ESAIM: COCV 18 (2012) 941–953
DOI: 10.1051/cocv/2011199

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

VARIATIONAL ANALYSIS FOR A NONLINEAR ELLIPTIC PROBLEM ON THE SIERPIŃSKI GASKET

Gabriele Bonanno[1], Giovanni Molica Bisci[2] and Vicenţiu Rădulescu[3,4]

Abstract. Under an appropriate oscillating behaviour either at zero or at infinity of the nonlinear term, the existence of a sequence of weak solutions for an eigenvalue Dirichlet problem on the Sierpiński gasket is proved. Our approach is based on variational methods and on some analytic and geometrical properties of the Sierpiński fractal. The abstract results are illustrated by explicit examples.

Mathematics Subject Classification. 35J20, 28A80, 35J25, 35J60, 47J30, 49J52.

Received April 14, 2011. Revised November 11, 2011.
Published online 16 January 2012.

1. Introduction

In the last three decades of the 20th century it started to become clear that many phenomena in the real world are best modeled by some exotic geometric structures with a nonsmooth appearance. The theory of fractals, as Mandelbrot [21–23] has so forcefully argued, seeks to provide the mathematical framework for such powerful development. Analysis of PDEs on fractal domains has shown an explosive development, due to numerous applications to problems arising in various fields, including physics, chemistry and biology. In this paper we study the following Dirichlet problem

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda g(x) f(u(x)) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases} \qquad (S^{f,g}_{a,\lambda})$$

where V stands for the Sierpiński gasket, V_0 is its intrinsic boundary, Δ denotes the weak Laplacian on V and λ is a positive real parameter. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that the variable potentials a, $g : V \to \mathbb{R}$ satisfy the following conditions:

(h_1) $a \in L^1(V, \mu)$ and $a \leq 0$ almost everywhere in V;
(h_2) $g \in C(V)$ with $g \leq 0$ and such that the restriction of g to every open subset of V is not identically zero.

Keywords and phrases. Sierpiński gasket, nonlinear elliptic equation, Dirichlet form, weak Laplacian.

[1] Department of Science for Engineering and Architecture (Mathematics Section) Engineering Faculty, University of Messina, 98166 Messina, Italy. bonanno@unime.it

[2] Dipartimento MECMAT, University of Reggio Calabria, Via Graziella, Feo di Vito, 89124 Reggio Calabria, Italy. gmolica@unirc.it

[3] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania.

[4] Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania. vicentiu.radulescu@imar.ro

Article published by EDP Sciences

Here μ denotes the restriction to V of normalized $\log N/\log 2$-dimensional Hausdorff measure on V, so that $\mu(V)=1$.

In our main result just requiring an oscillating behaviour of the non-linearity f either at zero or at infinity, we prove that problem $(S_{a,\lambda}^{f,g})$ admits a sequence of pairwise distinct weak solutions; see Theorems 3.4 and 3.5 below.

The Sierpiński gasket has the origin in a paper by Sierpiński [29]. In a very simple manner, this fractal domain can be described as a subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of $1/4$ of the area, removing the corresponding open triangle from each of the three constituent triangles and continuing in this way. This fractal can also be obtained as the closure of the set of vertices arising in this construction.

Over the years, the Sierpiński gasket showed both to be extremely useful in representing roughness in nature and man's works. This geometrical object is one of the most familiar examples of fractal domains and it gives insight into the turbulence of fluids. According to Kigami [19], this notion was introduced by Mandelbrot [22] in 1977 to design a class of mathematical objects which are not collections of smooth components. We refer to Strichartz [30] for an elementary introduction to this subject and to Strichartz [32] for important applications to differential equations on fractals.

The study of the Laplacian on fractals was originated in physics literature, where so-called *spectral decimation method* was developed in Alexander [1] and Rammal *et al.* [26, 27]. The Laplacian on the Sierpiński gasket was first constructed as the generator of a diffusion process by Kusuoka [20] and Goldstein [14]. Among the contributions to the theory of nonlinear elliptic equations on fractals we mention [6, 10, 12, 15, 16, 31].

The main tools used in these papers to prove the existence of at least one nontrivial solution or of multiple solutions for nonlinear elliptic equations with zero Dirichlet boundary conditions are certain mini-max results (mountain pass theorems, saddle-point theorems), respectively, minimization procedures.

In this note, starting from the seminal paper by Falconer and Hu [12], we study the Dirichlet problem $(S_{a,\lambda}^{f,g})$ through variational methods. In particular, in order to prove our multiplicity result we use the following critical points theorem obtained by Bonanno and Molica Bisci in [3] that we recall here in a convenient form; see also the variational principle of Ricceri, contained in [28].

Theorem 1.1 ([3], Thm. 2.1)**.** *Let X be a reflexive real Banach space, and $\Phi,\Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semi-continuous and coercive and Ψ is sequentially weakly upper semi-continuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r-\Phi(u)},$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *If $\gamma < +\infty$ then, for each $\lambda \in]0, 1/\gamma[$, the following alternative holds:*
either
(a_1) *$I_\lambda := \Phi - \lambda\Psi$ possesses a global minimum,*
or
(a_2) *there is a sequence $\{v_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to\infty} \Phi(v_n) = +\infty$;*

(b) *if $\delta < +\infty$ then, for each $\lambda \in]0, 1/\delta[$, the following alternative holds:*
either
(b_1) *there is a global minimum of Φ which is a local minimum of I_λ,*
or
(b_2) *there is a sequence $\{v_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with $\lim_{n\to\infty} \Phi(v_n) = \inf_X \Phi$.*

The above theoretical result assures the existence of a sequence of pairwise distinct critical points for Gâteaux differentiable functionals under assumptions that, when we consider the energy functional associated to ($S^{f,g}_{a,\lambda}$), are satisfied assuming an appropriate oscillating behaviour on the potential of the nonlinearity either at infinity or at the origin. In this order of ideas, very recently, Theorem 1.1 has been used to prove the existence of sequences of solutions for different classes of elliptic problems with Dirichlet boundary condition; see, for example, Bonanno and Molica Bisci in [4]; Bonanno *et al.* in [5] and D'Aguì and Molica Bisci [8].

