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Published in:
EUROPEAN JOURNAL OF MECHANICS A: SOLIDS

DOI:
10.1016/j.euromechsol.2017.11.012

Published: 01/05/2018

Document Version
Peer reviewed version

Please cite the original version:
Tahaei Yaghoubi, S., Balobanov, V., Mousavi, S. M., & Niiranen, J. (2018). Variational formulations and isogeometric analysis for the dynamics of anisotropic gradient-elastic Euler-Bernoulli and shear deformable beams. *EUROPEAN JOURNAL OF MECHANICS A: SOLIDS*, *69*, 113-123. https://doi.org/10.1016/j.euromechsol.2017.11.012

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Article *in* European Journal of Mechanics - A/Solids · December 2017

DOI: 10.1016/j.euromechsol.2017.11.012

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Variational formulations and isogeometric analysis for the dynamics of anisotropic gradient-elastic Euler-Bernoulli and shear-deformable beams

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Abstract

The strain and velocity gradient framework is formulated for centrosymmetric anisotropic Euler-Bernoulli and third-order shear deformable (TSD) beam models, reducible to Timoshenko beams. The governing equations and boundary conditions are obtained by using a variational approach. The strain energy is generalized to include strain gradients and a tensor of anisotropic static length scale parameters. The kinetic energy includes the velocity gradients and a tensor of anisotropic length scale parameters and hence the static and kinetic quantities of the centrosymmetric anisotropic material are distinguished in microscale and in the macroscale. Furthermore, the external work is written in the general form. Free vibration of a simply supported centrosymmetric anisotropic TSD beam is studied by using analytical solution as well as isogeometric analysis.

Keywords: anisotropic strain and velocity gradient, shear deformable beam, centrosymmetric, Isogeometric analysis

1. Introduction

Recent developments in nanotechnology necessitates the analysis of structural elements in ultra-small scales. Micro and nanobeams are frequently used in nano-and micro-sized systems and devices such as sensors (Takamatsu et al., 2014; Shaat and Abdelkefi, 2015), resonators (Kacem et al., 2009; Chen et al., 2011) or actuators (Tian et al., 2016) and the accurate prediction of their behavior in micro/nano scales is of utmost importance. However, it is well-known that the classical theories of continuum mechanics fail to describe the behavior of micro- or nano-sized structures. The reason for this problem is that the equations of the standard elasticity theories do not include parameters characteristic for the underlying microstructure, named as the internal length scale parameters. In order to improve this deficiency, higher-order continuum theories such as couple stress theory (Mindlin and Tiersten, 1962; Toupin, 1964), non-local elasticity theory (Eringen, 1972, 1983) and gradient elasticity theory (Mindlin, 1964) were developed. In these size-dependent continuum theories, one or more internal length scale parameters appear in the constitutive equations and make the interpretation of the size-effect in the behavior of the structures possible. In this paper, the focus is on the gradient theory proposed by Mindlin (1964).

Classical Euler-Bernoulli beam theory provides inaccurate interpretation of the statical and dynamical behavior of the beams when their thickness to length ratio is relatively large (Wang et al., 2000). This deficiency was first demonstrated by Timoshenko (Timoshenko, 1921). Since in the Timoshenko beam theory the transverse shear strain and stress are constant on the thickness of the beam, a shear correction factor is introduced in the equations. Levinson (1981), introduced a third-order shear deformable theory (TSD) for beams of rectangular cross-section. In this theory, the shear-free conditions on the upper and

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lower surfaces of the beam are satisfied for isotropic beams with no requirement for the shear correction factor. Bickford (Bickford, 1982) and Reddy (Reddy, 1984) applied the displacement field proposed by Levinson (1981) and obtained variationally consistent equations of motion for isotropic beams and third-order laminated composite plates, respectively.

Bending, buckling and vibration of isotropic and functionally graded beams and plates have been investigated within generalized continuum theories by several investigators (Ansari et al., 2011; Sahmani and Ansari, 2013; Akgöz and Civalek, 2013; Wang et al., 2014; Mousavi et al., 2015; Yaghoubi et al., 2015; Ansari et al., 2016; Khodabakhshi and Reddy, 2017; Sahmani and Aghdam, 2017).

Recently, authors such as Gitman et al. (2010); Auffray et al. (2013, 2015) and Lazar and Po (2015a) generalized the simplified isotropic gradient elasticity for anisotropic materials. In the special case of centrosymmetic anisotropy, the general anisotropic gradient elasticity formulation is simplified such that the strain energy includes merely even-order tensors. Furthermore, the sixth-rank tensor incorporating material anisotropy and anisotropic length-scale effect is simplified as a combination of the classical fourth-rank tensor of elastic constants and a second-rank tensor of anisotropic length scale effects (Gitman et al., 2010; Lazar and Po, 2015a,b). The static bending and buckling of centrosymmetric anisotropic shear deformable plates and beams within strain gradient elasticity theory were previously formulated and analysed by Mousavi et al. (2016) and Yaghoubi et al. (2016, 2017). Recently, Reda et al. (2017) constructed first and second order gradient models in order to study the size effects in dynamical behaviour of homogenized anisotropic media for textile composite structures.

In this work, a complete gradient theory is considered which includes the velocity gradients in the generalized kinetic energy as well as the gradients of strain in the generalized strain energy. A generalization of the kinetic energy for centrosymmetic anisotropic materials is proposed.

