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# Variation of Discrete Spectra

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**Abstract.** A formula [see (1) below] estimating collectively the variation of eigenvalues of a symmetric matrix under a perturbation is extended to the case of discrete eigenvalues of a selfadjoint operator in Hilbert space, under the assumption that the perturbation is compact. For this purpose, the notion of an extended enumeration of discrete eigenvalues is introduced.

1. The following result is known (see [1, Theorem II-6.11]).

**Theorem I.** *If $A$, $B$ are $n \times n$ hermitian matrices, their eigenvalues $\alpha_j$ and $\beta_j$ can be enumerated, with multiple eigenvalues repeated according to the multiplicity, in such a way that for any real-valued convex function $\Phi$ on $\mathbb{R}$ we have*

$$\sum_j \Phi(\beta_j - \alpha_j) \leqq \sum_k \Phi(\gamma_k), \tag{1}$$

*where the $\gamma_k$ are the eigenvalues of $C = B - A$, similarly repeated.*

In what follows we shall generalize (1), with slight modifications, to the infinite-dimensional case. To this end we introduce several definitions.

Let $A$ be a selfadjoint operator in a separable Hilbert space $H$. An isolated point of the spectrum of $A$ is automatically an eigenvalue; if it has finite multiplicity, we shall call it a *discrete eigenvalue* of $A$. The complement of the set of all discrete eigenvalues relative to the spectrum is the *essential spectrum*. The essential spectrum is closed in $\mathbb{R}$; its complement in $\mathbb{R}$ consists of at most countably many intervals $I_n$. Each discrete eigenvalue belongs to one of these intervals.

By an *extended enumeration of discrete eigenvalues* for $A$ we mean a sequence $\{\alpha_j\}$ with the following properties. (a) Every discrete eigenvalue of $A$ with multiplicity $m$ appears exactly $m$-times in the values $\alpha_j$. We refer to these values as *proper* values of the sequence. (b) All other values of the $\alpha_j$, referred to as *improper values*, belong to the countable set consisting of all the boundary points of the intervals $I_n$ stated above. Improper values may or may not be eigenvalues, and their number may be finite or infinite. If there are no improper values, we simply speak of an *enumeration*.

**Theorem II.** *Let $A$, $B$ be a selfadjoint operators in $H$ such that $B=A+C$, where $C$ is a compact selfadjoint operator. Let $\{\gamma_k\}$ be an enumeration of nonzero eigenvalues of $C$. Then there exist extended enumerations $\{\alpha_j\}$, $\{\beta_j\}$ of discrete eigenvalues for $A$, $B$, respectively, such that the inequality* (1) *holds for any nonnegative convex function $\Phi$ on $\mathbb{R}$ such that $\Phi(0)=0$. In particular,*

$$\Big(\sum_j |\beta_j-\alpha_j|^p\Big)^{1/p} \leqq \|C\|_p \equiv \Big(\sum_k |\gamma_k|^p\Big)^{1/p}, \qquad 1 \leqq p \leqq \infty. \tag{1a}$$

**Corollary.** *If $C \geqq 0$ in Theorem II, we have automatically $\alpha_j \leqq \beta_j$ for all $j$.*

The corollary follows immediately by choosing a special $\Phi$ given by $\Phi(\mu)=0$ for $\mu \geqq 0$ and $\Phi(\mu)=|\mu|$ for $\mu<0$.

*Remark.* Introduction of the extended enumerations appears to be necessary since $A$ and $B$ in general have different numbers of discrete eigenvalues. In (1) it may happen, for example, that $\alpha_j$ is proper and $\beta_j$ is improper. Then the eigenvalue $\alpha_j$ of $A$ is matched with a boundary point $\beta_j$ of the essential spectrum (which is common to $A$ and $B$). If both $\alpha_j$ and $\beta_j$ are improper, the term involving them may be removed from (1) without affecting the inequality.

2. To prove the theorem, we introduce a family $A(t)$ of operators by

$$A(t)=A+tC, \qquad t \in \mathbb{R}, \tag{2}$$

so that $A(0)=A$, $A(1)=B$. Although we are primarily interested in $t \in [0,1]$, it is convenient to consider all $t \in \mathbb{R}$. Since $C$ is compact, all the $A(t)$ have common essential spectrum $\Sigma_e$. As mentioned above, the complement in $\mathbb{R}$ of $\Sigma_e$ consists of at most countably many open disjoint intervals $I_n$, $n=1,2,\ldots$. Every discrete eigenvalue of $A(t)$ for any $t \in \mathbb{R}$ belongs to one of the $I_n$.

Since all discrete eigenvalues are isolated with finite multiplicities, it follows from the analytic perturbation theory (see [1, Chap. VII], for example) that they are represented by a sequence of analytic functions. More precisely, there exist sequences $\{\lambda_j(t)\}$, $\{E_j(t)\}$ of real-analytic functions, the former real-valued and the latter operator-valued, with the following properties.

(i) Each $\lambda_j(t)$ is defined on a maximal interval $\Delta_j \subset \mathbb{R}$, and takes values in one of the $I_n$. The graph of $\lambda_j$ runs from boundary to boundary in $\Delta_j \times I_n$. In other words, if $\Delta_j$ is not all of $\mathbb{R}$, $\lambda_j(t)$ has a limit at each boundary of $\Delta_j$, and the limit belongs to the boundary of $I_n$.

(ii) Each discrete eigenvalue of $A(t)$ for any $t \in \mathbb{R}$ is one of the $\lambda_j(t)$.

