# VARIATONAL CONSTRUCTION OF CONNECTING ORBITS 

by John N. MATHER ${ }^{(*)}$

To Bernard Malgrange on his $65^{\text {th }}$ Birthday

## Introduction.

In Hamiltonian mechanics, various questions and results concern whether there exists an orbit which in the infinite past tends to one region of phase space and in the infinite future tends to another region of phase space. Other questions and results concern the possibility of finding an orbit which visits a prescribed sequence of regions of phase space in turn.

In [Ma5], I obtained results of this type for a class $\mathcal{P}^{1}$ of $C^{1}$ diffeomorphisms of the infinite cylinder $\mathbb{T} \times \mathbb{R}$ (where $\mathbb{T}=\mathbb{R} / \mathbb{Z}$ ). This is the class of diffeomorphisms which can be represented as $f_{1} \ldots f_{n}$, where each $f_{i}$ is an exact area preserving positive monotone twist diffeomorphism of the infinite cylinder which preserves the ends and twists each end infinitely. (See $[\mathrm{Ma} 5, \S 1]$ for a detailed definition.)

It is well known, by KAM theory, that invariant simple closed curves for such diffeomorphims often exist. Of course, any such invariant curve divides the cylinder into two regions and any orbit must stay in one region or the other. Thus, the existence of invariant curves limits the possibility of the construction of orbits of the type we seek.

Simple closed curves (i.e., Jordan curves, or homeomorphs of the circle) in the cylinder come in two varieties : those which separate the

[^0]top end of the cylinder from the bottom end, and those which don't. These two varieties may also be described as the homotopically non-trivial and the homotopically trivial Jordan curves in the cylinder.
G.D. Birkhoff proved that when $f \in \mathcal{P}^{1}$, any $f$-invariant homotopically non-trivial Jordan curve $\Gamma$ in the infinite cylinder is the graph of a Lipschitz function $u: \mathbb{T} \longrightarrow \mathbb{R}$. Moreover, his argument provides an a priori upper bound for the Lipschitz constant of $u$, which depends only on $f$. Explicitly, if both $f$ and $f^{-1}$ twist the vertical by an angle of at least $\theta$, then $\cot \theta$ is an upper bound for the Lipschitz constant of $u$. (See, e.g., the Appendix of [Ma5].)

Let $K=K_{f}$ denote the union of all $f$-invariant homotopically nontrivial Jordan curves $\Gamma$ in the infinite cyclinder. From Birkhoff's a priori bound, it follows that $K$ is closed. For a generic $f$, each component of the complement of $K$ is topologically an annulus, which goes around the cylinder. (The only exceptions occur when there are two $f$-invariant homotopically non-trivial Jordan curves of the same rotation number. In such cases, the rotation number is rational. Such cases do not occur for generic $f$.) A component of the complement of $K$ which is topologically an annulus is called a Birkhoff region of instability.

The results in [Ma5] concern the existence of orbits in a Birkhoff region of instability. To state these results, it is necessary to introduce the notion of the average action (or average Poincaré- Cartan invariant) of an invariant probability measure. The notion of the action (or PoincaréCartan invariant) for a periodic orbit of a Hamiltonian system is well known [Cart]. Dividing the action by the period, we obtain the average action of an orbit. This notion has an obvious generalization to invariant measures. (See $\S 1$ for the definition.)

It is also necessary to introduce the notion of the rotation number of an invariant probability measure $\mu$. This is the average advance in the $T$ coordinate of a point of the cylinder under iteration by $f$, the average being taken with respect to $\mu$. For what we will do, it is necessary to define the rotation number as a real number (not a number mod. 1). For this, it is necessary to choose a lift $\tilde{f}$ of $f$ to the universal cover $\mathbb{R}^{2}$ of the cylinder. Replacing the lift $\tilde{f}$ with another lift $\tilde{f}+(k, 0), k \in \mathbb{Z}$, adds $k$ to the rotation number of every $f$-invariant probability measure, and thus does not change anything important in the subsequent discussion. In the sequel, we will suppose that a lift $\tilde{f}$ of $f$ has been chosen once and for all. Then for any invariant probability measure $\mu$ whose average action $A(\mu)$ is finite,
the rotation number $\rho(\mu)$ is a well defined real number. (See $\S 1$ for precise definitions.)

The set $\{(\rho(\mu), A(\mu): \mu$ is an $f$-invariant probability measure with $A(\mu)<+\infty\}$ is a convex subset of $\mathbb{R}^{2}$, since the set of $f$-invariant probability measures of finite action is convex, and $\rho(\mu)$ and $A(\mu)$ are affine functions of $\mu$. In fact, this set is the epigraph of a convex real valued function $\beta=\beta_{f}$ of a real variable. We call $\beta(\omega)$ the minimal average action associated to the rotation number $\omega$ (and the diffeomorphism $f$ ). By definition, $A(\mu) \geqslant \beta(\rho(\mu))$, for any invariant probability measure $\mu$ for which $A(\mu)<+\infty$. We say that an invariant probability measure $\mu$ is action minimizing if $A(\mu)=\beta(\rho(\mu))$.

If $\Gamma$ is an $f$-invariant homotopically non-trivial Jordan curve in the cylinder, then any invariant measure supported in $\Gamma$ is action minimizing [Ma5, Prop. 2.8]. There are two cases. When the rotation number of $\Gamma$ is irrational, then $f \mid \Gamma$ is uniquely ergodic, i.e., there is just one invariant probability measure with support in $\Gamma$. This is part of the well known Denjoy theory. (See, e.g., [Her1].)

When the rotation number of $\Gamma$ is rational, each periodic orbit in $\Gamma$ supports a unique ergodic invariant probability measure. In either case, these are all the action minimizing ergodic probability measure of $f$ whose rotation number $\omega$ is the same as that of $f \mid \Gamma$.

These results generalize to arbitrary real numbers $\omega$ (not necessarily the rotation number of an invariant curve), as long as $f \in \mathcal{P}^{1}$. Thus, if $\omega$ is irrational, there is a unique action minimizing $f$-invariant probability measure $\mu_{\omega}$ of rotation number $\omega$, and this measure is ergodic. If $\omega$ is rational, $\omega=p / q$ in lowest terms, then every periodic orbit of period $q$ and rotation number $p / q$ carries a unique ergodic probability measure. If the periodic orbit minimizes the action over orbits of period $q$ and rotation number $p / q$, then the corresponding measure is action minimizing. All ergodic action minimizing measures are obtained in this way. In particular, one has the existence, and, for generic $f$, the uniqueness of action minimizing measures of rotation number $p / q$. These results were proved in [Ma4] and again in [Ma6] by a different method and provide a slight refinement (in the sense that the action minimizing measures are unique) of previous results in the theory developed by Aubry and myself (independently).

Let $M_{\omega}$ denote the union of the supports of all the action minimizing measures $\mu$ of rotation number $\omega$. In the case that $\omega$ is irrational, there is
just one such measure $\mu_{\omega}$ and $M_{\omega}=\operatorname{supp} \mu_{\omega}$. In this case, $M_{\omega}$ is called the Aubry-Mather set of rotation number $\omega$, provided that there is no $f$ invariant homotopically non-trivial Jordan curve of rotation number $\omega$. If, to the contrary, there is such a curve, it contains a unique minimal set (in the sense of topological dynamics), by Denjoy theory (see, e.g., [Herl]), and $M_{\omega}$ is that set. In the case that $\omega$ is rational, $M_{\omega}$ is the union of all action minimizing periodic orbits of period $q$ and rotation number $p / q$.

Now we may state the main results of [Ma5]. Let $R$ be a Birkhoff region of instability bounded by $f$-invariant homotopically non-trivial Jordan curves $\Gamma_{-}$and $\Gamma_{+}$with $\rho\left(\Gamma_{-}\right)<\rho\left(\Gamma_{+}\right)$.

Theorem A. - Suppose $\rho\left(\Gamma_{-}\right)<\alpha, \omega<\rho\left(\Gamma_{+}\right)$. Then there is an orbit of $f$ whose $\alpha$-limit set lies in $M_{\alpha}$ and whose $\omega$-limit set lies in $M_{\omega}$. Furthermore, if $\rho\left(\Gamma_{-}\right)$(resp. $\rho\left(\Gamma_{+}\right)$) is irrational, then this conclusion still holds with the weaker hypothesis $\rho\left(\Gamma_{-}\right) \leqslant \alpha, \omega\left(\right.$ resp. $\left.\alpha, \omega \leqslant \rho\left(\Gamma_{+}\right)\right)$.

Theorem B. -- Consider for each $i \in \mathbb{Z}$ a real number $\rho\left(\Gamma_{-}\right) \leqslant$ $\omega_{i} \leqslant \rho\left(\Gamma_{+}\right)$and a positive number $\epsilon_{i}$. Then there exists an orbit $\left(\ldots, P_{j}, \ldots\right)$ and an increasing bi-infinite sequence of integers $j(i)$ such that dist. $\left(P_{j(i)}, M_{\omega(i)}\right)<\epsilon_{i}$.

These are Theorems 4.1 and 4.2 of [Ma5]. Our purpose in this paper is to state and prove a version of these results in more degrees of freedom, specifically to the setting considered in [Ma5]. Our results are far less than what one might hope for in more degrees of freedom. We regard this paper as (hopefully) the first step in a program which could lead to interesting results in more degrees of freedom. We will state a conjecture in $\S 13$ to indicate what we have in mind.

We state our results in $\S 9$. They are too technical to state in this introduction, as they depend on a generalization of Peierls's barrier to more degrees of freedom (§6). This generalization perhaps deserves attention as a new idea.

In [Ma5], the orbits we constructed were constrained minima. The main technical difficulty was to construct the constraints so that the constrained minima do not bump up against the constraints. The proof that the constrained minima do not bump up against the constraints depended heavily on order properties of the real line. For the generalization, no order properties are available, and we have had to find a new method. When we originally planned this paper, we planned to construct appropriate constraints and show that constrained minima do not bump up against
the constraints, as in [Ma5], but in a way that did not depend on the order properties of the real line. In writing the proof down, we found that it was simpler to introduce a new variational principle and show that the action minimizing configurations for the new variational principle have the required properties. Although we use no constraints, the main technical difficulty is similar to that of [Ma5]. Here, the main technical difficulty is to show that the action minimizing configurations for the new variational principle correspond to trajectories of the Euler-Lagrange flow of the original variational principle. In view of the way that the new variational principle is constructed, this amounts to showing that the action minimizing configurations are restricted to appropriate regions of the configuration space $M$. This is analogous to showing that the constrained minima do not bump up against the constraints.

As a historical remark, I would like to point out that the methods of [Ma5] extend those of [Ma1], and that I spoke of the results of [Ma5] in Oberwolfach in 1985. Although the results of [Ma1] are very different from the results of [Ma5] and this paper, the methods have something in common, and may perhaps be taken as an indication that the methods of [Ma1], [Ma5] and this paper have many possibilities, which are yet to be exploited.

As a further historical remark, I would like to mention that Bernstein and Katok $[B-K]$ were the first to use the general sort of variational method which we discuss in this paper in more degrees of freedom. They proved the existence of periodic orbits near invariant tori. I also note the article by Katok [Kat] which contains results about minimal orbits in more degrees of freedom, and the article by Bangert [Ban2], in which he studied minimal (or "class A") geodesics on higher dimensional manifolds.

The results of Herman [Her2] give a very important complement to the results of [Ma6]. He gives examples showing that the Lipschitz graph property of invariant tori holds only for positive (or negative) definite invariant tori, thus showing that the positive definiteness condition in not just a convenience for the proof, but actually makes a difference in the dynamics.

In a recent paper [Bol], Bolotin constructs connecting orbits by a variational method similar to the one we use, but under very different hypotheses.

Since this paper is aimed at a general audience, I have included a great deal of expository material. $\S \S 1-5$ is a summary of previous work I
have done, with occasional small modifications. The new material begins in $\S 6$, where I explain how to generalize Peierls's barrier to more degrees of freedom. In $\S 7$, I show that this generalization reduces to Peierls's barrier in the case that $M=\mathbb{T}$. For the statement of the main theorem of this paper, I needed a variant on the barrier introduced in $\S 6$. I define this in $\S 8$, where I also discuss the nature of both barriers in the twist map case.

In $\S 9$, I state the main theorems and discuss their application to the twist map case. I prove these theorems in $\S \S 10,11$. In $\S 12$, I state generalizations of these theorems. In $\S 13$, I state a conjecture. This is intended to suggest what I hope to do with this theory.

## 1. The setting.

To generalize Peierls's barrier to more degrees of freedom and state our main results, we use the setting of [Ma6]. In this section, we recall this setting.

We consider a smooth compact manifold $M$, and a $C^{2}$ real valued function $L$ defined on $T M \times \mathbb{T}$, where $T M$ denotes the tangent bundle of $M$ and $\mathbb{T}=\mathbb{R} / \mathbb{Z}$. In the usual terminology, $L$ is a periodic Lagrangian (of period one) on $M$. For our methods to apply, we need the following two hypotheses:

Positive Definiteness. For each $m \in M$ and each $\theta \in \mathbb{T}$, the restriction of $L$ to $T M_{m} \times \theta$ is strictly convex in the sense that its Hessian second derivative is everywhere positive definite.

Superlinear Growth. Let || \| denote a Riemannian metric on $M$. Then

$$
L(v, \theta) /\|v\| \longrightarrow+\infty, \quad \text { as }\|v\| \longrightarrow \infty
$$

where $v$ ranges over $T M$ and $\theta \in \mathbb{T}$.
In other words, for every $C>0$, there exists $C_{1}>0$ such that $\|v\| \geqslant C_{1}$ implies $L(v, \theta) \geqslant C\|v\|$. This condition is plainly independent of the choice of Riemannian metric, since $M$ is compact.

Under these conditions, the Legendre transformation $\mathcal{L}$ is defined : if $m \in M, v \in T M_{m}, \theta \in \mathbb{T}$, then $\mathcal{L}(v, \theta)=\left(d_{v}\left(L \mid T M_{m} \times \theta\right), \theta\right) \in T^{*} M_{m} \times \theta$, where $T^{*} M$ denotes the cotangent bundle of $M$ and $T^{*} M_{m}$ denotes the
fiber over $m$. If $L$ is $C^{r}(r \geqslant 2)$, then $\mathcal{L}$ is a $C^{r-1}$ diffeomorphism of $T M \times \mathbb{T}$ onto $T^{*} M \times \mathbb{T}$ which commutes with the projections on $M \times \mathbb{T}$. We will write $\mathcal{L}_{m, \theta}$ for the restriction of $\mathcal{L}$ to the fiber $T M_{m} \times \theta$.

