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VARIATION NORM CONVERGENCE OF FUNCTION SEQUENCES[1]

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ABSTRACT. We prove that a pointwise convergent sequence of convex functions with a continuous limit converges with respect to the total variation norm. This yields a theorem on convexity-preserving operators which has as a corollary the result that a complex function f is absolutely continuous on $[0, 1]$ if and only if the sequence $B_{\bullet}(f)$ of Bernstein polynomials of f converges to f with respect to the total variation norm.

In this paper a theorem which is analogous to Dini's theorem is proved;

Theorem 1. *If $f_{\bullet}$ is a pointwise convergent sequence of real-valued functions, each of which is convex on $[a, b]$ and the limit function F is continuous on $[a, b]$, then the sequence $f_{\bullet}$ converges to F with respect to the total variation norm on $[a, b]$.*

This is then used to prove

Theorem 2. *Suppose $T_{\bullet}$ is a sequence of linear operators from $AC[a, b]$ into $AC[a, b]$ such that for each $f \in AC[a, b]$, (1) $T_{\bullet}(f)$ converges pointwise to f on $[a, b]$; (2) if f is convex on $[a, b]$ and n is a nonnegative integer, $T_n(f)$ is convex on $[a, b]$; and (3) there is a number $M \geq 0$ such that for each nonnegative integer n, $\int_a^b |d(T_n(f))| \leq M\int_a^b |df|$. Then, for each $f \in AC[a, b]$, the function sequence $T_{\bullet}(f)$ converges to f with respect to the total variation norm.*

Corollary. *A complex-valued function f is absolutely continuous on*

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$[0, 1]$ *if and only if the sequence* $B_{\cdot}f$ *of Bernstein polynomials of* f *converges to* f *with respect to the total variation norm.*

An example is given to show that Theorem 1 does not extend to differences of convex functions.

Definitions and notation. A real-valued function f on $[a, b]$ is said to be convex on $[a, b]$ (or simply, convex) provided that for each $[u, v] \subseteq [a, b]$ and each number t, $0 < t < 1$,

$$f((1-t)u + tv) \geq (1-t)f(u) + tf(v).$$

I denotes the identity function on the complex plane, and we employ the convention that I^0 is the constant function 1 so that for each nonnegative integer n and each complex number x, $I^n(x) = x^n$. Hence if f is a complex function on $[0, 1]$, the Bernstein polynomial sequence of f is defined by

$$B_0 f = f(0) \quad \text{and} \quad B_n f = \sum_{p=0}^{n} \binom{n}{p} f\left(\frac{p}{n}\right) I^p (1 - I)^{n-p}$$

for n a positive integer. For a complex function f from a subset of the real numbers, $f(x-)$ and $f(x+)$ respectively denote the left and right hand limits of f at x in case the limit exists; if S is a subset of the domain of f and $f(S)$ is a bounded set, $|f|_S = \sup\{|f(x)|: x \text{ in } S\}$; if f is of bounded variation on $[a, b]$, $\int_a^b |df|$ denotes the total variation of f on $[a, b]$. The notation $]a, b[$ denotes the open interval $\{x: a < x < b\}$ and (a, b) is reserved for an ordered pair.

1. **Convex functions.** We note without proof the following properties of convex functions:

If f is a convex function on $[a, b]$, then

(1) f is continuous on $]a, b[$;

(2) each of $f(a+)$ and $f(b-)$ exists and $f(a) \leq f(a+)$ and $f(b-) \geq f(b)$;

(3) if, in addition, f is nonconstant on $]a, b[$, then only one of the following statements is true:

(a) f is nondecreasing on $[a, b[$,

(b) f is nonincreasing on $]a, b]$,

(c) there is a number x_0 in $]a, b[$ such that f is nondecreasing on $[a, x_0]$, f is nonincreasing on $[x_0, b]$ and f is nonconstant on $]a, x_0[$ and on $]x_0, b[$;

(4) if in addition f is continuous at a and at b, then f is absolutely continuous on $[a, b]$;

(5) a continuous polygonal function is a difference of continuous convex polygonal functions.