In the cited papers, for technical reasons, a necessary condition applying our abstract result is the presence of suitable classes of test functions. For instance, one admissible set of maps was introduced by Bonanno and Livrea in [2] and successfully used in several works. It is worth noticing that the different test functions introduced here are fundamental proving Theorems 3.4 and 3.5 since the usual cannot be used in elliptic problems on Sierpiński gasket due to the particular geometry of the fractal set.

Moreover, we just mention that most results for classical Dirichlet problems on a bounded domain $\Omega \subset \mathbb{R}^N$ assume that the datum f is odd in order to apply some variant of the classical Lusternik-Schnirelmann theory. Only a few papers deal with non-linearities having no symmetry properties; see, for instance, the papers of Omari and Zanolin [24,25] and the recent work of Kristály and Moroşanu [17] for perturbed Dirichlet equations. In analogy with the contributions obtained in [4, 5, 9] in our approach here we don't require any symmetry hypothesis.

Finally, we recall that Breckner *et al.* [7] proved the existence of infinitely many solutions of problem ($S^{f,g}_{a,\lambda}$) under the key assumption, among others, that the non-linearity f is non-positive in a sequence of positive intervals. We point out that our results are mutually independent compared to those achieved in the above mentioned manuscript; see Remark 4.2.

This paper is organized as follows. In Section 2 we recall the geometrical construction of the Sierpiński gasket and our variational framework. Successively, Section 3 is devoted to the main theorem and finally, in the last section, we give two applications of the presented results. We cite the very recent monograph by Kristály *et al.* [18] as general reference for the basic notions used here.

2. Abstract framework

Let $N \geq 2$ be a natural number and let $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, \dots, N\}$, the map $S_i \colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}\, x + \frac{1}{2}\, p_i\, .$$

Let $\mathcal{S} := \{S_1, \dots, S_N\}$ and denote by $G \colon \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$G(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the attractor of the family $\mathcal{S}$, such that $G(V) = V$; see, Theorem 9.1 in Falconer [11].

The set V is called the *Sierpiński gasket* in $\mathbb{R}^{N-1}$. It can be constructed inductively as follows:

Put $V_0 := \{p_1, \dots, p_N\}$, $V_m := G(V_{m-1})$, for $m \geq 1$, and $V_* := \bigcup_{m \geq 0} V_m$. Since $p_i = S_i(p_i)$ for $i \in \{1, \dots, N\}$, we have $V_0 \subseteq V_1$, hence $G(V_*) = V_*$. Taking into account that the maps S_i, $i \in \{1, \dots, N\}$, are homeomorphisms, we conclude that $\overline{V_*}$ is a fixed point of G. On the other hand, denoting by C the convex hull of the set $\{p_1, \dots, p_N\}$, we observe that $S_i(C) \subseteq C$ for $i = 1, \dots, N$. Thus $V_m \subseteq C$ for every $m \in \mathbb{N}$, so $\overline{V_*} \subseteq C$. It follows that $\overline{V_*}$ is non-empty and compact, hence $V = \overline{V_*}$.

In the sequel V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$. The set V_0 is called the *intrinsic boundary* of V. By Theorem 9.3 in Falconer [11], the Hausdorff (fractal) dimension d of V satisfies the equality

$$\sum_{i=1}^{N}\left(\frac{1}{2}\right)^{d}=1,$$

hence $d=\ln N/\ln 2$, and $0<\mathcal{H}^d(V)<\infty$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. We point out that the Hausdorff dimension of a set is a more refined notion than the topological dimension. In particular, the Hausdorff dimension is closely related to entropy. To see this, we consider the map $T(z)=z^2$ defined on the circle. If ν is any ergodic T-invariant measure, then the Kolmogorov entropy of T is (up to a constant depending on the base of the logarithm in the definition of entropy) the infimum of the Hausdorff dimension of subsets of the circle that have full ν measure. In the other direction, if E is a closed subset of the circle that is invariant under the map T, then E supports an invariant measure whose entropy is (again up to that constant) the Hausdorff dimension of the set E.

Let μ be the normalized restriction of $\mathcal{H}^d$ to the subsets of V, so $\mu(V)=1$. Finally, the following property of μ will be useful in the sequel:

$$\mu(B)>0,\ \text{for every non-empty open subset } B \text{ of } V. \tag{2.1}$$

In other words, the support of μ coincides with V; see, for instance, Breckner *et al.* [7].