The Hamiltonian $H: T^{*} M \times \mathbb{T} \longrightarrow \mathbb{R}$ is defined by $H(m, \xi, \theta)=$ $\left\langle\xi, \mathcal{L}_{m, \theta}^{-1}(\xi)\right\rangle-L\left(\mathcal{L}_{m, \theta}^{-1}(\xi)\right)$. If we introduce local coordinates $x=\left(x_{1}, \ldots, x_{n}\right)$ in $M$ and let $(x, \dot{x})=\left(x_{1}, \ldots, x_{n}, \dot{x}_{1}, \ldots \dot{x}_{n}\right)$ denote the corresponding local coordinates in $T M$, and $(q, p)=(x, p)=\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{n}\right)=$ $\left(x_{1}, \ldots, x_{n}, p_{1}, \ldots, p_{n}\right)$ the corresponding local coordinates in $T^{*} M$, then we may express the Hamiltonian in its classical form

$$
H(q, p)=p \cdot \dot{x}-L(x, \dot{x})
$$

where $(x, \dot{x})$ and ( $q, p$ ) are related by the Legendre transformation :

$$
q=x, \quad \quad p=L_{\dot{x}}
$$

One easily computes

$$
H_{q}=-L_{x}, \quad H_{p}=\dot{x}
$$

If $L$ is $C^{r}(r \geqslant 2)$, then $\mathcal{L}$ is $C^{r-1}$. The equations above show that the first derivatives of $H$ are $C^{r-1}$. Consequently $H$ is $C^{r}$. Similarly if $H$ is $C^{r}(r \geqslant 2)$, then $L$ is $C^{r}$.

In the Lagrangian formulation of classical mechanics, the evolution of the system is described by the flow of the Euler-Lagrange vector field $E_{L}$ associated to $L$. Its trajectories correspond to the solutions of the variational equation (fixed endpoint problem) :

$$
\delta \int_{a}^{b} L(\gamma(t), d \gamma(t) / d t, t) d t=0
$$

In other words, a curve in $T M \times \mathbb{T}$ is a trajectory of $E_{L}$ if and only if it can be represented in the form

$$
t \longrightarrow(\gamma(t), d \gamma(t) / d t, t(\bmod .1))
$$

where $\gamma$ is a curve on $M$ which satisfies the variational equation.
In local coordinates, the Euler-Lagrange vector field is defined by the Euler-Lagrange equations

$$
d x / d t=\dot{x}, \quad d\left(L_{\dot{x}}\right) / d t=L_{x}
$$

The Euler-Lagrange vector field is $\mathcal{L}$-related to the symplectic gradient of $H$, defined by Hamilton's equations

$$
d q / d t=H_{p}, \quad d p / d t=-H_{q} .
$$

Caratheodory [Cara, p. 207] made the following remark concerning differentiability classes. If $L$ is $C^{r}(r \geqslant 2)$, then, as we have seen, $H$ is $C^{r}$, so the corresponding Hamiltonian vector field is $C^{r-1}$, and so is the flow that it generates. Since the Legendre transformation is $C^{r-1}$, the flow generated by $E_{L}$ is also $C^{r-1}$, even though $E_{L}$ itself may be only $C^{r-2}$. This applies even in the case $r=2$, and we obtain that the conclusions of the fundamental existence and uniqueness theorem for ordinary differential equations holds for $E_{L}$, even though it may be only $C^{0}$.

Since a trajectory $t \rightarrow(\gamma(t), d \gamma(t) / d t, t(\bmod .1))$ of $E_{L}$ is $C^{r-1}$, the curve $\gamma$ on $M$ is $C^{r}$.

In the classical theory of the calculus of variations, one also has the following basic result concerning the boundary value problem, under the two hypotheses that we have imposed above.

Tonelli Theorem. - Let $a<b \in \mathbb{R}$, and let $m_{0}, m_{1} \in M$. Among the absolutely continuous curves $\gamma:[a, b] \rightarrow M$ such that $\gamma(a)=m_{0}$ and $\gamma(b)=m_{1}$, there is one which minimizes the action $\int_{a}^{b} L(\gamma(t), d \gamma(t) / d t, t) d t$.

As Mañé pointed out [Mañ1], it is not necessary to assume compactness of $M$, if the superlinear growth condition is satisfied with respect to some complete Riemannian metric on $M$.

A curve which minimizes $\int_{a}^{b} L(\gamma(t), d \gamma(t) / d t, t) d t$ subject to the fixed endpoint condition $\gamma(a)=m_{0}, \gamma(b)=m_{1}$, is called a Tonelli minimizer. Ball and Mizel [Bal] have constructed examples of Tonelli minimizers which are not $C^{1}$, even though $L$ satisfies the hypotheses we have stated above. A Tonelli minimizer which is $C^{1}$ is $C^{r}$ (if $L$ is $C^{r}$ ) and satisfies the EulerLagrange equation, by the usual elementary arguments in the calculus of variations, together with Caratheodory's remark on differentiability.

The Ball and Mizel examples may be excluded by imposing the following additional hypothesis :

Completeness of the Euler-Lagrange Flow. Every maximal trajectory of $E_{L}$ is defined for all time.

The fundamental existence and uniqueness theorem for ordinary differential equations says that for any initial condition $\gamma\left(t_{0}\right)=m_{0}$, $d \gamma\left(t_{0}\right) / d t=v_{0}$, there is a unique maximal trajectory $\gamma:(a, b) \longrightarrow M$, where $-\infty \leqslant a<b \leqslant+\infty$. The completeness hypothesis is that $a=-\infty$ and $b=+\infty$, for any initial condition.

Even without the completeness hypothesis, a Tonelli minimizer is $C^{1}$ on an open and dense set of full measure in the interval in which it is defined, and the velocity goes to infinity on the exceptional set. In view of this, the completeness hypothesis implies that a Tonelli minimizer is $C^{1}$ (and hence $C^{r}$ ).

For every $E_{L}$-invariant probability measure $\mu$ on $T M \times \mathbb{T}$ we may define its average action

$$
A(\mu)=\int L d \mu
$$

Since $L$ is bounded below, this integral is defined, although it may be $+\infty$. If $A(\mu)<+\infty$, we may also associate to $\mu$ its rotation vector $\rho(\mu) \in H_{1}(M, \mathbb{R})$. This may be uniquely characterized as follows. Consider a cohomology class $c \in H^{1}(M, \mathbb{R})$ and let $\lambda_{c}$ be a closed smooth 1-form on $M$ whose de Rham cohomology class is $c$. Usually one thinks of $\lambda_{c}$ as a section of $T^{*} M$, but one may also think of $\lambda_{c}$ as a real valued function on $T M$ which is linear on the fibers. One may then pull back $\lambda_{c}$ to $T M \times \mathbb{T}$ by composing with the projection on the first factor. The rotation vector $\rho(\mu)$ is uniquely characterized by the following equation :

$$
\langle c, \rho(\mu)\rangle=\int \lambda_{c} d \mu
$$

The bracket on the left is the canonical pairing of $H^{1}(M, \mathbb{R})$ and $H_{1}(M, \mathbb{R})$. The convergence of the integral on the right follows from the assumption $A(\mu)<+\infty$ and the superlinear growth hypothesis. It is elementary to show that addition of an exact one form to $\lambda_{c}$ does not change the integral on the right. (See [Ma6].) Since this integral is clearly linear in $c$, the equation above defines $\rho(\mu) \in H_{1}(M, \mathbb{R})$.

The idea of such a rotation vector is classical, going back to Schwartzman's asymptotic cycles [Sch].

## 2. The basic theory.

Throughout this paper, we let $M$ be a fixed smooth compact manifold. In all examples that interest us, $M$ is a torus, but the theorem we will state in this section is true without any restriction on the topology of $M$. We let $L$ be a $C^{2}$ real valued function defined on $T M \times \mathbb{T}$, satisfying the three hypotheses given in $\S 1$ : positive definiteness, superlinear growth, and completeness of the Euler-Lagrange flow. We will also fix $L$ throughout the discussion. We call $L$ the Lagrangian.

We let $U_{L}=\left\{(\rho(\mu), A(\mu)): \mu\right.$ is an $E_{L}$-invariant probability measure on $T M \times \mathbb{T}$ satisfying $A(\mu)<+\infty\}$. Clearly, $U_{L}$ is a convex subset of $H_{1}(M, \mathbb{R}) \times \mathbb{R}$ : the set of invariant probability measures is convex and $A$ and $\rho$ are linear functions on it.

Moreover, the projection of $U_{L}$ on $H_{1}(M, \mathbb{R})$ is surjective. This is the content of the Proposition on p. 178 of [Ma6]. Here, we briefly outline the proof and refer to [Ma6] for details.

We let $\widetilde{M}$ be the covering space of $M$ defined by $\pi_{1}(\widetilde{M})=\operatorname{ker}(\mathcal{H}$ : $\left.\pi_{1}(M) \rightarrow H_{1}(M, \mathbb{R})\right)$ where $\mathcal{H}$ denotes the Hurewicz homomorphism. For example, if $M=\mathbb{T}^{n}$ then $\widetilde{M}=\mathbb{R}^{n}$. The group of deck transformations of this covering space is

$$
\begin{aligned}
\mathcal{D} & =\operatorname{im}\left(\mathcal{H}: \pi_{1}(M) \longrightarrow H_{1}(M, \mathbb{R})\right) \\
& =\operatorname{im}\left(H_{1}(M, \mathbb{Z}) \longrightarrow H_{1}(M, \mathbb{R})\right) .
\end{aligned}
$$

It is a lattice in the finite dimensional vector space $H_{1}(M, \mathbb{R})$, i.e., it is discrete and spans $H_{1}(M, \mathbb{R})$. For example, if $M=\mathbb{T}^{n}$, then $\mathcal{D}=\mathbb{Z}^{n}$.

Choose $h \in H_{1}(M, \mathbb{R})$. Let $T_{1}, \ldots, T_{n}, \ldots$ be a sequence of deck transformation such that

$$
n^{-1} T_{n} \longrightarrow h \in H_{1}(M, \mathbb{R}), \quad \text { as } n \longrightarrow+\infty
$$

Let $\widetilde{x}_{0} \in \widetilde{M}$. Let $\widetilde{x}_{n}=T_{n} \widetilde{x}_{0}$. Let $\widetilde{\alpha}_{n}:[0, n] \rightarrow \widetilde{M}$ minimize $\int_{0}^{n} L\left(d \alpha_{n}(t), t\right) d t$ subject to the boundary conditions $\widetilde{\alpha}_{n}(0)=\widetilde{x}_{0}$ and $\widetilde{\alpha}_{n}(n)=\widetilde{x}_{n}$, where $\alpha_{n}$ is the projection of $\widetilde{\alpha}_{n}$ on $M$. The existence of $\widetilde{\alpha}_{n}$ follows from a version of Tonelli's theorem on $\widetilde{M}$ (cf. [Ma6]). Moreover, $\widetilde{\alpha}_{n}$ is $C^{1}$, by the completeness hypothesis.

To obtain an invariant measure, we use a method which Kryloff and Bogoliuboff used [KB] to show that any flow on a compact metric space has an invariant measure.

For this purpose, it is useful to extend the Euler-Lagrange flow from $T M \times \mathbb{T}$ to its one point compactification $(T M \times \mathbb{T})^{*}$ by letting the point at infinity be a rest point. We extend the definition of the average action $A(\mu)$ of an invariant measure $\mu$, by letting $A(\mu)=+\infty$, if the point at infinity has positive $\mu$-mass.

We let $\gamma_{n}(t)=\left(d \alpha_{n}(t), t \bmod .1\right) \in T M \times \mathbb{T}$. We let $\mu_{n}$ denote the probability measure evenly distributed along $\gamma_{n}$ and let $\mu$ be a point of accumulation of $\mu_{n}$, with respect to the vague topology on measures on $(T M \times \mathbb{T})^{*}$. The Kryloff-Bogoliuboff argument shows that $\mu$ is an invariant measure.

It is easy to see that there exists $C>0$ and, for each positive integer $n$, a curve $\widetilde{\beta}_{n}:[0, n] \longrightarrow \widetilde{M}$ such that $\widetilde{\beta}_{n}(0)=\widetilde{x}_{0}, \widetilde{\beta}_{n}(n)=\widetilde{x}_{n}$ and $n^{-1} \int_{0}^{n} L\left(d \beta_{n}(t), t\right) d t \leqslant C$, where $\beta_{n}$ is the projection of $\widetilde{\beta}_{n}$ on $M$. Consequently,

$$
\int L d \mu_{n}=n^{-1} \int_{0}^{n} L\left(d \alpha_{n}(t), t\right) d t \leqslant n^{-1} \int_{0}^{n} L\left(d \beta_{n}(t), t\right) d t \leqslant C
$$

and it follows that $\int L d \mu \leqslant C$ and the point at infinity has zero measure with respect to $\mu$.

Thus $\mu$ is an $E_{L}$-invariant probability measure on $T M \times \mathbb{T}$. It is easily seen that $\rho(\mu)=\lim _{n \rightarrow+\infty} n^{-1} T_{n}=h \in H_{1}(M, \mathbb{R})$. This completes our outline of the proof that the projection of $U_{L}$ on $H_{1}(M, \mathbb{R})$ is surjective.

Clearly, $L$ is bounded below : there exists $B \in \mathbb{R}$ such that $L \geqslant B$. It follows from the definition that $U_{L} \subset H_{1}(M, \mathbb{R}) \times[B, \infty)$. Therefore, $U_{L}$ is the epigraph of a convex function $\beta=\beta_{L}: H_{1}(M, \mathbb{R}) \longrightarrow \mathbb{R}$. For $h \in H_{1}(M, \mathbb{R})$, we will call $\beta(h)$ the minimal average action of the rotation vector $h$. This generalizes the notion of minimal average action discussed in the introduction.

It is easy to see that $\beta$ has superlinear growth, i.e., $\beta(h) /\|h\| \longrightarrow+\infty$ as $\|h\| \longrightarrow \infty$, where we may choose $\|\|$ to be any norm on the finite dimensional vector space $H_{1}(M, \mathbb{R})$.

Let $\alpha: H^{1}(M, \mathbb{R}) \longrightarrow \mathbb{R}$ be the conjugate of $\beta$ in the sense of convex analysis. (See, e.g., [Roc].) In other words,

$$
\alpha(c)=\max \left\{<c, h>-\beta(h): h \in H_{1}(M, \mathbb{R})\right\}
$$

for $c \in H^{1}(M, \mathbb{R})$. Since $\beta$ has superlinear growth, the maximum is achieved and $\alpha$ is everywhere defined. Since $\beta$ is everywhere defined, $\alpha$ has superlinear growth.

We set $A_{c}(\mu)=A(\mu)-<c, \rho(\mu)>$, for $c \in H^{1}(M, \mathbb{R})$. We will say that an $E_{L}$-invariant measure $\mu$ is $c$-minimal if it minimizes $A_{c}(\mu)$ over all $E_{L}$-invariant measures. We will say that it is minimal or action minimizing if it is $c$-minimal for some $c \in H^{1}(M, \mathbb{R})$. We let $\mathcal{M}_{c}$ denote the family of all $c$-minimal measures. We let supp $\mathcal{M}_{c} \subset T M \times \mathbb{T}$ denote the closure of the union of the supports of $\mu$ for $\mu \in \mathcal{M}_{c}$. For brevity, we set $M_{c}=\operatorname{supp} \mathcal{M}_{c}$. Let $\pi: T M \times \mathbb{T} \longrightarrow M \times \mathbb{T}$ denote the projection.

