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Research Article

Variational Approach to Impulsive Differential Equations with Dirichlet Boundary Conditions

Huiwen Chen and Jianli Li

Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China

Correspondence should be addressed to Jianli Li, ljianli@sina.com

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We study the existence of n distinct pairs of nontrivial solutions for impulsive differential equations with Dirichlet boundary conditions by using variational methods and critical point theory.

1. Introduction

Impulsive effects exist widely in many evolution processes in which their states are changed abruptly at certain moments of time. Such processes are naturally seen in control theory [1, 2], population dynamics [3], and medicine [4, 5]. Due to its significance, a great deal of work has been done in the theory of impulsive differential equations. In recent years, many researchers have used some fixed point theorems [6, 7], topological degree theory [8], and the method of lower and upper solutions with monotone iterative technique [9] to study the existence of solutions for impulsive differential equations.

On the other hand, in the last few years, some researchers have used variational methods to study the existence of solutions for boundary value problems [10–16], especially, in [14–16], the authors have studied the existence of infinitely many solutions by using variational methods.

However, as far as we know, few researchers have studied the existence of n distinct pairs of nontrivial solutions for impulsive boundary value problems by using variational methods.

Motivated by the above facts, in this paper, our aim is to study the existence of n distinct pairs of nontrivial solutions to the Dirichlet boundary problem for the second-order impulsive differential equations

$$\begin{gathered} u''(t) + \lambda h(t, u(t)) = 0, \quad t \neq t_j, \text{ a.e. } t \in [0, T], \\ -\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \ldots, p, \\ u(0) = u(T) = 0, \end{gathered} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $\lambda > 0$, $h \in C([0,T] \times R, R)$, $I_j \in C(R, R)$, $j = 1, 2, \ldots, p$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, $u'(t_j^+)$ and $u'(t_j^-)$ denote the right and the left limits, respectively, of $u'(t_j)$ at $t = t_j$, $j = 1, 2, \ldots, p$.

2. Preliminaries

Definition 2.1. Suppose that E is a Banach space and $\varphi \in C^1(E, R)$. If any sequence $\{u_k\} \subset E$ for which $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$ as $k \to +\infty$ possesses a convergent subsequence in E, we say that φ satisfies the Palais-Smale condition.

Let E be a real Banach space. Define the set $\Sigma = \{A \mid A \subset E \setminus \{\theta\}$ as symmetric closed set$\}$.

Theorem 2.2 (see [17, Theorem 3.5.3]). *Let E be a real Banach space, and let $\varphi \in C^1(E, R)$ be an even functional which satisfies the Palais-Smale condition, φ is bounded from below and $\varphi(0) = 0$; suppose that there exists a set $K \subset \Sigma$ and an odd homeomorphism $h : K \to S^{n-1}$ ($n-$one-dimensional unit sphere) and $\sup_{x \in K} \varphi(x) < 0$, then φ has at least n distinct pairs of nontrivial critical points.*

To begin with, we introduce some notation. Denote by X the Sobolev space $H_0^1(0, T)$, and consider the inner product

$$(u, v) = \int_0^T u'(t) v'(t) dt \tag{2.1}$$

and the norm

$$\|u\| = \left(\int_0^T |u'(t)|^2 dt \right)^{1/2}. \tag{2.2}$$

Hence, X is reflexive. We define the norm in $C([0,T])$ as $\|x\|_\infty = \max_{t \in [0,T]} |x(t)|$.

For $u \in H^2(0,T)$, we have that u and u' are absolutely continuous and $u'' \in L^2(0,T)$. Hence, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ for every $t \in [0,T]$. If $u \in H_0^1(0,T)$, then u is absolutely continuous and $u' \in L^2(0,T)$. In this case, the one-sided derivatives $u'(t^-)$, $u'(t^+)$ may not exist. As a consequence, we need to introduce a different concept of solution. Suppose that $u \in C([0,T])$ such that for every $j = 1, 2, \ldots, p$, $u_j = u|_{(t_j, t_{j+1})}$ satisfies $u_j \in H^2(t_j, t_{j+1})$, and it satisfies the equation in problem (1.1) for $t \neq t_j$, a.e. $t \in [0,T]$, the limits $u'(t_j^+)$, $u'(t_j^-)$, and

$j = 1, 2, \ldots, p$ exist, and impulsive conditions and boundary conditions in problem (1.1) hold, we say it is *a classical solution* of problem (1.1).

Consider the functional

$$\varphi : X \longrightarrow R, \tag{2.3}$$

defined by

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \lambda \int_0^T H(t, u(t))dt - \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds, \tag{2.4}$$

where $H(t,u) = \int_0^u h(t,s)ds$. Clearly, φ is a Fréchet differentiable functional, whose Fréchet derivative at the point $u \in X$ is the functional $\varphi'(u) \in X^*$ given by

$$\varphi'(u)(v) = \int_0^T u'(t)v'(t)dt - \lambda \int_0^T h(t, u(t))v(t)dt - \sum_{j=1}^{p} I_j(u(t_j))v(t_j), \tag{2.5}$$

for any $v \in X$. Obviously, φ' is continuous.