The variational formulation of third-order shear deformable beams and plates within higher-order continuum theories results in complicated differential equations of motion. Furthermore, in addition to the classical boundary conditions, higher-order boundary conditions are obtained. Similar to the classical boundary conditions, the variational approach leads to two options for the nonclassical boundary conditions (Lam et al., 2003; Kong et al., 2009; Xu and Deng, 2016; Niiranen and Niemi, 2017). While the analytical solution is not feasible for some types of boundary conditions, application of numerical methods enables one to obtain the solution of the differential equations for any types of the boundary conditions. In this work, the equations of motion of a centrosymmetric anisotropic TSD strain and velocity gradient elastic beam are solved using the isogeometric analysis which can be considered as a generalization of finite element analysis.

Equations of motion within the framework of the gradient elasticity theory are partial differential equations with high-order derivatives. The most common way of solving continuum mechanics problems numerically is to use finite element methods (FEM). But classical FEM with Lagrange basis functions is not suitable because it can provide only C^0 continuity from element to element. Solution spaces for problems of gradient elasticity theory must provide at least C^1 continuity from element to element.

This paper is organized as follows. In the next section, a general three-dimensional variational formulation for a homogeneous and centrosymmetric anisotropic material in the framework of strain and velocity gradient elasticity theory is reviewed. In section (3), the dimension reduction is applied to the three-dimensional formulation and the equations of motion and boundary conditions for Euler-Bernoulli and TSD beams are determined. Moreover, the analytical solution for the free vibration of a simply supported TSD beam is derived. A numerical solution of the differential equations of motion is obtained in section (4) and the results obtained by the numerical and analytical solutions are compared. Finally, some concluding remarks are presented in section (5).

2. Theory and formulations

In the framework of first strain gradient theory proposed by Mindlin (1964), the strain energy density (U) of a homogeneous and centrosymmetric material takes the form (Auffray et al., 2013; Lazar and Po, 2015a)

$$U = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} D_{ijmkln} \varepsilon_{ij,m} \varepsilon_{kl,n}, \ i, j, k, l, m, n \in \{x, y, z\} \, . \tag{2.1}$$

where comma denotes the partial derivative, C_{ijkl} is the fourth rank tensor of elastic constants and ε_{ij} represents the infinitesimal elastic strain components in terms of displacement components u_j as

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right). \tag{2.2}$$

Furthermore, in equation (2.1), the sixth-rank constitutive tensor D_{ijmkln}, incorporates material anisotropy and static anisotropic length scale effects. For centrosymmetric materials, it is assumed that the tensor D_{ijmkln} can be decomposed into a product of the second-rank tensor of anisotropic length scale coefficients Ψ^s_{mn} (Gitman et al. (2010); Lazar and Po (2015a)) and the tensor of elastic parameters C_{ijkl}

$$D_{ijmkln} = C_{ijkl}\Psi^s_{mn}. \tag{2.3}$$

The tensor Ψ^s_{mn} is symmetric and positive definite and has the dimension of [m^2]. Appendix A presents the tensor Ψ^s_{mn} for different classes of crystal symmetry.

The decomposition (2.3), separates the two sources of anisotropy represented in the Mindlin's anisotropic gradient elasticity theory, that is the anisotropy of the elastic moduli and the anisotropy of gradient length scale parameters (Gitman et al. (2010); Lazar and Po (2015a)). It is noteworthy that the decomposition (2.3) is an approximate constitutive law which is proposed in order to simplify the formulation and reduces the maximum number of independent material parameters from 171 in tensor D_{ijmkln} to 27 in $C_{ijkl}\Psi^s_{mn}$ (Po et al., 2017).

In the framework of strain gradient elasticity, the Cauchy-like stress tensor components σ_{ij} and double-stress tensor components τ_{ijk} are given by

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}},\ \tau_{ijk} = \frac{\partial U}{\partial \varepsilon_{ij,k}}. \tag{2.4}$$

Considering equations (2.3) and (2.4), the variation of strain energy δU in a region Ω occupied by elastically deformed material reads

$$\delta U = \int_\Omega \left(\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\varepsilon_{ij,k}\right)\mathrm{d}v = \int_\Omega \left(\sigma_{ij}\delta u_{i,j} + \Psi^s_{kl}\sigma_{ij,l}\delta u_{i,jk}\right)\mathrm{d}v. \tag{2.5}$$

Moreover, the variation of the external work reads

$$\delta W = \int_\Omega f_i \delta u_i \mathrm{d}v + \int_{\partial\Omega}\left(t_i\delta u_i + q_i n_j \delta u_{i,j}\right)\mathrm{d}a \tag{2.6}$$

where $\partial\Omega$ is the bounding (closed) surface of Ω, f_i is body force and t_i and q_i are Cauchy traction vector and double stress traction vector on the boundary, respectively.

As pointed out by Rossi and Auffray (2016), the kinetic energy density is given by

$$K = \frac{1}{2}\left(\rho\delta_{ip}\dot{u}_p + \kappa_{ipq}\dot{u}_{p,q}\right)\dot{u}_i + \frac{1}{2}\left(\kappa_{ijp}\dot{u}_p + J_{ijpq}\dot{u}_{p,q}\right)\dot{u}_{i,j}, \tag{2.7}$$

where ρ is the mass density, upper dot denotes the time derivative and κ_{ijk}, J_{ijpq}, δ_{ij} are the components of coupling inertia, second order inertia and the unit second-order (i.e. Kronecker delta) tensors, respectively. For centrosymmetric media, the odd order-tensor κ is vanished. Hence the kinetic energy is reduced to

$$K = \frac{1}{2}\rho\delta_{ip}\dot{u}_p\dot{u}_i + \frac{1}{2}J_{ijpq}\dot{u}_{p,q}\dot{u}_{i,j}. \tag{2.8}$$