The principle result of [Ma6] may be stated as follows :
Theorem 2.2. $-M_{c}$ is a compact, non-empty subset of $T M \times \mathbb{T}$. The restriction of $\pi$ to $M_{c}$ is injective. The inverse mapping $\pi^{-1}: \pi\left(M_{c}\right) \longrightarrow M_{c}$ is Lipschitz.

Here, we have combined Proposition 4 and Theorems 1 and 2 of [Ma6].
Let $\Sigma_{c}=\pi\left(M_{c}\right) \subset M \times \mathbb{T}$. It follows from this theorem that the flow on $\Sigma_{c}$ which corresponds to the Euler-Lagrange flow on $M_{c}$ is Lipschitz, and is generated by a Lipschitz vector field.

From the point of view of existence theory, this theorem tells us nothing. For each $h \in H_{1}(M, \mathbb{Z})$, there exists, by Tonelli's theorem, a closed curve $\gamma$ in $M$, of period 1 , which minimizes the action $\int L(d \alpha(t), t) d t$ among the closed curves $\alpha$ of period 1 whose homology class is $h$. By the completeness hypothesis, $\gamma$ is $C^{1}$. Then $\gamma$ satisfies the Euler-Lagrange equation, and so is a periodic orbit of the Euler-Lagrange flow. Thus, one already has results which imply the existence of many compact invariant sets.

Our belief that this theorem should prove interesting is based on other considerations. These relate to results about twist maps which we believe should generalize to more degrees of freedom. Theorems A and B of the introduction are examples of one way that the basic theory described in this section may be used in the context of twist maps. In this paper, we make a beginning towards generalizing Theorems A and B to more degrees of freedom.

A few words on the proof of Theorem 2.2 may be helpful. The fact that $\mathcal{M}_{c}$ is non-empty is contained in Theorem 1 of [Ma6], specifically the fact that the minimum is achieved in the formula given there for $-\alpha(c)$. We may give a version of the proof given there, as follows : Let $\lambda_{c}$ be a closed smooth 1 -form on $M$ whose de Rham cohomology class is $c$. As in the end of $\S 1$, we think of $\lambda_{c}$ as a function on $T M \times \mathbb{T}$ (independent of the $\mathbb{T}$ variable). The function $L-\lambda_{c}$ is a Lagrangian, i.e., it satisfies the
conditions of positive definiteness, superlinear growth, and completeness we assumed at the beginning. Moreover, its Euler-Lagrange flow is the same as that of $L$. For an $E_{L}$-invariant measure, we have

$$
A_{c}(\mu)=\int\left(L-\lambda_{c}\right) d \mu
$$

The statement that $M_{c} \neq \varnothing$ amounts to the assertion that there exists a $c$-minimal measure $\mu$. The existence of such a measure may be proved by letting $\alpha_{n}:[0, n] \longrightarrow M$ be a curve which minimizes $\int\left(L-\lambda_{c}\right)\left(d \alpha_{n}(t), t\right) d t$ for the free endpoint problem. As before, one sets $\alpha_{n}(t)=\left(d \alpha_{n}(t), t(\bmod .1)\right) \in T M \times \mathbb{T}$. One lets $\mu_{n}$ be the probability measure uniformly distributed along $\alpha_{n}$ and lets $\mu$ be a point of accumulation of the $\mu_{n}$ with respect to the vague topology. By Kryloff-Bogoliuboff, $\mu$ is $E_{L}$-invariant. It is not at all difficult to see that $\mu$ minimizes $A_{c}$, for example by the argument used to prove Proposition 1 in [Ma6].

The fact that $M_{c}$ is compact is Proposition 4 in [Ma6]. By definition, $M_{c}$ is closed in $T M \times \mathbb{T}$. Hence if $M_{c}$ is not compact, $\|\xi\|$ is unbounded as $\left(\xi, t\right.$ (mod. 1)) ranges over $M_{c}$. From this, we were able [Ma6, pp. 185186] to construct an incomplete trajectory of the Euler-Lagrange flow, a contradiction.

The fact that $\pi: M_{c} \longrightarrow M \times \mathbb{T}$ is injective and its inverse is Lipschitz is Theorem 2 of [ Ma 6$]$. The intuitive idea of the proof is simple. There is a well known elementary curve shortening lemma in Riemannian geometry, as follows. Let $\alpha, \beta$ be curves on a Riemannian manifold joining points $P, P^{\prime}$ and $Q, Q^{\prime}$ resp. Suppose $\alpha$ and $\beta$ cross. Then there exist curves $a$, joining $P$ and $Q$, and $b$ joining $P^{\prime}$ and $Q^{\prime}$ such that

$$
\text { length }(a)+\text { length }(b)<\text { length }(\alpha)+\text { length }(\beta)
$$

(See Fig. 1.) In our setting, one may decrease the action in the same way. (See the lemma used in the proof of Theorem 2 in [Ma6].) If $\pi$ were not injective on $M_{c}$, or its inverse were not Lipschitz, it would be possible to construct an $E_{L}$-invariant measure $\mu$ on $T M \times \mathbb{T}$ for which $A_{c}(\mu)<\alpha(c)$, a contradiction. This would be done by cutting and pasting trajectories using the curve shortening lemma. Then the Tonelli theorem and the KryluffBogoliuboff argument would supply the required measure. See [Ma6] for the details, which are not simple.


Fig. 1

## 3. Twist mappings.

We define the Poincaré map $f=f_{L}: T M \longrightarrow T M$, associated to the Lagrangian $L$, as follows. Let $\xi \in T M$ and let $\gamma: \mathbb{R} \longrightarrow T M \times \mathbb{T}$ denote the trajectory of $E_{L}$ with initial condition $\gamma(0)=(\xi, 0(\bmod .1))$. We define $f$ by $\gamma(1)=(f(\xi), 0(\bmod .1))$.

Such a mapping is an example of an optical mapping in the sense of [Arn1]. In the case the $M=\mathbb{T}$, such mappings include twist mappings. To be precise, if $f \in \mathcal{P}^{1}$, Moser showed [Mo] that there is a Lagrangian on $T \mathbb{T} \times \mathbb{T}$ whose Poincaré map is $f$. Following an idea of Moser, Denzler [Den] showed how the basic results of Aubry-Mather theory can be obtained for Lagrangians on $T \mathbb{T} \times \mathbb{T}$. Our paper [Ma6] may be regarded as a generalization of [Den], with the idea of taking action minimizing measures as the basic notion being the new idea. This idea seems essential for the higher dimensional generalization.

In [Ma6, §6], we showed how the basic results of Aubry-Mather theory can be obtained from the basic theory described in $\S 2$. Here, we recall briefly the arguments.

In the case $M=\mathbb{T}$, we have $H_{1}(M, \mathbb{R})=\mathbb{R}$, of course, so the minimal average action may be regarded as a function $\beta: \mathbb{R} \longrightarrow \mathbb{R}$. The first point is that this function is strictly convex, i.e., its graph has no flat parts. For, suppose that graph $\beta$ intersects a line $l$ in $\mathbb{R}^{2}$ in a segment $\sigma$ (not
reduced to a point). Let $\left(h_{i}, \beta\left(h_{i}\right)\right), i=0,1$ be the endpoints of $\sigma$. Because these endpoints are extremal points of the epigraph of $\beta$, there exist action minimizing ergodic measures $\mu_{i}, i=0,1$ such that $\rho\left(\mu_{0}\right)=h_{0}$ and $\rho\left(\mu_{1}\right)=h_{1}$. Each has its support in $M_{c}$, where $c \in H^{1}(M, \mathbb{R})=\mathbb{R}$ is the slope of $l$. By Theorem 2.2, the projection $\pi$ of $M_{c}$ on the torus $\mathbb{T}^{1} \times \mathbb{T}^{1}$ in injective. But this leads to a contradiction : since $\mu_{0}$ and $\mu_{1}$ have different rotation numbers, trajectories in $\pi\left(\operatorname{supp} \mu_{0}\right)$ cross trajectories in $\pi$ (supp $\mu_{1}$ ), contradicting the injectivity of $\pi \mid M_{c}$. Thus, we have shown that $\beta$ is strictly convex. (For more detail, see [Ma6, Proposition 6].)

Let $h \in H_{1}(\mathbb{T}, \mathbb{R})=\mathbb{R}$, let $l \subset H^{1}(\mathbb{T}, \mathbb{R}) \times \mathbb{R}$ be a supporting hyperplane of the epigraph of $\beta$ which touches the epigraph of $\beta$ at $h$, and let $c$ be the slope of $l$. Let $M_{h}^{0}=(T \mathbb{T} \times 0) \cap M_{c}$. Note that $M_{h}^{0}$ is independent of the choice of $l$ : it is the support of the set of action minimizing invariant measures of Poincaré map $f$ of rotation number $h$.

The second point is that the projection $\pi$ of $M_{h}^{0} \subset T \mathbb{T}$ on $\mathbb{T}$ is injective and has Lipschitz inverse. This is immediate from Theorem 2.2.

Of course, $\pi\left(M_{h}^{0}\right)$ inherits a cyclic order from $\mathbb{T}$. The third point is that $f \mid M_{h}^{0}$ preserves this cyclic order. For, otherwise, the projection of $M_{c}$ on $\mathbb{T} \times \mathbb{T}$ would not be injective.

It is a well known result in the Denjoy theory of orientation preserving homeomorphisms of the circle that if $g: \mathbb{T} \longrightarrow \mathbb{T}$ is an orientation preserving homeomorphism of irrational rotation number, then $g$ is uniquely ergodic, i.e. there is exactly one $g$-invariant measure on $\mathbb{T}$. In the same way we may prove the fourth point: $f \mid M_{h}^{0}$ is uniquely ergodic when $h$ is irrational. This follows from the cyclic order preserving property of $f \mid M_{h}^{0}$.

It is easy to check that if $g: \mathbb{T} \longrightarrow \mathbb{T}$ is an orientation preserving homeomorphism of rational rotation number, then every ergodic measure is supported on a periodic orbit. Similarly, we have the fifth point : if $h$ is rational, say $h=p / q$ lowest terms, then $f \mid M_{h}^{0}$ is periodic of period $q$. This follows from the cyclic order preserving property of $f \mid M_{h}^{0}$, together with the definition of $M_{h}^{0}$ as the support of a set of invariant measures.

## 4. The variational principle.

As in $\S 2$, we let $\widetilde{M}$ be the covering space of $M$ such that $\pi_{1}(\widetilde{M})=$ $\operatorname{ker}\left(\mathcal{H}: \pi_{1}(M) \longrightarrow H_{1}(M, \mathbb{R})\right)$.

We may define a continuous function $h=h_{L}: \widetilde{M} \times \widetilde{M} \longrightarrow \mathbb{R}$, called the variational principle associated to $L$, as follows. For $\widetilde{m}, \widetilde{m}^{\prime} \in \widetilde{M}$, let

$$
h\left(\widetilde{m}, \widetilde{m}^{\prime}\right)=\min \int_{0}^{1} L(d \gamma(t), t) d t
$$

where the minimum is taken over all curves $\widetilde{\gamma}:[0,1] \longrightarrow \widetilde{M}$ such that $\widetilde{\gamma}(0)=\widetilde{m}, \widetilde{\gamma}(1)=\widetilde{m}^{\prime}$, and $\gamma$ denotes the projection of $\widetilde{\gamma}$ on $M$. By Tonelli's theorem, the minimum is achieved, and
$\left(H_{0}\right) \quad h$ is continuous.
Moreover,

$$
\begin{equation*}
h\left(T \tilde{m}, T \tilde{m}^{\prime}\right)=h\left(\tilde{m}, \tilde{m}^{\prime}\right), \quad \text { if } T \in \mathcal{D} \tag{1}
\end{equation*}
$$

(Recall that $\mathcal{D}$ denotes the group of deck transformations of $\widetilde{M}$.) If we provide $M$ with a Riemannian metric, lift it to $\widetilde{M}$, and let $d$ denote the corresponding metric on $\widetilde{M}$, then we have

$$
\begin{equation*}
h\left(\widetilde{m}, \widetilde{m}^{\prime}\right) \rightarrow+\infty, \text { as } d\left(\widetilde{m}, \widetilde{m}^{\prime}\right) \rightarrow \infty \tag{2}
\end{equation*}
$$

Actually, it follows from the superlinear growth condition that

$$
h\left(\widetilde{m}, \widetilde{m}^{\prime}\right) / d\left(\widetilde{m}, \widetilde{m}^{\prime}\right) \rightarrow+\infty \text { as } d\left(\widetilde{m}, \widetilde{m}^{\prime}\right) \rightarrow \infty
$$

However, the condition above is strong enough for the applications which have been given in [Ban1], [Ma2], and [Ma3].

In the case that $M=\mathbb{T}$, we have that there exists a positive continuous function $\rho$ on $\mathbb{R}^{2}$ such that

$$
\begin{equation*}
\partial_{12} h\left(x, x^{\prime}\right) \leqslant-\rho\left(x, x^{\prime}\right) \quad(\mathcal{D}) . \tag{5}
\end{equation*}
$$

Here, $(\mathcal{D})$ stands for in the sense of distributions, and $\partial_{12}$ denotes the mixed second partial derivative. In general, the function $h$ need not be differentiable, so this inequality makes sense only if it is understood in the distributional sense. See, for example, [Ma2] for a proof in the case $f \in \mathcal{P}^{1}$ : the proof in general works the same way. We do not know of any generalization of $\left(H_{5}\right)$ to manifolds of higher dimension. (See, however, [B-P] for some progress on related questions.)

In the case that $M=\mathbb{T}$, we also have that there exists on positive continuous function $\theta$ on $\mathbb{R}^{2}$ such that
$\left(H_{6}\right) \quad \partial_{11} h\left(x, x^{\prime}\right) \leqslant \theta\left(x, x^{\prime}\right), \quad \partial_{22} h\left(x, x^{\prime}\right) \leqslant \theta\left(x, x^{\prime}\right)(\mathcal{D})$.

See [Ma2] for a proof in the case $f \in \mathcal{P}^{1}$ (when $\theta$ can be taken to be constant) : the proof in general works the same way, although $\theta$ cannot be taken to be constant in general.

Until recently, Aubry-Mather theory for twist maps was based on the study of minimal configurations in $\mathbb{R}$ for a variational principle $h$ satisfying suitable conditions. The basic theory was developed in [Ban1] under conditions $\left(H_{0}\right)-\left(H_{4}\right)$ which are implied by our conditions $\left(H_{0}\right),\left(H_{1}\right),\left(H_{2}\right),\left(H_{5}\right)$ and $\left(H_{6}\right)$. The subsequent development in [Ma2] and [Ma3] was based on these latter conditions.