Lemma 2.3. *If $u \in X$ is a critical point of the functional φ, then u is a classical solution of problem* (1.1).

Proof. The proof is similar to the proof of [16, Lemma 2.4], and we omit it here. □

Lemma 2.4. *Let $u \in X$, then $\|u\|_\infty \leq \sqrt{T}\|u\|$.*

Proof. For $u \in X$, then $u(0) = u(T) = 0$. Hence, for $t \in [0,T]$, by Hölder's inequality, we have

$$|u(t)| = \left|\int_0^t u'(s)ds\right| \leq \int_0^T |u'(s)|ds \leq \sqrt{T}\left(\int_0^T |u'(s)|^2 ds\right)^{1/2} = \sqrt{T}\|u\|, \tag{2.6}$$

which completes the proof. □

3. Main Results

Theorem 3.1. *Suppose that the following conditions hold.*

(i) *There exist $a, b > 0$ and $\gamma \in [0,1)$ such that*

$$|h(t,u)| \leq a + b|u|^\gamma \quad \textit{for any} \ (t,u) \in [0,T] \times R. \tag{3.1}$$

(ii) *$h(t,u)$ is odd about u and $H(t,u) > 0$ for every $(t,u) \in [0,T] \times R \setminus \{0\}$.*

(iii) *$I_j(u)$ $(j = 1, 2, \ldots, p)$ are odd and $\int_0^u I_j(s)ds \leq 0$ for any $u \in R$ $(j = 1, 2, \ldots, p)$.*

Then for any $n \in N$, there exists λ_n such that $\lambda > \lambda_n$, and problem (1.1) *has at least n distinct pairs of nontrivial classical solutions.*

Proof. By (2.4), (ii), and (iii), $\varphi \in C^1(X, R)$ is an even functional and $\varphi(0) = 0$.

Next, we will verify that φ is bounded from below. In view of (i), (iii), and Lemma 2.4, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_0^T H(t, u(t))dt - \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds \\ &\geq \frac{1}{2}\|u\|^2 - \lambda \int_0^T \left(a|u(t)| + b|u(t)|^{\gamma+1}\right)dt \\ &\geq \frac{1}{2}\|u\|^2 - \lambda a T^{3/2}\|u\| - \lambda b T^{(\gamma+3)/2}\|u\|^{\gamma+1} \\ &> -\infty,\end{aligned} \tag{3.2}$$

for any $u \in X$. That is, φ is bounded from below.

In the following we will show that φ satisfies the Palais-Smale condition. Let $\{u_k\} \subset X$, such that $\{\varphi(u_k)\}$ is a bounded sequence and $\lim_{k \to \infty} \varphi'(u_k) = 0$. Then, there exists $M > 0$ such that

$$\left|\varphi(u_k)\right| \leq M. \tag{3.3}$$

In view of (3.2), we have

$$M \geq \frac{1}{2}\|u_k\|^2 - \lambda a T^{3/2}\|u_k\| - \lambda b T^{(\gamma+3)/2}\|u_k\|^{\gamma+1}. \tag{3.4}$$

So $\{u_k\}$ is bounded in X. From the reflexivity of X, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_k\}$, $u_k \rightharpoonup u$ in X. Next, we will verify that $\{u_k\}$ strongly converges to u in X. By (2.5), we have

$$\begin{aligned}\left(\varphi'(u_k) - \varphi'(u)\right)(u_k - u) = \|u_k - u\|^2 &- \lambda \int_0^T [h(t, u_k(t)) - h(t, u(t))](u_k(t) - u(t))dt \\ &+ \sum_{j=1}^{p} \left[I_j(u_k(t_j)) - I_j(u(t_j))\right]\left(u_k(t_j) - u(t_j)\right).\end{aligned} \tag{3.5}$$

By $u_k \rightharpoonup u$ in X, we see that $\{u_k\}$ uniformly converges to u in $C([0, T])$. So,

$$\begin{aligned}&\lambda \int_0^T [h(t, u_k(t)) - h(t, u(t))](u_k(t) - u(t))dt \longrightarrow 0, \\ &\sum_{j=1}^{p} \left[I_j(u_k(t_j)) - I_j(u(t_j))\right]\left(u_k(t_j) - u(t_j)\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.\end{aligned} \tag{3.6}$$

By $\lim_{k\to\infty}\varphi'(u_k)=0$ and $u_k \rightharpoonup u$, we have

$$(\varphi'(u_k)-\varphi'(u))(u_k-u)\longrightarrow 0 \quad \text{as } k\longrightarrow\infty. \tag{3.7}$$

In view of (3.5), (3.6), and (3.7), we obtain $\|u_k-u\|\to 0$ as $k\to\infty$. Then, φ satisfies the Palais-Smale condition.

