Open access • Journal Article • DOI:10.1007/BF02614622

Variation of cost functions in integer programming — Source link

Bernd Sturmfels, Rekha R. Thomas

Institutions: University of California, Berkeley, Texas A&M University

Published on: 01 Jun 1997 - Mathematical Programming (Springer-Verlag)

Topics: Graver basis

SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NY 14853-3801

TECHNICAL REPORT NO. 1106

September 1994

Variation of Cost Functions in Integer Programming[1]

by

Bernd Sturmfels
and Rekha R. Thomas

[1]Research was supported in part by the National Science Foundation and the David and Lucile Packard Foundation.

Variation of Cost Functions in Integer Programming*

Bernd Sturmfels Rekha R. Thomas

Cornell University, Ithaca, New York 14853.
bernd@math.cornell.edu rekha@cs.cornell.edu

August 23, 1994

Abstract

We study the problem of minimizing $c \cdot x$ subject to $A \cdot x = b$, $x \geq 0$ and x integral, for a fixed matrix A. Two cost functions c and c' are considered equivalent if they give the same optimal solutions for each b. We construct a polytope $St(A)$ whose normal cones are the equivalence classes. Explicit inequality presentations of these cones are given by the reduced Gröbner bases associated with A. The union of the reduced Gröbner bases as c varies (called the universal Gröbner basis) consists precisely of the edge directions of $St(A)$. We present geometric algorithms for computing $St(A)$, the Graver basis [Gra], and the universal Gröbner basis.

Introduction

In this paper we study the general integer programming problem

$$IP_{A,c}(b): \qquad \text{minimize } c \cdot x \text{ subject to } A \cdot x = b \text{ and } x \geq 0,\ x \in \mathbf{Z}^n,$$

where $c \in \mathbf{R}^n$, A is a $d \times n$ integral matrix of rank d, and $b \in \mathbf{Z}^d$. Let IP_A denote the family of all such programs with fixed matrix A. The convex hull P_b^I of the set of feasible solutions $\{x \in \mathbf{N}^n : Ax = b\}$ is called the *b-fiber* of IP_A. For simplicity we assume that each fiber is bounded. Two cost functions c and c' are *equivalent* if $IP_{A,c}(b)$ and $IP_{A,c'}(b)$ have the same set of optimal solutions, for each right hand side b. These equivalence classes are relatively open polyhedral cones

*Research partially supported by the National Science Foundation and the David and Lucile Packard Foundation.

in $\mathbf{R}^n$. They fit together to form a polyhedral fan, called the *Gröbner fan* of A. The study of the Gröbner fan is what we mean by "variation of cost functions in integer programming".

We shall construct a certain $(n-d)$-dimensional polytope $St(A)$ in $\mathbf{R}^n$, called the *state polytope* of A, which has the Gröbner fan as its normal fan. The state polytope can be represented as the *Minkowski integral* $\int_b P_b^I db$, where db is any suitable probability measure with support $pos_{\mathbf{Z}}(A)$. This provides an integral refinement of polyhedral results in [GKZ], [BiS], [BGS]. The concepts of Gröbner fans and state polytopes were first introduced in a more general algebraic setting by Mora-Robbiano [MR] and Bayer-Morrison [BM]. The new construction to be given in Section 3 is more explicit, self-contained and custom-tailored to integer programming.

Our discussion assumes familiarity with Gröbner bases as presented in e.g. [AL], [CT], [NTT], [Th]. We remark that software for computing them is readily available and surprisingly efficient [BaS]. The *reduced Gröbner basis* of A with respect to c is a finite subset $\mathcal{G}_c$ of $ker_{\mathbf{Z}}(A)$: it is the minimal *test set* (in the sense of [Schr],§17.3) for all programs $IP_{A,c}(\cdot)$. The union of all reduced Gröbner bases $\mathcal{G}_c$, as c varies over $\mathbf{R}^n$, is a finite set. It is denoted UGB_A and called the *universal Gröbner basis* of A. One of our main results (Theorem 5.1) states that the elements of UGB_A are precisely the edge directions of all fibers P_b^I.

This paper is organized as follows: In Section 1 we review known results for linear programming whose integer analogues are to be established later. Writing $LP_{A,c}(b)$ for the linear relaxation of $IP_{A,c}(b)$, we say that two cost functions c and c' are *equivalent (for LP_A)* if the linear programs $LP_{A,c}(b)$ and $LP_{A,c'}(b)$ have the same set of optimal solutions for each b. This is the case if and only if the *regular polyhedral subdivisions* Δ_c and $\Delta_{c'}$ coincide (Theorem 1.3). Each equivalence class is the relative interior of a cone in the *secondary fan* of A, which is the normal fan of the *secondary polytope* $\Sigma(A)$ (see [BFS], [BGS], [GKZ]). The *circuits* of A form a universal test set for LP_A, and these are precisely the set of edge directions of $\Sigma(A)$ (Theorem 1.8).

In Section 2 we examine the reduced Gröbner basis $\mathcal{G}_c$ and the universal Gröbner basis UGB_A. The set UGB_A is contained in the *Graver basis* of A, which is the well-known test set introduced in [Gra] (see also [BJ],[CGST]). Example 2.7 shows that the Graver basis can be much larger than the universal Gröbner basis. The integer analogue of the regular triangulation Δ_c is the *(initial) monomial ideal* $in_c(I_A)$. We show how the two are related (Theorem 2.4), and we present some enumerative and computational (cf. Algorithm 2.9) applications of monomial ideals.

The main result in Section 3 is the structure theorem (3.10) for the equivalence classes of IP_A, involving the Gröbner fan and the state polytope. Each equivalence class is shown to be the relative interior of a face of a *Gröbner cone* $\mathcal{K}_c = \{x \in \mathbf{R}^n : g_i \cdot x \geq 0,\, g_i \in \mathcal{G}_c\}$. Thus the local variation of cost functions is controlled by the reduced Gröbner basis $\mathcal{G}_c$. In Proposition 3.13 we introduce the *Graver arrangement* of A which is the collection of hyperplanes normal to the Graver basis. This is a natural refinement of the Gröbner fan, and it serves as an approximation to the latter.

The *Lawrence lifting* of A is the enlarged matrix $\Lambda(A) = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$ where $\mathbf{0}$ is the zero $d \times n$-matrix and $\mathbf{1}$ is the unit $n \times n$-matrix. The Lawrence lifting corresponds to programs in IP_A with upper bound constraints (as in equation (1.4)). Our main result in Section 4 states that *every* reduced Gröbner basis of $\Lambda(A)$ coincides with the Graver basis of $\Lambda(A)$. This leads to Algorithms 4.3 and 4.7 for computing the Graver basis of A and the universal Gröbner basis of A.

In Section 5 we give a geometric characterization of the vectors in the universal Gröbner basis: they are the edge directions of the state polytope (and hence of all fibers P_b^I). Algorithm 5.8 gives a geometric method for computing the Gröbner fan, and Theorem 5.9 relates the following three properties of a matrix: (i) A *unimodular*, (ii) $St(A) = \Sigma(A)$, (iii) $UGB_A = \{$ circuits of $A\}$.

The last section deals with the number of facets of a Gröbner cone or equivalently the valency of a vertex of $St(A)$. We conjecture that this number is bounded above by a function in the corank of A. Two examples that give lower bounds for this function are constructed.

The following table summarizes the interrelations between the main concepts in this paper. The symbol "$<$" denotes refinement for polyhedral fans and "is Minkowski summand of" for polytopes.

Linear programming		Integer programming
$LP_{A,c}(b)$: Min $cx : Ax = b,\ x \in \mathbf{R}^n_+$	programs	$IP_{A,c}(b)$: Min $cx : Ax = b,\ x \in \mathbf{N}^n$
$P_b = \{x \in \mathbf{R}^n_+ : Ax = b\}$ $\Sigma(A) = \int_b P_b\, db$ (secondary polytope)	polytopes $<$	$P_b^I = \text{conv}\,\{x \in \mathbf{N}^n : Ax = b\}$ $St(A) = \int_b P_b^I\, db$ (state polytope)
$\mathcal{N}(\Sigma(A))$ = secondary fan	normal fans $<$	$\mathcal{N}(St(A))$ = Gröbner fan
$\wedge$ circuit arrangement	hyperplane arrangements $<$	$\wedge$ $\wedge$ Gröbner arrgt. $<$ Graver arrgt.
circuits $\Vert$ (edges directions of P_b, for all b)	universal test sets $\subseteq$	universal Gröbner basis $\subseteq$ Graver basis $\Vert$ (edge directions of P_b^I for all b)

1 Variation of cost functions in linear programming.

The results on integer programming to be presented in this paper have known easier analogues in linear programming. In this section we give an exposition of these analogues. Our starting point is the Basis Decomposition Theorem for Linear Programming in [WW].

Theorem 1.1 (Walkup–Wets, 1969) *Let $LP_{A,c}(b)$ denote the linear program*

$$\textit{minimize } c \cdot x \textit{ subject to } A \cdot x \;=\; b \textit{ and } x \geq 0, \qquad (1.1)$$

where $c \in \mathbf{R}^n$ is fixed and $A = (a_1, a_2, \ldots, a_n)$ is a fixed $d \times n$-matrix of rank d. Then:
(i) $LP_{A,c}(b)$ is feasible if and only if b lies in the closed convex polyhedral d-cone $pos(A)$.
(ii) $LP_{A,c}(b)$ is bounded for all b in $pos(A)$ if and only if $ker(A) \cap \mathbf{R}^n_+ = \{0\}$.
(iii) If $LP_{A,c}(b)$ is bounded, then there exists a triangulation Δ of $pos(A)$ such that
(a) the d-dimensional cells of Δ have the form $C = pos(\{a_{i_1}, \ldots, a_{i_d}\})$, and
(b) the column vectors $a_{i_1}, \ldots, a_{i_d}$ constitute an optimal basis for all b in the cell C.

This theorem is best understood within the context of *secondary polytopes* (see [BFS],[BGS] and [GKZ]). For simplicity we shall assume that $LP_{A,c}(\,\cdot\,)$ is bounded. For every $c \in \mathbf{R}^n$ there is a polyhedral subdivision Δ_c of the cone $pos(A)$ defined as follows: A cone $pos(\{a_{i_1}, \ldots, a_{i_k}\})$ is a cell of Δ_c if and only if there exists a row vector $y \in \mathbf{R}^d$ such that $y \cdot a_j = c_j$ if $j \in \{i_1, \ldots, i_k\}$ and $y \cdot a_j < c_j$ if $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. In this situation, it is customary (and more precise) to say that $\{i_1, \ldots, i_k\}$ is a cell of Δ_c. Subdivisions obtained in this way are called *regular* (or *coherent*). For almost all $c \in \mathbf{R}^n$, the regular subdivision Δ_c is a triangulation, in which case we call c *generic*.

Part (iii) of the Walkup-Wets Theorem can be proved as follows: If c is generic, then $\Delta = \Delta_c$ is the desired triangulation. If c is not generic, then we may take Δ to be any regular triangulation which refines Δ_c. In other words, we may take $\Delta = \Delta_{c'}$ where c' is generic and very close to c.

Example 1.2 Consider the family of linear programs of the form (1.1) defined by the 3×6-matrix

$$A \;=\; \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

The vector $c = (1,0,0,1,0,0)$ is not generic. The corresponding subdivision of $pos(A) = \mathbf{R}^3_+$ consists of two triangular cones and one quadrangular cone: $\Delta_c = \{\{1,2,3\},\{2,4,5\},\{2,3,5,6\}\}$, where i indexes a_i. To get a regular triangulation refining Δ_c we may take $c' = (1,0,0,1,0,1)$. The cost function c' is generic, since $\Delta_{c'} = \{\{1,2,3\},\{2,3,5\},\{2,4,5\},\{3,5,6\}\}$. □

Let P_b denote the polytope $\{x \in \mathbf{R}^n_+ : Ax = b\}$. We call P_b the *b-fiber* of the family LP_A of all linear programs of the form (1.1) with fixed coefficient matrix A. This terminology is consistent with the usage in [BiS] and [Th]. For $x \in \mathbf{R}^n$ abbreviate $supp(x) = \{j \in \{1, \ldots, n\} : x_j \neq 0\}$.

Theorem 1.3 *Given two cost functions c and c' in $\mathbf{R}^n$, the following are equivalent:*
(i) The programs $LP_{A,c}(b)$ and $LP_{A,c'}(b)$ have the same set of optimal solutions, for every b.
(ii) The cost functions c and c' support the same optimal face in each fiber P_b of LP_A.
(iii) The vectors c and c' define the same polyhedral subdivision $\Delta_c = \Delta_{c'}$.

Proof: The conditions (i) and (ii) are equivalent because the set of optimal solutions of $LP_{A,c}(b)$ is the face of P_b supported by c. The equivalence of (i) and (iii) follows from Lemma 1.4 below. □

Lemma 1.4 *The optimal solutions x to $LP_{A,c}(b)$ are the solutions to the problem:*

$$\textit{Find } x \in \mathbf{R}^n \textit{ such that } A \cdot x = b,\ x \geq 0,\ \textit{and } supp(x) \textit{ is a subset of a face of } \Delta_c. \tag{1.2}$$

Proof: Consider the linear program dual to (1.1):

$$\text{Maximize } y \cdot b \text{ subject to } y \cdot A \leq c \text{ and } y \in \mathbf{R}^d. \tag{1.3}$$

Let x be an optimal solution of (1.1) and y an optimal solution of (1.3). By complementary slackness, $x_j > 0$ implies $y \cdot a_j = c_j$, which means that $supp(x)$ lies in a face of Δ_c. Conversely, let x be any solution to (1.2). Then there exists $y \in \mathbf{R}^d$ with $supp(x) \subseteq \{\, j \,:\, y \cdot a_j = c_j\}$. This implies $c \cdot x = y \cdot A \cdot x = y \cdot b$ and hence, x is an optimal solution of (1.1). □

Theorem 1.3 gives rise to a natural equivalence relation on cost functions: two vectors c and c' in $\mathbf{R}^n$ are *equivalent (with respect to LP_A)* if the conditions in Theorem 1.3 hold. We have the following structure theorem for the equivalence classes. Theorem 1.5 is a direct translation of results of Gel'fand-Kapranov-Zelevinsky ([GKZ], Chapter 7) and Billera-Gel'fand-Sturmfels [BGS].