We may define minimal configurations in complete generality, not just in the case $M=\mathbb{T}$. A configuration is a bi-infinite sequence $\left(\cdots, \widetilde{m}_{i}, \cdots\right), \widetilde{m}_{i} \in \widetilde{M}$. A segment of a configuration is a finite sequence $\left(\widetilde{m}_{a}, \cdots, \widetilde{m}_{i}, \cdots, \widetilde{m}_{b}\right), \widetilde{m}_{i} \in \widetilde{M}, a<b \in \mathbb{Z}$. For such a segment, we set $h\left(\widetilde{m}_{a}, \cdots, \widetilde{m}_{b}\right)=\sum_{i=a}^{b-1} h\left(\widetilde{m}_{i}, \widetilde{m}_{i+1}\right)$. Such a segment is said to be minimal if

$$
h\left(\widetilde{m}_{a}, \cdots, \widetilde{m}_{b}\right) \leqslant h\left(\widetilde{m}_{a}^{\prime}, \cdots, \widetilde{m}_{b}^{\prime}\right)
$$

for any other segment ( $\widetilde{m}_{a}^{\prime}, \cdots, \widetilde{m}_{i}^{\prime}, \cdots, \widetilde{m}_{b}^{\prime}$ ) such that $\widetilde{m}_{a}^{\prime}=\widetilde{m}_{a}$ and $\widetilde{m}_{b}^{\prime}=\widetilde{m}_{b}$ (but not necessarily $\widetilde{m}_{i}^{\prime}=\widetilde{m}_{i}$ for $a<i<b$ ). A configuration is said to be minimal if every segment of it is minimal.

Given a segment of a minimal configuration ( $\left.\widetilde{m}_{a}, \cdots, \widetilde{m}_{b}\right)$, we may construct a Tonelli minimizer $\widetilde{\gamma}:[a, b] \longrightarrow \widetilde{M}$ by letting $\widetilde{\gamma}(t), i \leqslant t \leqslant i+1$, be a Tonelli minimizer satisfying the boundary condition $\widetilde{\gamma}(i)=\widetilde{m}_{i}$, $\widetilde{\gamma}(i+1)=\widetilde{m}_{i+1}$. Conversely, if $\widetilde{\gamma}:[a, b] \longrightarrow \widetilde{M}$ is a Tonelli minimizer, $a<b \in \mathbb{Z}$, then $\left(\widetilde{m}_{a}, \cdots, \widetilde{m}_{b}\right)$ is a segment of a minimal configuration, if $\widetilde{m}_{i}=\widetilde{\gamma}(i)$. These assertions follow immediately from the definitions.

We will say that $\widetilde{\gamma}: \mathbb{R} \longrightarrow \widetilde{M}$ is a Tonelli minimizer if the restriction of it to each finite interval is a Tonelli minimizer. From the above discussion, it follows that there is a one-one correspondence between mappings $\mathbb{R} \longrightarrow \widetilde{M}$ which are Tonelli minimizers and minimal configurations.

## 5. Minimizers and minimal measures.

In what follows, we identify a curve $\gamma$ in $M$ (or $\widetilde{M}$ ) with the curve $t \longrightarrow(\gamma(t), t \bmod .1)$ in $M \times \mathbb{T}($ or $\widetilde{M} \times \mathbb{T})$.

We say that a curve in $M$ is an $\widetilde{M}$-minimizer if a lift of it to $\widetilde{M}$ is a Tonelli minimizer. There is a close relationship between $\widetilde{M}$-minimizers and
minimal measures, which is stated as Propositions 2 and 3 in [Ma6, §3]. In this section, we recall Proposition 2 and state a slightly more precise version of Proposition 3, which may be proved in the same way.

Let $\gamma: \mathbb{R} \longrightarrow M$ be a $C^{1}$ curve and let $\zeta(t)=(d \gamma(t), t \bmod .1) \in$ $T M \times \mathbb{T}$. Let $\mu$ be a probability measure on the one point compactification $(T M \times \mathbb{T})^{*}$ of $T M \times \mathbb{T}$. We say that $\mu$ is a limit measure of $\gamma$ (or $\zeta$ ) if there is a sequence $\left[a_{i}, b_{i}\right]$ of closed intervals in $\mathbb{R}$ with $b_{i}-a_{i}$ tending to $\infty$, such that $\mu_{i}$ tends vaguely to $\mu$, where $\mu_{i}$ is the probability measure evenly distributed along $\gamma \mid\left[a_{i}, b_{i}\right]$.

Let d denote the metric on $\widetilde{M}$ associated to the lift of a smooth Riemannian metric on $M$.

Proposition 5.1. - Let $\gamma: \mathbb{R} \rightarrow M$ be an $\widetilde{M}$-minimizer and suppose that

$$
\lim _{b \rightarrow+\infty} \inf _{a \rightarrow-\infty} d(\widetilde{\gamma}(a), \widetilde{\gamma}(b)) /(b-a)<\infty
$$

where $\widetilde{\gamma}$ denotes a lift of $\gamma$ to $\widetilde{M}$. Then there exists $c \in H^{1}(M, \mathbb{R})$ such that every limit measure of $\gamma$ minimizes $A_{c}$.

In particular, the point at infinity in $(T M \times \mathbb{T})^{*}$ has zero mass with respect to such a limit measure.

This is [Ma6, Proposition 2].
To state our refinement of [Ma6, Propositon 3], we introduce the following notion. We say a curve $\gamma$ in $M$ is a $c$-minimizer (where $c \in$ $\left.H^{1}(M, \mathbb{R})\right)$ if it satisfies the following condition. For any interval $[a, b]$ and any curve $\gamma_{1}:[c, d] \rightarrow M$ such that $c-a \in \mathbb{Z}$ and $d-b \in \mathbb{Z}$, we have

$$
\int_{a}^{b}\left(L-\eta_{c}-\alpha(c)\right)(d \gamma(t), t) d t \leqslant \int_{c}^{d}\left(L-\eta_{c}-\alpha(c)\right)\left(d \gamma_{1}(t), t\right) d t
$$

where $\alpha$ is as defined in $\S 2$, i.e., the conjugate of $\beta$ (the minimal average action).

We emphasize that segments of the curve minimize for curves in $M$ (not $\widetilde{M}$ ). Moreover, we have replaced the fixed endpoint problem by the requirement that the endpoints differ by an integer.

Note that if we replaced $L$ by $L-\eta_{c}-\alpha(c)$ in the definition of $\widetilde{M}$ minimizer, we would not change the class of curves we get. Subtraction of the closed one form $\eta_{c}$ would make no difference, by Stokes's theorem. Subtraction of the constant $\alpha(c)$ would make no difference for a fixed
endpoint problem. However, in the definition of $c$-minimizer, we no longer have a fixed endpoint problem, and the constant $-\alpha(c)$ is important.

It also follows from Stokes's theorem that the notion of $c$-minimizer is independent of the choice of $\eta_{c}$ representing $c$. Clearly, any $c$-minimizer is an $\widetilde{M}$-minimizer. On the other hand, any limit measure of a $c$-minimizer minimizes $A_{c}$, as may be seen from the proof of [Ma6, Proposition 2].

By Theorem 2.2, $\pi: M_{c} \longrightarrow \Sigma_{c}\left(=\pi\left(M_{c}\right) \subset M \times \mathbb{T}\right)$ is a bi-Lipschitz homeomorphism, for any $c \in H^{1}(M, \mathbb{R})$. The restriction of the EulerLagrange vector field to $M_{c}$ induces a Lipschitz vector field $E_{L}^{c}$ on $\Sigma_{c}$. This vector field generates a flow on $\Sigma_{c}$ which we will call the $c$-Euler-Lagrange flow.

Proposition 5.2. - Any trajectory of the $c$-Euler-Lagrange flow is a $c$-minimizer.

This is a slight refinement of [Ma6, Propositon 3] and may be proved in the same way.

## 6. A barrier.

In this section, we will define a partial generalization of Peierls's barrier to several degrees of freedom.

Throughout this section, we fix $c \in H^{1}(M, \mathbb{R})$. Recall that $\Sigma_{c} \subset$ $M \times \mathbb{T}$ and that there is a Lipschitz flow on $\Sigma_{c}$, which we called the $c$ -Euler-Lagrange flow. We set $\Sigma_{c}^{0}=\Sigma_{c} \cap(M \times 0)$ and let $f_{c}: \Sigma_{c}^{0} \longrightarrow \Sigma_{c}^{0}$ be the time one map associated to the $c$-Euler-Lagrange flow. We call $f_{c}$ the $c$-Poincaré map. Clearly, $f_{c}$ is Lipschitz.

We define a function $h=h_{c}: M \times M \longrightarrow \mathbb{R}$, as follows. We choose a closed smooth one form $\eta_{c}$ on $M$ whose de Rham cohomology class is $c$. For $m, m^{\prime} \in M$, we let

$$
h_{c}\left(m, m^{\prime}\right)=\min \int_{0}^{1}\left(L-\eta_{c}\right)(d \gamma(t), t) d t-\alpha(c)
$$

where $\alpha: H^{1}(M, \mathbb{R}) \rightarrow \mathbb{R}$ is as defined in $\S 2$, and the minimum is taken over all curves $\gamma:[0,1] \longrightarrow M$ such that $\gamma(0)=m$ and $\gamma(1)=m^{\prime}$. This is similar to the variational principle defined in $\S 4$, but differs in two respects : $L$ is replaced by $L-\eta_{c}$ and $h_{c}$ is defined on $M$, whereas the variational principle of $\S 4$ was defined on $\widetilde{M}$. In addition, we have subtracted $\alpha(c)$.

Note that $h_{c}$ depends on the choice of $\eta_{c}$. If $\eta_{c}^{\prime}=\eta_{c}+d u$, where $u$ is a smooth function on $M$, then $h_{c}^{\prime}\left(m, m^{\prime}\right)=h_{c}\left(m, m^{\prime}\right)+u\left(m^{\prime}\right)+u(m)$.

In this section, we will consider configurations in $M$ rather than in $\widetilde{M}$. Thus, we use the same definitions as in $\S 4$, but now the configurations are bi-infinite sequences $\left(\cdots, m_{i}, \cdots\right)$, with $m_{i} \in M$. When we wish to distinguish the two notions, we will refer to $\bar{M}$-configurations or $M$-configurations. We will say that a segment ( $m_{a}, \cdots, m_{b}$ ) of an $M$ configuration is $c$-minimal if

$$
h_{c}\left(m_{a}, \cdots, m_{b}\right) \leqslant h_{c}\left(m_{c}^{\prime}, \cdots, m_{d}^{\prime}\right)
$$

for any other segment ( $m_{c}^{\prime}, \cdots, m_{d}^{\prime}$ ) such that $m_{a}=m_{c}^{\prime}$ and $m_{b}=m_{d}^{\prime}$. Note that we do not require $d-c=b-a$, in contrast to the definition of minimal $\widetilde{M}$-configurations given in $\S 4$. We will say that an $M$-configuration is $c$-minimal if every segment of it is $c$-minimal.

In analogy to the one-one correspondence between $\widetilde{M}$-minimizers and minimal $\widetilde{M}$-configurations described at the end of $\S 4$, there is a one-one correspondence between $c$-minimizers and $c$-minimal $M$-configurations : starting with a $c$-minimal configuration $\left(\cdots, m_{i}, \cdots\right)$ one connects $m_{i}$ to $m_{i+1}$ by a curve $\gamma$ which minimizes $\int_{i}^{i+1}\left(L-\eta_{c}\right)(d \gamma(t), t) d t$. In this way, one obtains a $c$-minimizer.

Any orbit of the $c$-Poincaré map $f_{c}: \Sigma_{c}^{0} \longrightarrow \Sigma_{c}^{0}$ is a $c$-minimal $M$ configuration, by Proposition 5.2.

The $n$-fold conjunction $h_{c}^{n}$ of $h_{c}$ with itself (in the sense of [Ma2, §5]) is defined by the formula

$$
\begin{gathered}
h_{c}^{n}(\xi, \eta)=\min \left\{\sum_{i=0}^{n-1} h_{c}\left(m_{i}, m_{i+1}\right): m_{0}=\xi, m_{n}=\eta\right. \\
\text { and } \left.m_{i} \in M \text { for } 0 \leqslant i \leqslant n\right\} .
\end{gathered}
$$

We set

$$
h_{c}^{\infty}(\xi, \eta)=\lim _{n \rightarrow \infty} \inf _{c}^{n}(\xi, \eta)
$$

for $\xi, \eta \in M$. We set $B_{c}(\xi)=h_{c}^{\infty}(\xi, \xi)$. Note that $B_{c}$ is independent of the choice of $\eta_{c}$ representing $c$. We call $B_{c}$ the barrier (or $c$-barrier). We will show in $\S 7$ that when $M=\mathbb{T}$, this function reduces in many cases to Peierls's barrier.

The barrier $B_{c}$ is a Lipschitz non-negative function on $M$ which vanishes identically on $\Sigma_{c}^{0}$. In fact, we added the normalizing summand $-\alpha(c)$ in the definition of $h_{c}\left(m, m^{\prime}\right)$ to obtain this property.

To show that $B_{c}$ is non-negative, we first note that $\int\left(L-\eta_{c}\right) d \mu-$ $\alpha(c)=0$, for any $c$-minimal measure $\mu$, by the definition of $\alpha$. From this, it follows that $h_{c}^{n}(\xi, \xi) \geqslant 0$. For, otherwise, it would be possible to construct an invariant measure $\mu$ such that $\int\left(L-\eta_{c}\right) d \mu-\alpha(c)<0$, by using Tonelli's theorem and the Kryloff-Bogoliuboff argument as in [Ma6]. But this would contradict the definition of $\alpha$. From $h_{c}^{n}(\xi, \xi) \geqslant 0$, it follows that $B_{c}$ is non-negative.

Let $\xi_{0} \in \Sigma_{c}^{0}$. To show that $B_{c}\left(\xi_{0}\right)=0$, we suppose the contrary. Then there exists a large positive integer $N$ and a small positive number $\delta$ such that $h_{c}^{n}\left(\xi_{0}, \xi_{0}\right) \geqslant \delta$, for all $n \geqslant N$. Note that a Lipschitz constant for $h_{c}$ is also a Lipschitz constant for $h_{c}^{n}$, for all $n \geqslant 1$. Consequently, there is a neighborhood $U$ of $\xi_{0}$ in $M$ such that $h_{c}^{n}(\xi, \eta) \geqslant \delta / 2$, for all $\xi, \eta \in U$ and all $n \geqslant N$.

By the definition of $\Sigma_{c}^{0}$ and Proposition 5.2, $c$-minimal measures correspond one-one to $f_{c}$-invariant measures on $\Sigma_{c}^{0}$. In particular, $\Sigma_{c}^{0}$ is the closure of the union of the supports of all such measures. Thus, there exists an ergodic $f_{c}$-invariant measure $\mu$ whose support meets $U$. From the Birkhoff ergodic theorem, it follows that there exists $\xi \in U$ such that $f_{c}^{i}(\xi)$ returns to $U$ with positive frequency and

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} h_{c}\left(f_{c}^{i}(\xi), f_{c}^{i+1}(\xi)\right)=\int h_{c}\left(\xi, f_{c}(\xi)\right) d \mu(\xi)=0
$$

The last equation is a consequence of the fact that $\mu$ corresponds to a $c$-minimal measure.

