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Variational Approach to Differential Invariants of Rank 2 Vector Distributions

Igor Zelenko*

Abstract

In the present paper we construct differential invariants for generic rank 2 vector distributions on n-dimensional manifold. In the case $n = 5$ (the first case containing functional parameters) E. Cartan found in 1910 the covariant fourth-order tensor invariant for such distributions, using his "reduction-prolongation" procedure (see [12]). After Cartan's work the following questions remained open: first the geometric reason for existence of Cartan's tensor was not clear; secondly it was not clear how to generalize this tensor to other classes of distributions; finally there were no explicit formulas for computation of Cartan's tensor. Our paper is the first in the series of papers, where we develop an alternative approach, which gives the answers to the questions mentioned above. It is based on the investigation of dynamics of the field of so-called abnormal extremals (singular curves) of rank 2 distribution and on the general theory of unparametrized curves in the Lagrange Grassmannian, developed in [4],[5]. In this way we construct the *fundamental form* and the *projective Ricci curvature* of rank 2 vector distributions for arbitrary $n \geq 5$. For $n = 5$ we give an explicit method for computation of these invariants and demonstrate it on several examples. In the next paper [19] we show that in the case $n = 5$ our fundamental form coincides with Cartan's tensor.

Key words: nonholonomic distributions, Pfaffian systems, differential invariants, abnormal extremals, Jacobi curves, Lagrange Grassmannian.

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1 Introduction

Rank l vector distribution D on the n-dimensional manifold M or (l, n)-distribution (where $l < n$) is by definition a l-dimensional subbundle of the tangent bundle TM. In other words, for each point $q \in M$ a l-dimensional subspace $D(q)$ of the tangent space T_qM is chosen and $D(q)$ depends smoothly on q. Two vector distributions D_1 and D_2 are called equivalent, if there exists a diffeomorphism $F : M \mapsto M$ such that $F_*D_1(q) = D_2(F(q))$ for any $q \in M$. Two germs of vector distributions D_1 and D_2 at the point $q_0 \in M$ are called equivalent, if there exist neighborhoods U and $\tilde{U}$ of q_0 and a diffeomorphism $F : U \mapsto \tilde{U}$ such that

$$F_*D_1(q) = D_2(F(q)),\ \forall q \in U;$$
$$F(q_0) = q_0.$$

Our goal is to construct invariants of distributions w.r.t. this equivalence relation in order to see if two germs of distributions are equivalent or not. Distributions are associated with

*S.I.S.S.A., Via Beirut 2-4, 34014, Trieste, Italy; email: zelenko@sissa.it

Pfaffian systems and with control systems linear in the control. So invariants of distributions are also invariants of the corresponding Pfaffian systems and state-feedback invariants of the corresponding control systems.

An obvious (but very rough in the most cases) invariant of distribution D at q is so-called *small growth vector* at q: it is the tuple

$$\big(\dim D(q), \dim D^2(q), \dim D^3(q), \dots\big),$$

where D^j is the j-th power of the distribution D, i.e., $D^j = D^{j-1} + [D, D^{j-1}]$.

Let us roughly estimate the "number of parameters" in the considered equivalence problem. The set of l-dimensional subspaces in $\mathbb{R}^n$ forms $l(n-l)$-dimensional manifold. Therefore, if the coordinates on M are fixed then the rank l distribution can be defined by $l(n-l)$ functions of n variables. The group of the coordinate changes on M is parameterized by n functions of n variables. So, by a coordinate change one can "normalize", in general, only n functions among those $l(n-l)$ functions, defining the distribution. Thus, one may expect that the set of classes of equivalent germs of rank l distributions can be parameterized by $l(n-l)-n$ arbitrary germs of functions of n variables (see [15] or survey [16], subsection 2.7, for precise statements). According to this, in the case $l = 2$ the functional invariant should appear starting from $n = 5$. It is well known that in the low dimensions $n = 3$ or 4 all generic germs of rank 2 distributions are equivalent. (Darboux's theorem in the case $n = 3$, small growth vector $(2, 3)$ and Engel's theorem in the case $n = 4$, small growth vector $(2, 3, 4)$, see, for example, [10], [21]).

The case of $(2, 5)$-distributions with small growth vector $(2, 3, 5)$ was treated by E. Cartan in [12] by ingenious use of his "reduction-prolongation" procedure. In particular, he constructed invariant homogeneous polynomial of degree 4 on each plane $D(q)$ (we will call it Cartan's tensor). If the roots of the projectivization of this polynomial are different, then taking their cross-ratio one obtains functional invariant of the distribution D.

After the mentioned work of E. Cartan the following questions remained open: first the geometric reason for existence of Cartan's tensor was not clear (the tensor was obtained by very sophisticated algebraic manipulations) and the true analogs of this tensor in Riemannian geometry were not found; secondly it was not clear how to generalize this tensor to other classes of distributions ; finally there were no explicit formulas for computation of Cartan's tensor (in order to compute this tensor for concrete distribution, one had to repeat Cartan's "reduction-prolongation" procedure for this distribution from the very beginning, which is rather difficult task).

In the present paper we develop alternative, more geometric method for construction of functional invariants of generic germs of $(2, n)$-distribution for arbitrary $n \geq 5$, which allows to give the answers to the questions mentioned above. It is based on new, variational approach for constructing of differential invariants for geometric structures (feedback invariants for control systems) proposed recently by A. Agrachev (see [1], [2], and also Introduction to [4]). For rank 2 distributions this approach can be described as follows (the presentation here is closed to those given in Introduction to [4]):

First for $(2, n)$-distributions ($n \geq 4$) with small growth vector of the type $(2, 3, 4\,\text{or}\,5, \dots)$ one can distinguish special (unparametrized) curves in the cotangent bundle T^*M of M. For this let $\pi : T^*M \mapsto M$ be the canonical projection. Let σ be standard symplectic structure on T^*M, namely, for any $\lambda \in T^*M$, $\lambda = (p, q)$, $q \in M$,$p \in T_q^*M$ let

$$\sigma(\lambda)(\cdot) = -d\,p\big(\pi_*(\cdot)\big) \tag{1.1}$$

(here we prefer the sign "-" in the righthand side , although usually one defines the standard symplectic form on T^*M without this sign). Denote by $(D^l)^\perp \subset T^*M$ the annihilator of the lth power D^l, namely

$$(D^l)^\perp = \{(q,p) \in T^*M : \ p \cdot v = 0 \ \forall v \in D^l(q)\}. \tag{1.2}$$

The set $D^\perp$ is codimension 2 submanifold of T^*M. Consider the restriction $\sigma|_{D^\perp}$ of the form σ on $D^\perp$. It is not difficult to check that (see, for example [17], section 2): the set of points, where the form $\sigma|_{D^\perp}$ is degenerated, coincides with $(D^2)^\perp$; the set $(D^2)^\perp \backslash (D^3)^\perp$ is codimension 1 submanifold of $D^\perp$; for each $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ the kernel of $\sigma|_{D^\perp}(\lambda)$ is two-dimensional subspace of $T_\lambda D^\perp$, which is transversal to $T_\lambda (D^2)^\perp$. Hence $\forall \lambda \in (D^2)^\perp \backslash (D^3)^\perp$ we have

$$\ker \sigma|_{(D^2)^\perp}(\lambda) = \ker \sigma|_{D^\perp}(\lambda) \cap T_\lambda (D^2)^\perp$$

It implies that these kernels form line distribution in $(D^2)^\perp \backslash (D^3)^\perp$ and define a *characteristic 1-foliation* Ab_D of $(D^2)^\perp \backslash (D^3)^\perp$. Leaves of this foliation will be called *characteristic curves* of distribution D. Actually these characteristic curves are so-called regular abnormal extremals of D (see [14],[17] and also [7], where such curves are called abnormal extremals, satisfying the strong generalized Legendre-Clebsh condition).

Remark 1.1 The term abnormal extremal comes from Pontryagin Maximum Principle in Optimal Control. Defining on the set of all curves tangent to D some functional (for example, length w.r.t. some Riemannian metric on M), one can consider the corresponding optimal control problem with fixed endpoints. Abnormal extremals are the extremals of this problem with vanishing Lagrange multiplier near the functional, so they do not depend on the functional but on the distribution D itself. Projections of abnormal extremals to the base manifold M will be called *abnormal trajectories*. Conversely, an abnormal extremal projected to the given abnormal trajectory will be called its *lift*. If some lift of the abnormal trajectory is regular abnormal extremal, then this abnormal trajectory will be called regular. Again from Pontryagin Maximum Principle it follows that the set of all lifts of given abnormal trajectory can be provided with the structure of linear space. The dimension of this space is called *corank of the abnormal trajectory*.$\Box$

Further, for a given segment γ of characteristic curve one can construct a special (unparametrized) curve of Lagrangian subspaces, called Jacobi curve, in the appropriate symplectic space. For this for any $\lambda \in (D^2)^\perp$ denote by $\mathcal{J}(\lambda)$ the following subspace of $T_\lambda (D^2)^\perp$

$$\mathcal{J}(\lambda) = \big(T_\lambda(T^*_{\pi(\lambda)}M) + \ker \sigma|_{D^\perp}(\lambda)\big) \cap T_\lambda (D^2)^\perp. \tag{1.3}$$

Here $T_\lambda(T^*_{\pi(\lambda)}M)$ is tangent to the fiber $T^*_{\pi(\lambda)}M$ at the point λ (or vertical subspace of $T_\lambda(T^*M)$). Actually $\mathcal{J}$ is rank $(n-1)$ distribution on the manifold $(D^2)^\perp \backslash (D^3)^\perp$.

Let O_γ be a neighborhood of γ in $(D^2)^\perp$ such that

$$N = O_\gamma / (Ab_D|_{O_\gamma}) \tag{1.4}$$

is a well-defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with a symplectic structure $\bar\sigma$ induced by $\sigma|_{(D^2)^\perp}$. Let $\phi : O_\gamma \to N$ be the canonical projection on the factor. It is easy to check that $\phi_*\big(\mathcal{J}(\lambda)\big)$ is a Lagrangian subspace of the symplectic space $T_\gamma N$, $\forall \lambda \in \gamma$. Let $L(T_\gamma N)$ be the Lagrangian Grassmannian of the symplectic space $T_\gamma N$, i.e.,

$$L(T_\gamma N) = \{\Lambda \subset T_\gamma N : \Lambda^\angle = \Lambda\},$$

where $\Lambda^\angle$ is the skew-symmetric complement of the subspace Λ,

$$\Lambda^\angle = \{v \in T_\gamma N : \bar\sigma(v, \Lambda) = 0\}.$$

Jacobi curve of the characteristic curve (regular abnormal extremal) γ is the mapping

$$\lambda \mapsto J_\gamma(\lambda) \stackrel{def}{=} \phi_*\Big(\mathcal{J}(\lambda)\Big), \quad \lambda \in \gamma, \tag{1.5}$$

from γ to $L(T_\gamma N)$.

Remark 1.2 In [1] and [2] Jacobi curves of extremals were constructed in purely variational way using the notion of Lagrangian derivative ($\mathcal{L}$-derivative) of the endpoint map associated with geometric structure (control system). The reason to call these curves Jacobi curves is that they can be considered as generalization of spaces of "Jacobi fields" along Riemannian geodesics: in terms of these curves one can describe some optimality properties of corresponding extremals. Namely, if the Jacobi curve of the abnormal extremal is simple curve in Lagrange Grassmannian, then the corresponding abnormal trajectory is W^1_∞-isolated (rigid) curve in the space of all curves tangent to distribution D with fixed endpoints (the curve in Lagrange Grassmannian is called simple if one can choose Lagrangian subspace transversal to each Lagrange subspace belonging to the image of the curve). This result can be found in [17]. In different but equivalent form it is contained already in [7]. Moreover, if some Riemannian metric is given on M, then under the same conditions on the Jacobi curve the corresponding abnormal trajectory is the shortest among all curves tangent to distribution D, connecting its endpoints and sufficiently closed to this abnormal trajectory in W^1_1-topology (see [8]) and even in C-topology (see [9]).$\Box$

Jacobi curves are invariants of the distribution D. They are unparametrized curves in the Lagrange Grassmannians. In [4] for any curve of so-called constant *weight* in Lagrange Grassmannian we construct *the canonical projective structure* and the following two invariants w.r.t. the action of the linear Symplectic Group and reparametrization : a special degree 4 differential, *fundamental form*, and a special function, *projective Ricci curvature*. The next steps are to interpret the condition for Jacobi curve of regular abnormal extremal of distribution to be of constant weight in terms of distribution, to pass from the mentioned invariants defined on single Jacobi curve of each regular abnormal extremal of distribution to the corresponding invariants of distribution itself, and to investigate these invariants. These steps are the essence of the present work.

The paper is organized as follows. In section 2 first we find under what assumption on germ of $(2, n)$-distribution at q_0 one can apply the general theory of unparametrized curves in the Lagrange Grassmannians to the Jacobi curve of its regular abnormal extremals. In few words this assumption can be described as follows: there is at least one germ of regular abnormal trajectory of distribution passing through q_0 and having corank 1 (see Remark 1.1 for definitions). It is easy to see that the set of germs of $(2, n)$-distributions satisfying the last assumption is generic. In particular, in the case $n = 5$ and $n = 6$ the germs with the maximal possible small growth vector (namely, $(2, 3, 5)$ and $(2, 3, 5, 6)$ respectively) satisfy this assumption.

Further, for generic germ of $(2, n)$-distribution at q_0 we construct a fundamental form. By fundamental form at the point $q \in M$ we mean a special degree 4 homogeneous rational function defined, up to multiplication on positive constant, on the linear space

$$(D^2)^\perp(q) = (D^2)^\perp \cap T^*_q M. \tag{1.6}$$

For germs of distributions at q_0 satisfying our assumption the set of points, where the fundamental form is defined, is open and dense in some neighborhood of q_0. Later we show that for $(2, 5)$-distribution with small growth vector $(2, 3, 5)$ the fundamental form at any point is polynomial, while for $n > 5$ for generic $(2, n)$-distributions the fundamental form is a rational function, which is not a polynomial. Also, in the case of $(2,5)$-distribution with the small growth vector $(2,3,5)$ the fundamental form can be realized as degree 4 polynomial on the plane $D(q)$ for all $q \in M$ (we call it *tangential fundamental form*), i.e., it is an object of the same nature as Cartan's tensor. Further we describe the projective Ricci curvature of distribution, which is a function, defined on the subset of $(D^2)^\perp \backslash (D^3)^\perp$, where the fundamental form does not vanish. Note that the notion of projective Ricci curvature is new even for $n = 5$. Using this notion, we construct, in addition to fundamental form, a special degree 10 homogeneous rational function, defined, up to multiplication on positive constant, on $(D^2)^\perp(q)$ for any $q \in M$ (for $n = 5$ this function is again polynomial).

In section 3 we restrict ourselves to the case of $(2,5)$-distribution with small growth vector $(2,3,5)$. Using the notion of canonical moving frame of rank 1 curve in Lagrange Grassmannian introduced in [4] (section 6) and the structural equation for this frame derived in [5] (section 2) we obtain explicit formulas for fundamental form and projective curvature. It allows us to prove that in the considered case the fundamental form is a polynomial on each $(D^2)^\perp(q)$ (defined up to multiplication on positive constant). We apply the obtained formulas for several examples. In particular, we calculate our invariants for distribution generated by rolling of two spheres of radiuses r and $\hat r$ ($r \leq \hat r$) without slipping and twisting. We show that the fundamental form of such distribution is equal to zero iff $\frac{\hat r}{r} = 3$ and that the distributions with different ratios $\frac{\hat r}{r}$ are not equivalent. Also we give some sufficient conditions for rigidity of abnormal trajectories in terms of canonical projective structure and fundamental form on it.

Finally, in section 4 we demonstrate that for $n > 5$ generically the fundamental form is not polynomial on the fibers $(D^2)^\perp(q)$. It follows from the fact that fundamental form of the curve in Lagrange Grassmannian always has singularities at the points of jump of the weight.

In the next paper [19] we prove that for $n = 5$ our tangential fundamental form coincides (up to constant factor -35) with Cartan's tensor. For this we obtain another formula for fundamental form in terms of structural functions of any frame naturally adapted to distribution and express our fundamental form in terms of structural functions of special adapted frame, distinguished by Cartan during the reduction.

In the forthcoming paper [20] we construct the canonical frame of $(2,5)$- distribution on the subset of $(D^2)^\perp \backslash (D^3)^\perp$, where the fundamental form is not zero. It gives the way to check whether two different distribution are equivalent. Also we investigate distributions with constant projective Ricci curvature and big group of symmetries, giving models and proving uniqueness results (we announce some of these results at the end of the section 3, see Theorems 4).

Finally note that the approach of the present paper after some modifications can be applied to construction of invariant of other classes of distributions. In particular, the case of corank 2 distributions (i.e., when $n - l = 2$) will be treated in the nearest future.

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2 Fundamental form and projective Ricci curvature of rank 2 distribution

2.1 Preliminary Let W be $2m$-dimensional linear space and $G_m(W)$ be the set of all m-dimensional subspaces of W (i.e., the Grassmannian of half-dimensional subspaces). Below we give definitions of weight and rank of the curve in $G_m(W)$ and describe briefly the construction of fundamental form and projective Ricci curvature for the curve of constant weight in $G_m(W)$

(for the details see [4]), which are invariants w.r.t. the action of General Linear Group $GL(W)$. Since any curve of Lagrange subspaces w.r.t. some symplectic form on W is obviously the curve in $G_m(W)$, all constructions below are valid for the curves in Lagrange Grassmannian.

