdots & \text { on } U_{\xi(t)} .\end{cases}
$$

We set $w=f_{\xi}(t, z)$ on $\tilde{R}(t)$ and $\zeta=\phi(t, z)$ on $\tilde{R}(t)$. Then, $\zeta=L(t, w):=$ $\phi\left(t, f_{\xi}^{-1}(t, w)\right)$ is a univalent function on the domain $f_{\xi}(t, \tilde{R}(t))$ in $\mathbb{P}_{w}^{1}$. By Koebe's theorem concerning Schottky covering, we see that $L(t, w)$ is a fractional linear transformation on $\mathbb{P}_{w}^{1}$. Since $L(t, 0)=0$ and $L(t, w)=w+c_{0}(t) w^{2}+\cdots$ at $w=\infty$, it follows that $L(t, w) \equiv w$; that is,

$$
f_{\xi}(t, z)=\phi(t, z)=f(t, z)-f(t, \xi(t)), \quad z \in \tilde{R}(t)
$$

Thus, we have

$$
\frac{\partial f_{\xi}}{\partial z}=\frac{\partial\left(f_{1}-f_{1}(t, \xi(t))\right)}{\partial z}=\frac{\partial f_{1}}{\partial z}
$$

Moreover, we recall that $\lambda_{\xi}(t)=\log \left|\frac{\partial f_{\xi}}{\partial z}(t, \xi(t))\right|$ is subharmonic on $B$, which implies the assertion.

Finally, we show that $f$ is a holomorphic function for $t$ and $z$ on $\hat{\mathcal{S}}$. In fact, if we set $p(t, z):=\log \left|\frac{\partial f_{1}}{\partial z}\right|$ on $\hat{\mathcal{S}}$, then $p$ is harmonic for $z$ and hence we have $\frac{\partial^{2} p}{\partial z \partial \bar{z}} \equiv$ 0 . Furthermore,

$$
\operatorname{det}\left(\begin{array}{cc}
\frac{\partial^{2} p}{\partial t \partial \bar{t}} & \frac{\partial^{2} p}{\partial t \partial \bar{z}} \\
\frac{\partial^{2} p}{\partial \bar{t} \partial z} & \frac{\partial^{2} p}{\partial z \partial \bar{z}}
\end{array}\right) \geq 0
$$

because $p$ is a plurisubharmonic function on $\hat{\mathcal{S}}$. Therefore, $\frac{\partial^{2} p}{\partial \bar{t} \partial z} \equiv 0$; in other words, the holomorphic function $\frac{\partial p}{\partial z}$ for $z$ is also holomorphic for $t$ on $\hat{\mathcal{S}}$. Namely, $\frac{\partial^{2} f_{1}}{\partial z^{2}} / \frac{\partial f_{1}}{\partial z}$ is holomorphic for $(t, z)$.

Now we show that $\frac{\partial f_{1}}{\partial z}$ is holomorphic for $(t, z)$ on $B \times\{|z| \ll 1\}$. On the neighborhood $U_{0}$ of $z=0$, we have

$$
\begin{aligned}
f_{1} & =\frac{1}{z}+A_{0}(t)+\sum_{k=1}^{\infty} A_{k}(t) z^{k} \\
\frac{\partial f_{1}}{\partial z} & =-\frac{1}{z^{2}}+A_{1}(t)+\sum_{k=1}^{\infty} k A_{k}(t) z^{k-1} \\
\frac{\partial^{2} f_{1}}{\partial z^{2}} & =\frac{2}{z^{3}}+\sum_{k=2}^{\infty} k(k-1) A_{k}(t) z^{k-2}
\end{aligned}
$$

Thus,

$$
-\frac{z}{2} \frac{\frac{\partial^{2} f_{1}}{\partial z^{2}}}{\frac{\partial f_{1}}{\partial z}}=\frac{1+\sum_{k=2}^{\infty} \frac{k(k-1)}{2} A_{k}(t) z^{k+1}}{1-\sum_{k=2}^{\infty} k A_{k}(t) z^{k+1}}
$$

The left-hand side of this last equality is holomorphic for $(t, z)$, and hence so is the right-hand side. Forming the Taylor expansion of the right-hand side on $\{|z| \ll 1\}$, we can inductively see that each $A_{k}(t)$ is holomorphic for $t$. Therefore, $\frac{\partial f_{1}}{\partial z}$ is holomorphic for $(t, z)$ on $B \times\{|z| \ll 1\}$. By the identity theorem, $\frac{\partial f_{1}}{\partial z}$ is holomorphic for $(t, z)$ on $\hat{\mathcal{S}}$; hence $\int_{1}^{z} \frac{\partial f_{1}}{\partial z}(t, z) d z$ is holomorphic for $(t, z)$ on $\hat{\mathcal{S}}$. Since $\int_{1}^{z} \frac{\partial f_{1}}{\partial z}(t, z) d z=f_{1}(t, z)-f_{1}(t, 1)=f_{1}(t, z)$, we conclude that $f_{1}(t, z)$ is holomorphic for $(t, z)$ and obtain a simultaneous uniformization of $\tilde{\mathcal{S}}$ by $t=t, w=$ $f_{1}(t, z)$.

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[^0]:    Received July 30, 2009. Revision received September 22, 2009.
    Research partially supported by JSPS Grant-in-Aid for Young Scientists (B).