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V):=\{u\in C(V)\mid u|_{V_0}=0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $\|\cdot\|_\infty$. For a function $u\colon V\to\mathbb{R}$ and for $m\in\mathbb{N}$ let

$$W_m(u)=\left(\frac{N+2}{N}\right)^{m}\sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}}(u(x)-u(y))^2. \tag{2.2}$$

We have $W_m(u)\le W_{m+1}(u)$ for every natural m, so we can put

$$W(u)=\lim_{m\to\infty}W_m(u). \tag{2.3}$$

Define now

$$H_0^1(V):=\{u\in C_0(V)\mid W(u)<\infty\}.$$

It turns out that $H_0^1(V)$ is a dense linear subset of $L^2(V,\mu)$ equipped with the $\|\cdot\|_2$ norm. We now endow $H_0^1(V)$ with the norm

$$\|u\|=\sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: for $u,v\in H_0^1(V)$ and $m\in\mathbb{N}$ let

$$\mathcal{W}_m(u,v)=\left(\frac{N+2}{N}\right)^{m}\sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}}(u(x)-u(y))(v(x)-v(y)).$$

Put

$$\mathcal{W}(u,v)=\lim_{m\to\infty}\mathcal{W}_m(u,v).$$

Then $\mathcal{W}(u,v)\in\mathbb{R}$ and the space $H_0^1(V)$, equipped with the inner product $\mathcal{W}$, which induces the norm $\|\cdot\|$, becomes a real Hilbert space.

Moreover,

$$\|u\|_\infty\le(2N+3)\|u\|,\ \text{for every } u\in H_0^1(V), \tag{2.4}$$

and the embedding

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty) \tag{2.5}$$

is compact. We refer to Fukushima and Shima [13] for further details.

Remark 2.1. As pointed out by Falconer and Hu [12], we observe that if $a \in L^1(V)$ and $a \le 0$ in V then, from (2.4), the norm

$$\|u\|_* := \left(\mathcal{W}(u,u) - \int_V a(x)u^2 \mathrm{d}\mu\right)^{1/2},$$

is equivalent to $\sqrt{W(u)}$ in $H_0^1(V)$.

We now state a useful property of the space $H_0^1(V)$ which shows, together with the facts that $(H_0^1(V), \|\cdot\|)$ is a Hilbert space and $H_0^1(V)$ is dense in $L^2(V,\mu)$, that $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$.

Lemma 2.2. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with Lipschitz constant $L \ge 0$ and such that $h(0) = 0$. Then, for every $u \in H_0^1(V)$, we have $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \le L\|u\|$.*

Proof. It is clear that $h \circ u \in C_0(V)$. For every $m \in \mathbb{N}$ we have, by (2.2) and the Lipschitz property of h, that

$$W_m(h \circ u) \le L^2 W_m(u).$$

Hence $W(h \circ u) \le L^2 W(u)$, according to (2.3). Thus $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \le L\|u\|$. □

Following Falconer and Hu [12] we can define in a standard way a linear self-adjoint operator $\Delta\colon Z \to L^2(V,\mu)$, where Z is a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$ (and dense also in $(H_0^1(V), \|\cdot\|)$), such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v \mathrm{d}\mu, \text{ for every } (u,v) \in Z \times H_0^1(V).$$

The operator Δ is called the *(weak) Laplacian* on V.

Precisely, let $H^{-1}(V)$ be the closure of $L^2(V,\mu)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{\substack{h \in H_0^1(V) \\ \|h\|=1}} |\langle u, h\rangle|,$$

where

$$\langle v, h\rangle = \int_V vh \mathrm{d}\mu,$$

$v \in L^2(V,\mu)$ and $h \in H_0^1(V)$. Then $H^{-1}(V)$ is a Hilbert space. Then the relation

$$-\mathcal{W}(u,v) = < \Delta u, v >, \quad \forall v \in H_0^1(V),$$

uniquely defines a function $\Delta u \in H^{-1}(V)$ for every $u \in H_0^1(V)$.

Finally, fix $\lambda > 0$. Given $a\colon V \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ be as in Introduction. We say that a function $u \in H_0^1(V)$ is called a *weak solution* of $(S_{a,\lambda}^{f,g})$ if

$$\mathcal{W}(u,v) - \int_V a(x)u(x)v(x)\mathrm{d}\mu + \lambda \int_V g(x)f(u(x))v(x)\mathrm{d}\mu = 0,$$

for every $v \in H_0^1(V)$.

While we mainly work with the weak Laplacian, there is also a directly defined version. We say that Δ_s is the *standard Laplacian* of u if $\Delta_s u : V \to \mathbb{R}$ is continuous and

$$\lim_{m\to\infty} \sup_{x\in V\setminus V_0} |(N+2)^m (H_m u)(x) - \Delta_s u(x)| = 0,$$

where

$$(H_m u)(x) := \sum_{\substack{y\in V_m \\ |x-y|=2^{-m}}} (u(y)-u(x)),$$

for $x \in V_m$. We say that $u \in C_0(V)$ is a *strong solution* of ($S^{f,g}_{a,\lambda}$) if $\Delta_s u$ exists and is continuous for all $x \in V\setminus V_0$, and

$$\Delta u(x) + a(x)u(x) = \lambda g(x) f(u(x)), \quad \forall\, x \in V \setminus V_0.$$

The existence of the standard Laplacian of a function $u \in H^1_0(V)$ implies the existence of the weak Laplacian Δ; see, for completeness, Falconer and Hu [12].

Remark 2.3. If $a \in C(V)$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g \in C(V)$, then, using the regularity result Lemma 2.16 of Falconer and Hu [12], it follows that every weak solution of the problem ($S^{f,g}_{a,\lambda}$) is also a strong solution.