Theorem 1.5 *(i) There are only finitely many equivalence classes of cost functions for LP_A.*
(ii) Each equivalence class is the relative interior of a convex polyhedral cone in $\mathbf{R}^n$.
(iii) The collection of these cones defines a polyhedral fan which covers $\mathbf{R}^n$. This fan is called the secondary fan *of A.*
(iv) Let db denote any probability measure with support $pos(A)$. Then the Minkowski integral $\Sigma(A) = \int_b P_b\, db$ is an $(n-d)$-dimensional convex polytope, called the secondary polytope *of A. The inner normal fan of $\Sigma(A)$ equals the secondary fan of A.*

Example 1.2 (continued) The secondary polytope $\Sigma(A)$ is a simple 3-polytope with 14 vertices, 21 edges, and 9 facets; see Figure 34 in [GKZ] and Figure 3 in [St3]. It is known as the *associahedron* or *Stasheff polytope.* The family LP_A has $45 = 14 + 21 + 9 + 1$ equivalence classes of cost functions. The 14 distinct regular triangulations of $pos(A)$ are listed in Section 3.

Corollary 1.6 *For a cost function c in $\mathbf{R}^n$ the following are equivalent:*
(i) c is generic, i.e., the subdivision Δ_c is a triangulation.

(ii) c supports a vertex in each fiber P_b of LP_A.
(iii) For every $b \in pos(A)$, the program $LP_{A,c}(b)$ has a unique optimal solution.
(iv) c lies in the interior of an n-dimensional cell of the secondary fan of A.
(v) c supports a vertex of the secondary polytope $\Sigma(A)$.

An important tool used in this paper is a "test set" for integer programming (cf [Schr],[Th]). It is instructive to define the following analogue for linear programming: A *test set* for the family $LP_{A,c}(\,\cdot\,)$ is any finite subset $\mathcal{T}$ of the kernel of A such that, for every $b \in pos(A)$ and every $x \in P_b$, either x is an optimal solution of $LP_{A,c}(b)$ or there exists $t \in \mathcal{T}$ and $\epsilon > 0$ such that $x - \epsilon t \geq 0$ and $c \cdot t > 0$. A test set is *minimal* if it has minimal cardinality.

For the remainder of this section we shall assume that c is generic, so that Δ_c is a triangulation. We say that $I \subset \{1, \ldots, n\}$ is a *minimal non-face* of Δ_c if I is not a face of Δ_c but every proper subset of I is a face of Δ_c.

Proposition 1.7 *A finite subset $\mathcal{T} \subset ker(A)$ is a minimal test set for $LP_{A,c}(\,\cdot\,)$ if and only if for every minimal non-face I of Δ_c there is a unique vector $t \in \mathcal{T}$ such that $I = \{\, i \,:\, t_i > 0\}$.*

Example 1.2 (continued) Let $c' = (1,0,0,1,0,1)$ as above. The minimal non-faces of the triangulation $\Delta_{c'}$ are $\{1,4\}, \{1,5\}, \{1,6\}, \{2,6\}, \{3,4\}$ and $\{4,6\}$. A minimal test set for $LP_{A,c'}(\cdot)$ is $\mathcal{T} = \{e_1 + e_4 - 2e_2,\ e_1 + e_5 - e_2 - e_3,\ e_1 + e_6 - 2e_3,\ e_2 + e_6 - e_3 - e_5,\ e_3 + e_4 - e_2 - e_5,\ e_4 + e_6 - 2e_5\}$. Here and throughout this paper we write e_i for the i-th unit vector in $\mathbf{R}^n$.

We define a *universal test set* for the family LP_A to be any finite subset $\mathcal{T}$ of $ker(A)$ such that $\mathcal{T}$ is a test set for $LP_{A,c}(\,\cdot\,)$, for every generic $c \in \mathbf{R}^n$.

Theorem 1.8 *For a finite subset $\mathcal{C} \subset ker(A)$ the following are equivalent:*
(i) $\mathcal{C} \cup -\mathcal{C}$ is a minimal universal test set.
(ii) Every edge direction of any fiber P_b has a unique representative in $\mathcal{C}$.
(iii) Every edge direction of the secondary polytope $\Sigma(A)$ has a unique representative in $\mathcal{C}$.
(iv) Every equivalence class of circuits of A has a unique representative in $\mathcal{C}$.

Here a *circuit* of A is a non-zero vector of minimal support in $ker(A)$, and two circuits t and t' are equivalent if $t' = \lambda t$. (This is consistent with the usage in matroid theory.) The circuits $\mathcal{C}$ are precisely the possible directions taken by the simplex algorithm when solving any of the programs $LP_{A,c}(b)$. Thus the study of test sets can be viewed as an abstraction of the simplex algorithm.

Example 1.2 (continued) The matrix A has nine circuits. With $\mathcal{T}$ as above, we have

$$\mathcal{C} \quad = \quad \mathcal{T} \cup \{e_1 + 2e_5 - 2e_2 - e_6,\ 2e_2 + e_6 - 2e_3 - e_4,\ e_1 - e_4 - 2e_3 + 2e_5\}.$$

Each fiber P_b of IP_A is a polytope of dimension at most three and has its edge directions among these nine circuits. A fiber P_b is of dimension three if and only if b lies in the interior of $pos(A)$.

Corollary 1.9 *For every generic $c \in \mathbf{R}^n$, there exists a minimal test set for $LP_{A,c}(\cdot)$ which consists only of edges of certain fibers P_b.*

Sketch of proof: For every minimal non-face I of Δ_c there is a circuit t with $I = \{i : t_i > 0\}$. □

Remark 1.10 If the set of circuits is known, and a generic vector $c \in \mathbf{R}^n$ is given, then the regular triangulation Δ_c can be computed as follows. Direct each circuit $t \in \mathcal{C}$ such that $t \cdot c > 0$. Then the faces of Δ_c are those subsets of $\{1, \ldots, n\}$ which do not contain $\{i : t_i > 0\}$ for any $t \in \mathcal{C}$.

The *circuit arrangement* of A is the arrangement consisting of the hyperplanes in $\mathbf{R}^n$ which are orthogonal to the circuits of A. There are at most $\binom{n}{d+1}$ hyperplanes in the circuit arrangement, and all of them contain the row span of A. It is thus natural to consider the circuit arrangement in the $(n-d)$-dimensional space $ker(A) \simeq \mathbf{R}^n/rowspan(A)$.

Proposition 1.11 [BFS, Lemma 5.2], (see also [BLSWZ, Exercise 9.3.1])
(i) The circuit arrangement of A is a refinement of the secondary fan of A.
(ii) If A is of Lawrence type, then the circuit arrangement equals the secondary fan.

A matrix A is of *Lawrence type* if $A = \begin{pmatrix} A' & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$ where $\mathbf{0}$ is a zero matrix of the same format as A', and $\mathbf{1}$ denotes the unit matrix with the same number of columns. For this matrix and the right hand side $b = \binom{b'}{b''}$, the linear program (1.1) takes the special form

$$\text{minimize } c \cdot x \text{ subject to } A' \cdot x = b' \text{ and } 0 \leq x \leq b''. \tag{1.4}$$

The corresponding class of integer programs will play a special role in Section 4.

2 Test sets and monomial ideals in integer programming.

In this section we review some known *test sets* in integer programming, and we explain their connections with regular triangulations and with monomial ideals. For more details and proofs see [Th], [St1], and the references given there. Let $IP_{A,c}(b)$ denote the integer program

$$\text{minimize } c \cdot x \text{ subject to } A \cdot x = b \text{ and } x \geq 0,\ x \in \mathbf{Z}^n, \tag{2.1}$$

where $c \in \mathbf{R}^n$ is fixed and $A = (a_1, a_2, \ldots, a_n)$ is a fixed $d \times n$-integer matrix of rank d. We denote by IP_A the family of all integer programs (2.1) for which the coefficient matrix A is fixed. For simplicity we assume throughout that all programs in IP_A are bounded. Let P_b^I denote the polytope that is the convex hull of all feasible solutions to $IP_{A,c}(b)$. We call P_b^I the *b-fiber* of IP_A. A cost function c is said to be *generic (with respect to IP_A)* if the optimal solution of $IP_{A,c}(b)$ is a unique vertex of P_b^I for every b for which $IP_{A,c}(b)$ is feasible. In this section we shall assume that

c is generic. We remark that any given c can be made generic by refining the partial order on $\mathbf{N}^n$ given by the objective function value by the lexicographic order on $\mathbf{N}^n$.

We introduce the lattice $ker_{\mathbf{Z}}(A) := ker(A) \cap \mathbf{Z}^n$ and the semigroup $pos_{\mathbf{Z}}(A) := \{\sum_{i=1}^n m_i a_i : m_i \geq 0, m_i \in \mathbf{Z}\}$. A *test set* for the family of integer programs $IP_{A,c}(\cdot)$ is a finite subset $\mathcal{G}$ of $ker_{\mathbf{Z}}(A)$ such that, for every $b \in pos_{\mathbf{Z}}(A)$ and every $x \in P_b^I$, either x is the optimal solution of $IP_{A,c}(b)$ or there exists $g \in \mathcal{G}$ such that $x - g \geq 0$ and $c \cdot g > 0$. As before, a test set is minimal if it has minimal cardinality. We shall now construct a canonical minimal test set for $IP_{A,c}(\cdot)$.

Lemma 2.1 *There exists a unique minimal set of vectors $\alpha_1, \ldots, \alpha_t$ in $\mathbf{N}^n$ such that the set of all non-optimal solutions to all programs in $IP_{A,c}(\cdot)$ is of the form $\bigcup_{i=1}^t (\alpha_i + \mathbf{N}^n)$.*

Let $\mathcal{G}_c = \{\alpha_i - \beta_i : i := 1, \ldots, t\}$ where β_i is the unique optimal solution of $IP_{A,c}(A\alpha_i)$. Since α_i is a minimal element in the set of non-optimal solutions, it follows that $supp(\alpha_i) \cap supp(\beta_i) = \emptyset$ for each $\alpha_i - \beta_i$ in $\mathcal{G}_c$. (See Lemma 5.3 for the proof of a stronger statement.)

Proposition 2.2 [Th] *The set $\mathcal{G}_c$ is a minimal test set for the family of integer programs $IP_{A,c}(\cdot)$.*

The set $\mathcal{G}_c$ is called the *reduced Gröbner basis* of $IP_{A,c}(\cdot)$. The reduced Gröbner basis $\mathcal{G}_c$ was first introduced in an algebraic setting by Conti-Traverso [CT]. We briefly explain the relationship to our formulation, since general properties of Gröbner bases for polynomial ideals will be called upon repeatedly below. For an introduction to Gröbner bases with a view towards integer programming see also in Section 2.8 in [AL].

Let k be any field. The matrix $A = (a_1, \ldots, a_n)$ defines a k-algebra homomorphism

$$\phi_A \quad : \quad k[x_1, x_2, \ldots, x_n] \;\rightarrow\; k[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}], \; x_i \mapsto t^{a_i}.$$

Its kernel $I_A := ker(\phi_A)$ is the ideal generated by all "binomials" $x^u - x^v$ such that $Au = Av$, $u, v \in \mathbf{N}^n$, (i.e., u and v lie in the same fiber of IP_A). We call I_A the *toric ideal* of A. The subset $\{x^{\alpha_i} - x^{\beta_i} : i := 1, \ldots, t\}$, with x^{α_i} as leading term, is the reduced Gröbner basis of I_A with respect to c. We identify this set with $\mathcal{G}_c$. Under the usual identification of lattice points u in $\mathbf{N}^n$ and monomials x^u in $k[x_1, \ldots, x_n]$, the set $\cup_{i=1}^t (\alpha_i + \mathbf{N}^n)$ corresponds to the initial monomial ideal $in_c(I_A)$, i.e., a feasible solution u of $IP_{A,c}(b)$ is optimal if and only if x^u does not lie in $in_c(I_A)$.

An important motivation for the use of Gröbner bases in integer programming is the existence of computer programs for calculating them. The general purpose package MACAULAY [BaS] worked well for many non-trivial computations, such as the ones listed in the table in Example 2.8 below. More specialized Gröbner basis software for integer programming is currently being developed by Serkan Hosten at Cornell University. (For information contact `hosten@orie.cornell.edu`.)

Example 1.2 (continued) For $c' = (1, 0, 0, 1, 0, 1)$ the problems $LP_{A,c'}$ and $IP_{A,c'}$ are equivalent, since the triangulation $\Delta_{c'}$ consists of unimodular cones. Unimodularity is reflected by the fact that the minimal LP-test set $\mathcal{T}$ and the reduced Gröbner basis $\mathcal{G}_{c'}$ coincide. In this example, for sake of clarity, we denote the variables associated with the columns of A by a, b, c, d, e, f. In the notation used thus far, the variables should be identified as follows: $a = x_1$, $b = x_2$, $c = x_3$, $d = x_4$, $e = x_5$ and $f = x_6$. Therefore, in binomial notation,

$$\mathcal{G}_{c'} \quad = \quad \{\, \underline{ad} - b^2,\ \underline{ae} - bc,\ \underline{af} - c^2,\ \underline{bf} - ce,\ \underline{cd} - be,\ \underline{df} - e^2 \,\}.$$

The initial ideal $in_c(I_A)$ is generated by the six underlined monomials. □

Our discussion gives rise to the following integer analogue to the Walkup-Wets Theorem 1.1. The role of the regular triangulation Δ_c in Section 1 is now being played by the initial monomial ideal $in_c(I_A)$. We continue to identify lattice points in $\mathbf{N}^n$ and monomials in $k[x_1, \ldots, x_n]$.