Theorem 0. *If* $f_{\cdot}$ *is a pointwise convergent sequence of convex functions on* $[a, b]$ *and* F *denotes the limit function, then*

(1) F *is convex on* $[a, b]$ *and*

(2) *if* F *is continuous on* $[a, b]$, *then* $f_{\cdot}$ *converges uniformly on* $[a, b]$.

Proof. Part (1) follows from the facts that a pointwise convergent function sequence converges uniformly on a finite set, and, hence, for each $[u, v] \subset [a, b]$ and t, $0 < t < 1$, $F((1-t)u + tv) \geq (1-t)F(u) + tF(v)$ must be true since $f_n((1-t)u + tv) \geq (1-t)f_n(u) + tf_n(v)$ for each n.

Proof of (2). There is an $x_0 \in [a, b]$ such that F is monotone on each of $[a, x_0]$ and $[x_0, b]$. Hence it is enough to prove the theorem under the added assumption that F is nondecreasing. Suppose $c > 0$. There is an increasing sequence $\{t_i\}_0^k$ with $t_0 = a$ and $t_k = b$ such that $F(t_i) - F(t_{i-1}) < c$, $i = 1, \cdots, k$.

Let $s_i = (t_{i-1} + t_i)/2$, $i = 1, \cdots, k$. If $t_{i-1} \leq x \leq s_i$ then, since f_n is convex,

$$(f_n(s_i) - f_n(x))/(s_i - x) \geq (f_n(t_i) - f_n(s_i))/(t_i - s_i).$$

Since $(s_i - x)/(t_i - s_i) \leq (s_i - t_{i-1})/(t_i - s_i) = 1$, this implies

$$f_n(x) \leq f_n(s_i) + |f_n(t_i) - f_n(s_i)|;$$

and hence

$$\text{(A)} \qquad \sup_{t_{i-1} \leq x \leq s_i} \{f_n(x) - F(x)\} \leq f_n(s_i) - F(t_{i-1}) + |f_n(t_i) - f_n(s_i)|.$$

Similarly, if $s_i < x \leq t_i$ then

$$(f_n(x) - f_n(s_i))/(x - s_i) \leq (f_n(s_i) - f_n(t_{i-1}))/(s_i - t_{i-1}),$$

whence

$$\text{(B)} \qquad \sup_{s_i \leq x \leq t_i} \{f_n(x) - F(x)\} \leq f_n(s_i) - F(x_i) + |f_n(s_i) - f_n(t_{i-1})|.$$

Also, if $t_{i-1} \leq x \leq t_i$, then $f_n(x) \geq \min\{f_n(t_{i-1}), f_n(t_i)\}$, and

$$\text{(C)} \qquad \sup_{t_{i-1} \leq x \leq t_i} \{F(x) - f_n(x)\} \leq F(t_i) - \min\{f_n(t_{i-1}), f_n(t_i)\}.$$

As $n \to \infty$, the right-hand side of each of (A), (B) and (C) has a limit less than c, and the theorem is proved.

Lemma 1.1. *If f is convex on $[a, b]$, $e > 0$, g is convex on $[a, b]$ such that*

$$|f - g|_{[a,b]} \le e \quad \text{and} \quad P = \frac{f(b) - f(a)}{b - a}(I - a) + f(a),$$

then

$$\int_a^b |d(f - P)| = 2|f - P|_{[a,b]} \tag{1}$$

and

$$\int_a^b |d(g - P)| \le \int_a^b |d(f - P)| + 4e. \tag{2}$$

Proof. (1) follows immediately from the unproved assertion (3) about convex functions. To prove (2) we note that $g - P$ is convex and apply the same assertion (3) in the three separate cases. Let us consider only the case where there exists a number x_0 in $]a, b[$ such that $g - P$ is nondecreasing on $[a, x_0]$, nonincreasing on $[x_0, b]$ and nonconstant on each of $]a, x_0[$ and $]x_0, b[$. Thus

$$\begin{aligned}\int_a^b |d(g - P)| &= 2(g - P)(x_0) - (g - P)(a) - (g - P)(b)\\ &= 2g(x_0) - 2P(x_0) - g(a) + f(a) - g(b) + f(b)\\ &\le 2\{f(x_0) + e\} - 2P(x_0) + e + e\\ &= 2\{f(x_0) - P(x_0)\} + 4e \le 2|f - P|_{[a,b]} + 4e\\ &= \int_a^b |d(f - P)| + 4e.\end{aligned}$$

We omit proof of the other two cases.