Let $v_m(t)=(\sqrt{2T}/m\pi)\sin(m\pi/T)t$, $m=1,2,\ldots,n$, then

$$\|v_m\|^2=1=\frac{m^2\pi^2}{T^2}\int_0^T|v_m(t)|^2dt, \quad m=1,2,\ldots,n. \tag{3.8}$$

Define

$$K_n(r)=\left\{\sum_{m=1}^{n}c_mv_m \mid \sum_{m=1}^{n}c_m^2=r^2\right\}, \quad r>0. \tag{3.9}$$

Then, for any $r>0$, there exists an odd homeomorphism $f: K_n(r)\to S^{n-1}$. Let $0<r<1/\sqrt{T}$, then $\|u\|_\infty\le\sqrt{T}\|u\|=\sqrt{T}r<1$ for any $u\in K_n(r)$. By (ii), we have

$$H(t,u(t))=\int_0^{u(t)}h(t,s)ds>0 \quad \text{as } u(t)\neq 0, \tag{3.10}$$

then $\int_0^T H(t,u(t))dt>0$ for any $u\in K_n(r)$.

Let $\alpha_n=\inf_{u\in K_n(r)}\int_0^T H(t,u(t))dt$, $\beta_n=\inf_{u\in K_n(r)}\sum_{j=1}^{p}\int_0^{u(t_j)}I_j(s)ds$, then $\alpha_n>0$, $\beta_n\le 0$. Let $\lambda_n=((1/2)r^2-\beta_n)\alpha_n^{-1}>0$, then when $\lambda>\lambda_n$, for any $u\in K_n(r)$, we have

$$\begin{aligned}\varphi(u)&\le\frac{1}{2}r^2-\lambda\alpha_n-\beta_n\\&<\frac{1}{2}r^2-\lambda_n\alpha_n-\beta_n\\&=0.\end{aligned} \tag{3.11}$$

By Theorem 2.2, φ possesses at least n distinct pairs of nontrivial critical points. That is, problem (1.1) has at least n distinct pairs of nontrivial classical solutions. □

Corollary 3.2. *Let the following conditions hold:*

(i) *$h(t,u)$ is bounded,*

(ii) *$h(t,u)$ is odd about u and $H(t,u)>0$ for every $(t,u)\in[0,T]\times R\setminus\{0\}$,*

(iii) *$I_j(u)$ $(j=1,2,\ldots,p)$ are odd and $\int_0^u I_j(s)ds\le 0$ for any $u\in R$ $(j=1,2,\ldots,p)$.*

Then, for any $n\in N$, there exists λ_n such that $\lambda>\lambda_n$, and problem (1.1) *has at least n distinct pairs of nontrivial classical solutions.*

Proof. Let $\gamma = 0$ in Theorem 3.1, then Corollary 3.2 holds. □

Theorem 3.3. *Suppose that the following conditions hold.*

(i) *There exists* $a, b > 0$ *and* $\gamma \in [0,1)$ *such that*

$$|h(t,u)| \le a + b|u|^{\gamma} \quad \text{for any } (t,u) \in [0,T] \times R. \tag{3.12}$$

(ii) *There exists* $a_j, b_j > 0$ *and* $\gamma_j \in [0,1)$ $(j = 1,2,\ldots,p)$ *such that*

$$\left|I_j(u)\right| \le a_j + b_j|u|^{\gamma_j} \quad \text{for any } u \in R \ (j = 1,2,\ldots,p). \tag{3.13}$$

(iii) $h(t,u)$ *and* $I_j(u)$ $(j = 1,2,\ldots,p)$ *are odd about* u *and* $H(t,u) > 0$ *for every* $(t,u) \in [0,T] \times R \setminus \{0\}$.

Then, for any $n \in N$, *there exists* λ_n *such that* $\lambda > \lambda_n$, *and problem* (1.1) *has at least* n *distinct pairs of nontrivial classical solutions.*

Proof. By (2.4) and (iii), $\varphi \in C^1(X,R)$ is an even functional and $\varphi(0) = 0$.

Next, we will verify that φ is bounded from below. Let $M_1 = \max\{a_1, a_2, \ldots, a_p\}$, $M_2 = \max\{b_1, b_2, \ldots, b_p\}$. In view of (i), (ii), and Lemma 2.4, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_0^T H(t,u(t))dt + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds \\
&\ge \frac{1}{2}\|u\|^2 - \lambda \int_0^T \left(a|u(t)| + b|u(t)|^{\gamma+1}\right)dt \\
&\quad - \sum_{j=1}^{p}\left(a_j\left|u(t_j)\right| + b_j\left|u(t_j)\right|^{\gamma_j+1}\right) \\
&\ge \frac{1}{2}\|u\|^2 - \lambda a T^{3/2}\|u\| - \lambda b T^{(\gamma+3)/2}\|u\|^{\gamma+1} - pM_1\sqrt{T}\|u\| \\
&\quad - M_2 \sum_{j=1}^{p} T^{(\gamma_j+1)/2}\|u\|^{\gamma_j+1} \\
&> -\infty,
\end{aligned} \tag{3.14}$$

for any $u \in X$. That is, φ is bounded from below.