Since $f_{\mathrm{c}}^{i}(\xi)$ returns to $U$ with positive frequency, there exist $n_{1} \geqslant N$, $n_{2} \geqslant N, \ldots$ such that $f_{c}^{n_{1}}(\xi) \in U, f_{c}^{n_{1}+n_{2}}(\xi) \in U, \ldots$, and there exists a constant $C$ such that $\sum_{k=0}^{K-1} n_{k} \leqslant C K$ for all $K \geqslant 1$. Since $h_{c}^{n}(\xi, \eta) \geqslant \delta / 2$ for all $\xi, \eta \in U$ and $n \geqslant N$, we have

$$
\frac{1}{n} \sum_{i=0}^{n-1} h_{c}\left(f_{c}^{i}(\xi), f_{c}^{i+1}(\xi)\right) \geqslant \frac{K}{n} \frac{\delta}{2} \geqslant \frac{\delta}{2 C}
$$

when $n=n_{1}+\cdots+n_{K}$. This contradicts the previous equation. This contradiction proves that $B_{c}\left(\xi_{0}\right)=0$.

It is obvious that $h_{c}^{\infty}$ is Lipschitz as a function on $M \times M$ and that a Lipschitz constant for $h_{c}$ is also a Lipschitz constant for $h_{c}^{\infty}$. In particular, the barrier $B_{c}$ is a Lipschitz function on $M$. Clearly, $h_{c}^{m+n}(\xi, \nu) \leqslant$ $h_{c}^{m}(\xi, \eta)+h_{c}^{n}(\eta, \nu)$; hence, $h_{c}^{\infty}(\xi, \nu) \leqslant h_{c}^{\infty}(\xi, \eta)+h_{c}^{\infty}(\eta, \nu)$, for all $\xi, \eta, \nu \in M$.

For $\xi, \eta \in M$, we set $d_{c}(\xi, \eta)=h_{c}^{\infty}(\xi, \eta)+h_{c}^{\infty}(\eta, \xi)$. Obviously, $d_{c}(\xi, \xi)=2 B_{c}(\xi) \geqslant 0$, and $d_{c}(\xi, \nu) \leqslant d_{c}(\xi, \eta)+d_{c}(\eta, \nu)$ for all $\xi, \eta, \nu \in M$. We let $\Sigma_{c}^{0^{\prime}}$ be the set where $B_{c}$ vanishes. Thus, $\Sigma_{c}^{0} \subset \Sigma_{c}^{0^{\prime}}$. Clearly, the restriction of $d_{c}$ to $\Sigma_{c}^{0^{\prime}}$ is a pseudo-metric, i.e., $d_{c}(\xi, \xi)=$ $0, d_{c}(\xi, \eta)=d_{c}(\eta, \xi)$, and the triangle inequality holds.

Clearly,

$$
h_{c}^{\infty}(\xi, \eta)+h_{c}^{\infty}(\eta, \nu) \leqslant \min \left(d_{c}(\xi, \eta), d_{c}(\eta, \nu)\right)+h_{c}^{\infty}(\xi, \nu)
$$

Thus, $h_{c}^{\infty}(\xi, \nu)=h_{c}^{\infty}(\xi, \eta)+h_{c}^{\infty}(\eta, \nu)$ if either $d_{c}(\xi, \eta)=0$ or $d_{c}(\eta, \nu)=0$.
If $\left(\cdots, m_{i}, \cdots\right)$ is a $c$-minimal $M$-configuration, it follows from Proposition 5.1 that any limit measure of it has support in $\Sigma_{c}^{0}$. In particular, there exist $\alpha, \omega \in \Sigma_{c}^{0}$ such that $\alpha$ is an $\alpha$-limit point of $\left(\cdots, m_{i}, \cdots\right)$ and $\omega$ is an $\omega$-limit point of $\left(\cdots, m_{i}, \cdots\right)$, i.e., there exist $i_{k} \rightarrow-\infty$ such that $\alpha$ is the limit of $m_{i_{k}}$ and $j_{k} \rightarrow+\infty$ such that $\omega$ is the limit of $m_{j_{k}}$.

Note that $h_{c}^{k-i}\left(m_{i}, m_{k}\right)=h_{c}^{j-i}\left(m_{i}, m_{j}\right)+h_{c}^{k-j}\left(m_{j}, m_{k}\right)$, for $i<j<$ $k$. Since $\left(\cdots, m_{i}, \cdots\right)$ is $c$-minimal, we have $h_{c}^{k-i}\left(m_{i}, m_{k}\right)=\inf _{l} h_{c}^{l}\left(m_{i}, m_{k}\right)$ and similarly for $j$ in place of $i$. Hence, $\inf _{l} h_{c}^{l}\left(m_{i}, m_{k}\right)=h_{c}^{j-i}\left(m_{i}, m_{j}\right)+$ $\inf _{i} h_{c}^{l}\left(m_{j}, m_{k}\right)$. By passing to the limit, we have $h_{c}^{\infty}\left(m_{i}, \omega\right)=h_{c}^{j-i}\left(m_{i}, m_{j}\right)+$ $h_{c}^{\infty}\left(m_{j}, \omega\right)$, for $i<j$. Furthermore, $h_{c}^{\infty}\left(\omega, m_{j}\right) \leqslant h_{c}^{\infty}\left(\omega, m_{i}\right)+h_{c}^{j-i}\left(m_{i}, m_{j}\right)$. Adding, we obtain $d_{c}\left(m_{j}, \omega\right) \leqslant d_{c}\left(m_{i}, \omega\right)$, for $i<j$. In other words, $d_{c}\left(m_{i}, \omega\right)$ is a monotonically decreasing sequence. Since we already know that its liminf is zero, we obtain

$$
\lim _{i \rightarrow+\infty} d_{c}\left(m_{i}, \omega\right)=0
$$

Similarly, $d_{c}\left(m_{i}, \alpha\right)$ is a monotonically increasing sequence and

$$
\lim _{i \rightarrow-\infty} d_{c}\left(m_{i}, \alpha\right)=0
$$

From the triangle inequality, it follows that

$$
d_{c}(\alpha, \omega)=\lim _{i \rightarrow+\infty} d_{c}\left(\alpha, m_{i}\right)=\lim _{i \rightarrow-\infty} d_{c}\left(m_{i}, \omega\right)
$$

Also, if $\omega^{\prime}$ is a second $\omega$-limit point of $\left(\cdots, m_{i}, \cdots\right)$, we have $d_{c}\left(\omega, \omega^{\prime}\right) \leqslant$ $d_{c}\left(\omega, m_{i}\right)+d_{c}\left(m_{i}, \omega^{\prime}\right)$, so $d_{c}\left(\omega, \omega^{\prime}\right)=0$, by passage to the limit. Likewise, if $\alpha^{\prime}$ is a second $\alpha$-limit point of $\left(\cdots, m_{i}, \cdots\right)$, then $d_{c}\left(\alpha, \alpha^{\prime}\right)=0$.

If $d_{c}(\alpha, \omega)=0$, we will say that $\left(\cdots, m_{i}, \cdots\right)$ is a regular $c$-minimal configuration. In this case, $d_{c}\left(\alpha, m_{i}\right)=d_{c}\left(\omega, m_{i}\right)=0$, for all $i$, since these are monotonic bi-infinite sequences converging to zero at both ends. Consequently $d_{c}\left(m_{i}, m_{i}\right) \leqslant d_{c}\left(m_{i}, \omega\right)+d_{c}\left(\omega, m_{i}\right)=0$, so $m_{i} \in \Sigma_{c}^{0^{\prime}}$. By the triangle inequality, $d_{c}\left(m_{i}, m_{j}\right) \leqslant d_{c}\left(m_{i}, \alpha\right)+d_{c}\left(\alpha, m_{j}\right)=0$, so we have $d_{c}\left(m_{i}, m_{j}\right)=0$.

Conversely, for any $\xi \in \Sigma_{c}^{0^{\prime}}$, there exists a unique regular $c$-minimal configuration $\left(\cdots, m_{i}, \cdots\right)$ such that $\xi=m_{0}$. This may be shown as follows.

Existence. Choose an increasing sequence $n_{1}, n_{2}, \cdots, n_{k}, \cdots$ of positive integers such that $h_{\mathrm{c}}^{n_{k}}(\xi, \xi) \rightarrow B_{c}(\xi)=0$. For each $k$, choose a segment $m_{0}^{k}, \cdots, m_{i}^{k}, \cdots m_{n_{k}}^{k}$ of an $M$-configuration such that $\xi=m_{0}^{k}=m_{n_{k}}^{k}$ and $h_{c}^{n_{k}}(\xi, \xi)=\sum_{i=0}^{n_{k}-1} h_{c}\left(m_{i}^{k}, m_{i+1}^{k}\right)$. For every integer $j$, set $m_{j}^{k}=m_{i}^{k}$, where $0 \leqslant i<n_{k}$ is the remainder obtained by dividing $j$ by $n_{k}$. By passing to a subsequence, we may suppose that $m_{j}^{k} \longrightarrow m_{j}$. The resulting $M$-configuration $\left(\cdots, m_{j}, \cdots\right)$ is easily seen to be $c$-minimal and satisfy $m_{0}=\xi$.

To show that $\left(\cdots, m_{j}, \cdots\right)$ is regular, we observe that for $i>0$, $d_{c}\left(\xi, m_{i}\right)=\lim _{k \rightarrow \infty} d_{c}\left(\xi, m_{i}^{k}\right)$. Moreover, $d_{c}\left(\xi, m_{i}^{k}\right)=h_{c}^{\infty}\left(\xi, m_{i}^{k}\right)+h_{c}^{\infty}\left(m_{i}^{k}, \xi\right)$. Since $h_{c}^{\infty}(\xi, \xi)=B_{c}(\xi)=0$, we have $h_{c}^{\infty}\left(\xi, m_{i}^{k}\right) \leqslant h_{c}^{\infty}(\xi, \xi)+h_{c}^{i}\left(\xi, m_{i}^{k}\right)=$ $h_{c}^{i}\left(\xi, m_{i}^{k}\right)$. Likewise, $h_{c}^{\infty}\left(m_{i}^{k}, \xi\right) \leqslant h_{c}^{n_{k}-i}\left(m_{i}^{k}, \xi\right)$. Therefore,

$$
d_{c}\left(\xi, m_{i}^{k}\right) \leqslant h_{c}^{i}\left(\xi, m_{i}^{k}\right)+h_{c}^{n_{k}-i}\left(m_{i}^{k}, \xi\right)=h_{c}^{n_{k}}(\xi, \xi)
$$

Since the last quantity tends to zero as $k$ goes to infinity, it follows that $d_{c}\left(\xi, m_{i}\right)=0$, for $i>0$. A similar argument shows that we have this also for $i<0$. Passing to the limit, we obtain $d_{c}(\alpha, \omega)=0$.

Uniqueness. This is again the curve shortening argument. Given two regular $c$-minimal configurations ( $\cdots, m_{i}, \cdots$ ) and ( $\cdots, m_{i}^{\prime}, \cdots$ ) with $m_{0}=m_{0}^{\prime}=\xi$, we may use the curve shortening Lemma of $[\mathrm{Ma6}, \S 4]$ to construct two new configurations $\left(\cdots, m_{i}^{\prime \prime}, \cdots\right)$ and $\left(\cdots, m_{i}^{\prime \prime \prime}, \cdots\right)$ such that $m_{i}^{\prime \prime}=m_{i}$, for $i<0, m_{i}^{\prime \prime}=m_{i}^{\prime}$, for $i>0, m_{i}^{\prime \prime \prime}=m_{i}^{\prime}$, for $i<0, m_{i}^{\prime \prime \prime}=m_{i}$, for $i>0$, and

$$
\begin{aligned}
h_{c}\left(m_{-1}^{\prime \prime \prime}, m_{0}^{\prime \prime \prime}\right) & +h_{c}\left(m_{0}^{\prime \prime \prime}, m_{1}^{\prime \prime \prime}\right)+h_{c}\left(m_{-1}^{\prime \prime}, m_{0}^{\prime \prime}\right)+h_{c}\left(m_{0}^{\prime \prime}, m_{1}^{\prime \prime}\right) \\
& <h_{c}\left(m_{-1}, m_{0}\right)+h_{c}\left(m_{0}, m_{1}\right)+h_{c}\left(m_{-1}^{\prime}, m_{0}^{\prime}\right)+h_{c}\left(m_{0}^{\prime}, m_{1}^{\prime}\right)
\end{aligned}
$$

It follows that $d_{c}\left(\alpha^{\prime \prime \prime}, \omega^{\prime \prime \prime}\right)+d_{c}\left(\alpha^{\prime \prime}, \omega^{\prime \prime}\right)<d_{c}(\alpha, \omega)+d_{c}\left(\alpha^{\prime}, \omega^{\prime}\right)$, where $\alpha, \alpha^{\prime}$, etc. are the $\alpha$-limit points of $\left(\cdots, m_{i}, \cdots\right),\left(\cdots, m_{i}^{\prime}, \cdots\right)$ etc. and $\omega, \omega^{\prime}$, etc. are the corresponding $\omega$-limit points. However, $d_{c}(\alpha, \omega)=$ $d_{c}\left(\alpha^{\prime}, \omega^{\prime}\right)$, so we have obtained a contradiction. This contradiction proves uniqueness.

We may extend the $c$-Poincaré map $f_{c}$ to $\Sigma_{c}^{0^{\prime}}$, as follows. Given $\xi \in \Sigma_{c}^{0^{\prime}}$, we let $\left(\cdots, m_{i}, \cdots\right)$ be the unique regular $c$-minimal configuration such that $m_{0}=\xi$. We set $f_{c}(\xi)=m_{1}$. It is clear that this extends the previously defined $f_{c}$. Moreover, $f_{c}: \Sigma_{c}^{0^{\prime}} \longrightarrow \Sigma_{c}^{0^{\prime}}$ is a bi-Lipschitz homeomorphism. This follows from the proof of the uniqueness of the regular $c$-minimal configuration $\left(\cdots, m_{i}, \cdots\right)$ such that $\xi=m_{0}$ : the curve shortening Lemma of [Ma6, §4] contains the requisite Lipschitz result.

Now we may introduce some more terminology, using the one-one correspondence between $c$-minimal trajectories of the Euler-Lagrange flow and $c$-minimal configurations. We will say that a $c$-minimal trajectory of the Euler-Lagrange flow is regular if the corresponding $M$-configuration is regular. We let $M_{c}^{\prime} \subset T M \times \mathbb{T}$ denote the union of all $c$-minimal trajectories. The assertions of Theorem 2.2 extend to $M_{c}^{\prime}$.

Theorem 6.1. - $M_{c}^{\prime}$ is a compact, non-empty subset of $T M \times \mathbb{T}^{1}$ containing $M_{c}$. The restriction of $\pi$ to $M_{c}^{\prime}$ is injective. The inverse mapping $\pi^{-1}: \pi\left(M_{c}^{\prime}\right) \rightarrow M_{c}^{\prime}$ is Lipschitz.