3. Main results

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(\xi) = \int_0^\xi f(t)\mathrm{d}t$, for every $\xi \in \mathbb{R}$, and fix $\lambda > 0$. The functional $I : H^1_0(V) \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_V a(x)u^2(x)\mathrm{d}\mu + \lambda \int_V g(x)F(u(x))\mathrm{d}\mu,\ \forall\, u \in H^1_0(V), \tag{3.1}$$

will turns out to be the the energy functional attached to problem ($S^{f,g}_{a,\lambda}$). To see this we first recall, for completeness, a few basic notions.

Definition 3.1. Let $(E, \|\cdot\|)$ be a real Banach space, E^* its topological dual and $T : E \to \mathbb{R}$ a functional. We say that T is Fréchet differentiable at $u \in E$ if there exists a continuous linear map $T'(u) : E \to \mathbb{R}$, called the Fréchet differential of T at u, such that

$$\lim_{v\to 0} \frac{|T(u+v) - T(u) - T'(u)(v)|}{\|v\|} = 0.$$

The functional T is Fréchet differentiable on E if T is Fréchet differentiable at every point $u \in E$. A point $u \in E$ is a critical point of T if T is Fréchet differentiable at u and if $T'(u) = 0$. Moreover, if $T' : E \to E^*$ is continuous, then T is called a $C^1(E, \mathbb{R})$ functional.

Remark 3.2. Note that if the functional $T : E \to \mathbb{R}$ has in $u \in E$ a local extremum and if T is Fréchet differentiable at u, then u is a critical point of T.

We have the following result contained in [12], Proposition 2.19, that we recall here in a convenient form.

Lemma 3.3. *The functional $I_\lambda : H^1_0(V) \to \mathbb{R}$ defined by relation (3.1) is a $C^1(H^1_0(V), \mathbb{R})$ functional. Moreover,*

$$I'_\lambda(u)(v) = \mathcal{W}(u,v) - \int_V a(x)u(x)v(x)\mathrm{d}\mu + \lambda \int_V g(x)f(u(x))v(x)\mathrm{d}\mu,\ \forall\, v \in H^1_0(V)$$

for each point $u \in H^1_0(V)$. In particular, $u \in H^1_0(V)$ is a weak solution of problem ($S^{f,g}_{a,\lambda}$) if and only if u is a critical point of I_λ.

The aim of the paper is to prove the following result concerning the existence of infinitely many weak solutions of the problem ($S^{f,g}_{a,\lambda}$).

Theorem 3.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function. Assume that*

$$\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}<+\infty \quad \text{and} \quad \limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}=+\infty. \tag{h$_0$}$$

Then, for every

$$\lambda\in\left]0,-\frac{1}{2(2N+3)^2\left(\int_V g(x)\mathrm{d}\mu\right)\liminf\limits_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}}\right[,$$

there exists a sequence $\{v_n\}$ of pairwise distinct weak solutions of problem ($S^{f,g}_{a,\lambda}$) such that $\lim\limits_{n\to\infty}\|v_n\|=\lim\limits_{n\to\infty}\|v_n\|_\infty=0$.

Proof. Let us define the functionals $\Phi,\Psi:X\to\mathbb{R}$ by

$$\Phi(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}\int_V a(x)u^2(x)\mathrm{d}\mu \quad \text{and} \quad \Psi(u)=-\int_V g(x)F(u(x))\mathrm{d}\mu,$$

where X denotes the reflexive Banach space $H^1_0(V)$. Now, in order to achieve our goal, fix λ as in the conclusion. Clearly, with the above notations, one has that $I_\lambda=\Phi-\lambda\Psi$. Hence, we seek for weak solutions of problem ($S^{f,g}_{a,\lambda}$) by applying part (b) of Theorem 1.1. First of all we observe that, from Lemma 3.3, the functional $I_\lambda\in C^1(X,\mathbb{R})$. Moreover, Φ is obviously coercive and, by using Lemma 5.6 in Breckner *et al.* [7], the functionals Φ and Ψ are weakly sequentially lower semi-continuous on X. Now, let $\{c_n\}$ be a real sequence such that $\lim_{n\to\infty}c_n=0$ and

$$\lim_{n\to\infty}\frac{F(c_n)}{c_n^2}=\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}.$$

Put $r_n=\dfrac{c_n^2}{2(2N+3)^2}$ for every $n\in\mathbb{N}$. Due to the compact embedding into $C_0(V)$, from (2.4), we have

$$\{v\in X\mid \Phi(v)<r_n\}\subseteq\{v\in X\mid \|v\|_\infty\le c_n\}.$$

Therefore

$$\begin{aligned}\varphi(r_n)&=\inf_{\Phi(u)<r_n}\frac{\sup\limits_{\Phi(v)<r_n}\int_V(-g(x))F(v(x))\mathrm{d}\mu+\int_V g(x)F(u(x))\mathrm{d}\mu}{r_n-\Phi(u)}\\&\le\frac{\sup\limits_{\Phi(v)<r_n}\int_V(-g(x))F(v(x))\mathrm{d}\mu}{r_n}\le-\left(\int_V g(x)\mathrm{d}\mu\right)\frac{\max\limits_{|\xi|\le c_n}F(\xi)}{r_n}\\&=-\left(\int_V g(x)\mathrm{d}\mu\right)\frac{F(c_n)}{r_n}=-2(2N+3)^2\left(\int_V g(x)\mathrm{d}\mu\right)\frac{F(c_n)}{c_n^2}.\end{aligned}$$