In the following, we will show that φ satisfies the Palais-Smale condition. As in the proof of Theorem 3.1, by (3.3) and (3.14), we have

$$M \ge \frac{1}{2}\|u_k\|^2 - \lambda a T^{3/2}\|u_k\| - \lambda b T^{(\gamma+3)/2}\|u_k\|^{\gamma+1} - pM_1\sqrt{T}\|u_k\| - M_2 \sum_{j=1}^{p} T^{(\gamma_j+1)/2}\|u_k\|^{\gamma_j+1}. \tag{3.15}$$

It follows that $\{u_k\}$ is bounded in X. In the following, the proof of the Palais-Smale condition is the same as that in Theorem 3.1, and we omit it here.

Take the same $K_n(r)$ as in Theorem 3.1, then for any $r > 0$, there exists an odd homeomorphism $f : K_n(r) \to S^{n-1}$. Let $0 < r < 1/\sqrt{T}$, then $\|u\|_\infty \le \sqrt{T}\|u\| = \sqrt{T}r < 1$ for any $u \in K_n(r)$. By (iii), we have

$$H(t,u(t)) = \int_0^{u(t)} h(t,s)ds > 0 \quad \text{as } u(t) \neq 0. \tag{3.16}$$

Then, $\int_0^T H(t,u(t))dt > 0$ for any $u \in K_n(r)$.

Let $\alpha_n = \inf_{u\in K_n(r)} \int_0^T H(t,u(t))dt$, $\beta_n = \inf_{u\in K_n(r)} \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds$, then $\alpha_n > 0$. Let $\lambda_n = \max\{0, ((1/2)r^2 - \beta_n)\alpha_n^{-1}\}$, then when $\lambda > \lambda_n$, for any $u \in K_n(r)$, we have

$$\varphi(u) \le \frac{1}{2}r^2 - \lambda\alpha_n - \beta_n < \frac{1}{2}r^2 - \lambda_n\alpha_n - \beta_n \le 0. \tag{3.17}$$

By Theorem 2.2, φ possesses at least n distinct pairs of nontrivial critical points. That is, problem (1.1) has at least n distinct pairs of nontrivial classical solutions. □

Corollary 3.4. *Let the following conditions hold:*

(i) $h(t,u)$ *is bounded,*

(ii) $I_j(u)$ $(j = 1,2,\dots,p)$ *are bounded,*

(iii) $h(t,u)$ *and* $I_j(u)$ $(j = 1,2,\dots,p)$ *are odd about* u *and* $H(t,u) > 0$ *for every* $(t,u) \in [0,T] \times R \setminus \{0\}$.

Then, for any $n \in N$, *there exists* λ_n *such that* $\lambda > \lambda_n$, *and problem* (1.1) *has at least* n *distinct pairs of nontrivial classical solutions.*

Proof. Let $\gamma = 0$ and $\gamma_j = 0$ $(j = 1,2,\dots,p)$ in Theorem 3.3, then Corollary 3.4 holds. □

Theorem 3.5. *Suppose that the following conditions hold.*

(i) *There exist constants* $\sigma > 0$ *such that* $h(t,\sigma) = 0, h(t,u) > 0$ *for every* $u \in (0,\sigma)$.

(ii) $h(t,u)$ *is odd about* u.

(iii) $I_j(u)$ $(j = 1,2,\dots,p)$ *are odd and* $\int_0^u I_j(s)ds \le 0$ *for any* $u \in R$ $(j = 1,2,\dots,p)$.

Then, for any $n \in N$, *there exists* λ_n *such that* $\lambda > \lambda_n$, *and problem* (1.1) *has at least* n *distinct pairs of nontrivial classical solutions.*

Proof. Let

$$h_1(t,u) = \begin{cases} h(t,\sigma), & u > \sigma, \\ h(t,u), & |u| \le \sigma, \\ h(t,-\sigma), & u < -\sigma, \end{cases} \tag{3.18}$$

then $h_1(t,u)$ is continuous, bounded, and odd. Consider boundary value problem

$$\begin{aligned} &u''(t)+\lambda h_1(t,u(t))=0, \quad t\neq t_j, \text{ a.e. } t\in[0,T],\\ &-\Delta u'(t_j)=I_j(u(t_j)), \quad j=1,2,\dots,p,\\ &u(0)=u(T)=0. \end{aligned} \tag{3.19}$$

Next, we will verify that the solutions of problem (3.19) are solutions of problem (1.1). In fact, let $u_0(t)$ be the solution of problem (3.19). If $\max_{0\le t\le T}u_0(t)>\sigma$, then there exists an interval $[a,b]\subset[0,T]$ such that

$$u_0(a)=u_0(b)=\sigma, \quad u_0(t)>\sigma \quad \text{for any } t\in(a,b). \tag{3.20}$$

When $t\in[a,b]$, by (i), we have

$$u_0''(t)=-\lambda h_1(t,u)=-\lambda h(t,\sigma)=0. \tag{3.21}$$

Thus, there exist constants c such that $u_0'(t)=c$ for any $t\in[a,b]$. We consider the following two possible cases.