The tensor $\mathbf{J}$ is assumed as

$$J_{ijpq} = \rho\delta_{ip}\Psi^d_{jq}. \tag{2.9}$$

Above, Ψ^d_{jq} is a symmetric second rank anisotropic internal length scale tensor regarding the velocity gradient. The assumption (2.9)is motivated in line with the assumption (2.3). Of course such simplification should be validated against experimental results one available. Hence the density of the kinetic energy is written as

$$K = \frac{1}{2}\rho \dot{u}_i \dot{u}_i + \frac{1}{2}\rho \Psi^d_{jk} \dot{u}_{i,j} \dot{u}_{i,k}. \tag{2.10}$$

In equation (2.10), the term $\frac{1}{2}\rho \Psi^d_{jk} \dot{u}_{i,j} \dot{u}_{i,k}$ can be written in the form $\frac{1}{2}\rho \mathbf{c}^T \mathbf{M} \mathbf{c}$ where

$$\mathbf{c}^T = \left[\begin{array}{ccccccccc} \dot{u}_{x,x} & \dot{u}_{x,y} & \dot{u}_{x,z} & \dot{u}_{y,x} & \dot{u}_{y,y} & \dot{u}_{y,z} & \dot{u}_{z,x} & \dot{u}_{z,y} & \dot{u}_{z,z} \end{array}\right] \tag{2.11}$$

and

$$\mathbf{M} = \left[\begin{array}{ccc} \mathbf{\Psi}^d & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Psi}^d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Psi}^d \end{array}\right] \tag{2.12}$$

For the positive definiteness of the kinetic energy, matrix $\mathbf{M}$ and consequently the tensor $\mathbf{\Psi}^d$ must be positive definite (see Appendix A).

The variation of the kinetic energy in a region Ω occupied by the elastically deformed material is

$$\delta K = \int_\Omega \rho \left(\dot{u}_i \delta \dot{u}_i + \Psi^d_{jk} \dot{u}_{i,j} \delta \dot{u}_{i,k} \right) \mathrm{d}v. \tag{2.13}$$

According to Hamilton's principle

$$\int_{t_0}^{t_1} (\delta K - \delta U + \delta W)\, \mathrm{d}t = 0, \tag{2.14}$$

Substitution of the variations of the strain energy, external work and kinetic energy (2.5, 2.6, 2.13) into the Hamilton's principle (2.14) and application of the fundamental lemma of calculus of variation leads to the governing equilibrium equations and boundary conditions in three dimensional form. In order to simplify the 3-D formulation for a specific case of beam structures, the dimension reduction is applied to the general formulation.

3. Anisotropic beam models

Let us consider a prismatic body in 3D space which can be represented by a beam model:

$$\mathcal{B} = A \times \Omega, \tag{3.15}$$

with $\Omega = (0, L)$ denoting the central axis (or neutral fibre) piercing the middle points of cross sections with constant area A. We fix a Cartesian coordinate system such that x-axis coincide with the beam's central axis. Loadings and material parameters distribution are chosen to cause uni-axial bending in xz-plane. The beam length L is assumed to prevail over two other dimensions: $L \gg diam(A)$.

3.1. Euler-Bernoulli beam

Taking all the foregoing in this section into account, one can assume the displacement field $\boldsymbol{u} = (u_x, u_z)$ of the Euler-Bernoulli beam as

$$u_x(x, z) = -z w_{,x}(x), \quad u_z(x) = w(x). \tag{3.16}$$

Here, $u_x(x, z)$ and $u_z(x)$ denote the displacements along the coordinates x and z, respectively and $w(x)$ represents the transverse deflection of a point on the beam axis. According to equation (2.2), the only nonzero component of the strain tensor is

$$\varepsilon_{xx} = -z w_{,xx}, \tag{3.17}$$

and the only non-zero components of the gradient of stress tensor are

$$\varepsilon_{xx,x} = -z w_{,xxx}, \; \varepsilon_{xx,z} = -w_{,xx}. \tag{3.18}$$

The Cauchy and the double stress tensor components read

$$\begin{aligned} \sigma_{ij} &= -zC_{ijxx}w_{,xx}, \\ \tau_{ijk} &= -z\Psi^s_{kx}C_{ijxx}w_{,xxx} - \Psi^s_{kz}C_{ijxx}w_{,xx}. \end{aligned} \tag{3.19}$$

Using equations (3.17) and (3.19), the first variation of the strain energy (2.5) takes the form

$$\begin{aligned} \delta U = \int_\Omega & \{-z\sigma_{xx}\delta w_{,xx} - z\left[\Psi^s_{xx}\sigma_{xx,x} + \Psi^s_{xz}\sigma_{xx,z}\right]\delta w_{,xxx} \\ & - \left[\Psi^s_{xz}\sigma_{xx,x} + \Psi^s_{zz}\sigma_{xx,z}\right]\delta w_{,xx}\}\,\mathrm{d}v. \end{aligned} \tag{3.20}$$

In order to apply dimension reduction, the stress resultants are defined as

$$\{N_{xx}, M_{xx}\} = \int_A \{1, z\}\sigma_{xx}\mathrm{d}A \tag{3.21}$$

Moreover, the gradient-of-stress resultants are defined as

$$\{N^z_{xx}, M^z_{xx}\} = \int_A \{1, z\}\sigma_{xx,z}\mathrm{d}A \tag{3.22}$$

These resultants can be written in terms of the displacement field as

$$N_{xx} = M^z_{xx} = -A^*_{xx}w_{,xx},\ M_{xx} = -D_{xx}w_{,xx},\ N^z_{xx} = -A_{xx}w_{,xx} \tag{3.23}$$

where

$$(A_{xx}, D_{xx}) = \int_A C_{xxxx}\left(1, z^2\right)\mathrm{d}A,\ A^*_{xx} = \int_A C_{xxxx}z\mathrm{d}A. \tag{3.24}$$

It is noted that in equations (3.21) and (3.22), the only classical stress resultant is M_{xx}. Obviously, for a beam with a rectangular cross-section the resultants N_{xx} and M^z_{xx} vanish.