For given $\Lambda \in G_m(W)$ denote by $\Lambda^{\pitchfork}$ the set of all m-dimensional subspaces of W transversal to Λ,

$$\Lambda^{\pitchfork} = \{\Gamma \in G_m(W) : W = \Gamma \oplus \Lambda\} = \{\Gamma \in G_m(W) : \Gamma \cap \Lambda = 0\}$$

Fix some $\Delta \in \Lambda^{\pitchfork}$. Then for any subspace $\Gamma \in \Lambda^{\pitchfork}$ there exist unique linear mapping from Δ to Λ with graph Γ. We denote this mapping by $\langle \Delta, \Gamma, \Lambda \rangle$. So,

$$\Gamma = \{v + \langle \Delta, \Gamma, \Lambda \rangle v | v \in \Delta\}.$$

Choosing the bases in Δ and Λ one can assign to any $\Gamma \in \Lambda^{\pitchfork}$ the matrix of the mapping $\langle \Delta, \Gamma, \Lambda \rangle$ w.r.t these bases. In this way we define the coordinates on the set $\Lambda^{\pitchfork}$.

Remark 2.1 Assume that W is endowed with some symplectic form $\bar{\sigma}$ and Δ, Λ are Lagrange subspaces w.r.t. $\bar{\sigma}$. Then the map $v \mapsto \bar{\sigma}(v, \cdot)$, $v \in \Delta$, defines the canonical isomorphism between Δ and Λ^*. It is easy to see that Γ is Lagrange subspace iff the mapping $\langle \Delta, \Gamma, \Lambda \rangle$, considered as the mapping from Λ^* to Λ, is self-adjoint. $\Box$

Let $\Lambda(t)$ be a smooth curve in $G_m(W)$ defined on some interval $I \subset \mathbb{R}$. We are looking for invariants of $\Lambda(t)$ by the action of $GL(W)$. We say that the curve $\Lambda(\cdot)$ is *ample at* τ if $\exists s > 0$ such that for any representative $\Lambda_\tau^s(\cdot)$ of the s-jet of $\Lambda(\cdot)$ at τ, $\exists t$ such that $\Lambda_\tau^s(t) \cap \Lambda(\tau) = 0$. The curve $\Lambda(\cdot)$ is called *ample* if it is ample at any point. This is an intrinsic definition of an ample curve. In coordinates this definition takes the following form: if in some coordinates the curve $\Lambda(\cdot)$ is a curve of matrices $t \mapsto S_t$, then $\Lambda(\cdot)$ is ample at τ if and only if the function $t \mapsto det(S_t - S_\tau)$ has a root of *finite order* at τ.

Definition 1 *The order of zero of the function $t \mapsto det(S_t - S_\tau)$ at τ, where S_t is a coordinate representation of the curve $\Lambda(\cdot)$, is called a weight of the curve $\Lambda(\cdot)$ at τ.*

It is clear that the weight of $\Lambda(\tau)$ is integral valued upper semicontinuous functions of τ. Therefore it is locally constant on the open dense subset of the interval of definition I.

Now suppose that the curve has the constant weight k on some subinterval $I_1 \subset I$. It implies that for all two parameters t_0,t_1 in I_1 sufficiently such that $t_0 \neq t_1$, one has

$$\Lambda(t_0) \cap \Lambda(t_1) = 0.$$

Hence for such t_0, t_1 the following linear mappings

$$\frac{d}{ds}\langle \Lambda(t_0), \Lambda(s), \Lambda(t_1) \rangle \Big|_{s=t_0} : \Lambda(t_0) \mapsto \Lambda(t_1), \tag{2.1}$$

$$\frac{d}{ds}\langle \Lambda(t_1), \Lambda(s), \Lambda(t_0) \rangle \Big|_{s=t_1} : \Lambda(t_1) \mapsto \Lambda(t_0) \tag{2.2}$$

are well defined. Taking composition of mapping (2.2) with mapping (2.1) we obtain the operator from the subspace $\Lambda(t_0)$ to itself, which is actually the infinitesimal cross-ratio of two points $\Lambda(t_i)$, $i = 0, 1$, together with two tangent vectors $\dot{\Lambda}(t_i)$, $i = 0, 1$, at these points in $G_m(W)$ (see [4] for the details).

Theorem 1 *(see [4], Lemma 4.2) If the curve has the constant weight k on some subinterval $I_1 \subset I$, then the following asymptotic holds*

$$\operatorname{tr}\Big(\tfrac{d}{ds}\langle\Lambda(t_1),\Lambda(s),\Lambda(t_0)\rangle\Big|_{s=t_1}\circ\tfrac{d}{ds}\langle\Lambda(t_0),\Lambda(s),\Lambda(t_1)\rangle\Big|_{s=t_0}\Big) = \\ -\tfrac{k}{(t_0-t_1)^2} - g_\Lambda(t_0,t_1), \tag{2.3}$$

where $g_{\Lambda}(t,\tau)$ is a smooth function in the neighborhood of diagonal $\{(t,t)|t \in I_1\}$.

The function $g_{\Lambda}(t,\tau)$ is "generating" function for invariants of the parametrized curve by the action of $GL(2m)$. The first coming invariant of the parametrized curve, *the generalized Ricci curvature*, is just $g_{\Lambda}(t,t)$, the value of g_{Λ} at the diagonal.

In order to obtain invariants for unparametrized curves (i.e., for one-dimensional submanifold of $G_m(W)$) we use a simple reparametrization rule for a function g_{Λ}. Indeed, let $\varphi:\mathbb{R}\mapsto\mathbb{R}$ be a smooth monotonic function. It follows directly from (2.3) that

$$g_{\Lambda\circ\varphi}(t_0,t_1) = \dot\varphi(t_0)\dot\varphi(t_1)g_{\Lambda}(\varphi(t_0),\varphi(t_1)) + k\left(\frac{\dot\varphi(t_0)\dot\varphi(t_1)}{(\varphi(t_0)-\varphi(t_1))^2} - \frac{1}{(t_0-t_1)^2}\right). \tag{2.4}$$

In particular, putting $t_0 = t_1 = t$, one obtains the following reparametrization rule for the generalized Ricci curvature

$$g_{\Lambda\circ\varphi}(t,t) = \dot\varphi(t)^2 g_{\Lambda}(\varphi(t),\varphi(t)) + \frac{k}{3}\mathbb{S}(\varphi), \tag{2.5}$$

where $\mathbb{S}(\varphi)$ is a Schwarzian derivative of φ,

$$\mathbb{S}(\varphi) = \frac{1}{2}\frac{\varphi^{(3)}}{\varphi'} - \frac{3}{4}\Big(\frac{\varphi''}{\varphi'}\Big)^2 = \frac{d}{dt}\Big(\frac{\varphi''}{2\,\varphi'}\Big) - \Big(\frac{\varphi''}{2\,\varphi'}\Big)^2. \tag{2.6}$$

From (2.5) it follows that the class of local parametrizations, in which the generalized Ricci curvature is identically equal to zero, defines a *canonical projective structure* on the curve (i.e., any two parametrizations from this class are transformed one to another by Möbius transformation). This parametrizations are called projective. From (2.4) it follows that if t and τ are two projective parametrizations on the curve $\Lambda(\cdot)$, $\tau = \varphi(t) = \frac{at+b}{ct+d}$, and g_Λ is generating function of $\Lambda(\cdot)$ w.r.t. parameter τ then

$$\frac{\partial^2 g_{\Lambda\circ\varphi}}{\partial t_1^2}(t_0,t_1)\Big|_{t_0=t_1=t} = \frac{\partial^2 g_{\Lambda}}{\partial \tau_1^2}(\tau_0,\tau_1)\Big|_{\tau_0=\tau_1=\varphi(t)}(\varphi'(t))^4, \tag{2.7}$$

which implies that the following degree four differential

$$\mathcal{A} = \frac{1}{2}\frac{\partial^2 g_{\Lambda}}{\partial \tau_1^2}(\tau_0,\tau_1)\Big|_{\tau_0=\tau_1=\tau}(d\tau)^4$$

on the curve $\Lambda(\cdot)$ does not depend on the choice of the projective parametrization (by degree four differential on the curve we mean the following: for any point of the curve a degree 4 homogeneous function is given on the tangent line to the curve at this point). This degree four differential is called a *fundamental form* of the curve.

If t is an arbitrary (not necessarily projective) parametrization on the curve $\Lambda(\cdot)$, then the fundamental form in this parametrization has to be of the form $A(t)(dt)^4$, where $A(t)$ is a smooth function, called the *density* of the fundamental form w.r.t. parametrization t. The density $A(t)$ can be expressed by generating function g_Λ in the following way (see [4], Lemma 5.1):

$$A(t) = \frac{1}{2}\frac{\partial^2}{\partial t_0^2} g_\Lambda(t_0, t_1)\Big|_{t_0=t_1=t} - \frac{3}{5k} g_\Lambda(t,t)^2 - \frac{3}{20}\frac{d^2}{dt^2} g_\Lambda(t,t). \tag{2.8}$$

Remark 2.2 If $t \mapsto S_t$ is a coordinate representation of the curve $\Lambda(\cdot)$ in some parametrization t, then

$$g_\Lambda(t_0, t_1) = \frac{\partial^2}{\partial t_0 \partial t_1} \ln\left(\frac{\det(S_{t_0} - S_{t_1})}{(t_0 - t_1)^k}\right) \tag{2.9}$$

(the proof of the last formula follows from [4], see relations (4.9),(4.11), and Lemma 4.2 there). From this and (2.8) it follows that for any t_0 the density $A(t_0)$ w.r.t. parametrization t of the fundamental form of $\Lambda(\cdot)$ at t_0 is equal to a rational expression w.r.t. some entries of the matrices $\dot S(t_0), \ddot S(t_0), \ldots, S^{(j)}(t_0)$ for some $j > 0$. $\Box$

If the fundamental form $\mathcal{A}$ of the curve $\Lambda(\cdot)$ is not equal to zero at any point of $\Lambda(\cdot)$, then the canonical length element $\sqrt[4]{|\mathcal{A}|}$ is defined on $\Lambda(\cdot)$. The length with respect to this length element gives canonical, up to the shift, parametrization of the unparametrized curve. The Ricci curvature $\rho_n(\tau)$ w.r.t. this parametrization is a functional invariant of the unparametrized curve, which is called its *projective Ricci curvature*.

Remark 2.3 By construction, the density of the fundamental form in the canonical parameter is identically equal to 1. $\Box$

Remark 2.4 Note that in the case $m = 1$ the fundamental form is always identically equal to zero (see [4], Lemma 5.2 there): in this case the only invariant of unparametrized curve in the corresponding Lagrange Grassmannian is the canonical projective structure on it. $\Box$

Another important characteristic of the curve $\Lambda(\cdot)$ in $G_m(W)$ is the rank of its velocities (or simply rank). In order to define it let

$$\mathcal{D}^{(1)}\Lambda(\tau) \stackrel{def}{=} \Lambda(\tau) + \left\{ v \in W : \begin{array}{c} \exists \text{ a curve } l(\cdot) \text{ in } W \text{ such that} \\ l(t) \in \Lambda(t)\ \forall t \text{ and } v = \frac{d}{dt} l(t)|_{t=\tau} \end{array} \right\} \tag{2.10}$$

Then the rank $r(\tau)$ of $\Lambda(\cdot)$ at τ is defined as follows

$$r(\tau) \stackrel{def}{=} \dim \mathcal{D}^{(1)}\Lambda(\tau) - \dim \Lambda(\tau). \tag{2.11}$$

Remark 2.5 Note that the tangent space $T_\Lambda G_m(W)$ to any subspace $\Lambda \in G_m(W)$ can be identified with the space $\mathrm{Hom}\,(\Lambda, W/\Lambda)$ of linear mappings from Λ to W/Λ. Namely, take a curve $\Lambda(t) \in G_k(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all sufficiently small t and $l(0) = l$. Denote by $\mathrm{pr} : W \mapsto W/\Lambda$ the canonical projection on the factor. It is easy to see that the mapping $l \mapsto \mathrm{pr}\, l'(0)$ from Λ to W/Λ is linear mapping depending only on $\frac{d}{dt}\Lambda(0)$. In this way we identify $\frac{d}{dt}\Lambda(0) \in T_\Lambda G_k(W)$ with some element of $\mathrm{Hom}\,(\Lambda, W/\Lambda)$ (a simple counting of dimension shows that these correspondence between $T_\Lambda G_k(W)$ and $\mathrm{Hom}\,(\Lambda, W/\Lambda)$ is a bijection). By construction, the rank of the curve $\Lambda(t)$ at the point τ in $G_m(W)$ is actually equal to the rank of the linear mapping corresponding to its velocity $\frac{d}{dt}\Lambda(t)$ at τ.$\Box$

Remark 2.6 If W is endowed with some symplectic form $\bar{\sigma}$ and $L(W)$ is the corresponding Lagrange Grassmannian, then the tangent space $T_\Lambda L(W)$ to any $\Lambda \in L(W)$ can be identified with the space of quadratic forms $Q(\Lambda)$ on the linear space Λ. Namely, let $\Lambda(t)$ and $l(t)$ be as in the previous remark (where $G_m(W)$ is substituted by $L(W)$). It is easy to see that the quadratic form $l \mapsto \bar{\sigma}\big(l'(0), l\big)$ depends only on $\frac{d}{dt}\Lambda(0)$. In this way we identify $\frac{d}{dt}\Lambda(0) \in T_\Lambda G_k(W)$ with some element of $Q(\Lambda)$ (a simple counting of dimension shows that these correspondence between $T_\Lambda L(W)$ and $Q(\Lambda)$ is a bijection). $\Box$

Using the identification in the previous remark one can define the notion of monotone curves in the Lagrange Grassmannian: the curve $\Lambda(t)$ in $L(W)$ is called *nondecreasing (nonincreasing)* if its velocities $\frac{d}{dt}\Lambda(t)$ at any point are nonnegative (nonpositive) definite quadratic forms.

As we will see in the next subsection the rank of Jacobi curves of characteristic curves of rank 2 distribution is identically equal to 1. There is a simple criterion for rank 1 curves in Lagrange Grassmannian to be of constant weight. To formulate it let us introduce inductively the following subspaces $\mathcal{D}^{(i)}\Lambda(\tau)$ in addition to $\mathcal{D}^{(1)}\Lambda(\tau)$:

$$\mathcal{D}^{(i)}\Lambda(\tau) \stackrel{def}{=} \mathcal{D}^{(i-1)}\Lambda(\tau) + \left\{ v \in W : \begin{array}{l} \exists \text{ a curve } l(\cdot) \text{ in } W \text{ such that} \\ l(t) \in \mathcal{D}^{(i-1)}\Lambda(t)\ \forall t \text{ and } v = \frac{d}{dt}l(t)|_{t=\tau} \end{array} \right\} \tag{2.12}$$

(we set $Der^{(0)}\Lambda(t) = \Lambda(t)$)

Proposition 2.1 *The curve $\Lambda(\cdot)$ of constant rank 1 in Lagrange Grassmannian $L(W)$ of symplectic space W, $\dim W = 2m$, has the constant finite weight in a neighborhood of the point τ iff*

$$\dim \mathcal{D}^{(m)}\Lambda(\tau) = 2m \tag{2.13}$$

In this case the weight is equal to m^2.

The proof of the proposition can be easily obtained by application of some formulas and statements of section 6 and 7 of [4] (for example, formulas (6.15), (6.16), (6.18), (6.19), Proposition 4, and Corollary 2 there).

Note also that from the fact that the rank of the curve is equal to 1 it follows easily that

$$\dim \mathcal{D}^{(i)}\Lambda(\tau) - \dim \mathcal{D}^{(i-1)}\Lambda(\tau) \leq 1 \tag{2.14}$$

Therefore the condition (2.13) is equivalent to

$$\dim \mathcal{D}^{(i)}\Lambda(\tau) = m + i,\ \forall i = 1, \dots m \tag{2.15}$$

2.2 Properties of Jacobi curves of regular abnormal extremals of rank 2 distributions In this subsection we find under what assumption on germ of $(2,n)$-distribution ($n \geq 4$) with small growth vector of the type $(2, 3, 4 \text{ or } 5, \dots)$ one can apply the theory of previous subsection. First note that Jacobi curve J_γ of characteristic curve γ of distribution D defined by (1.5) is not ample, because all subspaces $J_\gamma(\lambda)$ have a common line. Indeed, let $\delta_a : T^*M \mapsto T^*M$ be the homothety by $a \neq 0$ in the fibers, namely,

$$\delta_a(p, q) = (ap, q), \quad q \in M,\ p \in T^*M. \tag{2.16}$$

Denote by $\vec{e}(\lambda)$ the following vector field called Euler field

$$\vec{e}(\lambda) = \frac{\partial}{\partial a}\delta_a(\lambda)\Big|_{a=1} \tag{2.17}$$

Remark 2.7 Obviously, if γ is characteristic curve of D, then also $\delta_a(\gamma)$ is. $\Box$

It implies that the vectors $\phi_*(\vec{e}(\lambda))$ coincide for all $\lambda \in \gamma$, so the line

$$E_\gamma \stackrel{def}{=} \{\mathbb{R}\phi_*(\vec{e}(\lambda))\} \tag{2.18}$$

is common for all subspaces $J_\gamma(\lambda)$, $\lambda \in \gamma$ (here, as in Introduction, $\phi : O_\gamma \to N$ is the canonical projection on the factor $N = O_\gamma/(Ab_D|_{O_\gamma})$, where O_γ is sufficiently small tubular neighborhood of γ in $(D^2)^\perp$).