Case 1. $c\ge 0$, then u_0' is nondecreasing in $[a,b]$. By $u_0'(a)\ge 0$ and $u_0'(b)\le 0$, we have

$$0\le u_0'(a)\le u_0'(t)\le u_0'(b)\le 0 \quad \text{for every } t\in[a,b]. \tag{3.22}$$

That is, $u_0'(t)\equiv 0$ for any $t\in[a,b]$. So, there exists a constant d such that $u_0(t)\equiv d$, which contradicts (3.20). Then, $\max_{0\le t\le T}u_0(t)\le\sigma$. Similarly, we can prove that $\min_{0\le t\le T}u_0(t)\ge-\sigma$.

Case 2. $c<0$, the arguments are analogous, then $u_0(t)$ is solution of problem (1.1).

For every $u\in X$, we consider the functional

$$\varphi_1 : X\longrightarrow R, \tag{3.23}$$

defined by

$$\varphi_1(u)=\frac{1}{2}\|u\|^2-\lambda\int_0^T H_1(t,u(t))dt-\sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds, \tag{3.24}$$

where $H_1(t,u)=\int_0^u h_1(t,s)ds$.

It is clear that φ_1 is Fréchet differentiable at any $u\in X$ and

$$\varphi_1'(u)(v)=\int_0^T u'(t)v'(t)dt-\lambda\int_0^T h_1(t,u(t))v(t)dt-\sum_{j=1}^{p}I_j(u(t_j))v(t_j), \tag{3.25}$$

for any $v \in X$. Obviously, φ_1' is continuous. By Lemma 2.3, we have the critical points of φ_1 as solutions of problem (3.19). By (3.24), (ii), and (iii), $\varphi_1 \in C^1(X, R)$ is an even functional and $\varphi_1(0) = 0$.

In the following, we will show that φ_1 is bounded from below. since $h_1(t, u) = 0$ for $|u| \geq \sigma$, thus

$$\int_0^T H_1(t, u(t))dt = \int_0^T \int_0^{u(t)} h_1(t, s)ds\, dt \leq \int_0^T \int_0^{\sigma} h_1(t, s)ds\, dt = e > 0. \tag{3.26}$$

By (iii), we have

$$\begin{aligned} \varphi_1(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_0^T H_1(t, u(t))dt - \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds \\ &\geq \frac{1}{2}\|u\|^2 - \lambda e \geq -\lambda e, \end{aligned} \tag{3.27}$$

for any $u \in X$. That is, φ_1 is bounded from below.

In the following we will show that φ_1 satisfies the Palais-Smale condition. Let $\{u_k\} \subset X$ such that $\{\varphi_1(u_k)\}$ is a bounded sequence and $\lim_{k\to\infty}\varphi_1'(u_k) = 0$. Then, there exists $M_3 > 0$ such that

$$\left|\varphi_1(u_k)\right| \leq M_3. \tag{3.28}$$

By (3.27), we have

$$\frac{1}{2}\|u_k\|^2 \leq M_3 + \lambda e. \tag{3.29}$$

It follows that $\{u_k\}$ is bounded in X. In the following, the proof of the Palais-Smale condition is the same as that in Theorem 3.1, and we omit it here.

Take the same $K_n(r)$ as in Theorem 3.1, then, for any $r > 0$, there exists an odd homeomorphism $f : K_n(r) \to S^{n-1}$. Let $0 < r < \sigma/\sqrt{T}$, then $\|u\|_\infty \leq \sqrt{T}\|u\| = \sqrt{T}r < \sigma$ for any $u \in K_n(r)$. By (i) and (ii), we have

$$H_1(t, u(t)) = \int_0^{u(t)} h_1(t, s)ds = \int_0^{u(t)} h(t, s)dt > 0 \quad \text{as } u(t) \neq 0. \tag{3.30}$$

Then, $\int_0^T H_1(t, u(t))dt > 0$ for any $u \in K_n(r)$.

Let $\alpha_n = \inf_{u\in K_n(r)} \int_0^T H_1(t,u(t))dt$, $\beta_n = \inf_{u\in K_n(r)} \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds$, then $\alpha_n > 0$, $\beta_n \le 0$. Let $\lambda_n = ((1/2)r^2 - \beta_n)\alpha_n^{-1} > 0$, then when $\lambda > \lambda_n$, for any $u \in K_n(r)$, we have

$$\begin{aligned}\varphi_1(u) &\le \frac{1}{2}r^2 - \lambda\alpha_n - \beta_n \\ &< \frac{1}{2}r^2 - \lambda_n\alpha_n - \beta_n \\ &= 0.\end{aligned} \tag{3.31}$$

By Theorem 2.2, φ_1 possesses at least n distinct pairs of nontrivial critical points. Then, problem (3.19) has at least n distinct pairs of nontrivial classical solutions, that is, problem (1.1) has at least n distinct pairs of nontrivial classical solutions □