Thus, since

$$\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}<+\infty,$$

we deduce that

$$\delta \leq \liminf_{n\to\infty} \varphi(r_n) \leq -2(2N+3)^2 \left(\int_V g(x)\mathrm{d}\mu\right) \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < +\infty.$$

At this point we will show that 0, that is the unique global minimum of Φ, is not a local minimum of the functional I_λ. Hence, fix a function $u \in X$ such that there is an element $x_0 \in V$ with $u(x_0) > 1$. It follows that

$$D := \{x \in V \mid u(x) > 1\}$$

is a non-empty open (from the continuity of u) subset of V. Define $h\colon \mathbb{R} \to \mathbb{R}$ as follows

$$h(t) = |\min\{t, 1\}|, \qquad \text{for all } t \in \mathbb{R}.$$

Then $h(0) = 0$ and h is a Lipschitz function whose Lipschitz constant L is equal to 1. Hence, by using Lemma 2.2, it follows that $v := h \circ u \in X$. Moreover, $v(x) = 1$ for every $x \in D$, and $0 \leq v(x) \leq 1$ for every $x \in V$. Bearing in mind that

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty,$$

there exists a sequence $\{\xi_n\}$ in $]0, \rho[$ such that $\lim_{n\to\infty} \xi_n = 0$ and

$$\lim_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2} = +\infty. \tag{3.2}$$

Consider the sequence of functions $\{\xi_n v\} \subset X$. Clearly $\|\xi_n v\| \to 0$ and

$$I_\lambda(\xi_n v) = \frac{\xi_n^2}{2}\|v\|^2 - \frac{\xi_n^2}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + \lambda F(\xi_n)\int_D g(x)\mathrm{d}\mu + \lambda \int_{V\setminus D} g(x)F(\xi_n v(x))\mathrm{d}\mu.$$

Taking into account that F is positive in $]0, +\infty[$, from hypothesis (h$_2$), the above equation becomes

$$I_\lambda(\xi_n v) \leq \frac{\xi_n^2}{2}\|v\|^2 - \frac{\xi_n^2}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + \lambda F(\xi_n)\int_D g(x)\mathrm{d}\mu, \quad \forall\, n \in \mathbb{N}.$$

Thus

$$\frac{I_\lambda(\xi_n v)}{\xi_n^2} \leq \frac{1}{2}\|v\|^2 - \frac{1}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + \lambda\frac{F(\xi_n)}{\xi_n^2}\int_D g(x)\mathrm{d}\mu,$$

for every $n \in \mathbb{N}$.

Moreover, conditions (h$_2$) and (2.1) imply that $\int_D g(x)\mathrm{d}\mu < 0$. Now, thanks to (3.2), the above computations ensure that

$$\lim_{n\to\infty} \frac{I_\lambda(\xi_n v)}{\xi_n^2} = -\infty.$$

Thus $I_\lambda(\xi_n v) < 0$ for every n sufficiently large. Since $I_\lambda(0_X) = \Phi(0_X) - \lambda\Psi(0_X) = 0$, this means that 0_X is not a local minimum of I_λ. Moreover, since Φ has 0_X as unique global minimum, Theorem 1.1 ensures the existence of a sequence $\{v_n\}$ of pairwise distinct critical points of the functional I_λ, such that

$$\lim_{n\to\infty} \left(\|v_n\|^2 - \int_V a(x)v_n^2(x)\mathrm{d}\mu\right) = 0.$$

Hence, one has that $\lim_{n\to\infty} \|v_n\| = 0$. In particular, by (2.5), it follows that $\lim_{n\to\infty} \|v_n\|_\infty = 0$. Then, the proof is complete bearing in mind Lemma 3.3. □

By the same method, applying part (a) instead of part (b) of Theorem 1.1, one can prove the following analogous result in the case when the nonlinear term $f\colon \mathbb{R} \to \mathbb{R}$ has an oscillating behaviour at infinity. In this setting one obtains a sequence $\{v_n\}$ of weak solutions to problem $(S^{f,g}_{a,\lambda})$ such that $\lim\limits_{n\to\infty} \|v_n\| = +\infty$.

Theorem 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < +\infty \qquad \text{and} \qquad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty. \tag{h_∞}$$

Then, for every

$$\lambda \in \left]0, -\frac{1}{2(2N+3)^2\left(\displaystyle\int_V g(x)\mathrm{d}\mu\right)\liminf\limits_{\xi\to+\infty}\dfrac{F(\xi)}{\xi^2}}\right[,$$

problem $(S^{f,g}_{a,\lambda})$ admits a sequence of weak solutions which is unbounded in $H^1_0(V)$.

Proof. The strategy of the proof is very similar to the previous one. Hence, in the sequel, we omit the details and we use the notations adopted showing Theorem 3.4. Then, from hypothesis

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} < +\infty,$$

by direct computations, it follows that $\gamma := \liminf\limits_{r\to+\infty} \varphi(r) < +\infty$. On the other hand by

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty,$$

there exists a sequence $\{\eta_n\}$ of positive constants such that $\lim\limits_{n\to\infty} \eta_n = +\infty$ and

$$\lim_{n\to\infty} \frac{F(\eta_n)}{\eta_n^2} = +\infty. \tag{3.3}$$

Now, consider the sequence of functions $\{\eta_n v\} \subset X$. Arguing as in Theorem 3.4, we obtain

$$\lim_{n\to\infty} \frac{I_\lambda(\eta_n v)}{\eta_n^2} = -\infty.$$

Hence, the functional I_λ is unbounded from below. Then, part (a) of Theorem 1.1 ensures the existence of a sequence $\{v_n\}$ of distinct critical points of I_λ such that

$$\lim_{n\to\infty} \left(\|v_n\|^2 - \int_V a(x)v_n^2(x)\mathrm{d}\mu\right) = +\infty.$$