Proposition 2.3
(i) The integer program $IP_{A,c}(b)$ is feasible if and only if b lies in the semigroup $pos_{\mathbf{Z}}(A)$.
(ii) $IP_{A,c}(b)$ is bounded for all b in $pos_{\mathbf{Z}}(A)$ if and only if $ker(A) \cap \mathbf{R}^n_+ = \{0\}$.
(iii) If $IP_{A,c}(\cdot)$ is bounded, then the set of all optimal solutions with respect to c, in the various fibers of IP_A, is the complement in $\mathbf{N}^n$ of the initial monomial ideal $in_c(I_A)$.

Theorem 2.4 below, shows that the faces of Δ_c can be recovered from the generators $\alpha_1, \ldots, \alpha_t$ of $in_c(I_A)$. This reflects the philosophy that integer programming is an arithmetic refinement of linear programming.

Theorem 2.4 *The faces of the regular triangulation Δ_c are those subsets of $\{1, \ldots, n\}$ which do not contain $supp(\alpha_i)$ for any minimal generator α_i of the initial monomial ideal $in_c(I_A)$.*

Sketch of proof: This result first appeared in Theorem 3.1 of [St1]. It can be derived from Remark 1.10 as follows. Every Gröbner basis element $\alpha_i - \beta_i$ can be written as a $\mathbf{Q}$-linear combination of circuits t such that $\{j : t_j > 0\} \subseteq supp(\alpha_i)$. At least one of these circuits satisfies $t \cdot c > 0$, since $(\alpha_i - \beta_i) \cdot c > 0$. On the other hand, for each circuit t with $t \cdot c > 0$ there exists a Gröbner basis element $\alpha_i - \beta_i$ such that $supp(\alpha_i) \subseteq \{j : t_j > 0\}$. □

A set $\mathcal{U} \subseteq ker_{\mathbf{Z}}(A)$ is a *universal test set* for A if $\mathcal{U}$ is a test set for $IP_{A,c}(\cdot)$ for every generic c in $\mathbf{R}^n$. Let UGB_A be the union of all reduced Gröbner bases $\mathcal{G}_c$ as c varies. Then UGB_A is a uniquely defined universal test set for A. We call it the *universal Gröbner basis* of A.

We describe another universal test set for A due to Graver [Gra]. For each $\sigma \in \{+, -\}^n$, consider the semigroup $S_\sigma = ker_{\mathbf{Z}}(A) \cap \mathbf{R}^n_\sigma$. Then $pos(S_\sigma)$ is a pointed closed polyhedral $(n-d)$-cone in $\mathbf{R}^n$. Let H_σ denote the unique Hilbert basis of S_σ (see Chapter 16, [Schr]). Then $\mathcal{H} := \cup_\sigma H_\sigma \backslash \{0\}$ is a universal test set for A. We call $\mathcal{H}$ the *Graver basis* of A. It is equivalent to the universal test set due to Blair and Jeroslow in [BJ] and under an appropriate transformation, to the universal test set due to Cook, Gerards, Schrijver and Tardos in [CGST] (see also Section 17.4 in [Schr]). The following theorem in [Th] relates the Graver basis of A to the universal Gröbner basis of A.

Theorem 2.5 *The Graver basis of A contains the universal Gröbner basis UGB_A.*

Corollary 2.6 *There exists only finitely many distinct reduced Gröbner bases associated with A as the cost function is varied. In particular, UGB_A is a finite set.*

The universal Gröbner basis is generally a proper subset of the Graver basis. Since the Graver basis is a symmetric set, its elements will be represented up to sign.

Example 1.2 (continued) The universal Gröbner basis UGB_A consists precisely of the nine circuits (cf. Lemma 3.15). The Graver basis consists of the nine circuits plus the one additional non-circuit $(1,-1,-1,1,-1,1)$. (We identify this vector with the binomial $adf - bce$.)

Example 2.7 Consider the $1 \times n$-matrix $A = [1, 1, \cdots, 1, D]$ where D is any non-negative integer. The Graver basis consists of $\binom{n-1}{2}$ elements of the form $e_i - e_j$, $1 \leq i < j \leq n-1$, and $\binom{n+D-2}{D}$ elements of the form $(i_1, i_2, \ldots, i_{n-1}, -1)$ where $i_1 + i_2 + \cdots i_{n-1} = D$. The universal Gröbner basis consists of the $\binom{n-1}{2}$ elements of the form $e_i - e_j, 1 \leq i < j \leq n-1$ and $n-1$ elements of the form $De_k - e_n$, $k = 1, \ldots, n-1$. The ratio of the cardinality of the Graver basis over the cardinality of UGB_A tends to infinity both in n and in D.

Example 2.8 *Knapsack problems* can be modeled using the family of matrices $A_n = [1, 2, 3, \cdots, n]$. The Graver basis of A_n consists of all binomials $x_{i_1}x_{i_2}\cdots x_{i_k} - x_{j_1}x_{j_2}\cdots x_{j_l}$ such that $i_1 + i_2 + \cdots + i_k = j_1 + j_2 + \cdots + j_l$ but no proper subsum of $i_1 + \cdots + i_k$ equals a subsum of $j_1 + \cdots + j_l$. Such binomials are called *primitive partition identities (ppi)*. It is proved in [DGS] that the degree of a ppi is at most $n \cdot (n-1)$. We list the number of ppi's for small values of n:

n	2	3	4	5	6	7	8	9	10	11	12	13
#	1	5	15	47	102	276	578	1261	2465	5362	9285	18900

For instance, for $n = 4$ the Graver basis equals $\{x_1^2 - x_2, x_1x_2 - x_3, x_1x_3 - x_2^2, x_1^3 - x_3, x_2^3 - x_3^2, x_1^4 - x_4, x_1^2x_4 - x_3^2, x_1x_4 - x_2x_3, x_1x_4^2 - x_3^3, x_1x_3 - x_4, \underline{x_1^2x_2 - x_4}, x_2x_3^2 - x_4^2, x_2^2 - x_4, x_2x_4 - x_3^2, x_3^4 - x_4^3\}$. The underlined binomial is the unique element of this Graver basis which is not in UGB_{A_4}. □

In remainder of Section 2 we take a glimpse at the wealth of information contained in $in_c(I_A)$. This material is independent from the rest of this paper and is intended to further motivate the use of Gröbner bases. We first recall some well-known facts about monomial ideals (see e.g. [Eis]).

Let M be any monomial ideal in $k[x_1, \ldots, x_n]$, or equivalently, let M be any subset of N^n such that $M + \mathrm{N}^n = M$. Given $I \subset \{1, \ldots, n\}$, we write N^I for the set of vectors with support in I. It is known that the complement of M in N^n can be written as a finite union of the form

$$\mathrm{N}^n \setminus M \quad = \quad \bigcup_{(v,I)} v + \mathrm{N}^I, \tag{2.2}$$

where (v, I) ranges over a finite subset of pairs $v \in \mathbf{N}^n$ and $I \subset \{1, \ldots, n\}$. The minimum number of pairs (v, I) needed in such a decomposition is called the *arithmetic degree* of M, and the sets I appearing in the minimum decomposition are the *associated primes* of M.

The *(multivariate) Hilbert series* of M is the formal power series

$$H(M; z_1, z_2, \ldots, z_n) \quad := \quad \sum \{ z_1^{u_1} z_2^{u_2} \cdots z_n^{u_n} \, : \, (u_1, u_2, \ldots, u_n) \in \mathbf{N}^n \setminus M \}.$$

It is possible to find a finite set of pairs (v, I) such that the union in (2.2) is disjoint. This shows that the Hilbert series is a rational generating function:

$$H(M; z_1, \ldots, z_n) \quad = \quad \sum_{(v,I)} \frac{z_1^{v_1} \cdots z_n^{v_n}}{\prod_{i \in I}(1 - z_i)}.$$

Practical algorithms for computing the Hilbert series $H(M; z)$ from generators of M are available in computer algebra systems: for instance, in MACAULAY the relevant command is `hilb_numer`.

We apply these techniques to the integer programs $IP_{A,c}(\cdot)$. Suppose a decomposition (2.2) is known for the initial ideal $in_c(I_A)$. It follows from Theorem 2.4 that, for each appearing pair (v, I), the set I is a face of the triangulation Δ_c and hence the columns of A indexed by I are linearly independent. This gives the following algorithm for solving $IP_{A,c}(b)$ for any right hand side b.

Algorithm 2.9 For each pair (v, I) in a decomposition (2.2) for $in_c(I_A)$, decide whether the linear system $A \cdot x = b - A \cdot v$ has a solution $x \in \mathbf{R}^n$ such that x has support in I. If yes, then $x \in \mathbf{R}^I$ is unique. If x is nonnegative and integral, then $x + v$ is the optimal solution of $IP_{A,c}(b)$.

The method of preprocessing a matrix A and cost function c and then solving several linear equations for each right hand side b was proposed by Kannan in [Kan]. We conjecture that the arithmetic degree of the initial ideal $in_c(I_A)$ satisfies the complexity bounds established in [Kan].

The Hilbert series $H(in_c(I_A); z_1, \ldots, z_n)$ is the formal sum over (monomials coding) all optimal solutions of the family $IP_{A,c}(\cdot)$. Three interesting rational functions can be obtained by specialization. The generating function of feasible right hand sides is $H(in_c(I_A); t^{a_1}, \ldots, t^{a_n}) = \sum\{ t^b \, : b \in pos_{\mathbf{Z}}(A) \}$. The generating function for the feasible right hand sides b together with their corresponding optimal solutions $opt(b)$ is $H(in_c(I_A); z_1 t^{a_1}, \ldots, z_n t^{a_n}) = \sum\{ t^b z^{opt(b)} \, : \, b \in pos_{\mathbf{Z}}(A) \}$. The value function of $IP_{A,c}(\cdot)$ is $val : pos_{\mathbf{Z}}(A) \to \mathbf{R}$ given by $b \mapsto c \cdot opt(b)$. The generating function of b along with $val(b)$ is $H(in_c(I_A); t^{a_1} s^{c_1}, \ldots, t^{a_n} s^{c_n}) = \sum\{ t^b s^{val(b)} : b \in pos_{\mathbf{Z}}(A) \}$.

Example 2.10 Let $A = [1, 2, 3, 4]$ and $c = (6, 4, 1, 0)$. From the 15 ppi's in Example 2.8 we can see that c is generic and $in_c(I_A) = \langle x_2 x_3^2, x_2 x_4, x_1 x_4, x_1 x_3, x_1 x_2, x_3^4, x_2^2, x_1^2 \rangle$. The minimal decomposition (2.2) of the set of optimal solutions is disjoint and has seven terms:

$$\mathbf{N}^4 \setminus in_c(I_A) \quad = \quad \mathbf{N}^{\{4\}} \cup (e_3 + \mathbf{N}^{\{4\}}) \cup (2e_3 + \mathbf{N}^{\{4\}}) \cup (3e_3 + \mathbf{N}^{\{4\}}) \cup \{e_1\} \cup \{e_2\} \cup \{e_2 + e_3\}.$$

The numerator of the Hilbert series equals $H(in_c(I_A); z_1, z_2, z_3, z_4) \cdot (1-z_1)(1-z_2)(1-z_3)(1-z_4) =$

$$1 - z_4 z_3 z_1^2 - z_3^4 z_2 z_1 - z_4 z_1 z_2^2 - z_4 z_1^2 z_2 + z_2 z_3 z_1 + z_4 z_3^2 z_2 + z_4 z_3 z_1 + 2 z_4 z_1 z_2 - z_2 z_3 z_1^2 + z_2 z_3^2 z_1$$
$$- z_2^2 z_3^2 z_1 - z_2^2 z_3^2 z_4 - z_4 z_3^2 z_1 z_2 - z_2 z_3^2 - z_2 z_4 - z_1 z_4 - z_1 z_3 - z_1 z_2 + z_1^2 z_4 + z_1^2 z_3 + z_1^2 z_2$$
$$+ z_2^2 z_3^2 + z_1 z_2^2 + z_2^2 z_4 + z_3^4 z_1 + z_3^4 z_2 - z_3^4 - z_2^2 - z_1^2 - z_4 z_3 z_1 z_2 + z_4 z_3 z_1^2 z_2 + z_2^2 z_3^2 z_4 z_1.$$

Note: $H(in_c(I_A); t, t^2, t^3, t^4) = 1/(1-t)$. The optimal solution in each fiber can be read off from

$$H(in_c(I_A); z_1 t, z_2 t^2, z_3 t^3, z_4 t^4) = 1 + z_1 t + z_2 t^2 + z_3 t^3 + z_4 t^4 + z_2 z_3 t^5 + z_3^2 t^6 + z_3 z_4 t^7 + z_4^2 t^8 + z_3^3 t^9$$
$$+ z_3^2 z_4 t^{10} + z_3 z_4^2 t^{11} + z_4^3 t^{12} + z_3^3 z_4 t^{13} + z_3^2 z_4^2 t^{14} + z_3 z_4^3 t^{15} + z_4^4 t^{16} + z_3^3 z_4^2 t^{17} + z_3^2 z_4^3 t^{18} + z_3 z_4^4 t^{19} + \cdots$$

It follows from Corollary 2.6 that the toric ideal I_A has only finitely many initial ideals $in_c(I_A)$ as c varies. Using the methods to be presented in the next sections we can enumerate all initial ideals, and hence we can compute their intersection $V_A := \bigcap_c in_c(I_A)$.

Observation 2.11 *A monomial x^u lies in V_A if and only if u is not a vertex of its fiber P^I_{Au}.*

Proof: A monomial x^u does not lie in $in_c(I_A)$ if and only if u is the unique vertex of P^I_{Au} supported by c. Taking the disjunction of this statement over all c, we obtain our claim. □.

The Hilbert function of the monomial ideal V_A enumerates all vertices of all fibers of IP_A. By specialization we obtain the following two rational generating functions: $H(V_A; t^{a_1}, \cdots, t^{a_n}) = \sum_b (\#\text{ vertices of } P^I_b) \cdot t^b$ and $H(V_A; z_1 t^{a_1}, \cdots, z_n t^{a_n}) = \sum_b (\sum\{ z^u : u \text{ vertex of } P^I_b\}) \cdot t^b$.