Theorem 3.6. *Let the following conditions hold.*

(i) *There exist constants $\sigma > 0$ such that $h(t,\sigma) = 0, h(t,u) > 0$ for every $u \in (0,\sigma)$.*

(ii) *There exist $a_j, b_j > 0$, and $\gamma_j \in [0,1)$ $(j = 1,2,\ldots,p)$ such that*

$$\left|I_j(u)\right| \le a_j + b_j|u|^{\gamma_j} \quad \text{for any } u \in R \ (j = 1,2,\ldots,p). \tag{3.32}$$

(iii) *$h(t,u)$ and $I_j(u)$ $(j = 1,2,\ldots,p)$ are odd about u.*

Then, for any $n \in N$, there exists λ_n such that $\lambda > \lambda_n$, and problem (1.1) *has at least n distinct pairs of nontrivial classical solutions.*

Proof. The proof is similar to the proof of Theorem 3.5, and we omit it here. □

Theorem 3.7. *Let the following conditions hold.*

(i) *There exist constants $\sigma_1 > 0$ such that $h(t,\sigma_1) \le 0$.*

(ii) *There exist $a_j, b_j > 0$, and $\gamma_j \in [0,1)$ $(j = 1,2,\ldots,p)$ such that*

$$\left|I_j(u)\right| \le a_j + b_j|u|^{\gamma_j} \quad \text{for any } u \in R \ (j = 1,2,\ldots,p). \tag{3.33}$$

(iii) *$h(t,u)$ and $I_j(u)$ $(j = 1,2,\ldots,p)$ are odd about u and $\lim_{u\to 0}h(t,u)/u = 1$ uniformly for $t \in [0,T]$.*

Then, for any $n \in N$, there exists λ_n such that $\lambda > \lambda_n$, and problem (1.1) *has at least n distinct pairs of nontrivial classical solutions.*

Proof. Let

$$h_2(t,u) = \begin{cases} h(t,\sigma_1), & u > \sigma_1, \\ h(t,u), & |u| \le \sigma_1, \\ h(t,-\sigma_1), & u < -\sigma_1, \end{cases} \tag{3.34}$$

then $h_2(t,u)$ is continuous, bounded, and odd. Consider boundary value problem

$$\begin{gathered} u''(t) + \lambda h_2(t,u(t)) = 0, \quad t \neq t_j, \text{ a.e. } t \in [0,T], \\ -\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\dots,p, \\ u(0) = u(T) = 0. \end{gathered} \tag{3.35}$$

Next, we will verify that the solutions of problem (3.35) are solutions of problem (1.1). In fact, let $u_0(t)$ be the solution of problem (3.35). If $\max_{0\le t\le T} u_0(t) > \sigma_1$, then there exists an interval $[a,b] \subset [0,T]$ such that

$$u_0(a) = u_0(b) = \sigma_1, \quad u_0(t) > \sigma_1 \quad \text{for any } t \in (a,b). \tag{3.36}$$

When $t \in [a,b]$, by (i), we have

$$u_0''(t) = -\lambda h_2(t,u) = -\lambda h(t,\sigma_1) \ge 0. \tag{3.37}$$

Thus, $u_0'(t)$ is nondecreasing in $[a,b]$. By $u_0'(a) \ge 0$ and $u_0'(b) \le 0$, we have

$$0 \le u_0'(a) \le u_0'(t) \le u_0'(b) \le 0 \quad \text{for every } t \in [a,b]. \tag{3.38}$$

That is, $u_0'(t) \equiv 0$ for any $t \in [a,b]$. So, there exists a constant d such that $u_0(t) \equiv d$, which contradicts (3.36). Then $\max_{0\le t\le T} u_0(t) \le \sigma_1$. Similarly, we can prove that $\min_{0\le t\le T} u_0(t) \ge -\sigma_1$. Then, $u_0(t)$ is solution of problem (1.1).

For every $u \in X$, we consider the functional

$$\varphi_2 : X \longrightarrow R, \tag{3.39}$$

defined by

$$\varphi_2(u) = \frac{1}{2}\|u\|^2 - \lambda \int_0^T H_2(t,u(t))dt - \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds, \tag{3.40}$$

where $H_2(t,u) = \int_0^u h_2(t,s)ds$.

It is clear that φ_2 is Fréchet differentiable at any $u \in X$ and

$$\varphi_2'(u)(v) = \int_0^T u'(t)v'(t)dt - \lambda \int_0^T h_2(t,u(t))v(t)dt - \sum_{j=1}^{p} I_j(u(t_j))v(t_j), \tag{3.41}$$

for any $v \in X$. Obviously, φ_2' is continuous. By Lemma 2.3, we have the critical points of φ_2 as solutions of problem (3.35). By (3.40) and (iii), $\varphi_2 \in C^1(X,R)$ is an even functional and $\varphi_2(0) = 0$.