In conclusion, by Remark 2.1, it follows that $\lim\limits_{n\to\infty} \|v_n\| = +\infty$. This completes the proof. □

Remark 3.6. If to the hypotheses of Theorem 3.4, respectively in Theorem 3.5, one adds the requirement that $a \in C(V)$, then our results and Remark 2.3 yield the existence of a sequence of pairwise distinct strong solutions of problem $(S^{f,g}_{a,\lambda})$ that either converges to zero or is unbounded in $H^1_0(V)$.

Remark 3.7. We explicitly observe that, exploiting the proof of Theorem 3.4, one can see that our result also holds for sign-changing functions $f : \mathbb{R} \to \mathbb{R}$ just requiring that

$$-\infty < \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2}, \quad \liminf_{\xi\to 0^+} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^2} < +\infty \quad \text{and} \quad \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty,$$

instead of condition (h_0). In this setting, for every

$$\lambda \in \left]0, -\frac{1}{2(2N+3)^2 \left(\int_V g(x)\mathrm{d}\mu\right) \liminf_{\xi\to 0^+} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^2}}\right[,$$

there exists a sequence $\{v_n\}$ of pairwise distinct weak solutions of problem ($S^{f,g}_{a,\lambda}$) such that $\lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|v_n\|_\infty = 0$.

An analogous conclusion can be achieved if the potential F has the same behaviour at infinity instead of at zero obtaining, in this case, the existence of a sequence of weak solutions which is unbounded in $H^1_0(V)$. Indeed, with the notations of Theorem 3.5, if

$$\liminf_{\xi\to +\infty} \frac{\max_{t\in[-\xi,\xi]} F(t)}{\xi^2} < +\infty,$$

direct computations ensure that $\gamma < +\infty$. On the other hand, consider the sequence of functions $\{\eta_n v\} \subset H^1_0(V)$. Bearing in mind that $\liminf_{\xi\to +\infty} \frac{F(\xi)}{\xi^2} > -\infty$, there exist $\varrho > 0$ and a real constant k such that

$$F(\xi) \geq k\xi^2, \ \text{ for every } \xi \in]\varrho, +\infty[. \tag{3.4}$$

Moreover, one has

$$I_\lambda(\eta_n v) = \frac{\eta_n^2}{2}\|v\|^2 - \frac{\eta_n^2}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + \lambda F(\eta_n)\int_D g(x)\mathrm{d}\mu + \lambda\int_{V\setminus D} g(x)F(\eta_n v(x))\mathrm{d}\mu,$$

for every $n \in \mathbb{N}$.

Hence

$$\begin{aligned} I_\lambda(\eta_n v) = {} & \frac{\eta_n^2}{2}\|v\|^2 - \frac{\eta_n^2}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + \lambda F(\eta_n)\int_D g(x)\mathrm{d}\mu + \lambda\int_{G_\varrho\cap(V\setminus D)} g(x)F(\eta_n v(x))\mathrm{d}\mu \\ & +\lambda\int_{G^\varrho\cap(V\setminus D)} g(x)F(\eta_n v(x))\mathrm{d}\mu, \end{aligned}$$

where

$$G_\varrho := \{x \in V : 0 \leq \eta_n v(x) \leq \varrho\} \quad \text{and} \quad G^\varrho := \{x \in V : \eta_n v(x) > \varrho\}.$$

Now, by using the mean value theorem, it easy to see that

$$\int_{G_\varrho\cap(V\setminus D)} g(x)F(\eta_n v(x))\mathrm{d}\mu \leq \|g\|_\infty \max_{t\in[0,\varrho]} |f(t)|\varrho. \tag{3.5}$$

Then, inequalities (3.4) and (3.5) yield

$$\begin{aligned} I_\lambda(\eta_n v) \leq {} & \frac{\eta_n^2}{2}\|v\|^2 - \frac{\eta_n^2}{2}\int_V a(x)v^2(x)\mathrm{d}\mu + \lambda F(\eta_n)\int_D g(x)\mathrm{d}\mu + \lambda\|g\|_\infty \max_{t\in[0,\varrho]} |f(t)|\varrho \\ & +\lambda k\eta_n^2\int_{V\setminus D} g(x)v^2(x)\mathrm{d}\mu, \end{aligned}$$

for every $n \in \mathbb{N}$.

Thus, condition (h_2) and (2.1) imply that $\int_D g(x)\mathrm{d}\mu < 0$. Finally, from (3.3) and the above inequality, we have that

$$\lim_{n\to\infty} \frac{I_\lambda(\eta_n v)}{\eta_n^2} = -\infty.$$

Thus I_λ is unbounded from below. The proof is attained from part (a) of our theoretical result. In conclusion, for every

$$\lambda \in \left]0, -\frac{1}{2(2N+3)^2\left(\int_V g(x)\mathrm{d}\mu\right)\liminf\limits_{\xi\to+\infty}\frac{\max_{t\in[-\xi,\xi]}F(t)}{\xi^2}}\right[,$$

there exists a sequence $\{v_n\}$ of weak solutions of problem ($S^{f,g}_{a,\lambda}$) which is unbounded in $H^1_0(V)$.