Using equations (3.21), (3.22), the variation of the strain energy of the Euler-Bernoulli beam takes the form

$$\begin{aligned} \delta U = \int_0^L & \{-M_{xx}\delta w_{,xx} - \Psi^s_{xx}M_{xx,x}\delta w_{,xxx} \\ & +\Psi^s_{xz}\left(-M^z_{xx}\delta w_{,xxx} - N_{xx,x}\delta w_{,xx}\right) - \Psi^s_{zz}N^z_{xx}\delta w_{,xx}\}\,\mathrm{d}x. \end{aligned} \tag{3.25}$$

Green's theorem is applied to equation (3.25) and the variation of the strain energy is written as

$$\begin{aligned} \delta U = \int_0^L & \left[-M_{xx,xx} + \Psi^s_{xx}M_{xx,xxxx} - \Psi^s_{zz}N^z_{xx,xx}\right]\delta w\mathrm{d}x \\ & + \left[M_{xx,x} - \Psi^s_{xx}M_{xx,xxx} + \Psi^s_{zz}N^z_{xx,x}\right]\delta w|_0^L \\ & + \left[-M_{xx} + \Psi^s_{xx}M_{xx,xx} - \Psi^s_{zz}N^z_{xx}\right]\delta w_{,x}|_0^L \\ & + \left[-\Psi^s_{xx}M_{xx,x} - \Psi^s_{xz}M^z_{xx}\right]\delta w_{,xx}|_0^L. \end{aligned} \tag{3.26}$$

where L is the length of the beam. Moreover, substituting equation (3.16) into the variation of kinetic energy (2.13), integrating over time and applying Green's theorem leads to

$$\begin{aligned} \int_{t_0}^{t_1}\delta K\mathrm{d}t =& \rho\int_{t_0}^{t_1}\int_0^L \left[I\ddot{w}_{,xx} - A\ddot{w} + \Psi^d_{xx}\left(-I\ddot{w}_{,xxxx} + A\ddot{w}_{,xx}\right) + \Psi^d_{zz}A\ddot{w}_{,xx}\right]\delta w\mathrm{d}x\mathrm{d}t \\ & + \rho\int_{t_0}^{t_1}\left[-I\ddot{w}_{,x} + \Psi^d_{xx}\left(I\ddot{w}_{,xxx} - A\ddot{w}_{,x}\right) - \Psi^d_{zz}A\ddot{w}_{,x}\right]\delta w|_0^L\mathrm{d}t \\ & + \rho\int_{t_0}^{t_1}\left[-\Psi^d_{xx}I\ddot{w}_{,xx} - \Psi^d_{xz}A^*\ddot{w}_{,x}\right]\delta w_{,x}|_0^L\mathrm{d}t \end{aligned} \tag{3.27}$$

where

$$(A^*, I) = \int_A \left(z, z^2\right) \mathrm{d}A. \tag{3.28}$$

In obtaining equation (3.27), the initial conditions are set equal to zero.

Using equation (2.6) the variation of external work is written as

$$\delta W = \int_0^L \int_A \left(f_x \delta u_x + f_z \delta u_z\right) \mathrm{d}A\mathrm{d}x + \int_A \left[t_x \delta u_x + t_z \delta u_z + q_x \delta u_{x,x} + q_z \delta u_{z,x}\right]|_0^L \mathrm{d}A. \tag{3.29}$$

Considering the displacement field (3.16) and using Green's theorem, the variation of external work can be expressed as

$$\begin{aligned}\delta W &= \int_0^L \left(F^1_{x,x} + F_z\right) \delta w \mathrm{d}x + \left[\left(T_z - F^1_x\right) \delta w + \left(-T^1_x + Q_z\right) \delta w_{,x} - Q^1_x \delta w_{,xx}\right]|_0^L \\ &= \int_0^L \left(F^1_{x,x} + F_z\right) \delta w \mathrm{d}x + \left[\bar{V}^E \delta w + \bar{M}^E \delta w_{,x} + \bar{M}^E_h \delta w_{,xx}\right]|_0^L,\end{aligned} \tag{3.30}$$

where

$$\left\{F^1_x, T^1_x, Q^1_x\right\} = \int_A \left\{f_x, t_x, q_x\right\} z \mathrm{d}A,\ \left\{F_z, T_z, Q_z\right\} = \int_A \left\{f_z, t_z, q_z\right\} \mathrm{d}A \tag{3.31}$$

In order to derive the equations of motion, Hamilton's principle (2.14) is used. Substituting (3.26), (3.27) and (3.30) into Hamilton's principle (2.14) and taking advantage of the fundamental lemma of calculus of variation yields the governing equation of motion and the boundary conditions. The governing equation of motion is

$$\begin{aligned}&- M_{xx,xx} + \Psi^s_{xx} M_{xx,xxxx} - \Psi^s_{zz} N^z_{xx,xx} \\ &- \rho I \ddot{w}_{,xx} + \rho A \ddot{w} + \rho \Psi^d_{xx} \left(I \ddot{w}_{,xxxx} - A \ddot{w}_{,xx}\right) - \rho \Psi^d_{zz} A \ddot{w}_{,xx} - \left(F^1_{x,x} + F_z\right) = 0\end{aligned} \tag{3.32}$$

which can be written in terms of displacement as

$$\begin{aligned}&\left(D_{xx} w_{,xx}\right)_{,xx} - \Psi^s_{xx} \left(D_{xx} w_{,xx}\right)_{,xxxx} + \Psi^s_{zz} \left(A_{xx} w_{,xx}\right)_{,xx} \\ &- \rho I \ddot{w}_{,xx} + \rho A \ddot{w} + \rho \Psi^d_{xx} \left(I \ddot{w}_{,xxxx} - A \ddot{w}_{,xx}\right) - \rho \Psi^d_{zz} A \ddot{w}_{,xx} - \left(F^1_{x,x} + F_z\right) = 0\end{aligned} \tag{3.33}$$