Lemma 1.2, which follows, was proved independently by the author for convex functions only. It follows immediately from a result of J. R. Edwards and S. G. Wayment [1, p. 254] on absolutely continuous functions and the fact that a continuous convex function is absolutely continuous.

Lemma 1.2. *If F is a continuous convex function on $[a, b]$ and $c > 0$, then there is an increasing sequence $\{t_p\}_0^n$ with $t_0 = a$ and $t_n = b$ such that if P is the function on $[a, b]$ defined by*

$$P(x) = \frac{F(t_{p+1}) - F(t_p)}{t_{p+1} - t_p}(x - t_p) + F(t_p) \quad \text{for } x \text{ in } [t_p, t_{p+1}],$$

then $\int_a^b |d(F - P)| < c$.

Theorem 1. *If $f_{\bullet}$ is a pointwise convergent sequence of functions on*

$[a, b]$ each of which is convex on $[a, b]$ and the limit function F is continuous on $[a, b]$, then the sequence $f_.$ converges to F in the total variation norm on $[a, b]$.

Proof. Suppose the hypothesis and let $c > 0$. There is an increasing sequence $\{t_p\}_0^n$ with $t_0 = a$ and $t_n = b$ such that if P is the function as defined in Lemma 1.2 for F, then $\int_a^b |d(F - P)| < c/4$. Let $e = c/(8n)$. There is a positive integer N such that if q is an integer, $q > N$, then $|f_q - F|_{[a,b]} < e$ by Theorem 0. For each integer $q > N$

$$\int_a^b |d(f_q - F)| \le \int_a^b |d(f_q - P)| + \int_a^b |d(P - F)|$$

$$< \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} |d(f_q - P)| + \frac{c}{4}.$$

But Lemma 1.1 and the fact that $|f_q - F|_{[t_p,t_{p+1}]} \le |f_q - F|_{[a,b]} < e$ imply that for each integer p, $0 \le p \le n - 1$,

$$\int_{t_p}^{t_{p+1}} |d(f_q - P)| \le \int_{t_p}^{t_{p+1}} |d(F - f_q)| + 4e.$$

Whence we see that

$$\sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} |d(f_q - P)| \le \sum_{p=0}^{n-1} \left\{\int_{t_p}^{t_{p+1}} |d(F - P)| + 4e\right\}$$

$$= \int_a^b |d(F - P)| + 4ne = \int_a^b |d(f - P)| + \frac{c}{2} < \frac{3c}{4}.$$

Thus $\int_a^b |d(f_q - F)| < c$ for each integer $q > N$.

Corollary. *If F is a continuous convex function on $[0, 1]$, then the sequence $B_.F$ of Bernstein polynomials of F converges to F with respect to the total variation norm on $[0, 1]$.*

Proof. This is an immediate consequence of the well-known facts that since F is continuous, $B_.F$ converges uniformly to F and that for each nonnegative integer, n, B_nF is convex on $[0, 1]$; cf. Lorentz [3, p. 5 and p. 23 resp.].

Remark. Theorem 1 does not extend to sequences of differences of convex functions, as may be seen from the following example: let $f_.$ be a sequence of functions on $[0, 1]$ such that for each positive integer n, and nonnegative integer $p < 2^n$

$$f_n(x) = \begin{cases} x - p/2^n & \text{if } x \in [p/2^n, (p+1)/2^n[\text{ and } p \text{ is even,} \\ 1/2^n - (x - p/2^n) & \text{if } x \in [p/2^n, (p+1)/2^n] \text{ and } p \text{ is odd.} \end{cases}$$

Each function f_n is a continuous polygonal function with $\int_a^b |df_n| = 1$, and the sequence $f_.$ converges uniformly to the constant function 0.