Therefore it is natural to make an appropriate factorization by this common line E_γ. Namely, by above all subspaces $J_\gamma(\lambda)$ belong to skew-symmetric complement $E_\gamma^\angle$ of E_γ in $T_\gamma N$. Denote by $p : T_\gamma N \mapsto T_\gamma N/E_\gamma$ the canonical projection on the factor-space. The mapping

$$\lambda \mapsto \widetilde{J}_\gamma(\lambda) \stackrel{def}{=} p(J_\gamma(\lambda)),\ \lambda \in \gamma \tag{2.19}$$

from γ to $L(E_\gamma^\angle/E_\gamma)$ is called *reduced Jacobi curve* of characteristic curve γ. Note that

$$\dim \widetilde{J}_\gamma(\lambda) = n - 3 \tag{2.20}$$

Now the question is at which points $\lambda \in \gamma$ the germ of reduced Jacobi curve has constant weight? The answer on this question can be easily done in terms of rank $(n-1)$ distribution $\mathcal{J}$ defined by (1.3) on $(D^2)^\perp \backslash (D^3)^\perp$.

First note that for any $\lambda \in \gamma$ one can make the following identification

$$T_\gamma N \sim T_\lambda (D^2)^\perp / \ker \sigma|_{(D^2)^\perp}(\lambda). \tag{2.21}$$

Take on O_γ any vector field H tangent to characteristic 1-foliation Ab_D and without stationary points, i.e., $H(\lambda) \in \ker\sigma|_{(D^2)^\perp}(\lambda)$, $H(\lambda) \neq 0$ for all $\lambda \in O_\gamma$. Then it is not hard to see that under identification (2.21) one has

$$\widetilde{J}_\gamma(e^{tH}\lambda) = (e^{-tH})_*\Big(\mathcal{J}(e^{tH}\lambda)\Big)/\mathrm{span}(\ker \sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)) \tag{2.22}$$

where e^{tH} is the flow generated by the vector field H. Recall that for any vector field ℓ in $(D^2)^\perp$ one has

$$\frac{d}{dt}\Big((e^{-tH})_*\ell\Big) = (e^{-tH})_*[H, \ell] \tag{2.23}$$

Set $\mathcal{J}^{(0)} = \mathcal{J}$ and define inductively

$$\mathcal{J}^{(i)}(\lambda) = \mathcal{J}^{(i-1)}(\lambda) + \{[H, V](\lambda) : H \in \ker \sigma|_{(D^2)^\perp}, V \in \mathcal{J}^{(i-1)} \text{ are vector fields}\} \tag{2.24}$$

or shortly $\mathcal{J}^{(i)} = \mathcal{J}^{(i-1)} + [\ker \sigma|_{(D^2)^\perp}, \mathcal{J}^{(i-1)}]$. Then by definition of operation $\mathcal{D}^{(i)}$ (see (2.12)) and formulas (2.22), (2.23) it follows that

$$\mathcal{D}^{(i)}\widetilde{J}_\gamma(e^{tH}\lambda)|_{t=0} = \mathcal{J}^{(i)}(\lambda)/\mathrm{span}(\ker \sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)), \tag{2.25}$$

which in turn implies that

$$\dim \mathcal{D}^{(i)}\widetilde{J}_\gamma(e^{tH}\lambda)|_{t=0} - \dim \mathcal{D}^{(i-1)}\widetilde{J}_\gamma(e^{tH}\lambda)|_{t=0} = \dim \mathcal{J}^{(i)}(\lambda) - \dim \mathcal{J}^{(i-1)}(\lambda). \tag{2.26}$$

Proposition 2.2 *The (reduced) Jacobi curve of characteristic curve of* $(2,n)$*-distribution* $(n \geq 4)$ *with small growth vector of the type* $(2, 3, 4 \,\mathrm{or}\, 5, \dots)$ *is of rank 1 at any point and nondecreasing.*

Proof. First show that the (reduced) Jacobi curve has rank 1 at any point. For this, according to (2.26), it is sufficient to prove that

$$\dim \mathcal{J}^{(1)}(\lambda) - \dim \mathcal{J}(\lambda) = 1 \tag{2.27}$$

Let X_1, X_2 be two vector fields, constituting the basis of distribution D, i.e.,

$$D(q) = \mathrm{span}(X_1(q), X_2(q)) \ \ \forall q \in M. \tag{2.28}$$

Since our study is local, we can always suppose that such basis exists, restricting ourselves, if necessary, on some coordinate neighborhood instead of whole M. Given the basis X_1, X_2 one can construct a special vector field tangent to characteristic 1-foliation Ab_D. For this suppose that

$$\begin{aligned} &X_3 = [X_1, X_2] \quad \mathrm{mod}\, D, \ \ X_4 = [X_1, [X_1, X_2]] = [X_1, X_3] \quad \mathrm{mod}\, D^2, \\ &X_5 = [X_2, [X_1, X_2]] = [X_2, X_3] \quad \mathrm{mod}\, D^2 \end{aligned} \tag{2.29}$$

Let us introduce "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $1 \leq i \leq 5$,

$$u_i(\lambda) = p \cdot X_i(q), \ \lambda = (p, q), \ q \in M, \ p \in T_q^*M \tag{2.30}$$

For given function $G : T^*M \mapsto \mathbb{R}$ denote by $\vec{G}$ the corresponding Hamiltonian vector field defined by the relation $\sigma(\vec{G}, \cdot) = d\,G(\cdot)$. Then it is easy to show (see, for example [17]) that

$$\ker \sigma\Big|_{D^\perp}(\lambda) = \mathrm{span}(\vec{u}_1(\lambda), \vec{u}_2(\lambda)), \quad \forall \lambda \in D^\perp, \tag{2.31}$$

$$\ker \sigma\Big|_{(D^2)^\perp}(\lambda) = \mathbb{R}\Big((u_4\vec{u}_2 - u_5\vec{u}_1)(\lambda)\Big), \quad \forall \lambda \in (D^2)^\perp \backslash (D^3)^\perp \tag{2.32}$$

The last relation implies that the following vector field

$$\vec{h}_{X_1, X_2} = u_4\vec{u}_2 - u_5\vec{u}_1 \tag{2.33}$$

is tangent to the characteristic 1-foliation (this field is actually the restriction on $(D^2)^\perp$ of the Hamiltonian vector field of the function $h_{X_1,X_2} = u_4 u_2 - u_5 u_1$).

Suppose that $\dim D^3(q) = 5$ for any q (the case, when $\dim D^3(q) = 4$ for some q can be treated similarly and it is left to the reader). Let us complete tuple $(X_1, X_2, X_3, X_4, X_5)$ to the local frame $X_1, \dots, X_n$ on M. Similar to (2.30) define "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $5 < i \leq n$.

Denote

$$\partial_\theta = u_4 \partial_{u_5} - u_5 \partial_{u_4}, \tag{2.34}$$

$$\mathcal{X} = u_5\vec{u}_2 + u_4\vec{u}_1 - (u_4^2 + u_5^2)\partial_{u_3}, \tag{2.35}$$

$$F = \vec{u}_3 + u_4\partial_{u_1} + u_5\partial_{u_2} \tag{2.36}$$

On $(D^2)^\perp\backslash(D^3)^\perp$ one has

$$\mathcal{J} = \operatorname{span}(\vec{h}_{X_1,X_2}, \vec{e}, \mathcal{X}, \partial_\theta, \partial_{u_6}, \ldots, \partial_{u_n}) \tag{2.37}$$

By direct computations, one can obtain that

$$[\vec{h}_{X_1,X_2}, \partial_{u_i}] \in \operatorname{span}(\vec{e}, \partial_\theta, \partial_{u_6}, \ldots, \partial_{u_n}),\ 6 \le i \le n \tag{2.38}$$

$$[\vec{h}_{X_1,X_2}, \partial_\theta] \equiv \mathcal{X}\ \Big(\mathrm{mod}\big(\operatorname{span}(\vec{h}_{X_1,X_2}, \vec{e}, \partial_\theta, \partial_{u_6}, \ldots, \partial_{u_n})\big)\Big), \tag{2.39}$$

$$[\vec{h}_{X_1,X_2}, \mathcal{X}] \equiv -(u_4^2 + u_5^2)F\ \ (\mathrm{mod}\ \mathcal{J}). \tag{2.40}$$

From this and definition of $\mathcal{J}^{(1)}$ it follows that

$$\mathcal{J}^{(1)} = \mathbb{R}F \oplus \mathcal{J}, \tag{2.41}$$

which implies (2.27).

Finally, from (2.35), (2.36), and (2.40), it follows easily that

$$\bar{\sigma}([h, \mathcal{X}], \mathcal{X}) = (u_4^2 + u_5^2)^2 > 0,$$

which implies that the (reduced) Jacobi curve is nondecreasing (see Remark 2.6 and the sentence after it). $\Box$

Proposition 2.1, relation (2.14), Proposition 2.2 and relation (2.26) imply immediately the following characterization of the points of $(D^2)^\perp\backslash(D^3)^\perp$ in which the germ of corresponding reduced Jacobi curve has a constant weight:

Proposition 2.3 *For any $\lambda \in (D^2)^\perp\backslash(D^3)^\perp$ the following relation holds*

$$\dim \mathcal{J}^{(i)}(\lambda) - \dim \mathcal{J}^{(i-1)}(\lambda) \le 1, \quad \forall\, i = 1, \ldots n-3 \tag{2.42}$$

The germ of reduced Jacobi curve $\widetilde{J}_\gamma$ at $\lambda \in \gamma$ has constant weight iff

$$\dim \mathcal{J}^{(n-3)}(\lambda) = 2n - 4 \tag{2.43}$$

In this case the weight is equal to $(n-3)^2$.

Remark 2.8 From (2.42) it follows that (2.43) is equivalent to the following relations

$$\dim \mathcal{J}^{(i)}(\lambda) = n - 1 + i, \quad \forall\, i = 1, \ldots n-3 \tag{2.44}$$

Denote by $\mathcal{R}_D$ the set of all $\lambda \in (D^2)^\perp\backslash(D^3)^\perp$ such that the germ of reduced Jacobi curve $\widetilde{J}_\gamma$ at $\lambda \in \gamma$ has constant weight. By the previous proposition ,

$$\mathcal{R}_D = \{\lambda \in (D^2)^\perp\backslash(D^3)^\perp : \dim \mathcal{J}^{(n-3)}(\lambda) = 2n - 4\}. \tag{2.45}$$

Also $\forall q \in M$ let

$$\mathcal{R}_D(q) = \mathcal{R}_D \cap T_q^*M \tag{2.46}$$

and $(D^2)^\perp(q)$ be as in (1.6). The question is whether for generic germ of rank 2-distribution at q the set $\mathcal{R}_D(q)$ is not empty so that we can apply the theory, presented in subsection 2.1.

For this first we will investigate the following question: suppose that the reduced Jacobi curve of the regular abnormal extremal γ has constant weight; what can be said about the corresponding abnormal trajectory $\xi = \pi(\gamma)$? Take some basis (X_1, X_2) in a neighborhood of the curve ξ such that ξ is tangent to the line distribution spanned by X_1 (since our considerations are local we always can do it, restricting ourselves, if necessary, to some subinterval of ξ). For any $q \in \xi$ denote by $\mathcal{T}_\xi^{(i)}(q)$ the following subspace of T_qM as follows:

$$\mathcal{T}_\xi^{(i)}(q) = \operatorname{span}\bigl(X_1(q), X_2(q), \operatorname{ad}X_1(X_2)(q), \dots, (\operatorname{ad}X_1)^i(X_2)(q)\bigr) \tag{2.47}$$

It is easy to see that the subspaces $\mathcal{T}_\xi^{(i)}(q)$ do not depend on the choice of the local basis (X_1, X_2) with the above property, but only on the germ of the distribution D and the curve ξ at q. The property of the curve ξ to be abnormal trajectory can be described in terms of $\mathcal{T}_\xi^{(i)}(q)$:

Proposition 2.4 *If γ is abnormal extremal in $(D^2)^\perp \backslash (D^3)^\perp$ and ξ is the corresponding abnormal trajectory , $\xi = \pi(\gamma)$, then $\forall \lambda \in \gamma$ the following relations hold*

$$\mathcal{T}_\xi^{(i)}\bigl(\pi(\lambda)\bigr) = \pi_* \mathcal{J}^{(i)}(\lambda), \tag{2.48}$$

$$\dim \mathcal{T}_\xi^{(i)}\bigl(\pi(\lambda)\bigr) = \dim \mathcal{J}^{(i)}(\lambda) - (n-3). \tag{2.49}$$

Proof. Let, as before, H be some vector field without stationary points tangent to characteristic 1-foliation Ab_D in a neighborhood of γ. Also, let $\widetilde{\mathcal{X}}$ be some vector field in a neighborhood of γ such that $\pi_*\Bigl(\operatorname{span}\bigl(H(\lambda), \widetilde{\mathcal{X}}(\lambda)\bigr)\Bigr) = D\bigl(\pi(\lambda)\bigr)$. Then from construction of $\mathcal{J}^{(i)}$ and relations (2.38)-(2.40) it follows easily that

$$\mathcal{J}^{(i)}(\lambda) = \operatorname{span}\Bigl(T_\lambda\Bigl((D^2)^\perp\bigl(\pi(\lambda)\bigr)\Bigr), H(\lambda), \widetilde{\mathcal{X}}(\lambda), \operatorname{ad}H(\widetilde{\mathcal{X}})(q), \dots, (\operatorname{ad}H)^i(\widetilde{\mathcal{X}})(q)\Bigr). \tag{2.50}$$

Take some n-dimensional submanifold Σ of $(D^2)^\perp$, passing through γ transversal to the fibers $(D^2)^\perp\bigl(\pi(\lambda)\bigr)$ for any $\lambda \in \gamma$. By construction, π projects some neighborhood $\widetilde{\Sigma}$ of γ in Σ bijectively to some neighborhood V of ξ in M. Taking

$$X_1\bigl(\pi(\lambda)\bigr) = \pi_* H(\lambda),\ X_2\bigl(\pi(\lambda)\bigr) = \pi_*\widetilde{\mathcal{X}}(\lambda), \quad \forall \lambda \in \widetilde{\Sigma}$$

and using equations (2.47), (2.50), one obtains (2.48). Relation (2.49) follows from (2.48) and the fact that the fiber $(D^2)^\perp(q)$ is $(n-3)$-dimensional. This concludes the proof. $\Box$

Corollary 1 *The reduced Jacobi curve of the regular abnormal extremal γ has constant weight iff*

$$\dim \mathcal{T}_\xi^{(n-3)}(q) = n-1, \quad \forall q \in \xi, \tag{2.51}$$

where $\xi = \pi(\gamma)$ is the abnormal trajectory corresponding to γ.

Remark 2.9 Note that a smooth curve ξ in M, satisfying (2.51) together with the following relation

$$\dim \mathcal{T}_\xi^{(n-2)}(q) = n-1, \quad \forall q \in \xi, \tag{2.52}$$

is corank 1 abnormal trajectory (see Remark 1.1 for definition of corank). If in addition to (2.51) and (2.52) the following relation holds

$$\mathcal{T}_\xi^{(n-3)}(q) + D^3(q) = T_qM, \quad \forall q \in \xi, \tag{2.53}$$

then the curve ξ is regular abnormal extremal. In term of a local basis (X_1, X_2) in the such that ξ is tangent to the line distribution spanned by X_1, the condition (2.53) is equivalent to the fact that $\forall q \in \xi$ the following condition holds

$$\operatorname{span}\big(X_1(q), X_2(q), \operatorname{ad}X_1(X_2)(q), \ldots, (\operatorname{ad}X_1)^{n-3}(X_2)(q), [X_2, [X_1, X_2]](q)\big) = T_qM. \tag{2.54}$$

The assertions of this remark can be deduced without difficulties from the fact that abnormal trajectories are critical points of certain endpoint mapping (or time$\times$ input/state mapping) and from the expression for the first differential for this mapping (one can use , for example, [7], section 4). $\square$

Remark 2.10 If the germ of regular abnormal trajectory ξ at some point q_0 has corank 1, then the set of $q \in \xi$, satisfying (2.51), is open and dense set in some neighborhood of q_0 in ξ. $\square$

Now we are ready to prove the following genericity result:

Proposition 2.5 *For generic germ of $(2,n)$-distribution D at $q_0 \in M$ ($n \geq 4$) the set $\mathcal{R}_D(q_0)$, defined in (2.46), is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q_0)$, i.e., $\mathcal{R}_D(q_0)$ is a complement to some proper algebraic variety of $(D^2)^\perp(q_0)$.*

Proof. First note that the set $(D^2)^\perp(q_0)\backslash\mathcal{R}_D(q_0)$ is an algebraic variety in the linear space $(D^2)^\perp(q_0)$. Indeed, choose again a local frame $\{X_i\}_{i=1}^n$ on M such that X_1, X_2 constitute a local basis of D and X_3, X_4, X_5 satisfy (2.29). Then from (2.37), definitions of subspaces $\mathcal{J}^{(i)}(\lambda)$ and vector field $\vec{h}_{X_1,X_2}$ it follows that as a basis of spaces $\mathcal{J}^{(i)}(\lambda)$ one can take some vector fields, which are linear combination of the fields $\vec{u}_k$, ∂_{u_l} with polynomial in u_j coefficients (here $k, l = 1, \ldots, n$, $j = 4, \ldots, n$). Therefore the set

$$(D^2)^\perp(q_0)\backslash\mathcal{R}_D(q_0) = \{\lambda \in (D^2)^\perp(q_0) : \mathcal{J}^{(n-3)}(\lambda) < 2n-4\} \tag{2.55}$$

can be represented as a zero level set of some polynomial in u_j, $j = 4, \ldots, n$.