4. Examples

Following Omari and Zanolin in [25], we give a concrete example of positive continuous function $q : \mathbb{R} \to \mathbb{R}$ such that its potential Q satisfies our growth conditions at zero. Precisely, let $\{s_n\}$, $\{t_n\}$ and $\{\delta_n\}$ be real sequences defined by

$$s_n := 2^{-\frac{n!}{2}}, \qquad t_n := 2^{-2n!}, \qquad \delta_n := 2^{-(n!)^2}.$$

Observe that, definitively, one has

$$s_{n+1} < t_n < s_n - \delta_n.$$

Example 4.1. With the above notations, let $q : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $q(t) = 0$ in $]-\infty, 0]$, $q(t) > 0$ for every $t > 0$ and

$$q(t) := 2^{-n!}, \quad \forall\, t \in [s_{n+1}, s_n - \delta_n],$$

for n sufficiently large. Define $Q : \mathbb{R} \to \mathbb{R}$ given by $Q(\xi) = \int_0^\xi q(t)\mathrm{d}t$, for every $\xi \in \mathbb{R}$. Then

$$\frac{Q(s_n)}{s_n^2} \le \frac{q(s_{n+1})s_n + q(s_n)\delta_n}{s_n^2} \to 0,$$

and

$$\frac{Q(t_n)}{t_n^2} \ge \frac{q(s_{n+1})(t_n - s_{n+1})}{t_n^2} \to +\infty.$$

Hence

$$\liminf_{\xi\to 0^+} \frac{Q(\xi)}{\xi^2} = 0, \qquad \text{and} \qquad \limsup_{\xi\to 0^+} \frac{Q(\xi)}{\xi^2} = +\infty.$$

Thus, for every $a, g : V \to \mathbb{R}$ satisfying the conditions (h_1) and (h_2) the following Dirichlet problem

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda g(x)q(u(x)) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases} \tag{$S^{q,g}_{a,\lambda}$}$$

for every $\lambda > 0$, admits a sequence $\{v_n\}$ of pairwise distinct weak solutions such that $\lim\limits_{n\to\infty} \|v_n\| = \lim\limits_{n\to\infty} \|v_n\|_\infty = 0$.

Remark 4.2. In [7] the authors proved the existence of infinitely many solutions for problem $(S^{f,g}_{a,1})$. Their result is achieved under suitable assumptions on the behaviour of the potential F at zero and requiring that there exist two real positive sequences $\{a_n\}$ and $\{b_n\}$ with $b_{n+1} < a_n < b_n$, $\lim_{n\to\infty} b_n = 0$ and such that $f(t) \leq 0$ for every $t \in [a_n, b_n]$. Clearly, the mentioned result cannot be applied to the cases in which the nonlinearity is strictly positive in $]0, +\infty[$.

Finally, the next example deals with a sign-changing function $h : \mathbb{R} \to \mathbb{R}$. The result is achieved taking into account a more general condition of (h_∞) in Theorem 3.5 as indicated in Remark 3.7.

Example 4.3. Set

$$a_1 := 2, \qquad a_{n+1} := (a_n)^{\frac{3}{2}},$$

for every $n \in \mathbb{N}$ and $S := \bigcup_{n\geq 0}]a_{n+1} - 1, a_{n+1} + 1[$. Define the continuous function $h : \mathbb{R} \to \mathbb{R}$ as follows

$$h(t) := \begin{cases} (a_{n+1})^3 \mathrm{e}^{\frac{1}{(t-(a_{n+1}-1))(t-(a_{n+1}+1))}+1} \frac{2(a_{n+1}-t)}{(t-(a_{n+1}-1))^2(t-(a_{n+1}+1))^2} & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$H(\xi) = \int_0^\xi h(t)\mathrm{d}t = \begin{cases} (a_{n+1})^3 \mathrm{e}^{\frac{1}{(\xi-(a_{n+1}-1))(\xi-(a_{n+1}+1))}+1} & \text{if } \xi \in S \\ 0 & \text{otherwise,} \end{cases}$$

and $H(a_{n+1}) = (a_{n+1})^3$ for every $n \in \mathbb{N}$. Hence, one has

$$\limsup_{\xi\to+\infty} \frac{H(\xi)}{\xi^2} = +\infty.$$

On the other hand, by choosing $x_n = a_{n+1} - 1$ for every $n \in \mathbb{N}$, one has $\max_{\xi\in[-x_n,x_n]} H(\xi) = (a_n)^3$ for every $n \in \mathbb{N}$. Moreover

$$\lim_{n\to\infty} \frac{\max_{\xi\in[-x_n,x_n]} H(\xi)}{{x_n}^2} = 1$$

and, by a direct computation, it follows that

$$\liminf_{\xi\to+\infty} \frac{\max_{t\in[-\xi,\xi]} H(t)}{\xi^2} = 1.$$

Hence,

$$0 \leq \liminf_{\xi\to+\infty} \frac{H(\xi)}{\xi^2} \leq \liminf_{\xi\to+\infty} \frac{\max_{t\in[-\xi,\xi]} H(t)}{\xi^2} = 1, \quad \limsup_{\xi\to+\infty} \frac{H(\xi)}{\xi^2} = +\infty.$$

Taking into account Remark 3.7, for each parameter λ belonging to

$$\left]0, \frac{1}{2(2N+3)^2}\right[,$$

the following problem

$$\begin{cases} \Delta u(x) + \lambda h(u(x)) = u(x) & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases} \qquad (S^h_\lambda)$$

possesses an unbounded sequence of strong solutions.