The proofs of all these assertions follow from the discussion above.
We set $\Sigma_{c}^{\prime}=\pi\left(M_{c}^{\prime}\right) \subset M \times \mathbb{T}$. It follows from this theorem that the flow on $\Sigma_{c}^{\prime}$ which corresponds to the Euler-Lagrange flow on $M_{c}^{\prime}$ is Lipschitz, and is generated by a Lipschitz vector field. Clearly, $\Sigma_{c}^{0^{\prime}}=\Sigma_{c}^{\prime} \cap(M \times 0)$ and $f_{c}: \Sigma_{c}^{0^{\prime}} \rightarrow \Sigma_{c}^{0^{\prime}}$ is the time one map of this flow.

## 7. Peierls's barrier.

In this section, we specialize to the case $M=\mathbb{T}$. We will show that if $(c, \alpha(c))$ is an extremal point of the epigraph of $\alpha$, then the barrier $B_{c}$ (defined in the previous section) is the same as Peierls's barrier $P_{\omega}$, for a suitable rotation symbol $\omega$. Peierls's barrier was defined originally in [A-LD-A] and again in [Ma1] and [Ma2]. See also [Ma5], which is perhaps the most convenient reference for this notion, and also for the notion of rotation symbol. Here we recall only that a rotation symbol is a Dedekind cut of
$\mathbb{Q}$. Thus, to each irrational number, there corresponds a unique rotation symbol. To each rational number, there correspond three rotation symbols, denoted $p / q-, p / q$ and $p / q+$.

To be more explicit, we have to recall some properties of the minimal average action $\beta: H_{1}(\mathbb{T}, \mathbb{R}) \longrightarrow \mathbb{R}$ and of its conjugate $\alpha: H^{1}(\mathbb{T}, \mathbb{R}) \longrightarrow \mathbb{R}$, which were proved independently by the author [Ma7] and Bangert [ Ba 3 ], and discussed much earlier from a physicist's viewpoint by Aubry [Aub]. In terms of the identification $H_{1}(\mathbb{T}, \mathbb{R})=\mathbb{R}$, we have that $\beta$ is differentiable at all irrational numbers. Moreover, if $p / q$ is a rational number in lowest terms, then $\beta$ is differentiable at $p / q$ if and only if there exists a homotopically non-trivial $f_{L}$-invariant curve $\Gamma \subset T \mathbb{T}(=\mathbb{T} \times \mathbb{R})$ of rotation number $p / q$, consisting entirely of periodic orbits of period $q$.

Of course, these results may be reinterpreted in terms of the conjugate function $\alpha: H^{1}(\mathbb{T}, \mathbb{R}) \longrightarrow \mathbb{R}$. We use the identification $H^{1}(\mathbb{T}, \mathbb{R})=\mathbb{R}$ and thus think of $\alpha$ as a real valued function of a real variable. The fact that $\beta$ is strictly convex translates to the fact that $\alpha$ is differentiable. The fact the $\beta$ is differentiable at irrational numbers translates to the fact that every flat piece of graph $\alpha$ has rational slope.

Proposition 7.1. -- When $\omega=\alpha^{\prime}(c)$ is irrational, we have $B_{c}=P_{\omega}$.
Strictly speaking, in [Ma1] and [Ma2], we defined $P_{\omega}$ to be a real valued function of a real variable. However, it is periodic of period one. Thus, we may think of it as a real valued function on $\mathbb{T}$. The equation $B_{c}=P_{\omega}$ above means equality of functions on $\mathbb{T}$.

For the proof, we recall that $P_{\omega}$ was defined to be identically zero on $\Sigma_{c}^{0}$ in [Ma1] and [Ma2]. In the previous section, we showed that $B_{c}$ is identically zero on $\Sigma_{c}^{0}$. Thus, it is enough to consider $a \epsilon \mathbb{T} \backslash \Sigma_{c}^{0}$ and show that $B_{c}(a)=P_{\omega}(a)$. The component of $\mathbb{T} \backslash \Sigma_{c}^{0}$ which contains $a$ is a segment whose endpoints we denote by $a_{-}$and $a_{+}$. Since $a_{ \pm} \in \Sigma_{c}^{0}$, there exist unique $c$-minimal configuration $\xi_{ \pm}=\left(\cdots, \xi_{i \pm}, \cdots\right)$ such that $\xi_{0 \pm}=a_{ \pm}$. We choose lifts $\widetilde{a}, \widetilde{a}_{ \pm}, \widetilde{\xi}_{i \pm}$ of $a, a_{ \pm}, \xi_{i \pm}$ to $\mathbb{R}$ such that $\widetilde{a}_{-}<\tilde{a}<\widetilde{a}_{+}<\widetilde{a}_{-}+1, \widetilde{\xi}_{0 \pm}=$ $\widetilde{a}_{ \pm}$, and such that $\widetilde{\xi}_{ \pm}=\left(\cdots, \widetilde{\xi}_{i \pm}, \cdots\right)$ is a minimal configuration. Peierls's barrier is defined as follows ([Ma1], [Ma2]) :

$$
P_{\omega}(a)=\min \sum_{i \in \mathbb{Z}} \widetilde{h}\left(\widetilde{\xi}_{i}, \widetilde{\xi}_{i+1}\right)-\widetilde{h}\left(\widetilde{\xi}_{i-}, \widetilde{\xi}_{i+1-}\right)
$$

Here, $\widetilde{h}$ is the variational principle associated to $L$ in the sense of $\S 4$, so it is a function defined on $\mathbb{R} \times \mathbb{R}$. The minimum is taken over all
$\tilde{\xi}=\left(\cdots, \widetilde{\xi}_{i}, \cdots\right) \in \mathbb{R}^{\mathbb{Z}}$ such that $\widetilde{\xi}_{i-} \leqslant \widetilde{\xi}_{i} \leqslant \widetilde{\xi}_{i+}$ and $\widetilde{\xi}_{0}=\widetilde{a}$. The condition $\widetilde{\xi}_{i-} \leqslant \widetilde{\xi}_{i} \leqslant \widetilde{\xi}_{i+}$ guarantees that the sum above is absolutely convergent, since $\sum \widetilde{\xi}_{i+1-}-\widetilde{\xi}_{i-} \leqslant 1$ in the case that $\omega$ is irrational. Note that if $\widetilde{\xi}_{-}$is replaced by $\widetilde{\xi}_{+}$in the formula above for $P_{\omega}(a)$, it is still valid.

To prove that $B_{c}(a)=P_{\omega}(a)$, we consider how the definition of $P_{\omega}(a)$ may be put in a form more closely resembling that of $B_{c}(a)$. If we replace $\widetilde{h}$ by the variational principle $\widetilde{h}_{c}$ associated to $L-\eta_{c}-\alpha(c)$ in the expression defining $P_{\omega}(a)$, we get the same quantity. From Aubry's crossing lemma, we then obtain

$$
P_{\omega}(a)=\lim _{k, l \rightarrow+\infty} h_{c}^{k}\left(\xi_{-k-}, a\right)+h_{c}^{l}\left(a, \xi_{l-}\right)-h_{c}^{k+l}\left(\xi_{-k-}, \xi_{l-}\right)
$$

Note that $h_{c}^{k+l}\left(\xi_{-k-}, \xi_{l-}\right)=h_{c}^{\infty}\left(\xi_{-k-}, \xi_{l-}\right)$, since $\left(\cdots, \xi_{l-}, \cdots\right)$ is $c$ minimal.

Since $\omega=\alpha^{\prime}(c)$ is irrational, it follows from the well known theory of twist maps that $\Sigma_{c}^{0}$ is a Denjoy minimal set for the Poincaré map $f_{c}$. Thus, every orbit of $f_{c}$ is dense in $\Sigma_{c}^{0}$. In particular, we may choose $\xi \in \Sigma_{c}^{0}$ and sequences $k_{j}, l_{j} \rightarrow+\infty, j=1,2 \cdots$ such that $\xi_{-k_{j}-} \rightarrow \xi$ and $\xi_{l_{j}-} \rightarrow \xi$ as $j \rightarrow \infty$. Clearly, $\lim _{j \rightarrow \infty} h_{c}^{\infty}\left(\xi_{-k_{j}-}, \xi_{l_{j}-}\right)=h_{c}^{\infty}(\xi, \xi)=0$. Moreover, $\lim _{j \rightarrow \infty} h_{c}^{k_{j}}\left(\xi_{-k_{j}-}, a\right)=h_{c}^{\infty}(\xi, a)$ and $\lim _{j \rightarrow \infty} h_{c}^{l_{j}}\left(a, \xi_{l_{j}}\right)=h_{c}^{\infty}(a, \xi)$. Therefore

$$
B_{c}(a)=h_{c}^{\infty}(a, \xi)+h_{c}^{\infty}(\xi, a)=P_{\omega}(a)
$$

Proposition 7.2. - If $\alpha^{\prime}(c)=p / q$ in lowest terms and $c=$ $\max \left\{c^{*}: \alpha^{\prime}\left(c^{*}\right)=p / q\right\}\left(\right.$ resp. $\left.c=\min \left\{c^{*}: \alpha^{\prime}\left(c^{*}\right)=p / q\right\}\right)$ then $B_{c}(a)=$ $P_{p / q+}(a)\left(\right.$ resp. $\left.P_{p / q-}(a)\right)$.

The proof is similar to the proof of Proposition 7.1 and we omit it.
Thus, we have related $B_{c}(a)$ to Peierls's barrier in all cases when ( $c, \alpha(c)$ ) is an extremal point of the epigraph of $\alpha$. When $(c, \alpha(c))$ is not an extremal point of the epigraph of $\alpha$, then it lies on a flat part of the graph of $\alpha$. Let $p / q$ be the slope of this flat part, expressed in lowest terms. Then $B_{c}$ and $P_{p / q}$ have the same zero set. However, in general they are not equal.

## 8. Another barrier.

In the next section, we will state versions of Theorems $A$ and $B$ of the introduction (Theorems 4.1 and 4.2 of [Ma5]) in our more general setting. For this we need a variant of the barrier defined in $\S 6$. In this section we define this variant and develop some of its properties.

We set

$$
B_{c}^{*}(m)=\min \left\{h_{c}^{\infty}(\xi, m)+h_{c}^{\infty}(m, \eta)-h_{c}^{\infty}(\xi, \eta): \xi, \eta \in \Sigma_{c}^{0}\right\}
$$

It is easily seen that $B_{c}^{*}(m)$ is a Lipschitz function of $m$, with a uniform Lipschitz constant for $c$ in a compact set. Note that

$$
B_{c}(m)=\min \left\{h_{c}^{\infty}(\xi, m)+h_{c}^{\infty}(m, \eta)-h_{c}^{\infty}(\xi, \eta): \xi, \eta \in \Sigma_{c}^{0}, d_{c}(\xi, \eta)=0\right\}
$$

It follows that $0 \leqslant B_{c}^{*} \leqslant B_{c}$. Clearly, $\left\{B_{c}=0\right\}$ is the union of the regular $c$-minimal configurations and $\left\{B_{c}^{*}=0\right\}$ is the union of the $c$-minimal configurations. Moreover, if $d_{c}$ vanishes on $\Sigma_{c}^{0} \times \Sigma_{c}^{0}$, then $B_{c}^{*}=B_{c}$.

In $\S 6$, we observed that if $\left(\cdots, m_{i}, \cdots\right)$ is a regular $c$-minimal configurations, then $d_{c}\left(m_{i}, m_{j}\right)=0$. Thus, if the Poincaré map $f_{c}: \Sigma_{c}^{0} \rightarrow \Sigma_{c}^{0}$ has a dense orbit, then $d_{c}$ vanishes identically on $\Sigma_{c}^{0} \times \Sigma_{c}^{0}$.

In the case of a twist map, if $\omega=\alpha^{\prime}(c)$ is irrational, then $f_{c}: \Sigma_{c}^{0} \longrightarrow$ $\Sigma_{c}^{0}$ has a dense orbit, so $d_{c}$ vanishes identically on $\Sigma_{c}^{0} \times \Sigma_{c}^{0}$, and $B_{c}=B_{c}^{*}$.

Let $d$ be a metric on $M$ associated to a smooth Riemannian metric. The pseudo-metric $d_{c}$ satisfies a Hölder condition of exponent 2 with respect to $d$, viz.,

$$
d_{c}(\xi, \eta) \leqslant C d(\xi, \eta)^{2}, \quad \xi \in \Sigma_{c}^{0^{\prime}}, \eta \in M
$$

Here, $C$ is a constant which depends only on the Lagrangian $L$ and the cohomology class $c$. Moreover, $C$ may be chosen to be independent of $c$ for $c$ in a compact subset of $H^{1}(M, \mathbb{R})$.

To prove this, we use the fact that there is a regular $c$-minimal configuration $\left(\cdots, m_{i}, \cdots\right)$ such that $\xi=m_{0}$. Let $\alpha$ be an $\alpha$-limit point and $\omega$ an $\omega$-limit point of $\left(\cdots, m_{i}, \cdots\right)$. Then

$$
\begin{aligned}
d_{c}(\xi, \eta) & =h_{c}^{\infty}(\xi, \eta)+h_{c}^{\infty}(\eta, \xi) \\
& =h_{c}^{\infty}(\alpha, \eta)+h_{c}^{\infty}(\eta, \omega)-h_{c}^{\infty}(\alpha, \xi)-h_{c}^{\infty}(\xi, \omega) \\
& \leqslant h_{c}\left(m_{-1}, \eta\right)+h_{c}\left(\eta, m_{1}\right)-h_{c}\left(m_{-1}, \xi\right)-h_{c}\left(\xi, m_{1}\right) \\
& \leqslant C d(\xi, \eta)^{2} .
\end{aligned}
$$

Here, the second equation is consequence of the equations $h_{c}^{\infty}(\alpha, \xi)+$ $h_{c}^{\infty}(\xi, \eta)=h_{c}^{\infty}(\alpha, \eta)$ and $h_{c}^{\infty}(\eta, \xi)+h_{c}^{\infty}(\xi, \omega)=h_{c}^{\infty}(\eta, \omega)$, which hold because $d_{c}(\alpha, \xi)=d_{c}(\xi, \omega)=0$. To prove the first inequality, we consider sequences $k_{j}, l_{j} \rightarrow+\infty$ such that $m_{-k_{j}} \rightarrow \alpha$ and $m_{l_{j}} \rightarrow \omega$. Then $h_{c}^{k_{j}}\left(m_{-k_{j}}, \xi\right)=\inf _{l} h_{c}^{l}\left(m_{-k_{j}}, \xi\right) \rightarrow h_{c}^{\infty}(\alpha, \xi)$ and $h_{c}^{l_{j}}\left(\xi, m_{l_{j}}\right)=$ $\inf _{l} h_{c}^{l}\left(\xi, m_{l_{j}}\right) \rightarrow h_{c}^{\infty}(\xi, \omega)$. The first inequality follows easily. The second inequality is elementary.

Since $d_{c}$ satisfies a Hölder condition of exponent 2, we have that $d_{c}(\xi, \eta)=0$, if $\xi, \eta \in \Sigma_{c}^{0}$ and $\xi$ and $\eta$ can be connected in $\Sigma_{c}^{0}$ by a rectifiable curve. Thus, it follows that $d_{c}$ vanishes identically if $\Sigma_{c}^{0}=M$. For example, in the twist map case, we have seen that if $p / q$ is a rational number and $\left\{c: \alpha^{\prime}(c)=p / q\right\}$ is reduced to one point, then $\Sigma_{c}^{0}=\mathbb{T}^{1}$. Thus $d_{c}$ vanishes identically in this case.