Example 2.10 (continued) The toric ideal I_{A_4} of the knapsack matrix $A_4 = [1, 2, 3, 4]$ has precisely 20 initial monomial ideals. Their intersection equals

$$V_{A_4} = \langle x_1^2 x_2,\ x_1 x_2 x_3,\ x_1^4 x_4,\ x_1^3 x_3,\ x_1 x_3 x_4,\ x_2^2 x_4,\ x_2^3 x_3^2,\ x_2 x_3^2 x_4,\ x_3^4 x_4^3 \rangle.$$

The number of vertices in each fiber of IP_{A_4} is given by the Hilbert series $H(V_{A_4}; t, t^2, t^3, t^4) =$

$$\frac{t^{23}+t^{22}+t^{21}+t^{20}+t^{19}+t^{18}+t^{17}+t^{16}+t^{15}+t^{14}+2t^{13}+2t^{12}+4t^{11}+4t^{10}+6t^9+8t^8+11t^7+10t^6+10t^5+8t^4+6t^3+4t^2+2t+1}{(1-t)(t+1)(t^2+1)(t^2+t+1)} =$$
$$1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 5t^6 + 7t^7 + 5t^8 + 6t^9 + 7t^{10} + 8t^{11} + 4t^{12} + 8t^{13} + \cdots + 9t^{19} + \cdots$$

It is easy to see that this series is eventually periodic of period $24 = 1 \cdot 2 \cdot 3 \cdot 4$. The maximum appearing coefficient and hence the maximum number of vertices of any fiber of IP_{A_4} is nine. □

3 The Gröbner fan and the state polytope.

Our objective is to study the variation of cost functions in integer programming using Gröbner bases methods. There is a natural equivalence relation on the space of all (not just generic) cost functions with respect to IP_A. It is analogous to the one for linear programming in Theorem 1.3.

Definition 3.1 Two cost vectors c and c' in $\mathbf{R}^n$ are *equivalent (with respect to IP_A)* if the integer programs $IP_{A,c}(b)$ and $IP_{A,c'}(b)$ have the same set of optimal solutions for all b in $pos_{\mathbf{Z}}(A)$.

The main result in this section is a structure theorem for these equivalence classes (Theorem 3.10). It is the integer analogue to Theorem 1.5. We note that Theorem 3.10 can also be derived from more general results of Mora-Robbiano [MR] and Bayer-Morrison [BM] on *Gröbner fans* and *state polytopes* for *graded polynomial ideals.* What we present here is an alternative construction for toric ideals, which is self-contained and provides more precise information for integer programming.

Recall that a cost vector c is *generic* for IP_A if the optimal solution with respect to c in every fiber P_b^I of IP_A is a unique vertex. Generic equivalence classes are characterized as follows:

Proposition 3.2 *Given two generic cost functions c and c' in $\mathbf{R}^n$, the following are equivalent:*
(i) For every $b \in pos_{\mathbf{Z}}(A)$, the programs $IP_{A,c}(b)$ and $IP_{A,c'}(b)$ have the same optimal solution.
(ii) The cost functions c and c' support the same optimal vertex in each fiber P_b^I of IP_A.
(iii) The reduced Gröbner bases $\mathcal{G}_c$ and $\mathcal{G}_{c'}$ associated with A are equal.

Proof: Conditions (i) and (ii) are equivalent since the optimal solution of $IP_{A,c}(b)$ is the vertex of P_b^I supported by c. The set of all non-optimal solutions to the programs $IP_{A,c}(\cdot)$ and $IP_{A,c'}(\cdot)$ are the monomial ideals $in_c(I_A)$ and $in_{c'}(I_A)$ respectively. Then (i) holds if and only if $in_c(I_A) = in_{c'}(I_A)$. This is equivalent to (iii) by Proposition 2.2. □

In what follows we view elements of the reduced Gröbner bases as vectors and not as binomials.

Lemma 3.3 *Let $\mathcal{G}_c \subset \mathbf{Z}^n$ be the reduced Gröbner basis of $IP_{A,c}(\cdot)$. Then $span_{\mathbf{Z}}(\mathcal{G}_c) = ker_{\mathbf{Z}}(A)$.*

Proof: Every vector $g_i = \alpha_i - \beta_i$ in $\mathcal{G}_c$ lies in $ker_{\mathbf{Z}}(A)$. Hence $span_{\mathbf{Z}}(\mathcal{G}_c) \subseteq ker_{\mathbf{Z}}(A)$. Let $\alpha \in ker_{\mathbf{Z}}(A)$. We can write α uniquely as $\alpha^+ - \alpha^-$ where α^+, α^- are vectors in $\mathbf{N}^n$ with disjoint supports. Further, $A\alpha^+ = A\alpha^-$, and hence α^+ and α^- lie in the same fiber of $IP_{A,c}(\cdot)$. Let β be the unique optimum in this fiber with respect to c. Since $\mathcal{G}_c$ is a test set for $IP_{A,c}(\cdot)$, there exist non-negative integral multipliers n_i and n_i' such that $\alpha^+ - \beta = \sum_{g_i \in \mathcal{G}_c} n_i g_i$ and $\alpha^- - \beta = \sum_{g_i \in \mathcal{G}_c} n_i' g_i$. Hence $\alpha = \sum_{g_i \in \mathcal{G}_c} (n_i - n_i') g_i$ which implies that $span_{\mathbf{Z}}(\mathcal{G}_c) = ker_{\mathbf{Z}}(A) \simeq \mathbf{Z}^{n-d}$. □

Let $u = u^+ - u^- \in ker_{\mathbf{Z}}(A)$. Both u^+ and u^- lie in the Au^+-fiber of IP_A, and we may think of u as the line segment $[u^+, u^-]$ in this fiber. We shall refer to the polytope $P_{Au^+}^I$ as *the fiber of* u. Recall the universal Gröbner basis UGB_A defined in Section 2. It is the union of all reduced Gröbner bases $\mathcal{G}_c$ as c varies. By a *Gröbner fiber* of IP_A we mean the fiber of an element $u \in UGB_A$. Let $St(A)$ denote the Minkowski sum of all Gröbner fibers. This is a well-defined polytope in $\mathbf{R}^n$ which we call the *state polytope* of A. Lemma 3.3 implies $dim(St(A)) = n - d$.

If P is any polytope in $\mathbf{R}^n$ and c is any (not necessarily generic) vector in $\mathbf{R}^n$, then we write $face_c(P)$ for the face of P at which c gets minimized. If F is any face of P, then $\mathcal{N}(F; P)$ denotes the cone of (inner) normals. In symbols, $\mathcal{N}(F;P) = \{ c \in \mathbf{R}^n : c \cdot x \leq c \cdot y \text{ for all } x \in F, y \in P\}$.

The collection of cones $\mathcal{N}(F;P)$ is denoted $\mathcal{N}(P)$ and called the *normal fan* of the polytope P. We say that two polytopes are *normally equivalent* if they have the same normal fan.

Lemma 3.4 *Every fiber of IP_A is a Minkowski summand of $St(A)$.*

Proof: It suffices to show that $\mathcal{N}(St(A))$ is a refinement of $\mathcal{N}(P_b^I)$ for all $b \in pos_{\mathbf{Z}}(A)$. Let c be a generic cost function and let $w \neq c$ belong to the interior of the cone $\mathcal{N}(face_c(St(A)); St(A))$. Then w lies in $\mathcal{N}(\beta_i; P^I_{A\beta_i})$ for each element $\alpha_i - \beta_i$ in the reduced Gröbner basis $\mathcal{G}_c$. This implies $w \cdot \alpha_i > w \cdot \beta_i$ for all i, and therefore $\mathcal{G}_w = \mathcal{G}_c$.

Now consider an arbitrary $b \in pos_{\mathbf{Z}}(A)$. Let u be the unique optimum of $IP_{A,c}(b)$. The equality of test sets $\mathcal{G}_w = \mathcal{G}_c$ implies that u is also the unique optimum of $IP_{A,w}(b)$. Hence w lies in the interior of $\mathcal{N}(u; P_b^I)$. We have shown that $\mathcal{N}(face_c(St(A)); St(A)) \subseteq \mathcal{N}(u; P_b^I)$, as desired. □

The operation of taking Minkowski sums of finitely many polytopes extends naturally to the operation of taking *Minkowski integrals* of infinitely many polytopes. See [BiS] for details.

Proposition 3.5 *Let db denote any probability measure with support $pos_{\mathbf{Z}}(A)$ such that $\int_b b\,db$ is finite. Then the Minkowski integral $\int_b P_b^I db$ is a polytope normally equivalent to $St(A)$.*

Proof: The hypothesis $\int_b b\,db < \infty$ guarantees that $\int_b P_b^I db$ is bounded. By Lemma 3.4, $\int_b P_b^I db$ is a summand of $St(A)$ and is hence a polytope. However, each Gröbner fiber is a summand of $\int_b P_b^I db$ and hence $\int_b P_b^I db$ is an $(n-d)$-polytope in $\mathbf{R}^n$ that has the same normal fan as $St(A)$.□

Corollary 3.6 *There exists only finitely many facet directions among the fibers of IP_A.*

From now on we shall use the term *state polytope* for any polytope normally equivalent to $\int_b P_b^I db$. We define the *Gröbner cone* associated with $\mathcal{G}_c$ to be the closed convex polyhedral cone

$$\mathcal{K}_c \quad := \quad \{\, x \in \mathbf{R}^n \, : \, g_i \cdot x \geq 0, \; g_i \in \mathcal{G}_c \,\} \qquad (3.1).$$

Observation 3.7 *The Gröbner cone $\mathcal{K}_c$ has full dimension n. Its lineality space $\mathcal{K}_c \cap -\mathcal{K}_c$ equals $rowspan(A) \simeq \mathbf{R}^d$.*

Proof: We have $dim(\mathcal{K}_c) = n$ because c lies in the interior of $\mathcal{K}_c$. The lineality space $\mathcal{K}_c \cap -\mathcal{K}_c$ equals the orthogonal complement of $\mathcal{G}_c$ in $\mathbf{R}^n$, which coincides with the row span of A by Lemma 3.3. □

Proposition 3.8 *The normal fan of the state polytope is the collection of all Gröbner cones $\mathcal{K}_c$ together with their faces, as c varies over all generic cost functions.*

Proof: The argument in the proof of Lemma 3.4 shows that, for c generic, the Gröbner cone $\mathcal{K}_c$ is the closure of $\mathcal{N}(face_c(St(A)); St(A))$. □

The complete polyhedral fan $\mathcal{N}(St(A))$ is called the *Gröbner fan* of A. We remark that each cone in the Gröbner fan has the same lineality space $rowspan(A) \simeq \mathbf{R}^d$ and it is often more convenient to work with its image in $ker(A) \simeq \mathbf{R}^n / rowspan(A) \simeq \mathbf{R}^{n-d}$.

Corollary 3.9 *The equivalence classes of cost functions with respect to IP_A (cf. Definition 3.1) are precisely the cells of the Gröbner fan.*

Proof: By Proposition 3.2, two cost vectors c and c' are equivalent if and only if they support the same optimal face in each fiber of IP_A. Using Propositions 3.5 and 3.8, it follows that c and c' are equivalent if and only if they lie in the relative interior of the same cell in $\mathcal{N}(St(A))$. □

A cost vector w lies in the interior of a Gröbner cone $\mathcal{K}_c$ if and only if w is generic and equivalent to c. Hence the interiors of the top-dimensional cells in the Gröbner fan are precisely the equivalence classes of generic cost functions. The following theorem summarizes the above discussion.

Theorem 3.10
(i) There are only finitely many equivalence classes of cost vectors with respect to IP_A.
(ii) Each equivalence class is the relative interior of a convex polyhedral cone in $\mathbf{R}^n$.
(iii) The collection of these cones defines a polyhedral fan that covers $\mathbf{R}^n$. This fan is called the Gröbner fan *of A.*
(iv) Let db denote any probability measure with support $pos_{\mathbf{Z}}(A)$ such that $\int_b b\,db < \infty$. Then the Minkowski integral $St(A) = \int_b P_b^I db$ is an $(n-d)$-dimensional convex polytope, called the state polytope *of A. The normal fan of $St(A)$ equals the Gröbner fan of A.*

The secondary polytope of A was defined in Section 1 as the Minkowski integral of all fibers of LP_A. Using Theorem 2.4 we obtain the following corollaries, which first appeared in [St1].

Corollary 3.11 *The Gröbner fan of A is a refinement of the secondary fan of A.*

Corollary 3.12 *The secondary polytope of A is a summand of the state polytope of A.*

Example 1.2 (continued) There are 29 distinct initial ideals for I_A, each corresponding to one of the 14 regular triangulations of $pos(A)$. Each initial ideal is listed in the table below along with its associated regular triangulation and a representative cost vector.