Further the coefficients of this polynomial are some polynomials in the space of l_n-jets of $(2,n)$- distributions for some natural l_n. We will denote this space by $\mathrm{Jet}_{2,n}(l_n)$. It implies that there exists an open set $\mathcal{U}_n$ in Zariski topology of $\mathrm{Jet}_{2,n}(l_n)$ such that the set $\mathcal{R}_D(q_0)$ is not empty iff the l_n-jet of D at q_0 belongs to $\mathcal{U}_n$. Note that if the set $\mathcal{U}_n$ is not empty, then it is dense in $\mathrm{Jet}_{2,n}(l_n)$. Therefore in order to prove our proposition it is sufficient to give an example of germ of $(2,n)$-distribution such that $\mathcal{R}_D$ is nonempty. As such example one can take distribution D_0 spanned by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + \sum_{i=1}^{n-3} \frac{x_1^i}{i!}\frac{\partial}{\partial x_{i+2}} + x_1 x_2 \frac{\partial}{\partial x_n}, \tag{2.56}$$

where $(x_1, \ldots, x_n)$ are some local coordinates on M, q_0=(0,. . . ,0). Using Remark 2.9 and Corollary 1, it is easy to see that the curve $(x_1, 0, \ldots, 0)$ is regular abnormal trajectory and its lift has the reduced Jacobi curve of constant weight. This implies that $\mathcal{R}_{D_0}(q_0) \neq \emptyset$. $\square$

Below we give an explicit description of the set $\mathcal{R}_D$ for $n = 4$, 5 and 6. In the case $n = 4$, small growth vector $(2,3,4)$, from (2.40) it follows immediately that $\mathcal{R}_D = (D^2)^\perp\backslash(D^3)^\perp$. A similar result holds in the case $n = 5$:

Proposition 2.6 *For* $(2,5)$*-distribution with small growth vector* $(2,3,5)$ *the following relation holds*

$$\mathcal{R}_D = (D^2)^\perp \backslash (D^3)^\perp. \tag{2.57}$$

Proof. Let vector fields $\vec{h}_{X_1,X_2}$ and F be as in (2.33) and (2.36) respectively. Then, using (2.41), one can obtain by direct computations that

$$[\vec{h}_{X_1,X_2}, F] = u_4\vec{u}_5 - u_5\vec{u}_4 \pmod{J^{(1)}} \tag{2.58}$$

(actually this formula holds for all $n \geq 5$). Hence $\dim J^{(2)}(\lambda) = \dim J^{(1)}(\lambda) + 1 = 6$ for all $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$, which implies (2.57).$\Box$

Remark 2.11 Let D be $(2,n)$-distribution $(n \geq 5)$ such that $\dim D^3(q) = 4$ for any q in some neighborhood U. Then from (2.58) it follows easily that $J^{(2)}(\lambda) = J^{(1)}(\lambda)$ for any $q \in \pi(U)$. It implies that $\mathcal{R}_D(q) = \emptyset$ for any such q and the theory of subsection 2.1 cannot be applied for the reduced Jacobi curves.$\Box$

In the case of $(2,6)$-distribution D with growth vector of the type $(2,3,5,\ldots)$ the set $\mathcal{R}_D$ can be described as follows: Take some $\bar{\lambda} = (\bar{p}, q) \in (D^3)^\perp(q)$ and some vector $v \in D(q)$. Let ν be some vector field tangent to D such that $\nu(q) = v$. Also, let (X_1, X_2) be a local basis of distribution. Then it is easy to see that the number $\bar{p}\cdot\Big[\nu, \big[\nu, [X_1, X_2]\big]\Big](q)$ does not depends on the choice of the vector field ν, so one has a quadratic form

$$v \mapsto Q_{\bar{\lambda},X_1,X_2}(v) \stackrel{def}{=} \bar{p}\cdot\Big[\nu, \big[\nu, [X_1, X_2]\big]\Big](q) \tag{2.59}$$

on $D(q)$. Besides, a change of the local basis of distribution causes to multiplication of this quadratic form on a nonzero constant (which is equal to the determinant of the transition matrix between the bases). For $(2,6)$-distribution the linear space $(D^3)^\perp(q)$ is one-dimensional. Therefore the zero level set

$$\mathcal{K}(q) = \{v \in D(q) : Q_{\bar{\lambda},X_1,X_2}(v) = 0\} \tag{2.60}$$

of $Q_{\bar{\lambda},X_1,X_2}$ is the same for all $\bar{\lambda} \in (D^3)^\perp(q) \backslash (0,q)$ and any local basis X_1, X_2 of the distribution.

Proposition 2.7 *For* $(2,6)$*-distribution* D *with the small growth vector of the type* $(2,3,5,\ldots)$ *the following relation holds*

$$\mathcal{R}_D(q) = \{\lambda \in (D^2)^\perp(q) : \pi_*\big(\ker\sigma|_{(D^2)^\perp}(\lambda)\big) \notin \mathcal{K}(q)\}. \tag{2.61}$$

The set $\mathcal{R}_D(q) \neq \emptyset$ *iff the small growth vector of* D *at* q *is equal to* $(2,3,5,6)$.

Proof. As before, complete some basis X_1, X_2 of D to the frame $\{X_i\}_{i=1}^6$ on M such that X_3, X_4, X_5 satisfy (2.29). Let c_{ji}^k be the structural functions of this frame, i.e., the functions, satisfying $[X_i, X_j] = \sum_{k=1}^6 c_{ji}^k X_k$. Then from (2.58) by straightforward calculation it follows that

$$[\vec{h}_{X_1,X_2}, u_4\vec{u}_5 - u_5\vec{u}_4] = [u_4\vec{u}_2 - u_5\vec{u}_1, u_4\vec{u}_5 - u_5\vec{u}_4] = \alpha_6\vec{u}_6 \ \big(\mathrm{mod\ span}(\vec{u}_4, \vec{u}_5, J^{(1)})\big), \tag{2.62}$$

where

$$\alpha_6 = c_{52}^6 u_4^2 - (c_{42}^6 + c_{51}^6)u_4u_5 + c_{41}^6 u_5^2 \tag{2.63}$$

From (2.58) and (2.62) it follows that if $J^{(3)}(\lambda) = J^{(2)}(\lambda)$, then $\alpha_6 = 0$. Conversely, if $\alpha_6 = 0$ then by (2.62) $J^{(3)}(\lambda) \subset \operatorname{span}(\vec{u}_4, \vec{u}_5, J^{(1)})$. But by construction $J^{(3)}(\lambda) \subset \vec{e}(\lambda)^{\angle}$ (where $\vec{e}(\lambda)^{\angle}$ is the skew-symmetric complement of $\vec{e}(\lambda)$ in $T_\lambda T^*M$). This together with (2.58) implies that $J^{(3)}(\lambda) = J^{(2)}(\lambda)$. So, $\lambda \in \mathcal{R}_D(q)$ iff $\alpha_6(\lambda) \neq 0$. To prove (2.61) it remains to note that

$$Q_{\bar{\lambda}, X_1, X_2}(\pi_*\big(\vec{h}_{X_1,X_2}(\lambda)\big) = C\alpha_6(\lambda),$$

where C is a nonzero constant. The last assertion of the proposition follows from the fact that $\alpha_6 \equiv 0$ iff $c_{ji}^6 = 0$,where $i = 1, 2$, $j = 4, 5$, or, equivalently, that $\dim D^4(q) = 5$. $\square$

2.3 Fundamental form of distribution and its properties. For any $\lambda \in \mathcal{R}_D$ take characteristic curve γ, passing through λ. Let $\mathcal{A}_\lambda$ be the fundamental form of the reduced Jacobi curve $\widetilde{J}_\gamma$ of γ at λ. By construction $\mathcal{A}_\lambda$ is degree 4 homogeneous function on the tangent line to γ at λ. In the previous subsection to any (local) basis (X_1, X_2) of distribution D we assigned the vector field $\vec{h}_{X_1,X_2}$ tangent to characteristic 1-foliation Ab_D (see (2.33)). Let

$$A_{X_1,X_2}(\lambda) = \mathcal{A}_\lambda(\vec{h}_{X_1,X_2}(\lambda)) \tag{2.64}$$

In this way to any (local) basis (X_1, X_2) of distribution D we assign the function A_{X_1,X_2} on $\mathcal{R}_D$.

Remark 2.12 If we consider parametrization $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}_{X_1,X_2}}\lambda)$ of the reduced Jacobi curve of γ, then $A_{X_1,X_2}(\lambda)$ is the density of fundamental form of this curve w.r.t. parametrization t at $t = 0$. $\square$

Let $\tilde{X}_1, \tilde{X}_2$ be another basis of the distribution D. Then there exist functions $\{\nu_{ij}\}_{i,j=1}^2$ on M such that

$$\begin{aligned} \tilde{X}_1 &= \nu_{11} X_1 + \nu_{12} X_2 \\ \tilde{X}_2 &= \nu_{21} X_1 + \nu_{22} X_2 \end{aligned}$$

By direct computation one has

$$\vec{h}_{\tilde{X}_1,\tilde{X}_2}(\lambda) = \Delta^2\big(\pi(\lambda)\big)\vec{h}_{X_1,X_2}(\lambda), \tag{2.65}$$

where Δ is equal to determinant of transition matrix from the basis (X_1, X_2) to the basis $(\tilde{X}_1, \tilde{X}_2)$, i.e., $\Delta = \nu_{11}\nu_{22} - \nu_{12}\nu_{21}$. From this and homogeneity of $\mathcal{A}$ it follows that

$$A_{\tilde{X}_1,\tilde{X}_2}(\lambda) = \Delta\big(\pi(\lambda)\big)^8 A_{X_1,X_2}(\lambda) \tag{2.66}$$

Therefore for any $q \in M$ such that $\mathcal{R}_D(q) \neq \emptyset$ the restriction of A_{X_1,X_2} to $\mathcal{R}_D(q)$ is well defined function, up to multiplication on positive constant, or well defined element of "positive projectivization" of the space of the functions on $\mathcal{R}_D(q)$. We will call it *fundamental form of the rank 2 distribution D at the point q*. From now on we will write $\vec{h}$ instead of $\vec{h}_{X_1,X_2}$ and A instead of A_{X_1,X_2} without special mentioning.

Remark 2.13 According to subsection 2.1 (see the sentence after formula (2.6)) any abnormal extremals of $(2, n)$-distribution D lying in $\mathcal{R}_D$ carries the canonical projective structure. It can be shown that in the case $n = 4$, small growth vector $(2, 3, 4)$, our canonical projective structure defined on abnormal extremals (and therefore also on abnormal trajectories) coincides with the projective structure on abnormal trajectories, introduced in [11] (see Proposition 5 there). Note also that by Remark 2.4 and relation (2.20) in the case $n = 4$ the fundamental form is identically equal to zero.

Remark 2.14 Using Remark 2.2 it is easy to see that the fundamental form $A(\lambda)$ is a smooth function for all $\lambda \in \mathcal{R}_D$: one can choose the coordinate representation of the curves $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ smoothly depending on λ and use the fact that the operation of differentiation by t in coordinates corresponds to the operation $\operatorname{ad}\vec{h}$ due to the relation (2.23).

In fact one can say much more about algebraic structure of the fundamental form of distribution.

Proposition 2.8 *For any $q \in M$ such that $\mathcal{R}_D(q) \neq \emptyset$ the fundamental form of the rank 2 distribution D at the point q is degree 4 homogeneous rational function on $(D^2)^\perp(q)$, defined up to multiplication on positive constant.*

Proof. First let us prove that the fundamental form at q is rational function on $(D^2)^\perp(q)$. From Remark 2.2 it follows that in order to do this it is sufficient to show that the parametrized reduced Jacobi curves $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ have coordinate representations $t \mapsto S_\lambda(t)$ such that for any natural l all entries of $S_\lambda^{(l)}(0)$, as functions of λ, are rational functions on the fibers $(D^2)^\perp(q)$. For this choose the following $(2n-3)$ vector fields on $(D^2)^\perp$:

$$\partial_\theta, X, \partial_{u_6}, \ldots, \partial_{u_n}, F, Y_4, \ldots Y_{n-1}, Z, \vec{e}, \vec{h}, \tag{2.67}$$

where

$$Y_k = u_{k+1}\vec{u_k} - u_k\vec{u_{k+1}} + \sum_{i=1}^{3}\Big(u_{k+1}\{u_i, u_k\} - u_k\{u_i, u_{k+1}\}\Big)\partial_{u_i}, \tag{2.68}$$

$$Z = u_4\vec{u_5} + u_5\vec{u_4} + \sum_{i=1}^{3}\Big(u_4\{u_i, u_5\} + u_4\{u_i, u_4\}\Big)\partial_{u_i} \tag{2.69}$$

(here $\{u_i, u_j\}$ are Poisson brackets of the Hamiltonians u_i and u_j, i.e., $\{u_i, u_j\} = du_j(\vec{u_i})$). Let

$$W_\lambda = \Big(\vec{e}(\lambda)^\angle \cap T_\lambda(D^2)^\perp\Big)/\operatorname{span}\big(\ker\sigma|_{(D^2)^\perp}(\lambda), \vec{e}(\lambda)\big) \tag{2.70}$$

(here by $\vec{e}(\lambda)^\angle$ we mean a skew-symmetric complement of $\vec{e}(\lambda)$ in $T_\lambda T^*M$). Then under identification (2.21) the reduced Jacobi curve $\widetilde{J}_\gamma$ lives in Lagrange Grassmannina $L(W_\lambda)$ of symplectic space W_λ. Denote by $\mathcal{P}$ the set of all $\lambda \in (D^2)^\perp$ such that the vector fields (2.67) at λ constitute a basis of $T_\lambda(D^2)^\perp$. Evidently, for any $q \in M$ the set $\mathcal{P} \cap (D^2)^\perp(q)$ is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q)$. For any $\lambda \in (D^2)^\perp$ the first $2(n-3)$ vectors in (2.67) belong to $\vec{e}(\lambda)^\angle$. Therefore, for any $\lambda \in \mathcal{P}$ the images of the first $2(n-3)$ vectors in (2.67) under the canonical projection from $\Big(\vec{e}(\lambda)^\angle \cap T_\lambda(D^2)^\perp\Big)$ to W_λ constitute the basis of the space W_λ. Introduce in W_λ the coordinates w.r.t. this basis and suppose that $t \mapsto S_\lambda(t)$ is the corresponding coordinate representation of the curve $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$, $\widetilde{J}_\gamma(e^{t\vec{h}}\lambda) = \{x, S_\lambda(t)) : x \in \mathbb{R}^{n-3}\}$. Then from (2.22) and (2.23) it follows that for any natural l all entries of the matrix $S_\lambda^{(l)}(0)$ are some rational combinations of some coordinates of the vectors of the type

$$(\operatorname{ad}\vec{h})^j(\partial_\theta)(\lambda),\ \ (\operatorname{ad}\vec{h})^j(X)(\lambda),\ \ (\operatorname{ad}\vec{h})^j(\partial_{u_i})(\lambda),\quad 6 \le i \le n,\quad 1 \le j \le l. \tag{2.71}$$

w.r.t. the basis (2.67). But from the form of the vector fields Y_i and Z it is clear that coordinates of the vectors from (2.71) w.r.t. the basis (2.67) are rational functions on the fibers $(D^2)^\perp(q)$. So, for any q the fundamental form at q is a rational function on the fiber $(D^2)^\perp(q)$.

Now let us show that the fundamental form is homogeneous of degree 4. Indeed, it is clear that

$$\delta_{a\,*}\mathcal{J}(\lambda) = J(\delta_a(\lambda)),$$

where δ_a is the homothety defined by (2.16). This together with Remark 2.7 implies that $\delta_{a\,*}$ induces the symplectic transformation from W_λ to $W_{\delta_a(\lambda)}$, which transforms the curve $\widetilde{J}_\gamma$ to the curve $\widetilde{J}_{\delta_a(\gamma)}$. Therefore the following identity holds

$$\mathcal{A}_{\delta_a(\lambda)}\big(\delta_{a\,*}\vec{h}(\lambda)\big) = \mathcal{A}_\lambda\big(\vec{h}(\lambda)\big). \tag{2.72}$$

On the other hand, one has

$$\vec{h}\big(\delta_a(\lambda)\big) = a\,\delta_{a\,*}\vec{h}(\lambda).$$

Hence

$$A\big(\delta_a\lambda\big) = \mathcal{A}_{\delta_a(\lambda)}\Big(\vec{h}\big(\delta_a(\lambda)\big)\Big) = a^4\mathcal{A}_{\delta_a(\lambda)}\big(\delta_{a\,*}\vec{h}(\lambda)\big) = a^4\,\mathcal{A}_\lambda\big(\vec{h}(\lambda)\big) = a^4\,A(\lambda).$$

So A is homogeneous of degree 4. □

In the case $n = 5$ and small growth vector $(2, 3, 5)$ one can look at the fundamental form of the distribution D from the different point of view. In this case (in contrast to generic $(2, n)$-distributions with $n > 5$) there is only one abnormal trajectory starting at given point $q \in M$ in given direction (tangent to $D(q)$). All lifts of this abnormal trajectory can be obtained one from another by homothety. So they have the same, up to symplectic transformation, Jacobi curve. It means that one can consider Jacobi curve and fundamental form of this curve on abnormal trajectory instead of abnormal extremal. Therefore, to any $q \in M$ one can assign a homogeneous degree 4 rational function $\mathring{A}_q$ on the plane $D(q)$ in the following way:

$$\mathring{A}_q(v) \stackrel{def}{=} \mathcal{A}_\lambda(H) \tag{2.73}$$

for any $v \in D(q)$, where

$$\pi(\lambda) = q, \quad \pi_* H = v, \quad H \in \ker \sigma|_{(D^2)^\perp}(\lambda). \tag{2.74}$$

and the righthand side of (2.73) is the same for any choice of λ and H, satisfying (2.74). $\mathring{A}_q$ will be called *tangential fundamental form* of the distribution D at the point q. We stress that the tangential fundamental form is the well defined function on $D(q)$ and not the class of functions under positive projectivization.