The total order of the governing differential equation (3.33) in terms of displacement is six. Therefore, three boundary conditions in terms of w are expected at the boundaries (at each end of the beam). These conditions are

$$\begin{aligned}&M_{xx,x} - \Psi^s_{xx} M_{xx,xxx} + \Psi^s_{zz} N^z_{xx,x} \\ &\quad + \rho I \ddot{w}_{,x} - \rho \Psi^d_{xx} \left(I \ddot{w}_{,xxx} - A \ddot{w}_{,x}\right) + \rho \Psi^d_{zz} A \ddot{w}_{,x} - \bar{V}^E = 0 \quad \text{or} \quad w = w_0 \\ &-M_{xx} + \Psi^s_{xx} M_{xx,xx} - \Psi^s_{zz} N^z_{xx} + \rho \Psi^d_{xx} I \ddot{w}_{,xx} + \rho \Psi^d_{xz} A^* \ddot{w}_{,x} - \bar{M}^E = 0 \quad \text{or} \quad w_{,x} = \bar{\beta} \\ &-\Psi^s_{xx} M_{xx,x} - \Psi^s_{xz} M^z_{xx} - \bar{M}^E_h = 0 \quad \text{or} \quad w_{,xx} = \bar{\kappa}\end{aligned} \tag{3.34}$$

In accordance to the conventional classification, the boundary conditions of a beam can be grouped such that they form 4 different types of boundaries. In order to obtain the *clamped* boundary Γ_C one needs to prescribe deflection $\bar{w}$ and rotation $\bar{\beta}$, for *simply supported* Γ_{SS} – deflection $\bar{w}$ and moment $\bar{M}^E$, for *elastically supported (sliding)* Γ_{ES} – rotation $\bar{\beta}$ and force $\bar{V}^E$, and finally for *free* Γ_F – force $\bar{V}^E$ and moment $\bar{M}^E$. Gradient elasticity theory introduces the non-classical boundary condition (3.34-3) which duplicates the number of possible boundary types. Following (Niiranen et al., 2017) we call them *singly* for applied double moment $\bar{M}^E_h$ and *doubly* if instead curvature $\bar{\kappa}$ is prescribed. In such a manner, we have the boundaries which are *singly* and *doubly clamped* (Γ_{C_S} and Γ_{C_D} resp.), *singly* and *doubly simply supported* (Γ_{SS_S} and Γ_{SS_D}), and so on. Selection of the non-classical boundary condition (3.34-3) affects on the behaviour of the beam near boundaries and can cause appearance of the so-called boundary layers in the solution.

3.2. Anisotropic third-order shear-deformable beam

According to the TSD beam theory (Levinson (1981), Bickford (1982), Reddy (1984)), the displacement field of the TSD beam is

$$\begin{aligned} u_x\left(x,z\right) &= z\beta\left(x\right) - \alpha z^3\left[\beta(x) + \frac{\partial w(x)}{\partial x}\right], \\ u_z\left(x,z\right) &= w\left(x\right). \end{aligned} \tag{3.35}$$

In equation (3.35), $\beta(x)$ denotes the rotation of the beam cross section and α is a constant (α is a constant for a rectangular cross section and is approximated as a constant for other types of cross sections). By substituting the displacement field (3.35) into the strain-displacement relation (2.2), the nonzero components of the strain tensor are obtained to be

$$\begin{aligned} \varepsilon_{xx} &= z\beta_{,x} - \alpha z^3\left(\beta_{,x} + w_{,xx}\right), \\ \varepsilon_{xz} &= \frac{1}{2}\left(1 - 3\alpha z^2\right)\left(w_{,x} + \beta\right). \end{aligned} \tag{3.36}$$

The nonzero components of the gradient of the strain tensor are

$$\begin{aligned} \varepsilon_{xx,x} &= \left(z - \alpha z^3\right)\beta_{,xx} - \alpha z^3 w_{,xxx},\ \varepsilon_{xx,z} = \left(1 - 3\alpha z^2\right)\beta_{,x} - 3\alpha z^2 w_{,xx}, \\ \varepsilon_{xz,x} &= \frac{1}{2}\left(1 - 3\alpha z^2\right)\left(w_{,xx} + \beta_{,x}\right),\ \varepsilon_{xz,z} = -3\alpha z\left(w_{,x} + \beta\right). \end{aligned} \tag{3.37}$$

According to equations (2.4) and (3.36), the Cauchy and higher stress components read

$$\begin{aligned} \sigma_{ij} &= C_{ijxx}\left[\left(z - \alpha z^3\right)\beta_{,x} - \alpha z^3 w_{,xx}\right] + C_{ijxz}\left(1 - 3\alpha z^2\right)\left(w_{,x} + \beta\right) \\ \tau_{ijk} &= \Psi^s_{kx} C_{ijxx}\left[\left(z - \alpha z^3\right)\beta_{,xx} - \alpha z^3 w_{,xxx}\right] \\ &\quad + \Psi^s_{kx} C_{ijxz}\left(1 - 3\alpha z^2\right)\left(w_{,xx} + \beta_{,x}\right) \\ &\quad + \Psi^s_{kz} C_{ijxx}\left[\left(1 - 3\alpha z^2\right)\beta_{,x} - 3\alpha z^2 w_{,xx}\right] \\ &\quad - 6\alpha z \Psi^s_{kz} C_{ijxz}\left(w_{,x} + \beta\right) \end{aligned} \tag{3.38}$$