The state polytope $St(A)$ therefore has 29 vertices. Figure 1 is a Schlegel diagram of $St(A)$ where the numbers on the nodes tally with the numbers of the initial ideals. It will be seen in Chapter 5 that each edge direction of $St(A)$ is given by an element in UGB_A. The binomials on the edges denote this correspondence. The polytope $St(A)$ has an axis of symmetry through the vertices 1 and 17. The thin edges in Figure 1 are contracted when passing to the secondary polytope of A. Theorem 2.4 may be verified using this example.

	cost vector	initial ideal	regular triangulation
1.	$(1,0,0,1,0,1)$	$\langle df, cd, bf, af, ae, ad\rangle$	$\{1,2,3\},\{2,3,5\},\{2,4,5\},\{3,5,6\}$
2.	$(10,0,6,10,6,7)$	$\langle df, ce, ad, af, ae, cd\rangle$	$\{1,2,3\},\{2,3,6\},\{2,4,5\},\{2,5,6\}$
3.	$(1,0,0,0,0,0)$	$\langle e^2, ce, cd, af, ae, ad\rangle$	$\{1,2,3\},\{2,3,6\},\{2,4,6\}$
4.	$(7,1,1,0,15,0)$	$\langle e^2, ce, c^2d, be, af, ae, ad\rangle$	"
5.	$(49,24,0,14,70,26)$	$\langle e^2, ce, be, b^2f, af, ae, ad\rangle$	$\{1,2,3\},\{2,3,4\},\{3,4,6\}$
6.	$(17,6,0,10,14,13)$	$\langle e^2, bf, be, af, ae, ad\rangle$	"
7.	$(10,6,0,7,6,10)$	$\langle df, bf, be, af, ae, ad\rangle$	$\{1,2,3\},\{2,3,4\},\{3,4,5\},\{3,5,6\}$
8.	$(7,6,6,10,0,10)$	$\langle df, cd, bf, bc, af, ad\rangle$	$\{1,2,5\},\{2,4,5\},\{1,3,5\},\{3,5,6\}$
9.	$(79,23,77,27,181,0)$	$\langle e^2, ce, c^2, be, ae, ad\rangle$	$\{1,2,6\},\{2,4,6\}$
10.	$(1,0,1,1,1,1)$	$\langle e^2, ce, cd, c^2, ae, ad\rangle$	"
11.	$(27,23,181,79,77,0)$	$\langle e^2, ce, cd, c^2, bc, ad\rangle$	"
12.	$(0,0,0,1,0,0)$	$\langle df, ce, cd, c^2, bc, ae^2, ad\rangle$	$\{1,2,6\},\{2,4,5\},\{2,5,6\}$
13.	$(26,0,22,29,16,18)$	$\langle df, ce, cd, c^2, ae, ad\rangle$	"
14.	$(14,24,70,49,0,26)$	$\langle df, ce, cd, c^2, bc, b^2f, ad\rangle$	$\{1,2,5\},\{2,4,5\},\{1,5,6\}$
15.	$(5,4,6,8,0,7)$	$\langle df, cd, c^2, bf, bc, ad\rangle$	"
16.	$(0,0,0,0,1,0)$	$\langle e^2, ce, c^2, be, b^2, ae\rangle$	$\{1,4,6\}$
17.	$(1,1,1,0,1,1)$	$\langle e^2, ce, c^2, be, bc, b^2\rangle$	"
18.	$(0,1,0,0,0,0)$	$\langle e^2, c^2, bf, be, bc, b^2\rangle$	"
19.	$(0,0,1,0,0,0)$	$\langle e^2, ce, cd, c^2, bc, b^2\rangle$	"
20.	$(0,181,77,27,23,29)$	$\langle df, c^2, bf, be, bc, b^2\rangle$	$\{1,4,5\},\{1,5,6\}$
21.	$(2,6,16,10,0,5)$	$\langle df, ce, cd, c^2, bc, b^2\rangle$	"
22.	$(1,1,1,1,0,1)$	$\langle df, cd, c^2, bf, bc, b^2\rangle$	"
23.	$(10,6,0,2,16,5)$	$\langle e^2, ce, be, b^2, af, ae\rangle$	$\{1,3,4\},\{3,4,6\}$
24.	$(1,1,0,1,1,1)$	$\langle e^2, bf, be, b^2, af, ae\rangle$	"
25.	$(5,16,0,2,6,10)$	$\langle e^2, bf, be, bc, b^2, af\rangle$	"
26.	$(7,6,0,5,4,8)$	$\langle df, bf, be, b^2, af, ae\rangle$	$\{1,3,4\},\{3,4,5\},\{3,5,6\}$
27.	$(26,70,0,14,24,49)$	$\langle df, bf, be, bc, b^2, af, ae^2\rangle$	"
28.	$(0,15,1,0,1,7)$	$\langle df, c^2d, bf, be, bc, b^2, af\rangle$	$\{1,3,5\},\{1,4,5\},\{3,5,6\}$
29.	$(0,0,0,0,0,1)$	$\langle b^2, af, df, cd, bf, bc\rangle$	"

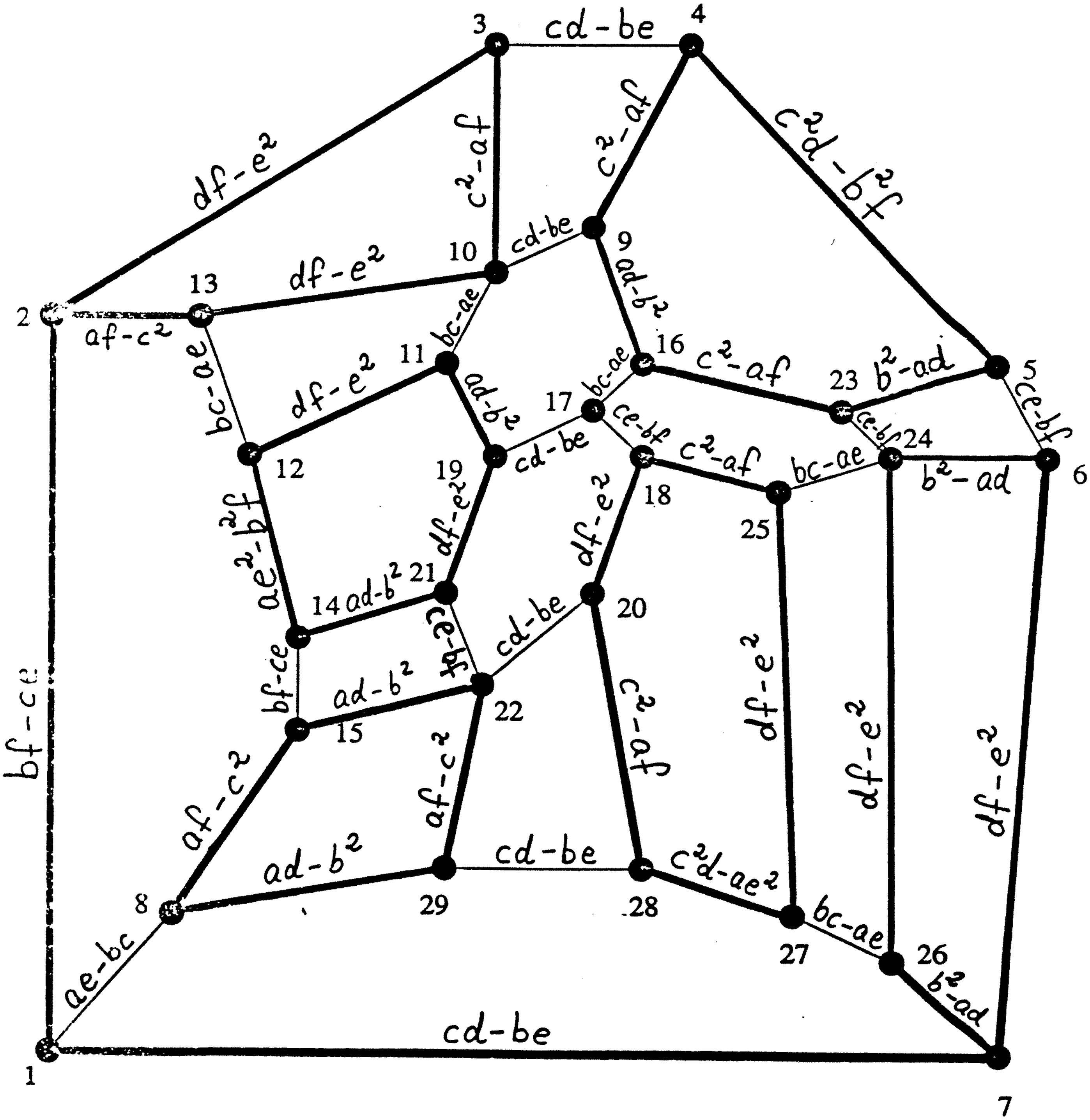

Figure 1: Schlegel diagram of the state polytope of A.

The Graver basis $\mathcal{H}$ introduced in Section 2 gives rise to a natural refinement of the Gröbner fan. The *Graver arrangement* of A is the arrangement consisting of the hyperplanes in $\mathbf{R}^n$ which are orthogonal to the elements in the Graver basis $\mathcal{H}$.

Proposition 3.13 *(i) The Graver arrangement of A is a refinement of the Gröbner fan of A.*
(ii) If the matrix is of Lawrence type, then the Graver arrangement equals the Gröbner fan.
(iii) The Graver arrangement of A is a refinement of the circuit arrangement of A.

Proof: Property (i) follows from Theorem 2.5. The proof of (ii) will be shown Corollary 4.2, and (iii) is a direct consequence of Lemma 3.15 below. □

Example 1.2 (continued) For this matrix, the secondary fan has 14 cells all of which are triangular and the Gröbner fan has 29 cells that are both triangular and quadrangular. The circuit arrangement and Graver arrangement have 48 and 60 triangular cells respectively.

In Section 1 we defined a circuit of A to be any non-zero vector of minimal support in $ker(A)$. For the rest of this paper we need a more specific definition, which is suitable for integer programming:

Definition 3.14 A *circuit* of A is a non-zero vector u in $ker_{\mathbf{Z}}(A)$ such that its support $supp(u)$ is minimal with respect to inclusion and u is a primitive lattice point, i.e., g.c.d.$(u_1, \ldots, u_n) = 1$.

Lemma 3.15 *The circuits of A are contained in UGB_A.*

Proof: Let $u = u^+ - u^-$ be a circuit of A. Consider the cost function $c := \sum\{e_i : i \notin supp(u)\}$. After refining c lexicographically to be generic we may suppose that $c \cdot u^+ > c \cdot u^-$. Then the monomial x^{u^+} lies in $in_c(I_A)$ and so there exists a binomial $x^{\alpha_i} - x^{\beta_i}$ in $\mathcal{G}_c$ such that x^{α_i} divides x^{u^+}. Since $c \cdot \alpha_i \geq c \cdot \beta_i$ and $supp(\alpha_i) \subseteq supp(u^+) \subseteq supp(u)$, we conclude that $supp(\beta_i) \subseteq supp(u)$. Since u is a circuit, these facts imply $u = \alpha_i - \beta_i \in UGB_A$. □.

Definition 3.16 An integral matrix A of full row rank is called *unimodular* if each of its maximal minors is one of $-c$, 0 or c, where c is a positive integral constant.

Theorem 3.17 [St2] *If A is unimodular, the circuits of A form the universal Gröbner basis of A.*

Sketch of proof: When A is unimodular, every coordinate of a circuit is one of $0, 1$ or -1. This implies that the Graver basis of A consists only of circuits of A. Therefore by Theorems 2.5 and 3.15 the circuits of A form the universal Gröbner basis of A. □

Corollary 3.18 *If the matrix A is unimodular and of Lawrence type, then its secondary fan. circuit arrangement, Gröbner fan and Graver arrangement all coincide.*

Example 3.19 Let A_G be the vertex-edge incidence matrix of a directed graph $G = (V, E)$, and consider the *capacitated transshipment problem:*

$$\text{minimize } c \cdot x \text{ subject to } A_G \cdot x \;=\; b \text{ and } 0 \leq x \leq b',\; x \in \mathbf{Z}^E.$$

When rewriting this integer program in the form (2.1), we get the enlarged coefficient matrix $\Lambda(A_G) = \begin{pmatrix} A_G & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$. This matrix has format $(|E| + |V|) \times 2|V|$ and it is unimodular and of Lawrence type. Hence, for the family of flow problems $IP_{\Lambda(A_G)}$, the secondary fan, the circuit arrangement, the Gröbner fan and the Graver arrangement all coincide.

4 Computing the Graver basis and universal Gröbner basis.

In this section we present algorithms for computing the Graver basis $\mathcal{H}$ and the universal Gröbner basis UGB_A of a matrix $A \in \mathbf{Z}^{d \times n}$ of rank d. To this end we consider the enlarged matrix $\Lambda(A)$ $= \begin{pmatrix} A & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$ where $\mathbf{1}$ is the $n \times n$-unit matrix and $\mathbf{0}$ is the $d \times n$-zero matrix. The $(d+n) \times 2n$-matrix $\Lambda(A)$ is called the *Lawrence lifting* of A. Any matrix of the form $\Lambda(A)$ is said to be of *Lawrence type* (cf. Proposition 1.11). This construction and terminology stems from the theory of oriented matroids (see Section 9.3 in [BLSWZ]). The matrices A and $\Lambda(A)$ have isomorphic kernels: $ker_{\mathbf{Z}}(\Lambda(A)) = \{(u, -u) : u \in ker_{\mathbf{Z}}(A)\}$. The toric ideal $I_{\Lambda(A)}$ is the homogeneous prime ideal

$$I_{\Lambda(A)} \quad = \quad \langle\, x^\alpha y^\beta - x^\beta y^\alpha \;:\; \alpha, \beta \in \mathrm{N}^n, A\alpha = A\beta \rangle$$

in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

We say that a one-dimensional fiber of IP_A is *primitive* if its two vertices have disjoint supports and are relatively prime. Clearly all one-dimensional Gröbner fibers are primitive. On the other hand, if the segment $[\alpha, \beta]$ is a primitive fiber of IP_A, then $x^\alpha - x^\beta$ (with either term as leading term) belongs to every reduced Gröbner basis of A and hence to UGB_A. In general, the set of primitive one-dimensional fibers of A is a proper subset of the set of Gröbner fibers of A. We call the fiber of an element in the Graver basis of A a *Graver fiber* of IP_A.

Theorem 4.1 *For a matrix $\Lambda(A)$ of Lawrence type, the following sets coincide:*
(i) the Graver basis of $\Lambda(A)$,
(ii) the universal Gröbner basis of $\Lambda(A)$,
(iii) any reduced Gröbner basis of $I_{\Lambda(A)}$,
(iv) any minimal generating set of $I_{\Lambda(A)}$ (up to scalar multiples), and
(v) the set of binomials $x^\alpha y^\beta - x^\beta y^\alpha$ supported on primitive one-dimensional fibers $[(\alpha, \beta), (\beta, \alpha)]$.