The analysis of the algebraic structure presented in the proof of Proposition 2.8 is rather rough. In the sequel we will show that for $n = 5$ fundamental form is always polynomial on $(D^2)^\perp(q)$ (defined up to multiplication on a positive constant), while for $n > 5$ it is nonpolynomial rational function for generic distribution.

2.4 Projective curvature of rank 2 distribution with nonzero fundamental form. Denote by

$$\aleph_D = \{\lambda \in \mathcal{R}_D : \mathcal{A}_\lambda \neq 0\} \tag{2.75}$$

Suppose that the set $\aleph_D$ is not empty.

Remark 2.15 For $n = 5$ the set $\aleph_D$ is empty iff distribution is locally equivalent to so-called free nilpotent $(2, 5)$-distribution (see Example 1 and Remark 3.6 in section 3). Our conjecture is that $\aleph_D$ is empty iff n is equal to the dimension of the free nilpotent r-step Lie algebra $g_{r,2}$ with two generators for some $r \geq 3$ and D is locally equivalent to the left-invariant distribution on a Lie group with Lie algebra $g_{r,2}$ such that this distribution is spanned by the generators of $g_{r,2}$.□

As was mentioned in subsection 2.1, for any characteristic leaf lying in $\aleph_D$, the corresponding Jacobi curve has canonical parameter. In other words, any such leaf has canonical parameter. Define the vector field $\overrightarrow{h_A}$ on $\aleph_D$ such that its integral curves are the characteristic leaves parameterized by their canonical parameter (i.e., the field $\overrightarrow{h_A}$ is given by velocities of the characteristic leaves parameterized by their canonical parameter). By construction, the vector field $\overrightarrow{h_A}$ is invariant of the distribution D. We call it *the canonical Hamiltonian vector field* of distribution D. Further, for any $\lambda \in \aleph_D$ take the characteristic leaf γ, passing through λ. Denote by $\rho_{_D}(\lambda)$ the projective Ricci curvature of the reduced Jacobi curve $\widetilde{J}_\gamma$ at λ. In other words, $\rho_{_D}(\lambda)$ is equal to the Ricci curvature of the curve $\tau \mapsto \widetilde{J}_\gamma(e^{\tau\overrightarrow{h_A}}\lambda)$ at the point $\tau = 0$. So, to given rank 2 distribution D we assign canonically the function

$$\rho_{_D} : \aleph_D \mapsto \mathbb{R} \tag{2.76}$$

This function is called *a projective Ricci curvature* of distribution D.

Now we give a method for computation of projective curvature $\rho_{_D}$. Take some local basis X_1, X_2 of D. Let again $\vec{h} = \vec{h}_{X_1 X_2}$ and $A = A_{X_1,X_2}$ be as in (2.33), and (2.64) respectively. Also denote by $\rho(\lambda)$ the Ricci curvature of the parameterized curve $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ at the point $t = 0$. Note that in contrast to $\rho_{_D}(\lambda)$, the function $\rho(\lambda)$ certainly depends on the local basis of distribution. Using the reparametrization rule (2.5) for Ricci curvature, one can easily express the projective curvature $\rho_{_D}(\lambda)$ by $\rho(\lambda)$ and $A(\lambda)$. Indeed, let τ be the canonical parameter on γ and t be parameter defined by the field $\vec{h}$. Then by Remark 2.3

$$d\tau = \sqrt[4]{|A(e^{t\vec{h}}\lambda)|}\; dt \tag{2.77}$$

Suppose that $t = \varphi(\tau)$. Then by (2.77)

$$\varphi'(\tau) = \frac{1}{\sqrt[4]{|A(e^{t\vec{h}}\lambda)|}} \tag{2.78}$$

Recall that the Jacobi curves under consideration have the weight equal to $(n-3)^2$. So, by (2.5)

$$\rho_{_D}\left(e^{\tau\vec{h}_A}\lambda\right) = \rho\left(e^{\varphi(\tau)\vec{h}}\lambda\right)\left(\varphi'(\tau)\right)^2 + \frac{(n-3)^2}{3}\mathbb{S}\big(\varphi(\tau)\big), \tag{2.79}$$

where $\mathbb{S}(\varphi)$ is Schwarzian of the function φ, defined by (2.6). One can check that Schwarzian satisfies the following relation

$$\mathbb{S}\big(\varphi(\tau)\big) = -\frac{y''(\tau)}{y(\tau)}, \tag{2.80}$$

where

$$y(\tau) = \frac{1}{\sqrt{\varphi'(\tau)}}.$$

By (2.78),

$$y(\tau) = \sqrt[8]{|A(e^{\tau\overrightarrow{h_A}}\lambda)|}$$

Substituting this in (2.80) and using (2.77) we obtain

$$\mathbb{S}\big(\varphi(\tau)\big) = -\frac{\frac{d^2}{d\tau^2}\left(\sqrt[8]{|A(e^{\tau\overrightarrow{h_A}}\lambda)|}\right)}{\sqrt[8]{|A(e^{\tau\overrightarrow{h_A}}\lambda)|}} = -\frac{1}{\sqrt[8]{|A(e^{t\,\vec{h}}\lambda)|^3}}\frac{d}{dt}\left(\frac{1}{\sqrt[4]{|A(e^{t\vec{h}}\lambda)|}}\frac{d}{dt}\left(\sqrt[8]{|A(e^{t\vec{h}}\lambda)|}\right)\right) =$$

$$= \frac{1}{\sqrt[8]{|(A(e^{t\vec{h}}\lambda)|^3}} \frac{d^2}{dt^2}\left(|A(e^{t\vec{h}}\lambda)|^{-\frac{1}{8}}\right) = \frac{\vec{h}\circ\vec{h}\left(|A(e^{t\vec{h}}\lambda)|^{-\frac{1}{8}}\right)}{\sqrt[8]{|(A(e^{t\vec{h}}\lambda)|^3}} \tag{2.81}$$

Finally, substituting (2.81) with $t = 0$ in (2.79) we get

$$\rho_D = \frac{\rho}{\sqrt{|A|}} + \frac{(n-3)^2}{3} \frac{\vec{h}\circ\vec{h}\big(|A|^{-\frac{1}{8}}\big)}{\sqrt[8]{|A|^3}}. \tag{2.82}$$

The last formula can be rewritten also as follows

$$\rho_D = \frac{\rho A^2 - \frac{(n-3)^2}{24}\vec{h}\circ\vec{h}(A)A + \frac{3(n-3)^2}{64}\big(\vec{h}(A)\big)^2}{|A|^{\frac{5}{2}}}. \tag{2.83}$$

Since ρ_D is well defined function on $\aleph_D$ and A is degree 4 homogeneous rational function on $(D^2)^\perp(q)$, defined up to multiplication on a positive constant, the numerator

$$\mathcal{C} \stackrel{def}{=} \rho A^2 - \frac{(n-3)^2}{24}\vec{h}\circ\vec{h}(A)A + \frac{3(n-3)^2}{64}\big(\vec{h}(A)\big)^2 \tag{2.84}$$

of (2.83) is degree 10 homogeneous function on $(D^2)^\perp(q)$, defined up to multiplication on a positive constant. This function will be called *a second fundamental form* of distribution D. The second fundamental form is rational function on $(D^2)^\perp(q)$, because A is rational and also ρ is rational, which follows from the same arguments as in Proposition 2.8. In the case $n = 5$ the second fundamental form is polynomial, which will follow from Theorem 3 below.

3 Calculation of invariants of (2,5)-distributions

In the present section we give explicit formulas for computation of the fundamental form and projective Ricci curvature in the case of rank 2 distribution on 5-dimensional manifold (as before we assume that the small growth vector is $(2, 3, 5)$). We demonstrate these formulas on several examples, showing simultaneously the efficiency of our invariants in proving that the rank 2 distributions are not equivalent.

3.1 Preliminaries. In order to obtain these formulas we need more facts from the theory of curves in Grassmannian $G_m(W)$ of half-dimensional subspaces (here $\dim W = 2m$) and in Lagrange Grassmannian $L(W)$ w.r.t. to some symplectic form on W, developed in [4], [5]). Below we present all necessary facts from the mentioned papers together with several new useful arguments.

Fix some $\Lambda \in G_m(W)$. As before , let $\Lambda^\pitchfork$ be the set of all m-dimensional subspaces of W transversal to Λ. Note that any $\Delta \in \Lambda^\pitchfork$ can be canonically identified with W/Λ. Keeping in mind this identification and taking another subspace $\Gamma \in \Lambda^\pitchfork$ one can define the operation of subtraction $\Gamma - \Delta$ as follows

$$\Gamma - \Delta \stackrel{def}{=} \langle \Delta, \Gamma, \Lambda\rangle \in \mathrm{Hom}\,(W/\Lambda, \Lambda).$$

It is clear that the set $\Lambda^\pitchfork$ provided with this operation can be considered as the affine space over the linear space $\mathrm{Hom}\,(W/\Lambda, \Lambda)$.

Consider now some ample curve $\Lambda(\cdot)$ in $G_m(W)$. Fix some parameter τ. By assumptions $\Lambda(t) \in \Lambda(\tau)^\pitchfork$ for all t from a punctured neighborhood of τ. We obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^\pitchfork$

in the affine space $\Lambda(\tau)^{\pitchfork}$ with the pole at τ. We denote by $\Lambda_\tau(t)$ the identical embedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. First note that the velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is well defined element of $\mathrm{Hom}(W/\Lambda, \Lambda)$. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda_\tau(t)$ a vector function with values in $\mathrm{Hom}\,(W/\Lambda, \Lambda)$ and with the pole at $t = \tau$. Obviously, only free term in the expansion of this function to the Laurent series at τ depends on the choice of the "origin" we did to identify the affine space with the linear one. More precisely, the addition of a vector to the "origin" results in the addition of the same vector to the free term in the Laurent expansion. In other words, for the Laurent expansion of a curve in an affine space, the free term of the expansion is an element of this affine space. Denote this element by $\Lambda^0(\tau)$. The curve $\tau \mapsto \Lambda^0(\tau)$ is called the *derivative curve* of $\Lambda(\cdot)$.

If we restrict ourselves to the Lagrange Grassmannian $L(W)$, i.e. if all subspaces under consideration are Lagrangian w.r.t. some symplectic form $\bar\sigma$ on W, then from Remark 2.1 it follows easily that the set $\Lambda_L^{\pitchfork}$ of all Lagrange subspaces transversal to Λ can be considered as the affine space over the linear space of all self-adjoint mappings from Λ^* to Λ, the velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is well defined self-adjoint mappings from Λ^* to Λ, and the derivative curve $\Lambda^0(\cdot)$ consist of Lagrange subspaces. Besides if the curve $\Lambda(\cdot)$ is nondecreasing rank 1 curve in $L(W)$, then $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is a nonpositve definite rank 1 self-adjoint map from Λ^* to Λ and for $t \neq \tau$ there exists a unique, up to the sign, vector $w(t,\tau) \in \Lambda(\tau)$ such that for any $v \in \Lambda(\tau)^*$

$$\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v\rangle = -\langle v, w(t,\tau)\rangle^2. \tag{3.1}$$

The properties of vector function $t \mapsto w(t,\tau)$ for a rank 1 curve of constant weight in $L(W)$ can be summarized as follows (see [4], section 7, Proposition 4 and Corollary 2):

Proposition 3.1 *If $\Lambda(\cdot)$ is a rank 1 curve of constant weight in L(W), then for any $\tau \in I$ the function $t \mapsto w(t,\tau)$ has a pole of order m at $t = \tau$. Moreover, if we write down the expansion of $t \mapsto w(t,\tau)$ in Laurent series at $t = \tau$,*

$$w(t,\tau) = \sum_{i=1}^{m} e_i(\tau)(t-\tau)^{i-1-l} + O(1),$$

then the vector coefficients $e_1(\tau), \dots, e_m(\tau)$ constitute a basis of the subspace $\Lambda(t)$.

The basis of the vectors $e_1(\tau), \dots, e_m(\tau)$ from the previous proposition is called a *canonical basis* of $\Lambda(\tau)$. Further for given τ take the derivative subspace $\Lambda^0(\tau)$ and let $f_1(\tau), \dots, f_m(\tau)$ be a basis of $\Lambda^0(\tau)$ dual to the canonical basis of $\Lambda(\tau)$, i.e. $\bar\sigma(f_i(\tau), e_j(\tau)) = \delta_{i,j}$. The basis

$$(e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau))$$

of whole symplectic space W is called *the canonical moving frame* of the curve $\Lambda(\cdot)$. Calculation of structural equation for the canonical moving frame is another way to obtain symplectic invariants of the curve $\Lambda(\cdot)$.

For the reduced Jacobi curves of abnormal extremals of $(2,5)$-distribution m is equal to 2. So we restrict ourselves to this case. For $m = 2$ the structural equation for the canonical moving frame has the following form (for the proof see [5] Section 2, Proposition 7):

$$\begin{pmatrix} e_1'(\tau) \\ e_2'(\tau) \\ f_1'(\tau) \\ f_2'(\tau) \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ \frac{1}{4}\rho(\tau) & 0 & 0 & 4 \\ -\left(\frac{35}{36}A(\tau) - \frac{1}{8}\rho(\tau)^2 + \frac{1}{16}\rho''(\tau)\right) & -\frac{7}{16}\rho'(\tau) & 0 & -\frac{1}{4}\rho(\tau) \\ -\frac{7}{16}\rho'(\tau) & -\frac{9}{4}\rho(\tau) & -3 & 0 \end{pmatrix} \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \\ f_1(\tau) \\ f_2(\tau) \end{pmatrix}, \tag{3.2}$$

where $\rho(\tau)$ and $A(\tau)$ are the Ricci curvature and the density of fundamental form of the parametrized curve $\Lambda(\tau)$ respectively.

One can express $e_2(\tau)$ by $e_1'(\tau)$ using the first equation of (3.2), then $f_2(\tau)$ by $e_1(\tau)$ and $e_1''(t)$ using the second equation of (3.2), then $f_1(\tau)$ by $e_1(\tau)$, $e_1'(\tau)$ and $e_1^{(3)}$ using the forth equation of (3.2). Finally substituting all this to the third equation of (3.2) one obtains the following useful Proposition

Proposition 3.2 *Suppose that $\Lambda(t)$ is rank 1 curve of the constant weight in $L(W)$ and $e_1(t)$ is the first vector in the canonical basis of $\Lambda(t)$. Then $e_1(t)$ satisfies the following relation:*

$$e_1^{(4)} = \left(35\,A - \frac{81}{16}\rho^2 - \frac{9}{4}\rho''\right) e_1 - \frac{15}{2}\rho' e_1' - \frac{15}{2}\rho e_1'' \tag{3.3}$$

The previous proposition says that in order to find ρ and A (which actually constitute a complete system of symplectic invariants of the parametrized curve $\Lambda(\cdot)$) it is sufficient to know the first vector $e_1(\tau)$ in the canonical basis of $\Lambda(\cdot)$. The following proposition gives a simple way to find the vector $e_1(\tau)$.

Proposition 3.3 *Let $\Lambda(\tau)$ be a rank 1 nondecreasing curve of constant weight in the Lagrange Grassmannian $L(W)$, where $\dim W = 4$. Then the first vector $e_1(\tau)$ of the canonical basis of $\Lambda(\tau)$ can be uniquely (up to the sign) determined by the following two conditions*

$$\mathbb{R}e_1(\tau) = \mathcal{D}^{(1)}\Lambda(\tau)^{\angle}, \tag{3.4}$$

$$\bar{\sigma}(e_1''(\tau), e_1'(\tau)) = 36, \tag{3.5}$$

where the subspace $\mathcal{D}^{(1)}\Lambda(\tau)$ is as in (2.10) and $\mathcal{D}^{(1)}\Lambda(\tau)^{\angle}$ is its skew-symmetric complement.

Proof. The relation (3.5) follows directly from the first two equations of (3.2). To prove (3.4) note that from (3.2) it is clear that

$$\mathcal{D}^{(1)}\Lambda(\tau) = \mathrm{span}\,\big(e_1(\tau), e_2(\tau), f_2(\tau)\big). \tag{3.6}$$

But from definition of canonical moving frame it follows that

$$\Big(\mathrm{span}\,\big(e_1(\tau), e_2(\tau), f_2(\tau)\big)\Big)^{\angle} = \mathbb{R}e_1(\tau),$$

which together with (3.6) implies (3.4). Finally, the vector $e_1(t)$ is determined by (3.4) and (3.5) uniquely, up to the sign: the first relation gives the direction of $e_1(t)$ and the second "normalizes" this direction. $\square$

3.2 Application to $(2,5)$-distributions. Choose some local basis (X_1, X_2) of $(2,5)$-distribution and complete it by the fields X_3 ,X_4, and X_5, satisfying (2.29), to the local frame on M. Such frame $(X_1, X_2, X_3, X_4, X_5)$ will be called *adapted* to the distribution D. If instead of (2.29) one has

$$X_3 = [X_1, X_2], \quad X_4 = [X_1, [X_1, X_2]] = [X_1, X_3], \quad X_5 = [X_2, [X_2, X_1]] = [X_3, X_2], \tag{3.7}$$

the frame $(X_1, X_2, X_3, X_4, X_5)$ will be called *strongly adapted* to D.