Using a similar procedure described in the previous section, the variation of the strain energy can be written in terms of the resultants as

$$\begin{aligned} \delta U = \int_0^L &[-\hat{M}_{xx,x} + \hat{N}_{xz} + \Psi^s_{xx}\left(\hat{M}_{xx,xxx} - \hat{N}_{xz,xx}\right) \\ &+ \Psi^s_{xz}\left(\hat{M}^z_{xx,xx} - \hat{N}^z_{xz,x} - \hat{N}_{xx,xx} - 6\alpha M_{xz,x}\right) \\ &- \Psi^s_{zz}\left(\hat{N}^z_{xx,x} + 6\alpha M^z_{xz}\right)]\delta\beta \mathrm{d}x \\ &+ \int_0^L [-\alpha P_{xx,xx} - \hat{N}_{xz,x} + \Psi^s_{xx}\left(\alpha P_{xx,xxxx} + \hat{N}_{xz,xxx}\right) \\ &+ \Psi^s_{xz}\left(\alpha P^z_{xx,xxx} + \hat{N}^z_{xz,xx} - 3\alpha R_{xx,xxx} + 6\alpha M_{xz,xx}\right) \\ &+ \Psi^s_{zz}\left(-3\alpha R^z_{xx,xx} + 6\alpha M^z_{xz,x}\right)]\delta w \mathrm{d}x + BC, \end{aligned} \tag{3.39}$$

where

$$
\begin{aligned}
BC =&[\hat{M}_{xx} + \Psi^s_{xx}\left(-\hat{M}_{xx,xx} + \hat{N}_{xz,x}\right) \\
&+ \Psi^s_{xz}\left(-\hat{M}^z_{xx,x} + \hat{N}^z_{xz} + \hat{N}_{xx,x}\right) + \Psi^s_{zz}\hat{N}^z_{xx}]\delta\beta|_0^L \\
&+ \left[\Psi^s_{xx}\hat{M}_{xx,x} + \Lambda_{xz}\hat{M}^z_{xx}\right]\delta\beta_{,x}|_0^L \\
&+ [\alpha P_{xx,x} + \hat{N}_{xz} + \Psi^s_{xx}\left(-\alpha P_{xx,xxx} - \hat{N}_{xz,xx}\right) \\
&+ \Psi^s_{xz}\left(-\alpha P^z_{xx,xx} - \hat{N}^z_{xz,x} + 3\alpha R_{xx,xx} - 6\alpha M_{xz,x}\right) \\
&+ \Psi^s_{zz}\left(3\alpha R^z_{xx,x} - 6\alpha M^z_{xz}\right)]\delta w|_0^L \\
&+ [-\alpha P_{xx} + \Psi^s_{xx}\left(\alpha P_{xx,xx} + \hat{N}_{xz,x}\right) \\
&+ \Psi^s_{xz}\left(\alpha P^z_{xx,x} + \hat{N}^z_{xz} - 3\alpha R_{xx,x}\right) - 3\alpha\Psi^s_{zz}R^z_{xx}]\delta w_{,x}|_0^L \\
&- \alpha\left[\Psi^s_{xx}P_{xx,x} + \Psi^s_{xz}P^z_{xx}\right]\delta w_{,xx}|_0^L .
\end{aligned} \tag{3.40}
$$

Above, L is the length of the beam and

$$
\begin{aligned}
&\hat{M}_{xx} = M_{xx} - \alpha P_{xx},\ \hat{N}_{xz} = N_{xz} - 3\alpha R_{xz},\ \hat{N}_{xx} = N_{xx} - 3\alpha R_{xx}, \\
&\hat{M}^z_{xx} = M^z_{xx} - \alpha P^z_{xx},\ \hat{N}^z_{xz} = N^z_{xz} - 3\alpha R^z_{xz},\ \hat{N}^z_{xx} = N^z_{xx} - 3\alpha R^z_{xx}.
\end{aligned} \tag{3.41}
$$

In equations (3.39)-(3.41), the stress resultants are defined as

$$
\begin{aligned}
&\{N_{xx}, M_{xx}, R_{xx}, P_{xx}\} = \int_A \{1, z, z^2, z^3\}\sigma_{xx}\mathrm{d}A, \\
&\{N_{xz}, M_{xz}, R_{xz}\} = \int_A \{1, z, z^2\}\sigma_{xz}\mathrm{d}A.
\end{aligned} \tag{3.42}
$$

Moreover, the gradient-of-stress resultants are defined as

$$
\begin{aligned}
&\{N^z_{xx}, M^z_{xx}, R^z_{xx}, P^z_{xx}\} = \int_A \{1, z, z^2, z^3\}\sigma_{xx,z}\mathrm{d}A, \\
&\{N^z_{xz}, M^z_{xz}, R^z_{xz}\} = \int_A \{1, z, z^2\}\sigma_{xz,z}\mathrm{d}A.
\end{aligned} \tag{3.43}
$$