2. **Absolutely continuous functions.** A function f is said to be absolutely continuous on $[a, b]$ provided that for each $c > 0$ there is a positive number d such that if $\{[u_p, v_p]\}_0^n$ is a sequence of nonoverlapping subintervals of $[a, b]$ with $\Sigma_{p=0}^n (v_p - u_p) < d$, then $\Sigma_{p=0}^n |f(v_p) - f(u_p)| < c$. It is well known [1] that the class $AC[a, b]$ of all absolutely continuous real-valued functions on $[a, b]$ is complete with respect to the total variation norm and that the polygonal functions form a dense subset thereof.

Theorem 2. *Suppose $T_.$ is a sequence of linear operators from $AC[a, b]$ into $AC[a, b]$ such that for each f in $AC[a, b]$,* (1) *$T_.(f)$ converges pointwise to f on $[a, b]$;* (2) *if f is convex on $[a, b]$ and n is a nonnegative integer, $T_n(f)$ is convex on $[a, b]$; and* (3) *there is a number $M \geq 0$ such that for each nonnegative integer n, $\int_a^b |d(T_n(f))| \leq M \int_a^b |df|$. Then, for each $f \in AC[a, b]$, the function sequence $T_.(f)$ converges to f with respect to the total variation norm.*

Proof. Let $\mathfrak{B}$ denote the set of all real-valued functions f on $[a, b]$ such that $T_. f$ converges to f with respect to the total variation norm. $\mathfrak{B}$ is closed with respect to the total variation norm, for if F is the limit with respect to the total variation norm of a sequence $f_.$ with values in $\mathfrak{B}$, then

$$\int_a^b |d(F - T_n F)| \leq \int_a^b |d(F - f_k)| + \int_a^b |d(f_k - T_n(f_k))| + \int_a^b |d(T_n(f_k) - T_n(F))|.$$

But from part (3) of the hypothesis we have that

$$\int_a^b |d(T_n(f_k) - T_n(F))| = \int_a^b |d(T_n(f_k - F))| \leq M \int_a^b |d(f_k - F)|.$$

Thus

$$\int_a^b |d(F - T_n(F))| \leq (M + 1) \int_a^b |d(F - f_k)| + \int_a^b |d(f_k - T_n(f_k))|,$$

from which it is clear that $\mathfrak{B}$ is closed with respect to the total variation norm.

If P is a polygonal function on $[a, b]$ then P is a difference of continuous convex functions, say $P = h - k$. But for each nonnegative integer n,

$T_n P = T_n(h - k) = T_n(h) - T_n(k)$; whence by Theorem 1 and parts (1) and (2) of the hypothesis, P must belong to $\mathcal{B}$. Thus $\mathcal{B} = AC[a, b]$ since $\mathcal{B}$ contains all polygonal functions and is closed with respect to the total variation norm.

Corollary. *A complex-valued function f is absolutely continuous on $[0, 1]$ if and only if the sequence $B_. f$ of Bernstein polynomials of f converges to f with respect to the total variation norm.*

Proof. Let us note that if f is a complex-valued absolutely continuous function on $[0, 1]$, then each of $\operatorname{Re} f$ and $\operatorname{Im} f$ is absolutely continuous; and if n is a nonengative integer, $B_n f = B_n \operatorname{Re} f + iB_n \operatorname{Im} f$. Thus it is sufficient to suppose f to be real valued, and we do so. Since any polynomial is absolutely continuous on $[0, 1]$, then any function f on $[0, 1]$ such that $B_. f$ converges to f with respect to the total variation norm must perforce be absolutely continuous. Theorem 2 yields the converse.

Comment. The corollary to Theorem 2 has been obtained independently by G. G. Johnson, who used methods different from ours. While the results herein give no estimate on the size of $\int_0^1 |d(F - B_n(F))|$, they offer an extension of a result of W. Hoeffding [2, p. 349] that: If f is a continuous convex function such that $\int_0^1 I^{1/2}(1 - I)^{1/2}\, d(f')$ exists, then $B_. f$ converges with respect to the total variation norm. Some applications of these results to moment problems will appear in a subsequent paper.

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