Proof: Let $\mathcal{H}$ be the Graver basis of A, and let $\mathcal{H}'$ be the Graver basis of $\Lambda(A)$. These two sets of binomials are related as follows: $\mathcal{H}' = \{\, x^\alpha y^\beta - x^\beta y^\alpha : \alpha, \beta \in \mathrm{N}^n, x^\alpha - x^\beta \in \mathcal{H} \,\}$. Clearly, $\mathcal{H}'$ is a generating set of $I_{\Lambda(A)}$ and a Gröbner basis with respect to every generic cost function. We will show that $\mathcal{H}'$ is the unique minimal generating set of $I_{\Lambda(A)}$. Choose any element $g := x^\alpha y^\beta - x^\beta y^\alpha$ of $\mathcal{H}'$, and fix $\sigma \in \{-,+\}^n$ such that $\alpha - \beta$ lies in $S_\sigma = ker_{\mathbf{Z}}(A) \cap \mathbf{R}^n_\sigma$. Let $\mathcal{B}$ be the set of all binomials $x^\gamma y^\delta - x^\delta y^\gamma$ in $I_{\Lambda(A)}$ except g. Suppose that $\mathcal{B}$ generates $I_{\Lambda(A)}$. Then $x^\alpha y^\beta - x^\beta y^\alpha$ can be written as a linear combination of elements in $\mathcal{B}$. But this is only possible if there exists a binomial $x^\gamma y^\delta - x^\delta y^\gamma$ in $\mathcal{B}$ such that $x^\gamma y^\delta$ divides $x^\alpha y^\beta$. This implies that $\gamma - \delta$ lies in the semigroup S_σ. Moreover, since $\gamma \leq \alpha$ and $\delta \leq \beta$, the non-zero vector $(\alpha - \beta) - (\gamma - \delta)$ lies in S_σ as well. Therefore $\alpha - \beta$ cannot be an element in the Hilbert basis of S_σ. This is a contradiction, and we conclude that every minimal generating set of $I_{\Lambda(A)}$ requires (a scalar multiple of) the binomial g.

We have shown the equality of the sets in (i),(ii),(iii) and (iv). For the equality of (i) and (v) we shall prove that every Graver fiber contains precisely two lattice points. Let $g \in \mathcal{H}'$ as above. Suppose that the common fiber of (α, β) and (β, α) contains a third point $(\gamma, \delta) \in \mathbf{N}^{2n}$. Then $\alpha, \beta, \gamma, \delta \in \mathbf{N}^n$ all lie in the same fiber of IP_A and $\alpha + \beta = \gamma + \delta$. This implies that $\alpha - \beta = (\gamma - \beta) + (\delta - \beta)$. We will show that the non-zero vectors $\gamma - \beta$ and $\delta - \beta$ are sign compatible with $\alpha - \beta$. This contradicts $\alpha - \beta \in \mathcal{H}$ and thus completes the proof. Let $j \in \{1, \ldots, n\}$. If $\alpha_j > 0$ then $\beta_j = 0$, and this implies $(\alpha - \beta)_j = \alpha_j > 0$, $(\gamma - \beta)_j = \gamma_j \geq 0$, and $(\delta - \beta)_j = \delta_j \geq 0$. If $\alpha_j = 0$ then $(\alpha - \beta)_j = -\beta_j \leq 0$, $(\gamma - \beta)_j = \gamma_j - \beta_j = -\delta_j \leq 0$, and $(\delta - \beta)_j = \delta_j - \beta_j = -\gamma_j \leq 0$. □

Corollary 4.2 *The state polytope of $\Lambda(A)$ is a zonotope (i.e., a Minkowski sum of line segments). The Gröbner fan of $\Lambda(A)$ coincides with the Graver arrangement of $\Lambda(A)$.*

This completes the proof of Proposition 3.13 (ii). It is the integer analogue of Proposition 1.11 (ii).

Algorithm 4.3 How to compute the Graver basis of A.

1. Compute the reduced Gröbner basis $\mathcal{G}$ of $I_{\Lambda(A)}$ with respect to any term order.
2. The Graver basis $\mathcal{H}$ of A consists of all elements $\alpha - \beta$ such that $x^\alpha y^\beta - x^\beta y^\alpha$ appears in $\mathcal{G}$.

The correctness of this algorithm is a corollary of Theorem 4.1. We found Algorithm 4.3 to be very useful for explicit computations. The main point is that, in order to compute the Graver basis of A, one only needs to compute a single reduced Gröbner basis for its Lawrence lifting $\Lambda(A)$. Algorithm 4.3 was first found in collaboration with Persi Diaconis. It can be applied to the problem of "sampling in the presence of prescribed zeros" as discussed in [DS].

By Theorem 2.5, the universal Gröbner basis UGB_A is contained in the Graver basis $\mathcal{H}$. We now present a geometric characterization of those binomials that belong to some reduced Gröbner basis of A. This gives an algorithm for computing UGB_A from $\mathcal{H}$. In Section 5 we shall provide yet another geometric characterization and algorithm for the universal Gröbner basis.

Let $\alpha, \beta \in \mathbf{N}^n$ such that $\alpha - \beta \in ker_{\mathbf{Z}}(A)$ and $supp(\alpha) \cap supp(\beta) = \emptyset$. Define $C_+ := \{w \in \mathbf{R}^n : w \cdot \alpha > w \cdot \beta$ and $\alpha - \beta \in \mathcal{G}_w\}$ and $C_- := \{w \in \mathbf{R}^n : w \cdot \beta > w \cdot \beta$ and $\beta - \alpha \in \mathcal{G}_w\}$, where $\mathcal{G}_w$ is the reduced Gröbner basis of $IP_{A,w}(\cdot)$. Then $\alpha - \beta$ lies in UGB_A if and only if $C_+ \cup C_- \neq \emptyset$.

For $u \in \mathbf{N}^n$, denote by $\mathcal{M}(u)$ the interior of the inner normal cone of the Au-fiber of IP_A at u. In symbols, $\mathcal{M}(u) = int\,\mathcal{N}(u; P^I_{Au})$. Thus $\mathcal{M}(u)$ is empty unless u is a vertex of P^I_{Au}.

Proposition 4.4 *The set C_+ equals the intersection $\mathcal{M}(\beta) \cap \bigcap_{i \in supp(\alpha)} \mathcal{M}(\alpha - e_i)$.*

Proof: A cost vector $w \in \mathbf{R}^n$ belongs to C_+ if and only if $x^\alpha - x^\beta$ appears with leading term x^α in the reduced Gröbner basis $\mathcal{G}_w$ if and only if $x^\beta \notin in_w(I_A)$, $x^\alpha \in in_w(I_A)$, and no proper factor of x^α is in $in_w(I_A)$. This is equivalent to $w \in \mathcal{M}(b)$ and $w \in \mathcal{M}(a - e_i)$ for all $i \in supp(a)$. □

Corollary 4.5 *The set of all cost functions which has a fixed binomial with fixed leading term in its reduced Gröbner basis is an open convex cone.*

Using the Graver basis $\mathcal{H}$, we get the following explicit inequality presentation for the cone $\mathcal{M}(u)$.

Lemma 4.6 *The interior of the inner normal cone of the Au-fiber of IP_A at $u \in \mathbf{N}^n$ equals*

$$\mathcal{M}(u) \;=\; \{\, w \in \mathbf{R}^n \,:\, wd > we \text{ for all } x^d - x^e \in \mathcal{H} \text{ such that } x^e \text{ divides } x^u \,\}.$$

Proof: We have $w \in \mathcal{M}(u)$ if and only if x^u does not lie in $in_w(I_A) = \langle in_w(h) : h \in \mathcal{H} \rangle$. □

Algorithm 4.7 How to compute the universal Gröbner basis UGB_A.

1. Compute the Graver basis $\mathcal{H}$ using Algorithm 4.3.

2. For each element $x^\alpha - x^\beta$ of $\mathcal{H}$:

- **2.1.** Compute the cones C_+ and C_- using Proposition 4.4 and Lemma 4.6.
- **2.2.** The binomial $x^\alpha - x^\beta$ is in UGB_A if and only if $C_+ \cup C_-$ is non-empty.

Example 1.2 (continued) The Graver basis equals $\mathcal{H} = \{ad - b^2, ae - bc, af - c^2, bf - ce, cd - be, df - e^2, ae^2 - b^2 f, b^2 f - c^2 d, ae^2 - c^2 d, adf - bce\}$. We shall prove that $\mathcal{H} \backslash UGB_A = \{adf - bce\}$. It suffices to show that $adf - bce$ is not in UGB_A, because the other nine binomials in $\mathcal{H}$ are all circuits (cf. Lemma 3.15).

Let $\alpha = (1,0,0,1,0,1)$ and $\beta = (0,1,1,0,1,0)$. By Lemma 4.6, we have $\mathcal{M}(\beta) = \{\, w \in \mathbf{R}^6 : w_1 + w_5 > w_2 + w_3,\ w_2 + w_6 > w_3 + w_5,\ w_3 + w_4 > w_2 + w_5\}$, $\mathcal{M}(\alpha - e_1) = \{w \in \mathbf{R}^6 : 2w_5 > w_4 + w_6\}$, $\mathcal{M}(\alpha - e_4) = \{w \in \mathbf{R}^6 : 2w_3 > w_1 + w_6\}$ and $\mathcal{M}(\alpha - e_6) = \{w \in \mathbf{R}^6 : 2w_2 > w_1 + w_4\}$. The intersection of these four cones is easily seen to be empty, so that $C_+ = \emptyset$. Reversing the roles of α and β we similarly find that $C_- = \emptyset$. Therefore $adf - bce \notin UGB_A$.

5 The geometry of the universal Gröbner basis.

The main result in this section is a geometric characterization of the universal Gröbner basis.

Theorem 5.1 *A vector $\alpha - \beta \in ker_{\mathbb{Z}}(A)$ lies in the universal Gröbner basis UGB_A if and only if $\alpha - \beta$ is primitive and the line segment $[\alpha, \beta]$ is an edge of the $A\alpha$-fiber of IP_A.*

We first recall a general fact about Minkowski sums of polytopes.

Lemma 5.2 *Let P be the Minkowski sum of the polytopes $P_1, \ldots, P_k$. Then the set of edge directions of P is the union of the sets of edge directions of P_i for $i = 1, \ldots, k$.*

In view of Proposition 3.5, this says that the edge directions of the fibers of IP_A are precisely the edge directions of the state polytope. If $[\alpha, \beta]$ is the primitive representative of an edge direction, then $[\alpha, \beta]$ is an edge of the $A\alpha$-fiber of IP_A. Therefore, Theorem 5.1 is equivalent to the following assertion: *the universal Gröbner basis consists of the edge directions of the state polytope.*

Proof of Theorem 5.1 (if): Suppose $g = \alpha - \beta$ is primitive and defines an edge direction of the state polytope $St(A)$. Then g is the normal vector to a facet of a maximal cone $\mathcal{K}_c$ in the Gröbner fan $\mathcal{N}(St(A))$. Therefore g appears in the inequality presentation of $\mathcal{K}_c$ given in (3.1). In other words, g is equal to one of the elements g_i of the reduced Gröbner basis $\mathcal{G}_c$. □

For the proof of the only-if direction we need two lemmas.

Lemma 5.3 *Let x^α be a minimal generator of the initial monomial ideal $in_c(I_A)$, and let δ be any lattice point in the $A\alpha$-fiber of IP_A such that $c \cdot \alpha \geq c \cdot \delta$. Then $supp(\delta) \cap supp(\alpha) = \emptyset$.*

Proof: Suppose $k \in supp(\alpha) \cap supp(\delta)$ for a lattice point δ in the $A\alpha$-fiber of IP_A for which $c \cdot \alpha \geq c \cdot \delta$. Then $\alpha - e_k$ and $\delta - e_k$ are lattice points in the same fiber of IP_A and $c \cdot (\alpha - e_k) \geq c \cdot (\delta - e_k)$. This implies that x^α / x_k lies in the initial monomial ideal $in_c(I_A)$, which is a contradiction to x^α being a minimal generator. □

Lemma 5.4 *For an element $\alpha - \beta$ of UGB_A, both α and β are vertices in the $A\alpha$-fiber of IP_A.*

Proof: By definition, β is the optimal vertex with respect to some cost function c in the $A\alpha$-fiber of IP_A. Recall our assumption that the integer programs $IP_{A,c}(b)$ are bounded. This implies the existence of an integral vector M with all coordinates positive in the row space of A. After replacing M by a multiple if necessary, we may assume that $M - c$ has all coordinates positive. Clearly, the cost function $\omega := M - c$ attains its **maximum** over $P^I_{A\alpha}$ at β. Let v denote the restriction of ω to the support of α (i.e, $v_i = w_i$ if $\alpha_i > 0$ and $v_i = 0$ if $\alpha_i = 0$). We claim that v attains a unique maximum over $P^I_{A\alpha}$ at α. If not, then there exists another lattice point δ in $P^I_{A\alpha}$ with $v \cdot \delta \geq v \cdot \alpha$. Since $v \cdot \alpha > 0$, the set $supp(v) \cap supp(\delta) = supp(\alpha) \cap supp(\delta)$ is not empty. By Lemma 5.3, this implies $c \cdot \alpha < c \cdot \delta$. In view of $M \cdot \alpha = M \cdot \delta$, we conclude that $v \cdot \alpha = w \cdot \alpha > w \cdot \delta \geq v \cdot \delta$, as desired. □

Proof of Theorem 5.1 (only-if): Let $\alpha - \beta \in UGB_A$ and choose $w, v \in \mathbf{N}^n$ as in the proof of Lemma 5.4. Consider the cost vector $u := (v \cdot (\alpha - \beta))\, w + (w \cdot (\beta - \alpha))\, v \in \mathbf{N}^n$. We have $u \cdot \alpha = u \cdot \beta$. It suffices to show that $u \cdot \alpha > u \cdot \gamma$ for all lattice points γ other than α and β in the $A\alpha$-fiber of IP_A. If $supp(\gamma) \cap supp(\alpha) = \emptyset$, then $supp(\gamma) \cap supp(v) = \emptyset$ which implies that $u \cdot \beta = (v \cdot (\alpha - \beta))(w \cdot \beta) > (v \cdot (\alpha - \beta))(w \cdot \gamma)$. If $supp(\gamma) \cap supp(\alpha) \neq \emptyset$, then by Lemma 5.3, $w \cdot \alpha > w \cdot \gamma$. This implies that $u \cdot \alpha = (v \cdot (\alpha - \beta))(w \cdot \alpha) + (w \cdot (\beta - \alpha))(v \cdot \alpha) > (v \cdot (\alpha - \beta))(w \cdot \gamma) + (w \cdot (\beta - \alpha))(v \cdot \gamma) = u \cdot \gamma$. Therefore the line segment $[\alpha, \beta]$ is an edge of the $A\alpha$-fiber of IP_A with outer normal vector u. □

Theorem 5.1 implies several interesting corollaries.