Next, we will show that φ_2 is bounded from below. Let $M_1 = \max\{a_1, a_2, \dots, a_p\}$, $M_2 = \max\{b_1, b_2, \dots, b_p\}$. since $uh_2(t,u) \le 0$ for $|u| \ge \sigma_1$, thus

$$\int_0^T H_2(t,u(t))dt = \int_0^T \int_0^{u(t)} h_2(t,s)ds\,dt \le \int_0^T \int_0^{\sigma_1} h_2(t,s)ds\,dt = e. \tag{3.42}$$

By (ii) and Lemma 2.4, we have

$$\begin{aligned}\varphi_2(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_0^T H_2(t,u(t))dt - \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds \\ &\ge \frac{1}{2}\|u\|^2 - \lambda e - pM_1\sqrt{T}\|u\| - M_2 \sum_{j=1}^{p} T^{(\gamma_j+1)/2}\|u\|^{\gamma_j+1} \\ &> -\infty,\end{aligned} \tag{3.43}$$

for any $u \in X$. That is, φ_2 is bounded from below.

In the following we will show that φ_2 satisfies the Palais-Smale condition. Let $\{u_k\} \subset X$ such that $\{\varphi_2(u_k)\}$ is a bounded sequence and $\lim_{k\to\infty}\varphi_2'(u_k) = 0$. Then, there exists $M_4 > 0$ such that

$$\left|\varphi_2(u_k)\right| \le M_4. \tag{3.44}$$

By (3.43), we have

$$\frac{1}{2}\|u_k\|^2 \le M_4 + \lambda e + pM_1\sqrt{T}\|u_k\| + M_2 \sum_{j=1}^{p} T^{(\gamma_j+1)/2}\|u_k\|^{\gamma_j+1}. \tag{3.45}$$

It follows that $\{u_k\}$ is bounded in X. In the following, the proof of the Palais-Smale condition is the same as that in Theorem 3.1, and we omit it here.

Take the same $K_n(r)$ as in Theorem 3.1, then for any $r > 0$, there exists an odd homeomorphism $f : K_n(r) \to S^{n-1}$. By (iii), for any $0 < \varepsilon < 1$, there exists $\delta > 0$, when $|u| \le \delta$, we have

$$h_2(t,u) \ge u - \varepsilon|u|. \tag{3.46}$$

Let $0 < r < \min\{\sigma_1/\sqrt{T}, \delta/\sqrt{T}\}$, then $\|u\|_\infty \le \sqrt{T}\|u\| = \sqrt{T}r < \min\{\sigma_1, \delta\}$ for any $u \in K_n(r)$. Then, $\int_0^T H_2(t,u(t))dt = \int_0^T \int_0^{u(t)} h_2(t,s)dt \ge \int_0^T (1/2)(1-\varepsilon)|u(t)|^2 dt > 0$ for any $u \in K_n(r)$.

Let $\alpha_n = \inf_{u\in K_n(r)} \int_0^T H_2(t,u(t))dt$, $\beta_n = \inf_{u\in K_n(r)} \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)ds$, then $\alpha_n > 0$. Let $\lambda_n = \max\{((1/2)r^2 - \beta_n)\alpha_n^{-1}, 0\}$, then when $\lambda > \lambda_n$, for any $u \in K_n(r)$, we have

$$\begin{aligned} \varphi_1(u) &\le \frac{1}{2}r^2 - \lambda\alpha_n - \beta_n \\ &< \frac{1}{2}r^2 - \lambda_n\alpha_n - \beta_n \\ &\le 0. \end{aligned} \tag{3.47}$$

By Theorem 2.2, φ_2 possesses at least n distinct pairs of nontrivial critical points. Then, problem (3.35) has at least n distinct pairs of nontrivial classical solutions, that is, problem (1.1) has at least n distinct pairs of nontrivial classical solutions. □

Theorem 3.8. *Let the following conditions hold.*

(i) *There exist constants $\sigma_1 > 0$ such that $h(t,\sigma_1) \le 0$.*

(ii) $\lim_{u\to 0} h(t,u)/u = 1$ *uniformly for $t \in [0,T]$.*

(iii) *$h(t,u)$ and $I_j(u)$ $(j = 1,2,\ldots,p)$ are odd about u and $\int_0^u I_j(s)ds \le 0$ for any $u \in R$ $(j = 1,2,\ldots,p)$.*

Then, for any $n \in N$, there exists λ_n such that $\lambda > \lambda_n$, and problem (1.1) *has at least n distinct pairs of nontrivial classical solutions.*

Proof. The proof is similar to the proof of Theorem 3.7, and we omit it here. □

4. Some Examples

Example 4.1. Consider boundary value problem

$$\begin{gathered} u''(t) + \lambda(1+t)\sqrt[3]{u(t)} = 0, \quad t \ne t_j, \text{ a.e. } t \in [0,\pi], \\ -\Delta u'(t_j) = -u(t_j), \quad j = 1,2, \\ u(0) = u(\pi) = 0. \end{gathered} \tag{4.1}$$

It is easy to see that conditions (i), (ii), and (iii) of Theorem 3.1 hold. Let

$$\begin{gathered} \alpha_n = \inf_{u\in K_n(r)} \frac{3}{4}\int_0^{\pi} (1+t)|u(t)|^{4/3}dt > \inf_{u\in K_n(r)} \frac{3}{4}\int_0^{\pi} |u(t)|^2 dt > \frac{3r^2}{4n^2}, \\ \beta_n = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\int_0^{u(t_j)} s ds = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\frac{|u(t_j)|^2}{2} \ge -\pi r^2, \end{gathered} \tag{4.2}$$

then $\lambda_n = ((1/2)r^2 - \beta_n)\alpha_n^{-1} < ((2+4\pi)/3)n^2$. Applying Theorem 3.1, then for any $n \in N$, when $\lambda > ((2+4\pi)/3)n^2$, problem (4.1) has at least n distinct pairs of nontrivial classical solutions.