Acknowledgements. V. Rădulescu acknowledges the support through Grant CNCS PNII–ID–PCE–47/2011.

References

[1] S. Alexander, Some properties of the spectrum of the Sierpiński gasket in a magnetic field. *Phys. Rev. B* **29** (1984) 5504–5508.

[2] G. Bonanno and R. Livrea, Multiplicity theorems for the Dirichlet problem involving the p-Laplacian. *Nonlinear Anal.* **54** (2003) 1–7.

[3] G. Bonanno and G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities. *Bound. Value Probl.* **2009** (2009) 1–20.

[4] G. Bonanno and G. Molica Bisci, Infinitely many solutions for a Dirichlet problem involving the p-Laplacian. *Proc. R. Soc. Edinb. Sect. A* **140** (2010) 737–752.

[5] G. Bonanno, G. Molica Bisci and D. O'Regan, Infinitely many weak solutions for a class of quasilinear elliptic systems. *Math. Comput. Model.* **52** (2010) 152–160.

[6] B.E. Breckner, D. Repovš and Cs. Varga, On the existence of three solutions for the Dirichlet problem on the Sierpiński gasket. *Nonlinear Anal.* **73** (2010) 2980–2990.

[7] B.E. Breckner, V. Rădulescu and Cs. Varga, Infinitely many solutions for the Dirichlet problem on the Sierpiński gasket. *Analysis and Applications* **9** (2011) 235–248.

[8] G. D'Aguì and G. Molica Bisci, Infinitely many solutions for perturbed hemivariational inequalities. *Bound. Value Probl.* **2011** (2011) 1–19.

[9] G. D'Aguì and G. Molica Bisci, *Existence results for an Elliptic Dirichlet problem*, *Le Matematiche* **LXVI**, Fasc. I (2011) 133–141.

[10] K.J. Falconer, Semilinear PDEs on self-similar fractals. *Commun. Math. Phys.* **206** (1999) 235–245.

[11] K.J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edition. John Wiley & Sons (2003).

[12] K.J. Falconer and J. Hu, Nonlinear elliptical equations on the Sierpiński gasket. *J. Math. Anal. Appl.* **240** (1999) 552–573.

[13] M. Fukushima and T. Shima, On a spectral analysis for the Sierpiński gasket. *Potential Anal.* **1** (1992) 1–35.

[14] S. Goldstein, Random walks and diffusions on fractals, in *Percolation Theory and Ergodic Theory of Infinite Particle Systems*, *IMA Math. Appl.* **8**, edited by H. Kesten. Springer, New York (1987) 121–129.

[15] J. Hu, Multiple solutions for a class of nonlinear elliptic equations on the Sierpiński gasket. *Sci. China Ser. A* **47** (2004) 772–786.

[16] C. Hua and H. Zhenya, Semilinear elliptic equations on fractal sets. *Acta Mathematica Scientica* **29** B (2009) 232–242.

[17] A. Kristály and G. Moroşanu, New competition phenomena in Dirichlet problems. *J. Math. Pures Appl.* **94** (2010) 555–570.

[18] A. Kristály, V. Rădulescu and C. Varga, *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*. Cambridge University Press, Cambridge (2010).

[19] J. Kigami, *Analysis on Fractals*. Cambridge University Press, Cambridge (2001).

[20] S. Kusuoka, *A diffusion process on a fractal*. Probabilistic Methods in Mathematical Physics, Katata/Kyoto (1985) 251–274; Academic Press, Boston, MA (1987).

[21] B.B. Mandelbrot, How long is the coast of Britain? Statistical self-similarity and fractional dimension, *Science* **156** (1967) 636–638.

[22] B.B. Mandelbrot, *Fractals: Form, Chance and Dimension*. W.H. Freeman & Co., San Francisco (1977).

[23] B.B. Mandelbrot, *The Fractal Geometry of Nature*. W.H. Freeman & Co., San Francisco (1982).

[24] P. Omari and F. Zanolin, Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential. *Comm. Partial Differential Equations* **21** (1996) 721–733.

[25] P. Omari and F. Zanolin, An elliptic problem with arbitrarily small positive solutions, *Proceedings of the Conference on Nonlinear Differential Equations* (Coral Gables, FL, 1999). *Electron. J. Differ. Equ. Conf.* **5**. Southwest Texas State Univ., San Marcos, TX (2000) 301–308.

[26] R. Rammal, A spectrum of harmonic excitations on fractals. *J. Phys. Lett.* **45** (1984) 191–206.

[27] R. Rammal and G. Toulouse, Random walks on fractal structures and percolation clusters. *J. Phys. Lett.* **44** (1983) L13–L22.

[28] B. Ricceri, A general variational principle and some of its applications. *J. Comput. Appl. Math.* **113** (2000) 401–410.

[29] W. Sierpiński, Sur une courbe dont tout point est un point de ramification. *Comptes Rendus (Paris)* **160** (1915) 302–305.

[30] R.S. Strichartz, Analysis on fractals. *Notices Amer. Math. Soc.* **46** (1999) 1199–1208.

[31] R.S. Strichartz, Solvability for differential equations on fractals. *J. Anal. Math.* **96** (2005) 247–267.

[32] R.S. Strichartz, *Differential Equations on Fractals, A Tutorial*. Princeton University Press, Princeton, NJ (2006).