(iii) $E_j(t)$, also defined on $\Delta_j$, is the eigenprojection associated with the discrete eigenvalue $\lambda_j(t)$ of $A(t)$ (except when "crossing" occurs at $t$, so that the associated eigenprojection is accidentally enlarged).

(iv) $\dim E_j(t)H = \operatorname{tr} E_j(t) = m_j$ is constant for $t \in \Delta_j$, $1 \leqq m_j < \infty$.

(v) $d\lambda_j(t)/dt = (1/m_j)\operatorname{tr}(CE_j(t))$, $t \in \Delta_j$ (see [1, II-(2.32)]).

We may add the following comments for the proof of (i). $\lambda_j(t)$ is uniformly Lipschitz continuous: $|d\lambda_j/dt| \leqq \|C\|$, as is implied by (v). Therefore it has a limit $\mu$ at the end, say $b$, of $\Delta_j$. $\mu$ must be on the boundary of $I_n$. Otherwise it is a discrete eigenvalue of $A(b)$, due to the continuity of the spectrum inside $I_n$. Then $\mu$ will split, in a neighborhood of $b$, into several analytic functions $\mu_k(t)$ representing the eigenvalues of $A(t)$, and $\lambda_j(t)$ must be one of them. Thus $\lambda_j(t)$ is analytically continuable to $t>b$, contrary to the maximality of $\Delta_j$.

3. The maximal interval $\Delta_j$ may or may not coincide with $\mathbb{R}$. If not, it is convenient to extend $\lambda_j(t)$ to all $t\in\mathbb{R}$ continuously as a constant function on each component of $\mathbb{R}\setminus\Delta_j$. According to (i), these constant values belong to the boundary of $I_n$. The function thus extended on $\mathbb{R}$ will be denoted by $\tilde{\lambda}_j(t)$. Similarly we extend $E_j(t)$ to all $t\in\mathbb{R}$ by setting $\tilde{E}_j(t)=0$ for $t\in\mathbb{R}\setminus\Delta_j$. Since for each fixed $t$ the set of the $E_j(t)$ with $t\in\Delta_j$ form an orthogonal system of projections, we have

(vi) $$0\leqq \sum_j \tilde{E}_j(t)\leqq 1, \qquad t\in\mathbb{R}.$$

4. We are now ready to prove Theorem II. $\tilde{\lambda}_j(t)$ and $\tilde{E}_j(t)$ are no longer analytic, but they are piecewise analytic and satisfy the differential equation (v). Since $\tilde{\lambda}_j(t)$ is continuous, we thus obtain

$$\tilde{\lambda}_j(1)-\tilde{\lambda}_j(0)=(1/m_j)\int_0^1 \operatorname{tr}(C\tilde{E}_j(t))\,dt. \tag{3}$$

Let $\{\phi_k\}$ be an orthonormal system of eigenvectors of $C$ that spans the closure of the range of $C$ (so that $C\phi_k=\gamma_k\phi_k$). Then

$$\operatorname{tr}(C\tilde{E}_j(t))=\sum_k \gamma_k(\tilde{E}_j(t)\phi_k,\phi_k). \tag{4}$$

It follows that

$$\tilde{\lambda}_j(1)-\tilde{\lambda}_j(0)=\sum_k \sigma_{jk}\gamma_k, \tag{5}$$

$$\sigma_{jk}=(1/m_j)\int_0^1 (\tilde{E}_j(t)\phi_k,\phi_k)\,dt\geqq 0, \tag{6}$$

$$\sum_k \sigma_{jk}\equiv\sigma_j\leqq 1, \qquad \sum_j m_j\sigma_{jk}\leqq 1. \tag{7}$$

Indeed, the first inequality in (7) follows from $\sum_k(\tilde{E}_j\phi_k,\phi_k)=\operatorname{tr}\tilde{E}_j\leqq m_j$, while the second follows from (vi).

Now let $\Phi$ be a nonnegative convex function with $\Phi(0)=0$. It follows from (5) and (7) that

$$\begin{aligned}\Phi(\tilde{\lambda}_j(1)-\tilde{\lambda}_j(0))&=\Phi\left(\sum_k(\sigma_{jk}/\sigma_j)\sigma_j\gamma_k\right)\leqq\sum_k(\sigma_{jk}/\sigma_j)\,\Phi(\sigma_j\gamma_k)\\&\leqq\sum_k\sigma_{jk}\Phi(\gamma_k)\end{aligned} \tag{8}$$

because $\Phi(\sigma\gamma)=\Phi((1-\sigma)0+\sigma\gamma)\leqq(1-\sigma)\,\Phi(0)+\sigma\Phi(\gamma)=\sigma\Phi(\gamma)$. Thus we have

$$\sum_j m_j\Phi(\tilde{\lambda}_j(1)-\tilde{\lambda}_j(0))\leqq\sum_k\Phi(\gamma_k) \tag{9}$$

by (7) again.

Formula (9) is what we wanted to prove. If we rewrite it by repeating $\Phi(\tilde{\lambda}_j(1)-\tilde{\lambda}_j(0))$ $m_j$-times and omitting the factor $m_j$, the result is exactly of the form (1). Indeed, $\{\tilde{\lambda}_j(0)\}$ thus repeated is an extended enumeration of discrete eigenvalues for $A$, and similarly for $\{\tilde{\lambda}_j(1)\}$.

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## Reference

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