We are going to show how to calculate our invariants starting from some adapted frame to distribution. Let again $\vec{h} = \vec{h}_{X_1,X_2}$ as in (2.33). For any $\lambda \in (D^2)^{\perp} \backslash (D^3)^{\perp}$ consider the characteristic curve γ of D passing through λ. Under identification (2.21) the reduced Jacobi curve $\widetilde{J}_\gamma$ lives in Lagrange Grassmannian $L(W_\lambda)$ of symplectic space W_λ, defined by (2.70). Let $\epsilon_1(\lambda)$ be the first vector in the canonical basis of the curve $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$ at the point $t=0$. Note that it is more convenient to work directly with vector fields of $(D^2)^\perp$, keeping in mind that the symplectic space W_λ belongs to the factor space $T_\lambda\big((D^2)^\perp\big)/\mathrm{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$. So, in the sequel by $\epsilon_1(\lambda)$ we will mean both the element of W_λ and some representative of this element in $T_\lambda\big((D^2)^\perp\big)$, depending smoothly on λ. In the last case all equalities, containing $\epsilon_1(\lambda)$, will be assumed modulo $\mathrm{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$. Now we are ready to prove the following

Proposition 3.4 *The vector $\epsilon_1(\lambda)$ can be chosen in the form*

$$\epsilon_1(\lambda) = 6\Big(\gamma_4(\lambda)\partial_{u_4} + \gamma_5(\lambda)\partial_{u_5}\Big), \tag{3.8}$$

where

$$\gamma_4(\lambda)u_5 - \gamma_5(\lambda)u_4 \equiv 1. \tag{3.9}$$

Proof. First note that by (2.41) one has

$$\mathrm{span}(\partial_{u_4}, \partial_{u_5}) \subset (\mathcal{J}^1)^\angle. \tag{3.10}$$

Hence by (3.4)

$$\epsilon_1 = 6\Big((\gamma_4\partial_{u_4} + \gamma_5\partial_{u_5}\Big) \Big(\mathrm{mod}\,\mathrm{span}(\vec{h}, \vec{e})\Big), \tag{3.11}$$

where $\gamma_4 u_5 - \gamma_5 u_4 \neq 0$. Further, denote by $e_1(t)$ the first vector in the canonical basis of the curve $t \mapsto \widetilde{J}_\gamma(e^{t\vec{h}}\lambda)$. Then

$$e_1(t) = (e^{-t\vec{h}})_*\epsilon\big(e^{t\vec{h}}(\lambda)\big). \tag{3.12}$$

Hence by (2.23)

$$\bar{\sigma}(e_1'(t), e_1''(t))\Big|_{t=0} = \sigma\Big(\big[\vec{h}, [\vec{h}, \epsilon_1]\big](\lambda), [\vec{h}, \epsilon_1](\lambda)\Big). \tag{3.13}$$

By direct computation one can show that

$$\begin{aligned} [\vec{h}, \epsilon_1] &= 6\Big(\gamma_5\vec{u}_1 - \gamma_4\vec{u}_2 + (\gamma_4 u_4 - \gamma_5 u_4)\partial_{u_3}\Big) \quad \Big(\mathrm{mod}\,\mathrm{span}(\vec{h}, \vec{e}, \epsilon_1)\Big), \\ \big[\vec{h}[\vec{h}, \epsilon_1]\big] &= 6(\gamma_4 u_4 - \gamma_5 u_4)(\vec{u}_3 + u_4\partial_{u_1} + u_5\partial_{u_2}) \quad \Big(\mathrm{mod}\,\mathrm{span}(\vec{h}, \vec{e}, \epsilon_1, [\vec{h}, \epsilon_1])\Big). \end{aligned} \tag{3.14}$$

From (3.14) it is easy to show that the righthand side of (3.13) is equal to $36(\gamma_4 u_5 - \gamma_5 u_4)^2$, which together with (3.5) implies (3.9). $\Box$

As a direct consequence of the previous Proposition, Proposition 3.3, and relations (2.23), (3.12), (2.83) we obtain

Theorem 2 *Let $\epsilon_1(\lambda)$ be as in (3.8) and (3.9). Then there exist functions A_0 ,A_1 on $(D^2)^\perp$ such that*

$$(\mathrm{ad}\vec{h})^4(\epsilon_1) = A_0\epsilon_1 + \vec{h}(A_1)\mathrm{ad}\vec{h}(\epsilon_1) + A_1(\mathrm{ad}\vec{h})^2(\epsilon_1)\ \mathrm{mod}\Big(\mathrm{span}\big(\vec{h}, \vec{e}\big)\Big) \tag{3.15}$$

The fundamental form $A(\lambda)$ and the projective Ricci curvature $\rho_D(\lambda)$ of the distribution D satisfy:

$$35A = A_0 + \frac{9}{100}A_1^2 - \frac{3}{10}(\vec{h})^2(A_1), \tag{3.16}$$

$$\rho_D = \frac{-\frac{2}{15}A_1 A^2 - \frac{1}{6}\vec{h}\circ\vec{h}(A)A + \frac{3}{16}\big(\vec{h}(A)\big)^2}{|A|^{\frac{5}{2}}}. \tag{3.17}$$

Remark 3.1 It is clear that in the previous theorem we can take ϵ_1 satisfying (3.8) and the following relation

$$\gamma_4(\lambda)u_5 - \gamma_5(\lambda)u_4 \equiv const \tag{3.18}$$

along any characteristic curve of D (instead of (3.9)). In particular one can take as ϵ_1 one of the following vector fields:

$$\frac{1}{u_5}\partial_{u_4},\ \frac{1}{u_4}\partial_{u_5},\ \frac{(u_5\partial_{u_4} - u_4\partial_{u_5})}{u_4^2 + u_5^2},\ \frac{(u_5\partial_{u_4} + u_4\partial_{u_5})}{u_5^2 - u_4^2} \tag{3.19}$$

□

The formulas (3.15),(3.16), and (3.17) give an explicit way to calculate the fundamental form and projective Ricci curvature of distribution, starting from some adapted frame of the distribution. We will demonstrate later these formulas on several examples. The previous theorem allows to prove also the following theorem about the algebraic structure of $(2,5)$-distributions

Theorem 3 *In the case $n = 5$ for any $q \in M$ the restriction of the densities $A(\cdot)$ of the fundamental form to the fibers $(D^2)^\perp(q)$ are degree 4 homogeneous polynomials on $(D^2)^\perp(q)$.*

Proof. Let $\epsilon_1 = \frac{1}{u_5}\partial_{u_4}$. Also denote

$$\begin{aligned}\tilde{\mathcal{X}} &= \vec{u}_2 - u_5\partial_{u_3}\\ \tilde{Y}_j &= \vec{u}_j + \textstyle\sum_{i=1}^{3}\{u_i, u_j\}\partial_{u_i}, \quad j = 4, 5\end{aligned}$$

and let F be as in (2.36). Then the tuple of vector fields

$$\epsilon_1, \tilde{\mathcal{X}}, F, Y_4, Y_5, \vec{h}, \vec{e} \tag{3.20}$$

constitute a frame on $(D^2)^\perp$. By direct calculations

$$[\vec{h}, \epsilon_1] = -\frac{1}{u_5}\tilde{\mathcal{X}} + p_1\epsilon_1 \mod \mathbb{R}\vec{e} \tag{3.21}$$

$$(\mathrm{ad}\vec{h})^2(\epsilon_1) = F + p_2X + p_3\epsilon_1 \mod \Big(\mathrm{span}(\vec{h}, \vec{e})\Big), \tag{3.22}$$

where p_i, $i = 1, 2, 3$, are some rational functions in u_4, u_5 with denominator of the form u_5^l. From the form of vector fields $\vec{h}$ and e_1 it follows that the coordinates of vector field $(\mathrm{ad}\vec{h})^4(\epsilon_1)$ w.r.t. the frame (3.20) are also rational functions in u_4, u_5 with denominator of the form u_5^l. But from (3.15), (3.21), and (3.22) it follows that

$$(\mathrm{ad}\vec{h})^4\epsilon_1 \subset \mathrm{span}(\epsilon_1, \tilde{\mathcal{X}}, F, \vec{h}, \vec{e}).$$

Expressing $\tilde{\mathcal{X}}$ and F by ϵ_1, $[\vec{h}, \epsilon_1]$, and $(\mathrm{ad}\vec{h})^2(\epsilon_1)$ from (3.21) and (3.22) mod(span$(\vec{h}, \vec{e})$), one obtains that coefficients A_0, A_1 from (3.15) and so also the fundamental form A are rational functions in u_4, u_5 with denominator of the form u_5^l. But by Remark 2.14 A is smooth at the points with $u_5 = 0$, $u_4 \neq 0$. It implies that A has to be polynomial. (Another argument is as follows: if at the beginning one takes $\epsilon_1 = \frac{1}{u_4}\partial_{u_5}$, then similarly to above one obtains that A is a rational function in u_4, u_5 with denominator of the form u_4^l, which implies that A has to be polynomial). □

As a direct consequence of the previous theorem we obtain

Corollary 2 *For any $q \in M$ the tangential fundamental form $\mathring{A}_q$ is degree 4 homogeneous polynomial on $D(q)$.*

Remark 3.2 From the previous corollary it follows that the tangential fundamental form has the same algebraic nature, as the *covariant binary biquadratic form*, constructed by E.Cartan in [12] (chapter VI, paragraph 33). We call it Cartan's tensor. In the next paper [19] we prove that the tangential fundamental form coincides (up to constant factor -35) with Cartan's tensor. $\square$

Remark 3.3 In terms of canonical projective structure on abnormal extremal (see Remark 2.13) and fundamental form one can obtain sufficient conditions for rigidity of the corresponding abnormal trajectory of $(2,5)$-distribution: A smooth curve ξ tangent to distribution D and connecting two points q_0 and q_1 is called rigid, if in some C^1-neighborhood of ξ the only curves tangent to D and connecting q_0 with q_1 are reparametrizations of ξ. Rigid curves are automatically abnormal trajectories of D. From result formulated in Remark 1.2 (see [17], Theorem 4.2 for precise statement) and comparison theorems from [5] (Theorem 5, item 1 there) one has the following: *For abnormal trajectory ξ of $(2,5)$-distribution to be rigid it is sufficient the existence of a global projective parameter on ξ together with the nonpositivity of the fundamental form along ξ (equivalently nonnegativity of Cartan's tensor along ξ).* Moreover, if some Riemannian metric is given on M, then under the same conditions the corresponding abnormal trajectory is the shortest among all curves tangent to distribution D, connecting its endpoints and sufficiently closed to this abnormal trajectory in C-topology. It follows again from the mentioned comparison theorem and from the fact that simplicity of the Jacobi curve of the abnormal extremal implies minimality of the length of the corresponding abnormal trajectory in C-topology (see [8] and [9]).$\square$

3.3 Examples. Now we will give five examples of concrete distributions or families of distributions, for which we have computed the fundamental form and projective Ricci curvature using Theorem 2. We will present the computations only in Example 4, while in other case we will give only the results (in fact Examples 2 and 3 are included in Example 4; Examples 1-3 and other examples with the detailed computations can be found in [18]).

But before let us introduce some notations. Let X_1, X_2, X_3, X_4, X_5 be an adapted frame to the distribution and u_i, $1 \leq i \leq 5$, be the corresponding quasi-impulses, defined by (2.30). Suppose that this frame satisfies the following commutative relations

$$[X_i, X_j] = \sum_{i=1}^{n} c_{ji}^k X_k \tag{3.23}$$

Then the Hamiltonian vector fields $\vec{u}_i$, corresponding to the functions u_i satisfy

$$\vec{u}_i = X_i + \sum_{j=1}^{5}\sum_{k=1}^{5} c_{ji}^k u_k \partial_{u_j} \tag{3.24}$$

Therefore, the restriction of the vector field $\vec{h}$ on

$$\left(D^2\right)^\perp \cap T^*M = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0\} \tag{3.25}$$

satisfies

$$\vec{h} = u_4\vec{u}_2 - u_5\vec{u}_1 = u_4X_2 - u_5X_1 + \Big(c^4_{42}u_4^2 + (c^5_{42} - c^4_{41})u_4u_5 - c^5_{41}u_5^2\Big)\partial_{u_4} + \\ +\Big(c^4_{52}u_4^2 + (c^5_{52} - c^4_{51})u_4u_5 - c^5_{51}u_5^2\Big)\partial_{u_5}. \tag{3.26}$$

Example 1. Free nilpotent (2,5)- distribution. Let L_1 be the 5-dimensional nilpotent Lie algebra with the following commutation rules in some basis $X_1, \dots, X_5$:

$$\begin{aligned} &[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_2, X_3] = X_5 \\ &adX_4 = 0,\ adX_5 = 0 \end{aligned} \tag{3.27}$$

Actually L_1 is the free nilpotent 3-step Lie algebra with two generators. Let M_1 be the Lie group with the Lie algebra L_1. We consider $X_1, \dots, X_5$ as left-invariant vector fields on M_1. Let $D_1 = span(X_1, X_2)$. Such distribution is called *free nilpotent* $(2,5)$*-distribution.*

By (3.27) the tuple of left invariants fields $(X_1, X_2, X_3, X_4, X_5)$ constitutes a strong adapted frame to distribution D_1. Applying Theorem 2 to this frame, it is easy to show that the fundamental form A_{D_1} of distribution of D_1 satisfies

$$A_{D_1} \equiv 0 \tag{3.28}$$

Example 2. Left-invariant rank 2 distribution on $SO(3) \times \mathbb{R}^2$. Denote by M_2 the Lie group $SO(3) \times \mathbb{R}^2$. Let $L_2 = so(3) \oplus \mathbb{R}^2$ be Lie algebra corresponding to Lie group M_2 . Suppose that E_{ij} is 3×3 matrix such that its (i, j)th entry is equal to 1 and all other entries equal to 0. Take the following basis a_1, a_2, a_3 in $so(3)$:

$$a_1 = E_{12} - E_{21}, a_2 = E_{13} - E_{31}, a_3 = E_{32} - E_{23} \tag{3.29}$$

Then a_1, a_2, a_3 satisfy the following commutative relations:

$$[a_1, a_2] = a_3,\ [a_2, a_3] = a_1,\ [a_3, a_1] = a_2 \tag{3.30}$$

Let b_1, b_2 be some basis of $\mathbb{R}^2$. Denote

$$X_1 = (a_1, b_1),\ X_2 = (a_2, b_2), \tag{3.31}$$

$$D_2 = \mathrm{span}\,(X_1, X_2) \tag{3.32}$$

We consider X_1, X_2, as left-invariant vector fields on Lie group M_2. Consequently, D_2 defined by (3.32) can be considered as left-invariant rank 2 distribution on M_2.

Remark 3.4 It can be shown easily (see [18]) that the distribution D_2 is unique, up to group automorphism of M_2, left-invariant completely nonholonomic rank 2 distribution on M_2 and its small growth vector is $(2, 3, 5)$.

Remark 3.5 Note that distribution D_2 appears, when one studies the problem of rolling ball on the plane without slipping and twisting (see Example 4 below and also [13] for the details).

Completing the basis (X_1, X_2), defined by (3.31), to the strong adapted frame to D_2, and applying Theorem 2 to this frame, one has easily that the fundamental form A_{D_2} and projective Ricci curvature ρ_{D_2} of D_2 satisfy

$$A_{D_2} \sim (u_4^2 + u_5^2)^2, \tag{3.33}$$

$$\rho_{D_2} = \frac{4\sqrt{35}}{9} \tag{3.34}$$

(here as in the sequel we use the sign $\sim$ to emphasize that the fundamental form at the point is defined up to multiplication on a positive constant).

Conclusion 1 *From (3.28) and (3.33) it follows that germs of distributions D_1 and D_2 are not equivalent*

Remark 3.6 Actually, the rank 2 distribution on 5-dimensional manifold has the identically zero fundamental form iff it is locally equivalent to the distribution D_1. It follows from the fact that our fundamental form coincides with Cartan's tensor (see [19]) and the fact that Cartan's tensor of distribution is identically zero iff it is locally equivalent to the distribution D_1 (see chapter VIII of [12]).□

Example 3. Left-invariant rank 2 distributions on $SL(2,\mathbb{R}) \times \mathbb{R}^2$. Let E_{ij} be 2×2 matrix such that its (i,j)th entry is equal to 1 and all other entries equal to 0. Take the following basis a_1, a_2, a_3 in $sl(2,\mathbb{R})$:

$$a_1 = \frac{1}{2}(E_{11} - E_{22}), a_2 = \frac{1}{2}(E_{12} - E_{21}), a_3 = \frac{1}{2}(E_{12} + E_{21}) \tag{3.35}$$

Then a_1, a_2, a_3 satisfy the following commutative relations:

$$[a_1, a_2] = a_3, \ [a_2, a_3] = a_1, \ [a_3, a_1] = -a_2, \tag{3.36}$$

Denote by

$$P_h = \mathrm{span}(a_1, a_2), \ \ P_e = \mathrm{span}(a_1, a_3) \tag{3.37}$$

Note that the restriction of the Killing form on P_h is indefinite nondegenerated and on P_e is positive definite quadratic form. Let b_1, b_2 be some basis of $\mathbb{R}^2$. Suppose that

$$D_{3,h} = \mathrm{span}\big((a_1, b_1), (a_2, b_2)\big), \ \ D_{3,e} = \mathrm{span}\big((a_1, b_1), (a_3, b_2)\big), \tag{3.38}$$

We consider $D_{3,h}$ and $D_{3,e}$, as left-invariant rank 2 distribution on the Lie group $SL(2,\mathbb{R}) \times \mathbb{R}^2$.

Remark 3.7 *It can be shown easily (see [18]) that distributions $D_{3,h}$ and $D_{3,e}$ are the only two different left-invariant rank 2 completely nonholonomic distributions on $SL(2,\mathbb{R}) \times \mathbb{R}^2$, up to Lie group automorphisms of $SL(2,\mathbb{R}) \times \mathbb{R}^2$, and their small growth vector is* $(2,3,5)$.