Example 4.2. Consider boundary value problem

$$u''(t) + \lambda(1+t)\sqrt[3]{u(t)} = 0, \quad t \neq t_j, \text{ a.e. } t \in [0,\pi],$$
$$-\Delta u'(t_j) = -\sqrt[3]{u(t_j)}, \quad j = 1,2, \tag{4.3}$$
$$u(0) = u(\pi) = 0.$$

It is easy to see that conditions (i), (ii), and (iii) of Theorem 3.3 hold. Let $r = 1/2\sqrt{\pi}$,

$$\alpha_n = \inf_{u\in K_n(r)} \frac{3}{4}\int_0^{\pi} (1+t)|u(t)|^{4/3}dt > \inf_{u\in K_n(r)} \frac{3}{4}\int_0^{\pi} |u(t)|^2 dt > \frac{3r^2}{4n^2},$$
$$\beta_n = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\int_0^{u(t_j)} s^{1/3}ds = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\frac{3}{4}\left|u(t_j)\right|^{4/3} > -\frac{3}{2}, \tag{4.4}$$

then $\lambda_n = ((1/2)r^2 - \beta_n)\alpha_n^{-1} < ((2+24\pi)/3)n^2$. Applying Theorem 3.3, then for any $n \in N$, when $\lambda > ((2+24\pi)/3)n^2$, problem (4.3) has at least n distinct pairs of nontrivial classical solutions.

Example 4.3. Consider boundary value problem

$$u''(t) + \lambda\left(1+t^2\right)\left[u(t) - (u(t))^3\right] = 0, \quad t \neq t_j, \text{ a.e. } t \in [0,\pi],$$
$$-\Delta u'(t_j) = -u(t_j), \quad j = 1,2, \tag{4.5}$$
$$u(0) = u(\pi) = 0.$$

Let $\sigma = 1$, it is easy to see that conditions (i), (ii), and (iii) of Theorem 3.5 hold. Let

$$\alpha_n = \inf_{u\in K_n(r)} \int_0^{\pi}\left(1+t^2\right)\left(\frac{1}{2}|u(t)|^2 - \frac{1}{4}|u(t)|^4\right)dt > \inf_{u\in K_n(r)} \frac{1}{4}\int_0^{\pi} |u(t)|^2 dt > \frac{r^2}{4n^2},$$
$$\beta_n = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\int_0^{u(t_j)} s\,ds = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\frac{\left|u(t_j)\right|^2}{2} \geq -\pi r^2, \tag{4.6}$$

then $\lambda_n = ((1/2)r^2 - \beta_n)\alpha_n^{-1} < (2+4\pi)n^2$. Applying Theorem 3.5, then for any $n \in N$, when $\lambda > (2+4\pi)n^2$, problem (4.5) has at least n distinct pairs of nontrivial classical solutions.

Example 4.4. Consider boundary value problem

$$u''(t) + \lambda\left[u(t) - (1+t)(u(t))^3\right] = 0, \quad t \neq t_j, \text{ a.e. } t \in [0,\pi],$$
$$-\Delta u'(t_j) = -\sqrt[3]{u(t_j)}, \quad j = 1,2, \tag{4.7}$$
$$u(0) = u(\pi) = 0.$$

Let $\sigma_1 = 1$, it is easy to see that conditions (i), (ii), and (iii) of Theorem 3.7 hold. Let $r = 1/4\sqrt{\pi}$,

$$\alpha_n = \inf_{u\in K_n(r)} \int_0^{\pi} \left[\frac{1}{2}|u(t)|^2 - \frac{1}{4}(1+t)|u(t)|^4\right] dt > \inf_{u\in K_n(r)} \frac{1}{4}\int_0^{\pi} |u(t)|^2 dt > \frac{r^2}{4n^2},$$
$$\beta_n = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\int_0^{u(t_j)} s^{1/3} ds = \inf_{u\in K_n(r)} -\sum_{j=1}^{2}\frac{3}{4}\left|u(t_j)\right|^{4/3} > -\frac{3}{2}, \tag{4.8}$$

then $\lambda_n = ((1/2)r^2 - \beta_n)\alpha_n^{-1} < (2 + 24\pi)n^2$. Applying Theorem 3.7, then for any $n \in N$, when $\lambda > (2 + 24\pi)n^2$, problem (4.7) has at least n distinct pairs of nontrivial classical solutions.

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