Continuing with the twist map case, we next consider the generic situation, viz. $\left\{c: \alpha^{\prime}(c)=p / q\right\}$ is an interval $\left[c_{0}, c_{1}\right]$, with $c_{0}<c_{1}$. For any $c \in\left[c_{0}, c_{1}\right], \Sigma_{c}^{0}$ is the union of the minimal configurations of rotation symbol $p / q$. If, furthermore, $c \in\left(c_{0}, c_{1}\right)$, then $\Sigma_{c}^{0^{\prime}}=\Sigma_{c}^{0}$. On the other hand, if $c=c_{0}$ (resp. $c=c_{1}$ ), then $\Sigma_{c}^{0^{\prime}}$ is the union of the minimal configurations of rotation symbol $p / q-$ (resp. $p / q+$ ) and those of rotation symbol $p / q$, and properly contains $\Sigma_{c}^{0}$ (c.f. [Ma2]).

In the cases $c=c_{0}$ and $c=c_{1}$, the pseudo-metric $d_{c}$ vanishes on $\Sigma_{c}^{0^{\prime}} \times \Sigma_{c}^{0^{\prime}}$. In the case $c_{0}<c<c_{1}$, two points in $\Sigma_{c}^{0}$ are at positive distance with respect to $d_{c}$ if and only if their images in the quotient space $\Sigma_{c}^{0} / f_{c}$ are in distinct connected components. For a generic twist diffeomorphism, $\Sigma_{c}^{0} / f_{c}$ is one point (in the case $\alpha^{\prime}(c) \in \mathbb{Q}$ ), but in exceptional cases, it has several points and $d_{c}$ does not vanish on $\Sigma_{c}^{0} \times \Sigma_{c}^{0}$. In the case that $\Sigma_{c}^{0} / f_{c}$ is one point, $B_{c}^{*}=B_{c}$, but in the remaining cases, $B_{c}^{*}<B_{c}$, and $\left\{B_{c}^{*}=0\right\}$ properly contains $\left\{B_{c}=0\right\}$.

Continuing with the twist map case with the restriction that $\{c: \alpha(c)=$ $p / q\}$ is an interval $\left[c_{0}, c_{1}\right]$, we now consider the situation when $\Sigma_{c}^{0} / f_{c}(c \in$ [ $\left.c_{0}, c_{1}\right]$ ) has two or more points. (Note that this set is independent of $c \in\left[c_{0}, c_{1}\right]$, since $\Sigma_{c}^{0}$ is the union of the minimal configurations of rotation symbol $p / q$.) Consider $\xi, \eta \in \Sigma_{c}^{0}$ which have positive distance with respect to $d_{c}$ and hence represent different elements in $\Sigma_{c}^{0} / f_{c}$, and let

$$
\Sigma_{c}^{0}(\xi, \eta)=\left\{m \in \mathbb{T}^{1}: h_{c}^{\infty}(\xi, m)+h_{c}^{\infty}(m, \eta)-h_{c}^{\infty}(\xi, \eta)=0\right\}
$$

Since $\Sigma_{c}^{0}$ inherits a cyclic order from $\mathbb{T}^{1}$, there is an induced cyclic order on $\Sigma_{c}^{0} / f_{c}$, and also on the quotient of the set of complementary intervals of
$\Sigma_{c}^{0}$ by $f_{c}$. Now $\Sigma_{c}^{0}(\xi, \eta)$ may be described in the following way. There is a critical value $c^{\prime}$ with $c_{0}<c^{\prime}<c_{1}$ such that for $c^{\prime}<c<c_{1}$ (resp. $c_{0}<c<$ $\left.c^{\prime}\right), \Sigma_{c}^{0}(\xi, \eta)$ is the set of all $m \in \mathbb{T}^{1}$ such that either $m$ is in the orbit of $\xi$ or $\eta$ under $f_{c}$, or $m$ is in a configuration of rotation symbol $p / q+$ (resp. $p / q-$ ) and $(\xi, m, \eta$ ) is positively (resp. negatively) oriented with respect to the quotient cyclic order. Moreover $\Sigma_{c^{\prime}}^{0}(\xi, \eta)$ is the union of the two sets just described.

Since $\left\{B_{c}^{*}=0\right\}$ is, by definition, the union of all the sets $\Sigma_{c}^{0}(\xi, \eta)$ just described, this provides a description of $\left\{B_{c}^{*}=0\right\}$.

## 9. Versions of Theorems A and B in more degrees of freedom.

In this section, we state versions of Theorems $A$ and $B$ of the introduction (Theorems 4.1 and 4.2 of [ Ma 5$]$ ) in more degrees of freedom. We will also discuss the extent to which these generalize Theorems A and B.

We let $W_{L}=\left\{c \in H^{1}(M, \mathbb{R})\right.$ : there exists an open neighborhood $U$ of $\left\{B_{c}^{*}=0\right\}$ in $M$ such that the inclusion map $H_{1}(U, \mathbb{R}) \longrightarrow H_{1}(M, \mathbb{R})$ is the zero map $\}$. From the fact that $\left\{B_{c}^{*}=0\right\}$ is the union of the $c$-minimal configurations, it follows that the set function $c \rightarrow\left\{B_{c}^{*}=0\right\}$ is upper semi-continuous. Consequently, $W_{L}$ is open in $H^{1}(M, \mathbb{R})$.

Theorem 9.1. - Suppose $c_{0}$ and $c_{1}$ are in the same connected component of $W_{L}$. Then there is a trajectory of the Euler-Lagrange flow whose $\alpha$-limit set lies in $M_{c_{0}}^{\prime}$ and whose $\omega$-limit set lies in $M_{c_{1}}^{\prime}$.

Theorem 9.2. - Consider a bi-infinite sequence ( $\cdots, c_{i}, \cdots$ ) of cohomology classes, each of which lies in the same connected component of $W_{L}$. Let $\left(\cdots, \epsilon_{i}, \cdots\right)$ be a sequence of positive numbers. Then there is a trajectory of the Euler-Lagrange flow which passes within a distance of $\epsilon_{i}$ of each of the sets $M_{c_{i}}^{\prime}$ in turn.

These are our versions of Theorems A and B in more degrees of freedom.

To see to what extent these generalize Theorems A and B, we examine the relationship between the connected components of $W_{L}$ and the Birkhoff regions of instability.

Thus, we consider a twist mapping $f$. According to Moser [Mo], there is a Lagrangian $L: T \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{R}$, satisfying the hypotheses we considered in $\S 1$, whose time one map is $f$. Clearly, a cohomology class $c$ is a member of $W_{L}$ if and only if $\left\{B_{c}^{*}=0\right\}$ is properly contained in $\mathbb{T}$. In the case that $\alpha^{\prime}(c)$ is irrational, this holds if and only if there is no homotopically non-trivial $f$-invariant curve of rotation number $\alpha^{\prime}(c)$. However, in the case that $\alpha^{\prime}(c)$ is rational, the situation is more complicated.

Let $p / q$ be a rational number, expressed in lowest terms. Let $\left[c_{0}, c_{1}\right]=$ $\left\{c \in H^{1}(M, \mathbb{R}): \alpha^{\prime}(c)=p / q\right\}$. If $\left[c_{0}, c_{1}\right]$ is reduced to one point $c$, then there exists a homotopically non-trivial invariant curve of rotation number $p / q$, consisting entirely of periodic orbits of period $q$ (cf. [Aub], [Ban3], [Ma7]). In this case, $\left\{B_{c}^{*}=0\right\}=\left\{B_{c}=0\right\}=\mathbb{T}$, so $c \notin W_{L}$.

Thus, we restrict our attention to the case when $c_{0}<c_{1}$. This case divides into several subcases, depending on how many action minimizing orbits of rotation number $p / q$ there are.

In the generic situation there is just one. When there is just one, we see from the description in $\S 8$ of $\left\{B_{c}^{*}=0\right\}$ that $c \in W_{L}$ if $c \in\left(c_{0}, c_{1}\right)$. Moreover, in the case $c=c_{0}$ (resp. $c=c_{1}$ ), $c \in W_{L}$ if and only if there does not exist a homotopically non-trivial $f$-invariant curve of rotation number $p / q$ consisting entirely of orbits of symbol $p / q$ or $p / q-($ resp. $p / q+$ ).

When there are two action minimizing orbits of rotation number $p / q$, the situation is the same as before for $c=c_{0}, c_{1}$, but more complicated for $c_{0}<c<c_{1}$. The assumption that there are two minimizing orbits means that $\Sigma_{c}^{0} / f_{c}$ has two elements. Let $\xi, \eta \in \Sigma_{c}^{0}$ represent the two different elements of $\Sigma_{c}^{0} / f_{c}$. As discussed in $\S 8$, we have $\left\{B_{c}^{*}=0\right\}=$ $\Sigma_{c}^{0}(\xi, \eta) \cup \Sigma_{c}^{0}(\eta, \xi)$. After some thought, the reader should be able to see that if there is a homotopically non-trivial $f$-invariant curve $\Gamma$ of rotation number $p / q$, then $\left\{B_{c}^{*}=0\right\}=\mathbb{T}$ in at least one of the following cases : $c=c_{0}, c=c_{1}, c=c^{\prime}$, or $c=c^{\prime \prime}$, where $c^{\prime}$ is the bifurcation value of $\Sigma_{c}^{0}(\xi, \eta)$ (i.e. the unique value of $c$ between $c_{0}$ and $c_{1}$ where it changes) and $c^{\prime \prime}$ is the bifurcation value of $\Sigma_{c}^{0}(\eta, \xi)$. More specifically, there are the following possibilities : $\Gamma$ consists of orbits of rotation symbol $p / q+$ (resp. $p / q-$ ) and orbits of rotation symbol $p / q$, in which case $\left\{B_{c_{1}}^{*}=0\right\}=\mathbb{T}$ (resp. $\left\{B_{c_{0}}^{*}=\right.$ $0\}=\mathbb{T}$ ); or it contains orbits both of rotation symbol $p / q$ - and of rotation symbol $p / q+$, in which case $\Sigma_{\mathrm{c}^{\prime}}^{0}(\xi, \eta)=\mathbb{T}$ or $\Sigma_{\mathrm{c}^{\prime \prime}}^{0}(\eta, \xi)=\mathbb{T}$.

In the case of two action minimizing orbits of rotation symbol $p / q$, it may happen that $\left\{B_{c}^{*}=0\right\}=\mathbb{T}$ for some $c_{0}<c<c_{1}$ even though there is no homotopically non-trivial $f$-invariant curve of rotation number $p / q$. If
this happens at all, it must happen when $c$ is one of the critical values $c^{\prime}$ or $c^{\prime \prime}$.

To see how this can happen, we consider for simplicity the case when $p / q=0$. Then $\Sigma_{c}^{0}$ has two points $\xi$ and $\eta$. Let $I$ (resp. $J$ ) denote the arc in $\mathbb{T}$ consisting of all points $\theta$ for which $(\xi, \theta, \eta)$ (resp. $(\eta, \theta, \xi)$ ) is positively oriented with respect to the cyclic order in $\mathbb{T}$. We could consider, e.g., a twist mapping $f$, for which every element of $I$ is part of a configuration of rotation symbol $0+$ and every element of $J$ is part of a configuration of rotation symbol $0+$ or $0-$, but there exist an element in $J$ which is not part of a configuration of rotation symbol $0+$ and an element in $J$ which is not part of a configuration of rotation symbol $0-$. Such twist maps are easily constructed. We may do this in such a way that $c^{\prime \prime}<c^{\prime}$. For such mappings, there is no homotopically non-trivial invariant curve of rotation number 0 , but $\left\{B_{c^{\prime}}=0\right\}=\mathbb{T}$.

Such examples are very exceptional, but they do show that Theorems 9.1 and 9.2 do not generalize Theorems A and B. Such examples give a kind of extraneous obstruction to finding connecting orbits - extraneous in the sense that the connecting orbits exist in these examples, even though their existence does not follow from Theorems 9.1 and 9.2.

Presumably, it should be possible to improve Theorems 9.1 and 9.2 by weakening the hypothesis, so that there are no such extraneous obstructions. However, we have not done so until now.

## 10. Proof of Theorem 9.1 .

Let $c_{0}, c_{1}$ be in the same connected component of $W_{L}$. Since $W_{L}$ is an open subset of the finite dimensional vector space $H^{1}(M, \mathbb{R})$, we may choose a simple smooth curve $\Gamma$ in $W_{L}$ joining $c_{0}$ and $c_{1}$. For each $c \in \Gamma$ we choose a smooth closed one form $\eta_{c}$ whose de Rham cohomology class is $c$. We choose $\eta_{c}$ so that it depends smoothly on $c$ and the other variables jointly and so that for any $c^{*} \in \Gamma$, we have a neighborhood $U_{c^{*}}$ of $\left\{B_{c^{*}}^{*}=0\right\}$ in $M$ and a neighborhood $J_{c^{*}}$ of $c^{*}$ in $\Gamma$ such that $\eta_{c}\left|U_{c^{*}}=\eta_{c^{*}}\right| U_{c^{*}}$ for $c \in J_{c^{*}}$. The possibility of choosing $\eta_{c}$ in this way is a consequence of the assumption that $\left\{B_{\mathrm{c}^{*}}^{*}=0\right\}$ has a neighborhood $V$ in $M$ such that the inclusion $\operatorname{map} H_{1}(V, \mathbb{R}) \longrightarrow H_{1}(M, \mathbb{R})$ is the zero map, together with the fact that the set function $c \longrightarrow\left\{B_{c}^{*}=0\right\}$ is upper semi-continuous.

Given a sequence $\vec{c}=\left(c^{0}, \cdots, c^{N}\right)$ of elements of $\Gamma$, an increasing sequence $\vec{j}=\left(j_{1}<\cdots<j_{N}\right)$ of integers, an $M$-configuration $\left(\cdots, m_{i}, \cdots\right)$, and integers $a<j_{1}$ and $b>j_{N}$, we define

$$
h_{\vec{c}, \vec{j}}\left(m_{a}, \cdots, m_{b}\right)=\sum_{i=0}^{N} \sum_{k=j_{i}}^{j_{i+1}-1} h_{c^{i}}\left(m_{k}, m_{k+1}\right)
$$

where we set $j_{0}=a$ and $j_{N+1}=b$. Note that $h_{c^{i}}$ depends on the choice of $\eta_{c^{i}}$. For what follows, it is essential that $\eta_{c}$ be chosen as indicated above.