These resultants can be written in terms of the displacement and rotation as

$$
\begin{aligned}
&N_{xx} = \bar{A}^*_{xx}\beta_{,x} - \alpha D^*_{xx}w_{,xx} + \hat{A}_{zz}\left(w_{,x} + \beta\right),\ M_{xx} = \bar{D}_{xx}\beta_{,x} - \alpha F_{xx}w_{,xx} + \hat{A}^*_{zz}\left(w_{,x} + \beta\right), \\
&R_{xx} = \bar{D}^*_{xx}\beta_{,x} - \alpha F^*_{xx}w_{,xx} + \hat{D}_{zz}\left(w_{,x} + \beta\right),\ P_{xx} = \bar{F}_{xx}\beta_{,x} - \alpha H_{xx}w_{,xx} + \hat{D}^*_{zz}\left(w_{,x} + \beta\right), \\
&N_{xz} = \bar{A}^*_{zz}\beta_{,x} - \alpha D^*_{zz}w_{,xx} + \hat{A}_{xz}\left(w_{,x} + \beta\right),\ M_{xz} = \bar{D}_{zz}\beta_{,x} - \alpha F_{zz}w_{,xx} + \hat{A}^*_{xz}\left(w_{,x} + \beta\right), \\
&R_{xz} = \bar{D}^*_{zz}\beta_{,x} - \alpha F^*_{zz}w_{,xx} + \hat{D}_{xz}\left(w_{,x} + \beta\right),\ N^z_{xx} = \hat{A}_{xx}\beta_{,x} - 3\alpha D_{xx}w_{,xx} - 6\alpha A^*_{zz}\left(w_{,x} + \beta\right), \\
&M^z_{xx} = \hat{A}^*_{xx}\beta_{,x} - 3\alpha D^*_{xx}w_{,xx} - 6\alpha D_{zz}\left(w_{,x} + \beta\right),\ R^z_{xx} = \hat{D}_{xx}\beta_{,x} - 3\alpha F_{xx}w_{,xx} - 6\alpha D^*_{zz}\left(w_{,x} + \beta\right), \\
&P^z_{xx} = \hat{D}^*_{xx}\beta_{,x} - 3\alpha F^*_{xx}w_{,xx} - 6\alpha F_{zz}\left(w_{,x} + \beta\right),\ N^z_{xz} = \hat{A}_{zz}\beta_{,x} - 3\alpha D_{zz}w_{,xx} - 6\alpha A^*_{xz}\left(w_{,x} + \beta\right), \\
&M^z_{xz} = \hat{A}^*_{zz}\beta_{,x} - 3\alpha D^*_{zz}w_{,xx} - 6\alpha D_{xz}\left(w_{,x} + \beta\right),\ R^z_{xz} = \hat{D}_{zz}\beta_{,x} - 3\alpha F_{zz}w_{,xx} - 6\alpha D^*_{xz}\left(w_{,x} + \beta\right).
\end{aligned} \tag{3.44}
$$

In resultants (3.44), the coefficients are

$$\begin{aligned}
&\hat{A}_{xx} = A_{xx} - 3\alpha D_{xx},\ \hat{A}_{xz} = A_{xz} - 3\alpha D_{xz},\ \hat{A}_{zz} = A_{zz} - 3\alpha D_{zz},\\
&\bar{D}_{xx} = D_{xx} - \alpha F_{xx},\ \hat{D}_{xz} = D_{xz} - 3\alpha F_{xz},\ \bar{D}_{zz} = D_{zz} - \alpha F_{zz},\\
&\hat{D}_{zz} = D_{zz} - 3\alpha F_{zz},\ \hat{D}_{xx} = D_{xx} - 3\alpha F_{xx},\ \bar{F}_{xx} = F_{xx} - \alpha H_{xx},\\
&\bar{A}^*_{xx} = A^*_{xx} - \alpha D^*_{xx},\ \hat{A}^*_{xx} = A^*_{xx} - 3\alpha D^*_{xx},\ \hat{A}^*_{zz} = A^*_{zz} - 3\alpha D^*_{zz},\\
&\bar{A}^*_{zz} = A^*_{zz} - \alpha D^*_{zz},\ \hat{A}^*_{xz} = A^*_{xz} - 3\alpha D^*_{xz},\ \hat{D}^*_{xx} = D^*_{xx} - 3\alpha F^*_{xx},\\
&\bar{D}^*_{xx} = D^*_{xx} - \alpha F^*_{xx},\ \hat{D}^*_{zz} = D^*_{zz} - 3\alpha F^*_{zz},\ \bar{D}^*_{zz} = D^*_{zz} - \alpha F^*_{zz}
\end{aligned} \tag{3.45}$$

where

$$\begin{aligned}
&(A_{xx}, A^*_{xx}, D_{xx}, D^*_{xx}, F_{xx}, F^*_{xx}, H_{xx}) = \int_A C_{xxxx}\left(1, z, z^2, z^3, z^4, z^5, z^6\right) \mathrm{d}A,\\
&(A_{zz}, A^*_{zz}, D_{zz}, D^*_{zz}, F_{zz}, F^*_{zz}) = \int_A C_{xxxz}\left(1, z, z^2, z^3, z^4, z^5\right) \mathrm{d}A,\\
&(A_{xz}, A^*_{xz}, D_{xz}, D^*_{xz}, F_{xz}) = \int_A C_{xzxz}\left(1, z, z^2, z^3, z^4\right) \mathrm{d}A.
\end{aligned} \tag{3.46}$$

In equation (3.42), M_{xx}, P_{xx}, N_{xz} and R_{xz} are the only classical stress resultants. Furthermore, the resultants of equation (3.44) and the variation of the strain energy (3.39) reduce to those of a Timoshenko beam by setting $\alpha = 0$. It is noted that the equations (3.35), were originally developed for a beam with a rectangular cross-section (Levinson, 1981) where

$$\alpha = \frac{4}{3L_Z^2}. \tag{3.47}$$

Above, L_Z is the height of the beam. Obviously in equation (3.46), the terms related to the integrals of the odd powers of z vanish for a beam with a rectangular cross-section.