Remark 3.8 Note that distribution $D_{3,e}$ appears, when one studies the problem of rolling hyperbolic plane on the Euclidean plane without slipping and twisting, (see geometric model in Example 4 below).

Completing the bases, chosen in (3.38), to the strong adapted frames of $D_{3,h}$ and $D_{3,e}$ and applying Theorem 2 to these frames, one has easily that the fundamental form $A_{D_{3,h}}$ and the projective curvature $\rho_{D_{3,h}}$ of $D_{3,h}$ satisfy

$$A_{D_{3,h}} \sim (u_4^2 - u_5^2)^2, \tag{3.39}$$

$$\rho_{D_{3,h}} = \begin{cases} -\frac{4\sqrt{35}}{9} & u_4 > u_5 \\ \frac{4\sqrt{35}}{9} & u_4 < u_5 \end{cases}, \tag{3.40}$$

while the fundamental form $A_{D_{3,e}}$ and the projective curvature $\rho_{D_{3,e}}$ of $D_{3,e}$ satisfy

$$A_{D_{3,e}} \sim (u_4^2 + u_5^2)^2, \tag{3.41}$$

$$\rho_{D_{3,e}} = -\frac{4\sqrt{35}}{9} \tag{3.42}$$

Conclusion 2 *From (3.33) and (3.39) it follows that the germs of distributions $D_{3,h}$ and D_2 are not equivalent; from (3.39) and (3.41) it follows that the germs of distributions $D_{3,h}$ and $D_{3,e}$ are not equivalent; finally, from (3.34) and (3.42) it follows that the germs of distributions D_2 and $D_{3,e}$ are not equivalent.*

Remark 3.9 Note that for distributions D_2 and $D_{3,e}$ the information about their fundamental forms does not imply their local nonequivalence: fundamental forms in both cases are squares of sign definite quadratic forms. This is the case when the projective Ricci curvature helps to distinct the distributions.

Example 4. Rolling of two spheres without slipping and twisting. Rank 2 distributions on 5-dimensional manifold appear naturally when one studies the possible motions of two surfaces S and $\widehat{S}$ in $\mathbb{R}^3$, which roll one on another without slipping and twisting. Here we follow the geometric model of this problem given in [6] (this model ignores the state constraints that correspond to the admissibility of contact of the bodies embedded in $\mathbb{R}^3$). The state space of the problem is the 5-dimensional manifold

$$M_4 = \{B : T_x S \mapsto T_{\hat{x}}\widehat{S} |\, B \text{ is an isometry}\}.$$

Let $B(t) \subset M_4$ be an admissible curve, corresponding to the motion of the rolling surfaces. Let $x(t)$ and $\hat{x}(t)$ be trajectories of the contact points in S and $\widehat{S}$ respectively (so $B(t)$ can be considered as an isometry from $T_{x(t)}S$ to $T_{\hat{x}(t)}\hat{S}$).The condition of absence of slipping means that

$$B(t)\dot{x}(t) = \dot{\hat{x}}(t), \tag{3.43}$$

while the condition of absence of twisting can be written as follows

$$B(t)(\text{vector field parallel along } x(t)) = \text{vector field parallel along } \hat{x}(t). \tag{3.44}$$

From conditions (3.43) and (3.44) it follows that a curve $x(t) \in S$ determines completely the whole motion $B(t) \in M_4$ and the velocities of admissible motions define a $(2,5)$-distribution $D_{4,S,\hat{S}}$ on M_4. If (v_1, v_2) and $(\hat{v}_1, \hat{v}_2)$ are some local orthonormal frames on S and $\hat{S}$ respectively and β is the angle of rotation from the frame $(Bv_1(x), Bv_2(x))$ to the frame $(\hat{e}_1(\hat{x}), \hat{e}_2(\hat{x}))$, then the points of M_4 are parametrized by $(x, \hat{x}, \beta)$ and one can choose a local basis of distribution $D_{4,S,\hat{S}}$ as follows

$$X_1 = v_1 + \cos\beta\hat{v}_1 + \sin\beta\hat{v}_2 - (-\sigma_1 + \hat{\sigma}_1\cos\beta + \hat{\sigma}_2\sin\beta)\partial_\beta,$$
$$X_2 = v_2 - \sin\beta\hat{v}_1 + \cos\beta\hat{v}_2 + (-\sigma_2 - \hat{\sigma}_1\sin\beta + \hat{\sigma}_2\cos\beta\partial_\beta, \tag{3.45}$$

where σ_i,$\hat{\sigma}_i$ are structural functions of the frames:

$$[v_1, v_2] = \sigma_1 v_1 + \sigma_2 v_2, \quad [\hat{v}_1, \hat{v}_2] = \hat{\sigma}_1\hat{v}_1 + \hat{\sigma}\hat{v}_2 \tag{3.46}$$

Let us restrict ourselves to the case, when S and $\widehat{S}$ are spheres of radiuses r and $\hat{r}$ respectively. We will denote the corresponding $(2,5)$-distribution by $D_{4,r,\hat{r}}$.

Remark 3.10 Obviously, distributions $D_{4,r,\hat{r}}$ with the same ratio $\frac{\hat{r}}{r}$ are equivalent and distributions $D_{4,r,\hat{r}}$ and $D_{4,\hat{r},r}$ are equivalent too. The question is whether distributions $D_{4,r,\hat{r}}$ with different ratios $\frac{\hat{r}}{r} \geq 1$ are equivalent.

It turns out that the calculation of fundamental form and projective Ricci curvature of $D_{4,r,\hat{r}}$ gives the answer to the question in the previous remark. Taking the spherical coordinates (φ, ψ) on the sphere S , where φ is the "altitude" (with values between 0 and π) and ψ is the "longitude", one can choose the following orthonormal frame on S

$$v_1 = \frac{1}{r}\partial_\varphi, \quad v_2 = \frac{1}{r\sin\varphi}\partial_\psi. \tag{3.47}$$

In the same way take the spherical coordinates on $\widehat{S}$ and the orthonormal frame $(\hat{v}_1, \hat{v}_2)$, defined by putting the sign ˆ over r, φ, and ψ in (3.47). Then the structural functions σ_i, $\widehat{\sigma}_i$ satisfy

$$\sigma_1 = 0,\ \sigma_2 = -\frac{\cot\varphi}{r}; \quad \widehat{\sigma}_1 = 0,\ \widehat{\sigma}_2 = -\frac{\cot\hat{\varphi}}{\hat{r}}. \tag{3.48}$$

Substituting (3.48) into (3.49) one gets

$$X_1 = v_1 + \cos\beta\hat{v}_1 + \sin\beta\hat{v}_2 - \frac{\cot\hat{\varphi}}{\hat{r}}\sin\beta\partial_\beta,$$
$$X_2 = v_2 - \sin\beta\hat{v}_1 + \cos\beta\hat{v}_2\left(\frac{\cot\varphi}{r} - \frac{\cot\hat{\varphi}}{\hat{r}}\cos\beta\right)\partial_\beta. \tag{3.49}$$

Let X_3, X_4, X_5 be as in (3.7). Then from (3.49) one can obtain by direct computation

$$\begin{aligned}
X_3 &= -\frac{\cot\varphi}{r}X_2 + \left(\frac{1}{\hat{r}^2} - \frac{1}{r^2}\right)\partial_\beta \\
X_4 &= -\frac{\cot\varphi}{r}X_3 + \frac{1}{r^2\sin^2\varphi}X_2 - \left(\frac{1}{\hat{r}^2} - \frac{1}{r^2}\right)\left(X_2 - v_2 - \frac{\cot\varphi}{r}\right)\partial_\beta \\
X_5 &= \left(\frac{1}{\hat{r}^2} - \frac{1}{r^2}\right)(X_1 - v_1)
\end{aligned} \tag{3.50}$$

Hence for $r \neq \hat{r}$ the tuple of the fields $(X_1, X_2, X_3, X_4, X_5)$ constitutes a strong adapted frame of the distribution $D_{4,r,\hat{r}}$.

Remark 3.11 It is clear that for $r = \hat{r}$ the distribution $D_{4,r,\hat{r}}$ is integrable.$\Box$

Let us calculate the fundamental form and the projective Ricci curvature in this frame. First, from (3.49) and (3.50) one can obtain by direct calculations that

$$
\begin{aligned}
&[X_1, X_4] = -\left(\frac{1}{\hat{r}^2} + \frac{2}{r^2\sin^2\varphi}\right)\frac{\cot\varphi}{r}X_2 - \left(\frac{1}{\hat{r}^2} - \frac{2}{r^2\sin^2\varphi}\right)X_3 - \frac{\cot\varphi}{r}X_4,\\
&[X_2, X_5] = -\left(\frac{1}{\hat{r}^2} - \frac{1}{r^2\sin^2\varphi}\right)\frac{\cot\varphi}{r}X_2 - \left(\frac{1}{\hat{r}^2} + \frac{\cot^2\varphi}{r^2}\right)X_3 - \frac{\cot\varphi}{r}X_4,\\
&[X_2, X_4] = [X_1, X_5] = 0.
\end{aligned} \tag{3.51}
$$

Then by (3.23) and (3.26)

$$\vec{h} = u_4X_2 - u_5X_1 - u_4\frac{\cot\varphi}{r}\partial_\theta, \tag{3.52}$$

where ∂_θ is as in (2.34). Take the polar coordinates $u_4 = R\cos\theta$, $u_5 = R\sin\theta$ on the fibers $(D^2)^\perp(q)$. From (3.52) it is clear that $\vec{h}(R) = 0$. Hence by Remark 3.1 one can take as ϵ_1 in Theorem 2 the field ∂_θ. Besides, by homogeneity of fundamental form it is sufficient to restrict our calculations to the set $\{R = 1\}$. The vector field $\vec{h}$ has on this set the form

$$\vec{h} = \cos\theta X_2 - \sin\theta X_1 - \cos\theta\frac{\cot\varphi}{r}\partial_\theta, \tag{3.53}$$

Further, by direct computation, it is not hard to get

$$
\begin{aligned}
&[\vec{h}, \partial_\theta] = \sin\theta X_2 + \cos\theta X_1 - \sin\theta\frac{\cot\varphi}{r}\partial_\theta,\\
&(\mathrm{ad}\vec{h})^2(\partial_\theta) = -\left(X_3 - \frac{\cot\varphi}{r} + \frac{1}{r^2}\partial_\theta\right),\\
&(\mathrm{ad}\vec{h})^3(\partial_\theta) = -\cos\theta X_5 + \sin\theta X_4 + \sin\theta\frac{\cot\varphi}{r}X_3 - \frac{\sin\theta}{r^2\sin^2\varphi}X_2 - \frac{1}{r^2}[\vec{h}, \partial_\theta].
\end{aligned} \tag{3.54}
$$

Finally, from (3.51) and (3.54) one can obtain without difficulties that

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = -\frac{1}{r^2\hat{r}^2}\partial_\theta - \left(\frac{1}{r^2} + \frac{1}{\hat{r}^2}\right)(\mathrm{ad}\vec{h})^2(\partial_\theta), \tag{3.55}$$

which together with (3.16) and (3.17) implies that the fundamental for $A_{D_{4,r,\hat{r}}}$ and projective Ricci curvature $\rho_{D_{4,r,\hat{r}}}$ of $D_{4,r,\hat{r}}$ satisfy

$$A_{D_{4,r,\hat{r}}} \sim \mathrm{sgn}\Big((9\hat{r}^2 - r^2)(\hat{r}^2 - 9r^2)\Big)(u_4^2 + u_5^2)^2, \tag{3.56}$$

$$\rho_{D_{4,r,\hat{r}}} = \frac{4\sqrt{35}}{3}\frac{r^2 + \hat{r}^2}{\sqrt{|(9\hat{r}^2 - r^2)(\hat{r}^2 - 9r^2)|}}. \tag{3.57}$$

Proposition 3.5 *Distributions $D_{4,r,\hat{r}}$ with different ratios $\frac{\hat{r}}{r} \geq 1$ are not equivalent.*

Proof. Let

$$I_1 = (1, 3), \quad I_2 = \{3\}, \quad I_3 = (3, +\infty).$$

From (3.56) distributions $D_{4,r,\hat{r}}$ with ratios, taking values in different I_j, are not equivalent. Note that the function $\alpha \mapsto \frac{\alpha^2+1}{\sqrt{|(\alpha^2-9)(9\alpha^2-1)|}}$ is monotone on both intervals I_1 and I_3. This together with (3.57) implies that distributions $D_{4,r,\hat{r}}$ with ratios, taking values in one of I_j, $j = 1$ or 3, are not equivalent, which together with Remark 3.11 concludes the proof of the Proposition. $\Box$

Remark 3.12 From (3.56) and Remark 3.6 it follows that if $\frac{\hat{r}}{r} = 3$ or $\frac{1}{3}$, then $D_{4,r,\hat{r}}$ is locally equivalent to the free nilpotent $(2,5)$-distribution D_1 from Example 1.$\square$

Remark 3.13 From (3.56) and (3.57) it follows that for $\frac{\hat{r}}{r} > 3$ the fundamental form is square of sign definite quadratic form and the projective Ricci curvature varies from $\frac{4\sqrt{35}}{9}$ to ∞ ($\frac{4\sqrt{35}}{9}$ corresponds to $\frac{\hat{r}}{r} = \infty$, i.e., to the rolling of the sphere on the plane) , while for $1 < \frac{\hat{r}}{r} < 3$ the fundamental form is $-$ square of sign definite quadratic form and the projective Ricci curvature varies from $\frac{\sqrt{35}}{3}$ to ∞ (not including $\frac{\sqrt{35}}{3}$). $\square$

More generally, if S and $\widehat{S}$ are surfaces of constant curvatures k and $\hat{k}$ respectively, then the fundamental form and projective Ricci curvature of the distribution, generated by the rolling of one surface on another without slipping and twisting, satisfy

$$A_{D_{4,r,\hat{r}}} \sim \operatorname{sgn}\Big((9\hat{k} - \hat{k})(\hat{k} - 9k)\Big)(u_4^2 + u_5^2)^2, \tag{3.58}$$

$$\rho_D = \frac{4\sqrt{35}}{3} \frac{k + \hat{k}}{\sqrt{|(9\hat{k} - k)(\hat{k} - 9k)|}}. \tag{3.59}$$

Example 5. Distributions generated by curves of constant torsion on 3-dimensional manifold of constant curvature. These distributions were mentioned already in [12] (chapter XI, paragraphs 52, 53). Let Q be an oriented 3-dimensional Riemannian manifold. Then for given τ the curves of constant torsion τ together with their binormals are admissible curves of a rank 2 distribution on 5-dimensional manifold $M_5 = Q \times S^2$. Indeed, let $\gamma(t)$ be the curve in Q without inflection points, and let $n(t) \in S^2$ be the corresponding binormal. Then γ has a constant torsion τ iff

$$\dot{\gamma}(t) = \frac{1}{\tau} n(t) \times \nabla_{\dot{\gamma}(t)} n(t), \tag{3.60}$$

where by $\times$ we mean the vector product induced on each (oriented) tangent space $T_{\gamma(t)}Q$ by the Riemannian metric and ∇ denotes the covariant derivative, corresponding to this metric. Obviously, relation (3.60) defines the rank 2 distribution on M_5. We restrict ourselves to the case when Q has constant curvature K and denote by $D_{5,\tau,K}$ the corresponding $(2,5)$-distribution. It can be shown that the corresponding fundamental form $A_{D_{5,\tau,K}}$ and the projective Ricci curvature $\rho_{D_{5,\tau,K}}$ satisfy

$$A_{D_{5,\tau,K}} \sim \operatorname{sgn}\Big((\frac{\tau^2}{K} - 4)(1 - 4\frac{\tau^2}{K})\Big)\Big(\text{sign definite quadratic form}\Big)^2 \tag{3.61}$$

$$\rho_{D_{5,\tau,K}} = \frac{2\sqrt{35}}{3} \frac{\frac{\tau^2}{K} + 1}{\sqrt{|(\frac{\tau^2}{K} - 4)(1 - 4\frac{\tau^2}{K})|}} \tag{3.62}$$

Suppose that S is three-dimensional sphere of radius R. Then $K = \frac{1}{R^2}$. Note that the expressions in (3.61) and (3.62) are invariant w.r.t. transformation $\tau \mapsto \frac{1}{R^2 \tau}$.

Remark 3.14 If $\tau R = 2$ or $\frac{1}{2}$, then from (3.61) the fundamental form is equal to zero. Hence by Remark 3.6 the corresponding distribution is locally equivalent to the free nilpotent $(2,5)$-distribution D_1 and by Remark 3.12 it is locally equivalent to the distribution $D_{4,r,\hat{r}}$ with $\frac{\hat{r}}{r} = 3$ or $\frac{1}{3}$.$\square$

Remark 3.15 If $0 < \tau R < \frac{1}{2}$ or $\tau R > 2$ then the fundamental form is $-$ square of sign definite quadratic form and the projective Ricci curvature varies monotonically on both intervals from $\frac{\sqrt{35}}{3}$ to ∞ ($\frac{\sqrt{35}}{3}$ corresponds to the case, when Q is Euclidean space). If $\frac{1}{2} < \tau R < 1$ or $1 < \tau R < 2$ then the fundamental form is square of sign definite quadratic form and the projective Ricci curvature varies monotonically on both intervals from $\frac{4\sqrt{35}}{9}$ to ∞ (not including $\frac{4\sqrt{35}}{9}$).□

Note that in the case $\tau = 0$ the distribution is integrable, while in the case $\tau R = 1$ the square of the distribution is rank 3 integrable.