Corollary 5.5 *For an element $\alpha - \beta$ in UGB_A, there exists two cost functions c and c' in $\mathbf{R}^n$ such that $\alpha - \beta \in \mathcal{G}_c$ and $\beta - \alpha \in \mathcal{G}_{c'}$.*

Proof: Every element in UGB_A appears as a facet normal of some cell in the Gröbner fan. Take as $\mathcal{G}_c$ and $\mathcal{G}_{c'}$ the Gröbner bases associated with the two Gröbner cones that share this facet. □

Corollary 5.6 *For every generic cost function $c \in \mathbf{R}^n$, the reduced Gröbner basis of $IP_{A,c}(\cdot)$ consists only of edges of certain fibers P_b^I.*

This is the integer programming analogue to Corollary 1.9. Theorem 5.1 implies that we can trace a monotone edge path from every non-optimal vertex of $IP_{A,c}(b)$ to the optimal vertex, using only elements in UGB_A. Thus reduction with respect to the universal Gröbner basis can be viewed as an integer analogue to the simplex method for linear programming.

Theorem 5.1 gives rise to the following algorithm for computing the universal Gröbner basis.

Algorithm 5.7 How to compute the universal Gröbner basis UGB_A.

1. Compute the Graver basis $\mathcal{H}$ using Algorithm 4.3.

2. For each element $x^\alpha - x^\beta$ of $\mathcal{H}$ decide whether $[\alpha, \beta]$ is an edge of its fiber.

To check whether a segment is an edge in its fiber amounts to solving a linear programming problem. When applying Algorithm 5.7 to examples, we often found the following method sufficient. Given $\alpha - \beta \in \mathcal{H}$, we first list **all** feasible solutions to the integer program $IP_{A,w}(A\alpha)$ where w is any cost function. This can be done by a *reverse search* method starting at the optimum of $IP_{A,w}(A\alpha)$, using the Gröbner basis $\mathcal{G}_w$. See §3.1 in [NTT] for details. We then check whether there exists $c \in \mathbf{R}^n$ such that $c \cdot \alpha = c \cdot \beta = 0$ and $c \cdot \gamma \geq 1$ for all lattice points γ different from α and β in the $A\alpha$-fiber. This is the case if and only if the segment $[\alpha, \beta]$ is an edge of its fiber.

Example 1.2 (continued) The fiber of the binomial $adf - bce \in \mathcal{H}\backslash UGB_A$ contains five lattice points. They form the vertices of a 3-dimensional bipyramid. The line segment $[(1,0,0,1,0,1), (0,1,1,0,1,0)]$ is the diagonal of this bipyramidal fiber. □

Example 2.8 (continued) In contrast to Lemma 5.4, it can happen that neither term of a Graver basis element corresponds to a vertex of its fiber. The ppi $x_2^2x_7x_9 - x_5^2x_{10}$ in $\mathcal{H}\backslash UGB_{A_{10}}$ has this property. To see this note $2e_2 + e_7 + e_9 \in conv\{3e_2 + 2e_7, e_2 + 2e_9\}$ and $2e_5 + e_{10} \in conv\{4e_5, 2e_{10}\}$.

The universal Gröbner basis of A can be used to devise a geometric method to construct the Gröbner fan of A and hence the state polytope of A. The *Gröbner arrangement* of A, denoted $Gr(A)$, is the arrangement consisting of the hyperplanes in $\mathbf{R}^n$ that are orthogonal to the elements in UGB_A. The Gröbner arrangement of A is a refinement of the Gröbner fan of A, since a hyperplane is in $Gr(A)$ if and only if it is the linear span of a facet of some Gröbner cone. The Graver arrangement of A is a refinement of the Gröbner arrangement of A.

Algorithm 5.8 A geometric construction of the Gröbner fan.
Input: The universal Gröbner basis UGB_A
Output: The maximal cells in the Gröbner fan $\mathcal{N}(St(A))$.

Compute the Gröbner arrangement $Gr(A)$ from its set UGB_A of normals (cf. [EOS]).
Let $G(A)$ be the set of maximal cells in $Gr(A)$, each represented by a vector in its interior.
WHILE $G(A) \neq \emptyset$ DO
Select a cell $C \in G(A)$.
Let $I_C = \{\underline{x^\alpha} - x^\beta \in UGB_A : C \subseteq \{c \in \mathbf{R}^n : \alpha \cdot c \geq \beta \cdot c\}\}$. Auto-reduce the set I_C with respect to the underlined leading terms to get the reduced Gröbner basis $\mathcal{G}_C$.
Let C' be the new cell obtained by erasing all facets of C whose normals are not in $\mathcal{G}_C$.
Remove from $G(A)$ all cells which are contained in C'. Output C'.

Proof of correctness: If $c \in int(C')$ then the cone C' coincides with the Gröbner cone $\mathcal{K}_c$ in (3.1). Therefore each cell output by Algorithm 5.8 lies in the Gröbner fan. Conversely, each maximal cell in the Gröbner fan will eventually be generated in the WHILE-loop since $Gr(A)$ covers $\mathbf{R}^n$. □

We now relate certain properties of a matrix that have been discussed in earlier sections.

Theorem 5.9 *Conder the following properties of a matrix $A \in \mathbf{Z}^{d\times n}$ of maximal row rank:*
(i) A is unimodular.
(ii) The state polytope $St(A)$ coincides with the secondary polytope $\Sigma(A)$.
(iii) The circuits of A constitute the universal Gröbner basis UGB_A.
Then (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) but (iii) $\not\Rightarrow$ (ii) and (ii) $\not\Rightarrow$ (i).

Proof: If A is unimodular then the integer programming fiber P_b^I coincides with the linear programming fiber P_b for all $b \in pos_{\mathbf{Z}}(A)$ ([Schr], Theorem 19.2). Moreover, if $b \in pos(A) \setminus pos_{\mathbf{Z}}(A)$, then there exists $b' \in pos_{\mathbf{Z}}(A)$ such that P_b and $P_{b'}$ are normally equivalent. Therefore the Minkowski integrals in Theorems 1.5 (iv) and 3.10 (iv) coincide, which proves the implication $(i) \Rightarrow (ii)$.

By Theorem 5.1 the edge directions of $St(A)$ are the elements of UGB_A and by Theorem 1.8 the edges of $\Sigma(A)$ are the circuits of A. Hence if $St(A) = \Sigma(A)$, then the circuits of A constitute UGB_A. This proves the implication $(ii) \Rightarrow (iii)$.

To see that $(iii) \not\Rightarrow (ii)$ consider our running Example 1.2. In the end of Section 4 we proved that the circuits constitute the universal Gröbner basis. However, the secondary polytope $\Sigma(A)$ has 14 vertices (it is the 3-dimensional associahedron) while the state polytope $St(A)$ has 29 vertices (it is depicted in Figure 1).

The fact that $(ii) \not\Rightarrow (i)$ is shown by the example $A = [1, 2]$. This matrix is not unimodular, but $\Sigma(A)$ and $St(A)$ are identical line segments parallel to $ker(A) = span\{(-2, 1)\}$. □

Example 5.10 The example in the proof of $(ii) \not\Rightarrow (i)$ is trivial since $St(A) = \Sigma(A)$ for every matrix A of corank 1. For an example with corank 2 consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is of Lawrence type. Its Graver basis consists precisely of the four circuits. There are eight distinct reduced Gröbner bases associated with this matrix each of which corresponds to a distinct triangulation. This implies implies that the state and secondary polytopes coincide. However, A is not unimodular since it has maximal minors of absolute value zero, one and two.

Recall that the fiber containing an element of the universal Gröbner basis UGB_A was called a Gröbner fiber of IP_A and the fiber containing an element of the Graver basis a Graver fiber of IP_A. By Theorem 2.5, the set of Gröbner fibers of A is contained in the set of Graver fibers of A. The following example shows that this containment may be strict.

Example 5.11 (Graver fibers versus Gröbner fibers)
For any non-negative integer i consider the 2×5-matrix $A_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3+6i & 4+6i & 6+6i \end{bmatrix}$.
Then the binomials $x_1^2 x_5^{2i+k+1} - x_2 x_3^{2i-2k+3} x_4^{3k-1}$, $k = 1, \ldots, i+1$, are contained in $\mathcal{H} \backslash UGB_{A_i}$. Each element comes from a distinct fiber of IP_{A_i}, and it can be shown that none of these is a

Gröbner fiber. (The proof is a lengthy case analysis and will be omitted.) We conclude that the matrix A_i has at least $i+1$ Graver fibers that are not Gröbner fibers.

Gröbner fibers can have arbitrarily many vertices even for 1×4-matrices:

Example 5.12 (Gröbner fibers with many vertices)
This example is based on Remark 18.1 in [Schr]. Let ϕ_k denote the *kth* Fibonacci number, and consider the 1×4-matrix $A_k := [\,\phi_{2k}\,,\,\phi_{2k+1}\,,\,1\,,\,\phi_{2k+1}^2 - 1\,]$. Consider the fiber of A_k over $b_k = \phi_{2k+1}^2 - 1$. This is a Gröbner fiber because it is the fiber of the circuit $(\,0,\,0,\,1-\phi_{2k+1}^2,\,1\,)$. The set of points with last coordinate zero is a facet of this fiber. It is a polygon isomorphic to the convex hull of all non-negative lattice points (x, y) with $\phi_{2k} \cdot x + \phi_{2k+1} \cdot y \leq \phi_{2k+1}^2 - 1$. This lattice polygon has $k+3$ vertices. We conclude that the b_k-fiber of A_k is a Gröbner fiber with at least $k+4$ vertices.

The encoding of the lattice polygon as a facet of a 3-polytope in Example 5.12 is a special case of the following general construction.

Proposition 5.13 *Every lattice polytope appears as a facet of some Gröbner fiber.*

Proof: Every $(n-d)$-dimensional lattice polytope can be written as a fiber P_b^I for some $A \in \mathbf{Z}^{d \times n}$ of maximal row rank and some $b \in \mathbf{Z}^d$. This polytope is isomorphic to the facet of points with zero last coordinate in the b-fiber of $IP_{[A,b]}$ where $[A, b] \in \mathbf{Z}^{d \times (n+1)}$. Moreover, the b-fiber of $IP_{[A,b]}$ is a Gröbner fiber since $x_{n+1} - x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ lies in $UGB_{[A,b]}$ for every vertex λ of P_b^I. □

6 On the complexity of Gröbner cones.

One direct application of the reduced Gröbner basis $\mathcal{G}_c$ is that it provides an inequality presentation for the equivalence class of cost functions containing c. This equivalence class is the interior of the normal cone of a vertex of the state polytope, or, equivalently, the interior of the Gröbner cone:

$$int\, \mathcal{K}_c \quad = \quad \{\, w \in \mathbf{R}^n : \alpha_i \cdot w > \beta_i \cdot w \text{ for all } \alpha_i - \beta_i \in \mathcal{G}_c \,\}. \tag{6.1}$$

While the Gröbner basis $\mathcal{G}_c$ can have arbitrarily many elements for fixed d and n, we observed in a large number of computations that a vast majority of the inequalities in (6.1) is redundant. Based on this experimental evidence we make the following two conjectures.

Conjecture 6.1 *There exists a function $\varphi : \mathbf{N} \longrightarrow \mathbf{N}$ such that, for every matrix $A \in \mathbf{Z}^{d \times n}$ of rank d, every Gröbner cone $\mathcal{K}_c$ associated with A has at most $\varphi(n-d)$ facets.*

Conjecture 6.2 *For a matrix A of corank three, every vertex of $St(A)$ has at most four neighbors.*

Obviously we have $\varphi(2) = 2$. Conjecture 6.2 asserts that $\varphi(3) = 4$, or, equivalently, that every vertex of a 3-dimensional state polytope is either 3-valent or 4-valent. A proof of Conjecture 6.2 for matrices of format 1×4 has been announced by Imre Barany (personal communication). His proof relies on a description of the "neighbors of the origin" as defined by H. Scarf [Sca],[Sha].

In this section we present two constructions which provide lower bounds for the function φ (assuming it exists). These constructions show in particular that $\varphi(4) \geq 8$, $\varphi(5) \geq 12$, $\varphi(6) \geq 18$, $\varphi(7) \geq 30$, and that $\varphi(n-d)$ is bounded below by an exponential function in $n-d$. To study the facets of a Gröbner cone we shall use the following general lemma about facets of polyhedra.

Lemma 6.3 *Let* $\mathcal{G}_c = \{\alpha_i - \beta_i \,:\, i = 1, \ldots, t\}$. *Then* $\alpha_j - \beta_j$ *defines a facet of the Gröbner cone* $\mathcal{K}_c$ *if and only if the system* $\{\alpha_i \cdot x > \beta_i \cdot x,\ i \in \{1, \ldots, t\} \setminus \{j\}\} \,\cup\, \{\beta_j \cdot x > \alpha_j \cdot x\}$ *is consistent.*

We say that a binomial $x^{\alpha_j} - x^{\beta_j}$ in $\mathcal{G}_c$ can be *flipped* if $\alpha_j - \beta_j$ defines a facet of the Gröbner cone $\mathcal{K}_c$. If $x^{\alpha_j} - x^{\beta_j}$ can be flipped and w is a solution of the linear system in Lemma 6.3, then the vertices of $St(A)$ in directions c and w are connected by an edge parallel to $\alpha_j - \beta_j$. Our first result concerns the family of knapsack problems in Example 2.8. We show that the complexity of their Gröbner cones grows at least quadratically in the dimension.