We will say that the segment ( $m_{a}, \cdots, m_{b}$ ) of an $M$-configuration is $(\vec{c}, \vec{j})$-minimal if for every increasing sequence $\vec{j}^{\prime}=\left(j_{1}^{\prime}<\cdots<j_{N}^{\prime}\right)$ of integers satisfying $j_{i+1}^{\prime}-j_{i}^{\prime} \leqslant j_{i+1}-j_{i}, i=1, \cdots, N$, any integers $c<j_{1}^{\prime}$ and $d>j_{N}^{\prime}$, and any segment of an $M$-configuration ( $m_{c}^{\prime}, \cdots, m_{d}^{\prime}$ ) satisfying the boundary condition $m_{a}=m_{c}^{\prime}, m_{b}=m_{d}^{\prime}$, we have

$$
h_{\vec{c}, \vec{j}}\left(m_{1}, \cdots, m_{b}\right) \leqslant h_{\vec{c}, \vec{j}}\left(m_{c}^{\prime}, \cdots, m_{d}^{\prime}\right)
$$

We will say that an $M$-configuration is $(\vec{c}, \vec{j})$-minimal if every segment of it is $(\vec{c}, \vec{j})$-minimal.

An easy compactness argument shows that for any sequence $\vec{c}=$ $\left(c^{0}, \cdots, c^{N}\right)$ of elements of $\Gamma$, and any increasing sequence $\vec{j}=\left(j_{1}<\cdots<\right.$ $j_{N}$ ) of integers, there exists a $(\vec{c}, \vec{j})$-minimal configuration.

The strategy of proof of Theorem 9.1 is to choose a sequence $\vec{c}=$ ( $c^{0}=c_{0}, c^{1}, c^{2}, \cdots, c^{N}=c_{1}$ ) of elements of $\Gamma$, with $c^{i+1}$ very close to $c^{i}$, an increasing sequence $\left(j_{1}<\cdots<j_{N}\right)$ of integers, with $j_{i+1}-j_{i}$ very large, and a $(\vec{c}, \vec{j})$-minimal configuration $\left(\cdots, m_{i}, \cdots\right)$. Then we construct a curve $\gamma: \mathbb{R} \rightarrow M$ by letting $\gamma(t), i \leqslant t \leqslant i+1$, be a Tonelli minimizer satisfying the boundary condition $\gamma(i)=m_{i}, \gamma(i+1)=m_{i+1}$.

Our assertion is that if $\vec{c}$ and $\vec{j}$ are chosen appropriately (i.e., if $c^{i+1}$ is sufficiently close to $c^{i}$, and $j_{i+1}-j_{i}$ is sufficiently large), then $\gamma$ satisfies the required conditions, i.e., it satisfies the Euler-Lagrange equation, and the curve $t \longrightarrow\left(d \gamma(t), t\right.$ mod. 1) in $T M \times \mathbb{T}$ has its $\alpha$-limit set in $M_{c_{0}}^{\prime}$ and its $\omega$-limit set in $M_{c_{N}}^{\prime}$.

The assertions about the $\alpha$ - and $\omega$-limit sets are easy consequences of the theory which we have already developed. We leave their verification to the reader.

The crux is the assertion that $\gamma$ satisfies the Euler-Lagrange equation. It is obvious that $\gamma$ satisfies the Euler-Lagrange equation except possibly at
$t=j_{i}, i=1, \cdots, N$. To verify that $\gamma$ satisfies the Euler-Lagrange equation at $t=j_{i}$, it is enough to check that $\eta_{c^{i-1}}=\eta_{c^{i}}$ in a neighborbood of $m_{j_{i}}$.

We may first choose $c^{* 0}, \cdots, c^{* k}$, ordered along $\Gamma$, starting at $c^{* 0}=c_{0}$ and ending at $c^{* k}=c_{1}$, such that for each $i$, there is an open set $U_{i} \subset M$ with the property that if $c$ is in the $\operatorname{arc}\left[c^{* i-1}, c^{* i+1}\right]$ in $\Gamma$, then $\eta_{c}=\eta_{c^{* i}}$ on $U_{i}$ and $\left\{B_{c}^{*}=0\right\} \subset U_{i}$. This is a consequence of the way that the $\eta_{c}$ were chosen and the upper semi-continuity of the set-function $c \rightarrow\left\{B_{c}^{*}=0\right\}$.

We think of $c^{* 0}, \cdots, c^{* k}$ as defining a partition of $\Gamma$. We choose $c^{0}, \cdots c^{N}$, ordered along $\Gamma$, to define a refinement of this partition. In other words, we choose the latter so that $\left\{c^{* 0}, \cdots, c^{* k}\right\} \subset\left\{c^{0}, \cdots, c^{N}\right\}$. It is easy to see that if $c^{i-1}$ and $c^{i}$ are sufficiently close and $j_{i}-j_{i-1}$ and $j_{i+1}-j_{i}$ are sufficiently large, then $m_{j_{i}} \in U_{l}$ where $l$ is chosen so that $c^{i-1}$ and $c^{i}$ are in $\left[c^{* l-1}, c^{* l}\right]$. Thus, $\eta_{c^{i-1}}=\eta_{c^{i}}$ in a neighborhood of $m_{j_{i}}$, as required.

## 11. Proof of Theorem 9.2.

This is very similar to the proof of Theorem 9.1.
Let $\left(\cdots, c_{i}, \cdots\right)$ be a bi-infinite sequence consisting of elements of the same connected component of $W_{L}$. This time, we choose a smooth parameterized curve $\Gamma: \mathbb{R} \longrightarrow W_{L}$ such that $\Gamma(i)=c_{i}$ and $\Gamma \mid[i, i+1]$ is an embedding. For each $t \in \mathbb{R}$, we choose a smooth closed one form $\eta_{t}$ whose de Rham cohomology class is $\Gamma(t)$. We choose $\eta_{t}$ so that it depends smoothly on $t$ and the other variables jointly and so that for any $t_{0} \in \mathbb{R}$, we have a neighborhood $U_{t_{0}}$ of $\left\{B_{\Gamma\left(t_{0}\right)}^{*}=0\right\}$ in $M$ and a neighborhood $J_{t_{0}}$ of $t_{0}$ in $\mathbb{R}$ such that $\eta U_{t_{0}}=\eta_{t_{0}} \mid U_{t_{0}}$ for $t \in J_{t_{0}}$. The possibility of choosing $\eta_{t}$ in this way is again a consequence of the assumption that $\left\{B_{\Gamma\left(t_{0}\right)}^{*}=0\right\}$ has a neighborhood $V$ and $M$ such that the inclusion $\operatorname{map} H_{1}(V, \mathbb{R}) \rightarrow H_{1}(M, \mathbb{R})$ is the zero map, together with the fact that the set function $c \rightarrow\left\{B_{c}^{*}=0\right\}$ is upper semi-continuous.

The strategy of proof of Theorem 9.2 follows the strategy of proof of Theorem 9.1, with appropriate modifications.

Since we may have $\eta_{t} \neq \eta_{t^{\prime}}$, even though $\Gamma(t)=\Gamma\left(t^{\prime}\right)$, we need to introduce the function $h_{t}: M \times M \rightarrow \mathbb{R}$ defined by

$$
h_{t}\left(m, m^{\prime}\right)=\min \int_{0}^{1}\left(L-\eta_{t}\right)(d \gamma(t), t) d t-\alpha(\Gamma(t))
$$

where the minimum is taken over all curves $\gamma:[0,1] \longrightarrow M$ such that $\gamma(0)=m$ and $\gamma(1)=m^{\prime}$. This is the same as the definition of $h_{c}$ in $\S 6$ (with $c=\Gamma(t)$ ), but now the dependence on $\eta_{c}$ is taken into account explicitly.

Given a bi-infinite sequence $\vec{t}=\left(\cdots, t_{i}, \cdots\right)$, a bi-infinite increasing sequence $\vec{j}=\left(\cdots<j_{i}<\cdots\right)$ of integers, an $M$-configuration ( $\cdots, m_{i}, \cdots$ ) and integers $a$, $b$, we define $h_{\vec{t}, \vec{j}}\left(m_{1}, \cdots, m_{b}\right)$ by an obvious modification of the definition of $h_{\vec{c}, \vec{j}}$ given in the previous section. Likewise, we introduce the notion of a segment $\left(m_{a}, \cdots, m_{b}\right)$ of an $M$-configuration being $(\vec{t}, \vec{j})$ minimal by an obvious modification of the notion of a $(\vec{c}, \vec{j})$-minimal configuration introduced in the previous section. Finally, we say that an $M$-configuration $\left(\cdots, m_{i}, \cdots\right)$ is $(\vec{t}, \vec{j})$-minimal if each finite segment of it is $(\vec{t}, \vec{j})$-minimal.

An easy compactness argument shows that for any sequence $\vec{t}=$ $\left(\cdots, t_{i}, \cdots\right)$ of real numbers, and any bi-infinite increasing sequence $\vec{j}=$ $\left(\cdots, j_{i}, \cdots\right)$ of integers, there exists a $(\vec{t}, \vec{j})$-minimal configuration.

To prove Theorem 9.2 , we choose an increasing sequence $\vec{t}=(\cdots<$ $t_{i}<\cdots$ ) of real numbers, with $t_{i+1}$ very close to $t_{i}$, an increasing sequence $\left(\cdots<j_{i}<\cdots\right)$ of integers, with $j_{i+1}-j_{i}$ very large, and a $(\vec{t}, \vec{j})$-minimal configuration $\left(\cdots, m_{i}, \cdots\right)$. Then we construct a curve $\gamma: \mathbb{R} \longrightarrow M$ by letting $\gamma(t), i \leqslant t \leqslant i+1$, be a Tonelli minimizer satisfying the boundary condition $\gamma(i)=m_{i}, \gamma(i+1)=m_{i+1}$.

Our claim is that if $\vec{t}$ and $\vec{j}$ are chosen appropriately (i.e. if $t_{i+1}$ is sufficiently close to $t_{i}$, and $j_{i+1}-j_{i}$ is sufficiently large), then $\gamma$ satisfies the required conditions. The proof of this is similar to the corresponding argument in the proof of Theorem 9.1, and we omit it.

## 12. A weaker hypothesis.

Just before the deadline for submitting this paper for the proceedings of the conference in honor of Malgrange's $65^{t h}$ birthday, I noticed that the proofs of Theorems 9.1 and 9.2 work under a weaker hypothesis, which I will explain in this section.

Given $c \in H^{1}(M, \mathbb{R})$, we define

$$
V_{c}=\bigcap_{U}\left\{i_{U *} H_{1}(U, \mathbb{R}): U \text { is a neighborhood of }\left\{B_{c}^{*}=0\right\}\right\}
$$

Here, $i_{U}: U \longrightarrow M$ denotes the inclusion map. Thus, $V_{c}$ is a vector subspace of $H_{1}(M, \mathbb{R})$. Moreover, $V_{c}=0$ if and only if $c \in W_{L}$.

We define $V_{c}^{\perp}$ to be the annihilator of $V_{c}$. In other words, if $c^{\prime} \in$ $H^{1}(M, \mathbb{R})$, then $c^{\prime} \in V_{c}^{\perp}$ if and only if $\left\langle c^{\prime}, h\right\rangle=0$ for all $h \in V_{c}$. Clearly,

$$
V_{c}^{\perp}=\bigcup_{U}\left\{\text { ker } i_{U}^{*}: U \text { is a neighborhood of }\left\{B_{c}^{*}=0\right\}\right\}
$$

Note that there exists a neighborhood $U$ of $\left\{B_{c}^{*}=0\right\}$ in $M$ such that $V_{c}=i_{U *} H_{1}(U, \mathbb{R})$ and $V_{c}^{\perp}=\operatorname{ker} i_{U}^{*}$.

We will say that $c_{0}, c_{1} \in H^{1}(M, \mathbb{R})$ are $C$-equivalent if there exists a continuous curve $\Gamma:[0,1] \longrightarrow M$ such that $\Gamma(0)=c_{0}$ and $\Gamma(1)=c_{1}$, and for each $t_{0} \in[0,1]$, there exists $\delta>0$ such that $\Gamma(t)-\Gamma\left(t_{0}\right) \in V_{\Gamma\left(t_{0}\right)}^{\perp}$ whenever $t \in[0,1]$ and $\left|t-t_{0}\right|<\delta$.

Theorem 9.1 remains true if the hypothesis that $c_{0}$ and $c_{1}$ are in the same connected component of $W_{L}$ is replaced by the hypothesis that $c_{0}$ and $c_{1}$ are $C$-equivalent. Likewise, Theorem 9.2 remains true if the hypothesis that the $c_{i}$ are all in the same connected component of $W_{L}$ is replaced by the hypothesis that the $c_{i}$ are all $C$-equivalent. The proofs are the same.

## 13. A conjecture.

To demonstrate the usefulness of our theory, we would need to give examples to which it applies. At present we have no real examples beyond twist maps. In this section, we will state a conjecture which we hope to prove at some future date by an extension of the methods of this paper. This conjecture gives an example of what we are aiming for in developing our theory.

Our conjecture concerns generic Lagrangians in the sense of Mañé [Man3]. We consider a smooth Lagrangian $L_{0}$ on a smooth compact manifold $M$, i.e., a $C^{\infty}$ mapping $L_{0}: T M \times \mathbb{T} \longrightarrow \mathbb{R}$ satisfying the hypotheses listed in §1, viz., positive definiteness, superlinear growth, and completeness of the Euler-Lagrange flow. We consider the family of Lagrangians of the form $L=L_{0}+\psi$, where $\psi: M \times \mathbb{T} \rightarrow \mathbb{R}$ is a $C^{\infty}$ function. Here, we identity $\psi$ with $\psi \circ \pi$, where $\pi: T M \times \mathbb{T} \rightarrow M \times \mathbb{T}$ denotes the projection. We will also assume that for any $L$ of this form, the Euler-Lagrange flow is complete.

We will say that a property of the Euler-Lagrange flow $E_{L}$ is generic (in the sense of Mañé) if for any $L_{0}$, the set of $\psi$ for which it is satisfied is residual, with respect to the $C^{\infty}$ topology on $C^{\infty}(M \times \mathbb{T})$. We conjecture that if $\operatorname{dim} M \geqslant 2$, then generically there exists an orbit $\gamma$ which escapes to infinity, in the sense that for every compact subset $K$ of $T M \times \mathbb{T}$, there exists $t_{0}$ such that $\gamma(t) \notin K$, for $t \geqslant t_{0}$.

Such a result is false when $\operatorname{dim} M=1$, by KAM theory. When $\operatorname{dim} M \geqslant 2$, the usual KAM tori do not separate phase space, so KAM theory does not tell whether our conjecture is true.

Our conjecture belongs to a class of speculations which go back to Boltzmann's quasi-ergodic hypothesis. Recently, Herman (see [Yoc]) has produced examples of Hamiltonian systems for which Boltzmann's quasiergodic hypothesis is false. It is noteworthy that in Herman's examples, variational methods do not apply.

Let us mention also the famous paper of Arnold [Arn2] who gave an example to show that certain results guaranteeing boundedness of orbits in Hamiltonain systems in two degrees of freedom in the autonomous case or one degree of freedom in the non-autonomous case have no analogue in more degrees of freedom. The method of [Arn2] is another method one might try to use to prove our conjecture, but it seems (at least to the author) that variational methods such as described here are more likely to succeed for proving the conjecture we have stated here.

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John N. MATHER,
Princeton University
Department of Mathematics Fine Hall
Washington Road
Princeton N.J. 08544 (USA).


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