Till now we used our invariants in order to prove the nonequivalence of distributions. But what to do, if both the fundamental form and the projective Ricci curvature do not distinct distributions? For example by Remarks 3.13 and 3.15 for any ratio $\frac{\hat{r}}{r}$ there exists distribution $D_{5,\tau,\frac{1}{R^2}}$, which has the fundamental form of the same type and the same projective Ricci curvature as $D_{4,r,\hat{r}}$. Does it imply that these distributions are equivalent? We will treat the questions of this kind in the forthcoming paper [20]. Below we formulate a theorem, which will be proved in this paper:

Theorem 4 *For given $s \in \{1, -1\}$ and $\rho \in R$ there exists unique, up to diffeomorphism, germ of $(2, 5)$-distribution satisfying the following three conditions:*

1. Its fundamental form is s multiplied by the square of a nondegenerated quadratic form Q;

2. Its symmetry group is 6-dimensional;

3. If Q is sign definite, then its projective Ricci curvature is identically equal to ρ, if Q is sign indefinite, then the absolute value of its projective Ricci curvature is identically equal to $|\rho|$.

Remark 3.16 It can be shown that if distribution D satisfies condition 1 of Theorems 4, then the dimension of the group of symmetries of D is not greater than 6.□

Remark 3.17 It can be shown that conditions 1 and 2 of Theorem 4 imply that projective Ricci curvature or its absolute value is identically equal to some constant.□

It is easy to see that the group of symmetries of distribution $D_{4,r,\hat{r}}$ contains a subgroup isomorphic to $SO(3) \times SO(3)$ and therefore by Remark 3.16 it is 6-dimensional for $\frac{\hat{r}}{r} \neq 3$ or $\frac{1}{3}$, while the group of symmetries of distribution $D_{5,\tau,\frac{1}{R^2}}$ contains a subgroup isomorphic to $SO(4)$ and therefore by Remark 3.16 it is also 6-dimensional for $\tau R \neq 2$ or $\frac{1}{2}$. Therefore Theorem 4 implies

Corollary 3 *If distributions $D_{4,r,\hat{r}}$ and $D_{5,\tau,\frac{1}{R^2}}$ have the fundamental forms, which are $\pm$ square of positive definite quadratic form, where the sign is the same for both distributions, and their projective Ricci curvatures are the same constant, then these distributions are locally equivalent.*

4 Algebraic structure of fundamental form in the case $n > 5$

In the present section we show that in the case $n > 5$ the fundamental form is in general a rational function, which is not a polynomial. By Remark 2.14 singularities of fundamental form could occur out of the set $\mathcal{R}_D$, i.e., at the points, where the weight of the corresponding Jacobi curve is not constant.

First take some curve $\Lambda(t)$ in the Grassmannian of half-dimensional subspaces $G_m(W)$ such that at some point $\bar{t}$ it has a jump of the weight and look more carefully what happens with Ricci curvature and fundamental form of this curve near this point. More precisely, we consider the following situation: in some punctured neighborhood of $\bar{t}$ the curve $\Lambda(t)$ has constant weight k, while at point $\bar{t}$ it has the weight $k+1$. In this case we will say that $\bar{t}$ is *a point of the weight jump one* of $\Lambda(t)$.

Lemma 4.1 *If the curve $\Lambda(t)$ in $G_m(W)$ has a point of the weight jump one at $\bar{t}$, then the Ricci curvature has a pole of order 2 at $t=\bar{t}$. If in addition the weight of $\Lambda(t)$ in the punctured neighborhood of $\bar{t}$ is greater than 1, then the density of fundamental form of this curve has a pole of order 4 at $t=\bar{t}$.*

Proof. Suppose that k is the weight of $\Lambda(t)$ in the punctured neighborhood of $\bar{t}$. Then the following function

$$X(t_0,t_1)=\frac{\det(S_{t_0}-S_{t_1})}{(t_0-t_1)^k} \tag{4.1}$$

is smooth. By assumptions about the jump of the weight at $\bar{t}$ one has

$$X(\bar{t},\bar{t})=0, \quad \frac{\partial}{\partial t_0}X(t_0,t_1)\Big|_{t_0=t_1=\bar{t}}=\frac{\partial}{\partial t_1}X(t_0,t_1)\Big|_{t_0=t_1=\bar{t}}\neq 0. \tag{4.2}$$

It implies that the function $X(t_0,t_1)$ is symmetric. Indeed by permuting t_0 and t_1 in (4.1) we obtain that X can be either symmetric or antisymmetric, but the last case is impossible, because $X(t,t)$ is not identically zero.

Without loss of generality it can be assumed that $\bar{t}=0$. From (4.2) and symmetricity it follows that

$$X(t_0,t_1)=(t_0+t_1)a(t_0,t_1), \tag{4.3}$$

where $a(t_0,t_1)$ is smooth. Using (4.3) and (2.9) we obtain that the generating function $g_\Lambda(t_0,t_1)$ of the curve $\Lambda(t)$ satisfies

$$g_\Lambda(t_0,t_1)=\frac{\partial^2 \ln X(t_0,t_1)}{\partial t_0\partial t_1}=-\frac{1}{(t_0+t_1)^2}+O(1) \tag{4.4}$$

It yields first that the Ricci curvature $\rho(t)=g_\Lambda(t,t)$ has the following expansion at $t=0$

$$\rho(t)=g_\Lambda(t,t)=-\frac{1}{4t^2}+O(1) \tag{4.5}$$

In order to find an asymptotic for the density $A(t)$ of fundamental form at $t=\bar{t}$, we use formula (2.8). Again from (4.4) it follows easily that

$$\frac{\partial^2}{\partial t_0^2}g_\Lambda(t_0,t_1)\Big|_{t_0=t_1=t}=-\frac{3}{8t^4}+O(1)$$

Substituting this into (2.8) and using (4.5) one gets

$$A(t)=\frac{3(k-1)}{80\,k}\frac{1}{t^4}+O\left(\frac{1}{t^2}\right), \tag{4.6}$$

which completes the proof of the Lemma. $\Box$

Remark 4.1 If at some point of the curve in $G_m(W)$ the weight has jump greater than 1, then in general the Ricci curvature and the density of fundamental form have also singularities, but the coefficients of the principal negative power in their Laurent expansion are not universal, as in (4.5) and (4.6). $\Box$

The following proposition gives a simple characterization of the points of the weight jump one of the rank 1 curve $\Lambda(t)$ in Lagrange Grassmannian in terms of the subspaces $\mathcal{D}^{(i)}\Lambda$ defined by (2.12):

Proposition 4.1 *The point $\hat{t}$ is the point of the weight jump one of rank 1 curve $\Lambda(t)$ in Lagrange Grassmannian $L(W)$, $\dim W = 2m$, iff the following relations holds*

$$\dim \mathcal{D}^{(m-1)}\Lambda(\bar{t}) = \dim \mathcal{D}^{(m)}\Lambda(\bar{t}) = 2m-1, \quad \dim \mathcal{D}^{(m+1)}\Lambda(\bar{t}) = 2m \tag{4.7}$$

The proof of the proposition can be easily obtained by application of some formulas and statements of section 6 and 7 of [4] (for example, formulas (6.15), (6.16), (6.18), (6.19), Lemma 6.1 and Proposition 3 there).

Let us apply Lemma 4.1 and Proposition 4.1 to distributions. For this let subspace $\mathcal{J}^{(i)}(\lambda)$ be as in (2.24). Denote by S^0_D the following subset of $(D^2)^\perp \backslash \Big((D^3)^\perp \cup \mathcal{R}_D\Big)$

$$S^0_D = \left\{ \lambda \in (D^2)^\perp \backslash (D^3)^\perp : \begin{array}{c} \dim \mathcal{J}^{(n-4)}(\lambda) = \dim \mathcal{J}^{(n-3)}(\lambda) = 2n-5, \\ \dim \mathcal{J}^{(n-3)}(\lambda) = 2n-4 \end{array} \right\}. \tag{4.8}$$

Then by Proposition 4.1 the set S^0_D coincides with the subset of $(D^2)^\perp \backslash (D^3)^\perp$, consisting of points, in which the corresponding reduced Jacobi curves have the weight jump one. Also, from Propositions 2.4 one has

Proposition 4.2 *The reduced Jacobi curve of the regular abnormal extremal γ has the weight jump one at point λ iff*

$$\dim \mathcal{T}^{(n-4)}_\xi(q) = \dim \mathcal{T}^{(n-3)}_\xi(q) = n-2, \quad \dim \mathcal{T}^{(n-2)}_\xi(q) = n-1 \tag{4.9}$$

where $\xi = \pi(\gamma)$ is the abnormal trajectory corresponding to γ and $q = \pi(\lambda)$.

Further note that by Proposition 2.3 the weight of these curves in the punctured neighborhoods of these points is equal to $(n-3)^2$ and therefore it is greater than 1 in the considered cases. Suppose also that

$$S^0_D(q) = S^0_D \cap T^*_q M, \quad q \in M \tag{4.10}$$

As a direct consequence of Lemma 4.1 and Proposition 2.8 we obtain the following

Proposition 4.3 *If the sets $\mathcal{R}_D(q)$ and $S^0_D(q)$ are not empty, then the fundamental form of distribution D at point q is a rational function, which is not a polynomial: all points of $S^0_D(q)$ are the points of discontinuity of it.*

Example. Consider distribution $\widetilde{D}$ in $\mathbb{R}^6$ spanned by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_4} + \Big(\frac{x_1^4}{4!} + \frac{x_1^2 x_2}{2}\Big)\frac{\partial}{\partial x_5} + x_1 x_2 \frac{\partial}{\partial x_6}. \tag{4.11}$$

Distribution $\widetilde{D}$ has the maximal possible small growth vector $(2,3,5,6)$ at any point. We claim that its fundamental form at 0 is a rational function, which is not a polynomial. Indeed, by

Proposition 2.7, the set $\mathcal{R}_{\widetilde{D}}(0)$ is not empty. It is not hard to show that the curve $(x_1, 0, \dots, 0)$ is regular abnormal trajectory of corank 1. Moreover, from Proposition 4.2 it follows that the reduced Jacobi curve of any its lift γ has the weight jump one at the point of intersection of γ with $(\widetilde{D}^2)^{\perp}(0)$, which implies that $S^0_{\widetilde{D}}(0)$ is not empty. Now our claim follows from Proposition 4.3. $\Box$

From Proposition 2.6 it is clear that for $(2,5)$-distributions with small growth vector $(2,3,5)$ the set S^0_D is empty. For $n > 5$ the situation is different: it turns out that after an appropriate complexification, one can prove that $S^0_D(q)$ is not empty for generic germ of $(2,n)$-distribution at q. Below we briefly describe this process, leaving the details to the reader. Note that for sufficiently big natural l the weight, the jump of the weight and the density of fundamental form of the curve in Lagrange Grassmannian at some point is completely determined by its l-jet at this point. So for our purposes it is sufficient to work with l-jets of the curves with sufficiently big l instead of the curves themselves. From the proof of Proposition 2.8 it follows that if for given $q \in M$ we consider a mapping Φ_l from $(D^2)^{\perp}(q)$ to the space of l-jets of curves in Lagrange Grassmannian, which assigns to any $\lambda \in (D^2)^{\perp}(q)$ the l-jet of the corresponding reduced Jacobi curve at λ, then the mapping Φ_l is rational on $(D^2)^{\perp}(q)$. Hence this mapping can be rationally continued from the real linear space $(D^2)^{\perp}(q)$ to its complexification $(D^2)^{\perp}(q)^{\mathbb{C}}$. After this continuation for any $\lambda \in (D^2)^{\perp}(q)^{\mathbb{C}}$ one has the l- jet of curve in the complex Lagrange Grassmannian (i.e., the set of (complex) half-dimensional subspace in complex even-dimensional space provided with skew-symmetric nondegenerated bilinear form). The theory of such curves is completely the same as in the case of real Lagrange Grassmannian. In particular, one has result analogous to Lemma 4.1. If we denote by $S^0_D(q)^{\mathbb{C}}$ the set of all $\lambda \in (D^2)^{\perp}(q)^{\mathbb{C}}$ such that the l-jet $\Phi_l(\lambda)$ corresponds to curve in the complex Lagrange Grassmannian with the weight jump one at λ, then we have the following generalization of Proposition 4.3:

Proposition 4.4 *If the sets $\mathcal{R}_D(q)$ and $S^0_D(q)^{\mathbb{C}}$ are not empty, then the fundamental form of distribution D at point q is a rational function, which is not a polynomial: all points of $S^0_D(q)^{\mathbb{C}}$ are singular points of the analytic continuation of the fundamental form to $(D^2)^{\perp}(q)^{\mathbb{C}}$.*

The set $S^0_D(q)^{\mathbb{C}}$ can be described in more constructive way. For this note that the mappings $\lambda \mapsto J^{(i)}(\lambda)$, $\lambda \in (D^2)^{\perp}(q)$, depend rationally on λ and therefore can be rationally continued to $(D^2)^{\perp}(q)^{\mathbb{C}}$ (after this continuation we look on $J^{(i)}(\lambda)$ as on complex linear spaces). Then from construction it follows that

$$S^0_D(q)^{\mathbb{C}} = \left\{ \lambda \in (D^2)^{\perp}(q)^{\mathbb{C}} : \begin{array}{c} \dim \mathcal{J}^{(n-4)}(\lambda) = \dim \mathcal{J}^{(n-3)}(\lambda) = 2n-5, \\ \dim \mathcal{J}^{(n-3)}(\lambda) = 2n-4 \end{array} \right\} \tag{4.12}$$

(here all dimensions are over $\mathbb{C}$).

Proposition 4.5 *In the case $n > 5$ for generic germ of $(2,n)$-distribution D at q_0 the set $S^0_D(q_0)^{\mathbb{C}}$ is not empty.*

Proof. Choose again a local frame $\{X_i\}_{i=1}^n$ on M such that X_1, X_2 constitute a local basis of D and X_3, X_4, X_5 satisfy (2.29). From Proposition 2.5 the set $(D^2)^{\perp}(q_0)\backslash\mathcal{R}_D(q_0)$ can be represented as a zero level set of some polynomial in u_j, $j = 4, \dots, n$. Denote this polynomial by $\mathcal{P}_D$. Using (2.55) and definition of subspaces $J^{(i)}$ one can show without difficulties that the polynomial $\mathcal{P}_D$ is either homogeneous polynomial of degree d_n or identically equal to zero. In the case $n > 5$ the degree $d_n > 0$, while $d_5 = 0$. For example, in the case $n = 6$ from Proposition

2.7 it follows that one can take $\mathcal{P}_D = \alpha_6$, where α_6 is as in (2.63). Hence $d_6 = 2$. In general $d_n = \frac{(n-4)(n-3)}{2} - 1$.

From definition of $\mathcal{P}_D$ and relations (2.55), (4.12) it is easy to see that

$$S^0_D(q_0)^{\mathbb{C}} = \left\{\lambda \in (D^2)^\perp(q_0)^{\mathbb{C}} : \mathcal{P}_D(\lambda) = 0,\ \vec{h}_{X_1,X_2}(\mathcal{P}_D)(\lambda) \neq 0\right\}, \tag{4.13}$$

where $\vec{h}_{X_1,X_2}$ is as in (2.33). In other words, the set $S^0_D(q_0)^{\mathbb{C}}$ is empty iff the following condition holds

$$\mathcal{P}_D(\lambda) = 0 \Rightarrow \vec{h}_{X_1,X_2}(\mathcal{P}_D)(\lambda) = 0.$$

From the form of the vector field $\vec{h}_{X_1,X_2}$ it follows that the polynomial $h_{X_1,X_2}(\mathcal{P}_D)(\lambda)$ is homogeneous of degree $d_n + 1$.

In general, if we consider the space of pairs of polynomials $\big(p_1(\lambda), p_2(\lambda)\big)$, where p_i are polynomials of the fixed degrees s_i, then the set of all pairs $\big(p_1(\lambda), p_2(\lambda)\big)$ such that $p_1(\lambda) = 0$ implies $p_2(\lambda) = 0$ is a finite union of algebraic varieties of the whole space of pairs. From this and the fact that coefficients of the polynomials $\mathcal{P}_D(\lambda)$, $\vec{h}_{X_1,X_2}(\mathcal{P}_D)(\lambda)$ are polynomials in some jets space of $(2,n)$- distributions, we conclude that there exists an open set $\widetilde{\mathcal{U}}_n$ in Zariski topology of this jets space such that the set $S^0_D(q_0)$ is not empty iff the corresponding jet of D at q_0 belongs to $\widetilde{\mathcal{U}}_n$. Therefore, in the same way as in Proposition 2.5, in order to prove our proposition it is sufficient to give an example of germ of $(2,n)$-distribution such that $S^0_D(q_0)$ is nonempty. As such example one can take distribution $\bar{D}$ spanned by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + \sum_{i=1}^{n-4} \frac{x_1^i}{i!}\frac{\partial}{\partial x_{i+2}} + \frac{x_1^{n-2}}{(n-2)!}\frac{\partial}{\partial x_{n-1}} + x_1 x_2 \frac{\partial}{\partial x_n}, \tag{4.14}$$

where $(x_1, \ldots, x_n)$ are some local coordinates on M, $q_0 = (0, \ldots, 0)$. It is easy to see that the curve $(x_1, 0, \ldots, 0)$ is regular abnormal trajectory of corank 1. Moreover, from Proposition 4.2 it follows that the reduced Jacobi curve of any its lift γ has the weight jump one at the point of intersection of γ with $(\bar{D}^2)^\perp(q_0)$, which implies that $S^0_{\bar{D}}(q_0)$ is not empty. This completes the proof. $\square$

Finally, as a direct consequence of Propositions 2.5, 4.4 and 4.5 one has the following

Theorem 5 *In the case $n > 5$ a generic germ of $(2,n)$-distribution D at q has the fundamental form, which is a nonpolynomial rational function on $(D^2)^\perp(q)$.*

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