Proposition 6.4 *The state polytope of the* $1 \times n$*-matrix* $A_n = [1, 2, 3, \ldots, n]$ *possesses a vertex with* $\binom{n}{2} - \lfloor \frac{n}{2} \rfloor$ *neighboring vertices. Hence* $\varphi(n-1) \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$.

Proof: Consider the cost vector $c = e_1 + e_2 + \cdots + e_{n-1}$. The corresponding integer program takes the form

$$\text{minimize } \gamma_1 + \gamma_2 + \ldots + \gamma_{n-1} \quad \text{subject to} \quad \gamma_1 + 2\gamma_2 + \cdots + (n-1)\gamma_{n-1} + n\gamma_n = \beta.$$

The vector c is generic because, for every positive integer β, there is a unique optimal solution γ^*: if n divides β then $\gamma^* = \frac{\beta}{n} \cdot e_n$, otherwise $\gamma^* = \lfloor \frac{\beta}{n} \rfloor \cdot e_n + e_i$, where $i \equiv \beta \pmod{n}$.

The corresponding Gröbner basis has $\binom{n}{2}$ elements:

$$\mathcal{G}_c \;=\; \{\, \underline{x_i x_j} \;-\; x_{i+j} \,:\, 1 \leq i \leq j \leq n-i \,\} \cup \{\, \underline{x_i x_j} \;-\; x_{i+j-n} x_n \,:\, n-j < i \leq j \leq n \,\}.$$

Here the leading terms are underlined. Thus the initial ideal of the toric ideal equals

$$in_c(I_A) \quad = \quad \langle x_1, x_2, \ldots, x_{n-1} \rangle^2.$$

We shall prove the proposition by establishing the following two claims:
Claim 1: The "diagonal" elements $\underline{x_i x_{n-i}} - x_n$, $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, cannot be flipped.
Claim 2: All other elements of $\mathcal{G}_c$ can be flipped.

To prove Claim 1 we assume on the contrary that $x_i x_{n-i} - x_n$ can be flipped in $\mathcal{G}_c$. That means there exists $\omega \in \mathbf{R}^n$ such that $\omega_i + \omega_{n-i} < \omega_n$ but ω selects the underlined leading term for all other binomials in $\mathcal{G}_c$.

Case (a): $i < \frac{n}{2}$. The binomial $\underline{x_{n-i}x_{2i}} - x_n x_i$ implies $\omega_{n-i} + \omega_{2i} - \omega_i - \omega_n > 0$, and the binomial $\underline{x_i^2} - x_{2i}$ implies $2\omega_i - \omega_{2i} > 0$. Summing both inequalities gives a contradiction.
Case (b): $i = \frac{n}{2}$. The binomial $\underline{x_{\frac{n}{2}} x_{\frac{n+2}{2}}} - x_1 x_n$ implies $\omega_{\frac{n}{2}} + \omega_{\frac{n+2}{2}} - \omega_1 - \omega_n > 0$, and $\underline{x_{\frac{n}{2}} x_1} - x_{\frac{n+2}{2}}$ implies $\omega_{\frac{n}{2}} + \omega_1 - \omega_{\frac{n+2}{2}} > 0$. Summing both inequalities gives a contradiction.

In our proof of Claim 2 we distinguish four cases.
Case (c): To flip a binomial $\underline{x_i x_j} - x_{i+j}$ with $i \neq j$ we take the cost function

$$\omega \quad = \quad 9e_i + 9e_j + 19e_{i+j} + \sum\{10e_k \, : \, k \in \{1, \ldots, n-1\} \setminus \{i, j, i+j\}\}.$$

Case (d): To flip a binomial $\underline{x_i^2} - x_{2i}$ we take the cost function

$$\omega \quad = \quad 9e_i + 19e_{2i} + \sum\{10e_k \, : \, k \in \{1, \ldots, n-1\} \setminus \{i, 2i\}\}.$$

Case (e): To flip a binomial $\underline{x_i x_j} - x_n x_{i+j-n}$ with $i \neq j$ we take the cost function

$$\omega \quad = \quad 12e_{i+j-n} + 9e_i + 9e_j + 7e_n + \sum\{10e_k \, : \, k \in \{1, \ldots, n-1\} \setminus \{i, j, i+j-n\}\}.$$

Case (f): To flip a binomial $\underline{x_i^2} - x_n x_{2i-n}$ we take the cost function

$$\omega \quad = \quad 12e_{2i-n} + 9e_i + 7e_n + \sum\{10e_k \, : \, k \in \{1, \ldots, n-1\} \setminus \{i, 2i-n\}\}.$$

In each case the cost functions c and ω select the same leading terms for all binomials in $\mathcal{G}_c$ except for the one which is to be flipped. By Lemma 6.3, this completes the proof. □

We conclude this section with an example in which the number of facets of a Gröbner cone is exponential in the corank of the matrix. This example uses a construction which was shown to us by Eric Babson. We first need to recall some general definitions and results from [BGS].

Definition 6.5 Given a matrix $A \in \mathbb{Z}^{d \times n}$, then the *chamber complex* $\Gamma(A)$ is the coarsest polyhedral complex that covers $pos(A)$ and refines all triangulations of A.

Proposition 6.6 [BGS] *Let $B \in \mathbb{Z}^{(n-d)\times n}$ be a Gale transform of the matrix $A \in \mathbb{Z}^{d\times n}$. Then the boundary complex of the secondary polyhedron $\Sigma(A)$ is antiisomorphic to the chamber complex $\Gamma(B)$ and the boundary complex of the secondary polyhedron $\Sigma(B)$ is antiisomorphic to $\Gamma(A)$.*

This shows that every chamber complex is a secondary fan and conversely. It is known that a matrix A is unimodular if and only if its Gale transform B is unimodular. In this case the Gröbner fan of A coincides with the secondary fan of A (cf. Theorem 5.9) and hence also with the chamber complex of B. To find a Gröbner cone of A with many facets, it therefore suffices to construct a chamber with many facets in the chamber complex of a unimodular matrix B.

Let B be the node-edge incidence matrix of the complete bipartite graph $K_{n,m}$ where $n = 2k-1$ and $m = 2k+1$. The columns of B are the vertices of the product of a regular $(n-1)$-simplex and a regular $(m-1)$-simplex. It is well known that B is unimodular ([Schr], §19.3).

Proposition 6.7 (E. Babson) *There exists a chamber in $\Gamma(B)$ with at least $\binom{n}{k}\binom{m}{k+1}$ facets.*

Proof: The cone $pos(B)$ has codimension 1 in $\mathbf{R}^n \times \mathbf{R}^m$. It consists of all non-negative vectors $(u_1, \ldots, u_n) \times (v_1, \ldots, v_m)$ such that $u_1 + \cdots + u_n = v_1 + \cdots + v_m$. By the *central ray* in $pos(B)$ we mean the one-dimensional cone generated by $(1/n, \ldots, 1/n) \times (1/m, \ldots, 1/m)$. The fact that n and m are relatively prime implies that the central ray lies in the interior of a chamber in $\Gamma(B)$. (Reason: it lies on none of the hyperplanes (6.2) below.) It is called the *central chamber* of $pos(B)$.

We will show that the central chamber has at least $\binom{n}{k}\binom{m}{k+1}$ facets. A facet of a chamber corresponds to a cut $(C_+, C_-; D_+, D_-)$ in $K_{n,m}$. Here (C_+, C_-) is a partition of $\{1, \ldots, n\}$ and (D_+, D_-) is a partition of $\{1, \ldots, m\}$. The hyperplane spanned by this facet is defined by

$$\sum_{i \in C_+} u_i - \sum_{i \in C_-} u_i - \sum_{j \in D_+} v_j + \sum_{j \in D_-} v_j = 0. \tag{6.2}$$

We call $(card(C_+), card(D_+))$ the *type* of this hyperplane.

The product of symmetric groups $S_n \times S_m$ fixes the central chamber and it acts transitively on the set of hyperplanes of fixed type. In what follows we shall determine which type(s) of hyperplanes appear(s) as facets of the central chamber. Starting at the point $(1/n, \ldots, 1/n) \times (1/m, \ldots, 1/m)$ on the central ray we move in the direction $(-1, -1, \ldots, -1, n-1) \times (0, 0, \ldots, 0)$ to a generic point $(a, a, \ldots, a, a+1-na) \times (1/m, \ldots, 1/m)$. By (6.2), we cross a facet of type (r, s) with $n \in C_-$ when

$$r \cdot a - (n-r-1) \cdot a - (a+1-na) - s/m + (m-s)/m = 2 \cdot r \cdot a - 2 \cdot s/m = 0.$$

Solving for a, we get $a = s/mr < 1/n$. We need to find $r \in \{1, \ldots, n\}$ and $s \in \{1, \ldots, m\}$ such that $1/n - s/mr$ is positive and as small possible. Equivalently, we wish to minimize the positive integer $m \cdot r - n \cdot s$. The unique solution that minimizes this expression is $r = k$ and $s = k+1$. We conclude that **every** hyperplane of type (r, s) is a facet of the central chamber. There exists $\binom{n}{k}\binom{m}{k+1}$ such facets. □

Corollary 6.8 *For all positive integers k we have $\varphi(4k-1) \geq \binom{2k-1}{k}\binom{2k+1}{k+1} > 2^k$.*

Proof: The matrix B has rank $m+n-1$. Its Gale transform A is a matrix of format $(m-1)(n-1) \times mn$ with maximal row rank. The kernel of A has dimension $mn-(m-1)(n-1) = m+n-1 = 4k-1$. By Proposition 6.6, the central chamber of $\Gamma(B)$ appears in the Gröbner fan of A. Corollary 6.8 now follows immediately from Proposition 6.7. □

We remark that the integer programs IP_A corresponding to the above Gale transform A are not bounded. This can be remedied by adding an extra column to B which is the negative of the sum of the other columns. This operation does not change $rank(B) = corank(A)$, and Corollary 6.8 holds for bounded integer programs as well.

References

[AL] W.W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, American Mathematical Society, Graduate Studies in Math., Vol. III, 1994.

[BM] D. Bayer and I. Morrison, Gröbner bases and geometric invariant theory I, *Journal of Symbolic Computation* **6** (1988) 209–217.

[BaS] D. Bayer and M. Stillman, MACAULAY: A computer algebra system for algebraic geometry. Available by anonymous ftp from `zariski.harvard.edu`.

[BiS] L.J. Billera and B. Sturmfels, Fiber polytopes, *Annals of Mathematics* **135** (1992) 527–549.

[BFS] L.J. Billera, P. Filliman and B. Sturmfels, Constructions and complexity of secondary polytopes, *Advances in Mathematics* **83** (1990) 155–179.

[BGS] L.J. Billera, I.M. Gel'fand and B. Sturmfels, Duality and minors of secondary polyhedra. *Journal of Combinatorial Theory* **B** **57** (1993) 258–268.

[BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler, *Oriented Matroids*. Cambridge University Press, 1993.

[BJ] C.E. Blair and R.G. Jeroslow, The value function of an integer program, *Mathematical Programming* **23** (1982) 237–273.

[CT] P. Conti and C. Traverso, Gröbner bases and integer programming, Proceedings AAECC-9 (New Orleans), Springer Verlag, LNCS **539** (1991) 130–139.

[CGST] W. Cook, A.M.H. Gerards, A. Schrijver and É. Tardos, Sensitivity theorems in integer linear programming, *Mathematical Programming* **34** (1986) 251–264.

[DS] P. Diaconis and B. Sturmfels, Algebraic algorithms for sampling from conditional distributions, *Annals of Statistics*, to appear.

[DGS] P. Diaconis, R. Graham and B. Sturmfels, Primitive partition identities, to appear in "Paul Erdös is 80", Vol. II, Bolyai Soc.

[EOS] H. Edelsbrunner, J. O'Rourke and R. Seidel, Constructing arrangements of lines and hyperplanes with applications, *SIAM J. Computing* **15** (1986) 341–363.

[Eis] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Springer Graduate Texts in Mathematics, 1994.

[GKZ] I.M. Gel'fand, M. Kapranov and A. Zelevinsky, *Multidimensional Determinants, Discriminants and Resultants*, Birkhäuser, Boston, 1994.

[Gra] J.E. Graver, On the foundations of linear and integer programming I, *Mathematical Programming* **8** (1975) 207–226.

[Kan] R. Kannan, Optimal solution and value of parametric integer programs, Manuscript, 1991.

[MR] T. Mora and L. Robbiano, The Gröbner fan of an ideal, *Journal of Symbolic Computation* **6** (1988) 183–208.

[NTT] N.R. Natraj, S.R. Tayur and R.R. Thomas, An algebraic geometry algorithm for scheduling in the presence of setups and correlated demands, *Mathematical Programming*, to appear.

[Sca] H.E. Scarf, Neighborhood systems for production sets with indivisibilities, *Econometrica* **54** (1986) 507-532.

[Schr] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1986.

[Sha] D. Shallcross, Neighbors of the origin for four by three matrices, *Mathematics of Operations Research* **17** (1992) 608–614.

[St1] B. Sturmfels, Gröbner bases of toric varieties, *Tôhoku Math. Journal* **43** (1991) 249–261.

[St2] B. Sturmfels, Asymptotic analysis of toric ideals, *Memoirs of the Faculty of Science*, Kyushu University Ser.A **46** (1992) 217–228.

[St3] B. Sturmfels, Sparse elimination theory, in *"Computational Algebraic Geometry and Commutative Algebra"* [D. Eisenbud and L. Robbiano, eds.], Proceedings Cortona (June 1991), Cambridge University Press, 1993, pp. 264–298.

[Th] R.R. Thomas, A geometric Buchberger algorithm for integer programming, *Manuscript*, Cornell University.

[WW] D.W. Walkup and R.J.-B. Wets, Lifting projections of convex polyhedra, *Pacific Journal of Math.* **28** (1969) 465–475.