Moreover, for a beam with rectangular cross-section and in a similar manner as the Euler-Bernoulli beam, substitution of equation (3.35) into the variation of the kinetic energy (2.13), integrating over the time domain (t_0, t_1) and applying Green's theorem leads to

$$\begin{aligned}
\int_{t_0}^{t_1} \delta K \mathrm{d}t = \rho \int_{t_0}^{t_1}\int_A\int_0^L &\{\left[-\left(z^2 + \alpha^2 z^6 - 2\alpha z^4\right)\ddot{\beta} - \left(\alpha^2 z^6 - \alpha z^4\right)\ddot{w}_{,x}\right]\delta\beta\\
&+ \left[\alpha^2 z^6 \ddot{w}_{,xx} + \left(\alpha^2 z^6 - \alpha z^4\right)\ddot{\beta}_{,x} - \ddot{w}\right]\delta w\}\mathrm{d}x\mathrm{d}A\mathrm{d}t\\
&+ \rho \int_{t_0}^{t_1}\int_A\int_0^L \{\Psi^d_{xx}\left[\left(z^2 + \alpha^2 z^6 - 2\alpha z^4\right)\ddot{\beta}_{,xx} + \left(\alpha^2 z^6 - \alpha z^4\right)\ddot{w}_{,xxx}\right]\\
&+ \Psi^d_{zz}\left[-\left(1 + 9\alpha^2 z^4 - 6\alpha z^2\right)\ddot{\beta} - \left(9\alpha^2 z^4 - 3\alpha z^2\right)\ddot{w}_{,x}\right]\}\delta\beta\mathrm{d}x\mathrm{d}A\mathrm{d}t\\
&+ \rho \int_{t_0}^{t_1}\int_A\int_0^L \{\Psi^d_{xx}\left[-\alpha^2 z^6 \ddot{w}_{,xxxx} + \ddot{w}_{xx} - \left(\alpha^2 z^6 - \alpha z^4\right)\ddot{\beta}_{,xxx}\right]\\
&+ \Psi^d_{zz}\left[\left(9\alpha^2 z^4 - 3\alpha z^2\right)\ddot{\beta}_{,x} + 9\alpha^2 z^4 \ddot{w}_{,xx}\right]\}\delta w\mathrm{d}x\mathrm{d}A\mathrm{d}t + BC,
\end{aligned} \tag{3.48}$$

where

$$
\begin{aligned}
BC = & -\rho \int_{t_0}^{t_1} \int_A \left[\alpha^2 z^6 \ddot{w}_{,x} + \left(\alpha^2 z^6 - \alpha z^4\right) \ddot{\beta}\right] \delta w|_0^L \mathrm{d}A\mathrm{d}t \\
& + \rho \int_{t_0}^{t_1} \int_A \Psi_{xx}^d \left[-\left(z^2 + \alpha^6 - 2\alpha z^4\right) \ddot{\beta}_{,x} - \left(\alpha^2 z^6 - \alpha z^4\right) \ddot{w}_{,xx}\right] \delta\beta|_0^L \mathrm{d}A\mathrm{d}t \\
& + \rho \int_{t_0}^{t_1} \int_A \{\Psi_{xx}^d \left[\left(\alpha^2 z^6 - \alpha z^4\right) \ddot{\beta}_{,xx} + \alpha^2 z^6 \ddot{w}_{,xxx} - \ddot{w}_{,x}\right] \\
& + \Psi_{zz}^d \left[-\left(9\alpha^2 z^4 - 3\alpha z^2\right) \ddot{\beta} - 9\alpha^2 z^4 \ddot{w}_{,x}\right]\} \delta w|_0^L \mathrm{d}A\mathrm{d}t \\
& + \rho \int_{t_0}^{t_1} \int_A \Psi_{xx}^d \left[-\left(\alpha^2 z^6 - \alpha z^4\right) \ddot{\beta}_{,x} - \alpha^2 z^6 \ddot{w}_{,xx}\right] \delta w_{,x}|_0^L \mathrm{d}A\mathrm{d}t.
\end{aligned} \tag{3.49}
$$

By using equation (3.29), considering the displacement field (3.35) and using Green's theorem, the variation of external work can be expressed as

$$
\begin{aligned}
\delta W = & \int_0^L \left[\left(F_z + \alpha F_{x,x}^3\right) \delta w + \left(F_x^1 - \alpha F_x^3\right) \delta\beta\right] \mathrm{d}x + [\left(T_z - \alpha F_x^3\right) \delta w \\
& + \left(-\alpha T_x^3 + Q_z\right) \delta w_{,x} - \alpha Q_x^3 \delta w_{,xx} + \left(T_x^1 - \alpha T_x^3\right) \delta\beta + \left(Q_x^1 - \alpha Q_x^3\right) \delta\beta_{,x}]|_0^L \\
= & \int_0^L \left[\left(F_z + \alpha F_{x,x}^3\right) \delta w + \left(F_x^1 - \alpha F_x^3\right) \delta\beta\right] \mathrm{d}x \\
& + \left[\bar{V}^S \delta w + \bar{M}^S \delta w_{,x} + \bar{M}_h^S \delta w_{,xx} + \bar{P}^S \delta\beta + \bar{P}_h^S \delta\beta_{,x}\right]|_0^L,
\end{aligned} \tag{3.50}
$$

where

$$
\{F_x^m, T_x^m, Q_x^m\} = \int_A \{f_x, t_x, q_x\} z^m \mathrm{d}A,\ m \in \{1, 3\} \tag{3.51}
$$

and F_z, T_z and Q_z are defined in equation (3.31).

Substitution of equations (3.50), (3.39) and (3.48) into Hamilton's principle (2.14) and the application of the fundamental lemma of calculus of variation, result in the governing equations of motion and boundary conditions for the anisotropic gradient elastic TSD beam. These governing equations are

$$
\begin{aligned}
& -\hat{M}_{xx,x} + \hat{N}_{xz} + \Psi_{xx}^s \left(\hat{M}_{xx,xxx} - \hat{N}_{xz,xx}\right) \\
& + \Psi_{xz}^s \left(\hat{M}_{xx,xx}^z - \hat{N}_{xz,x}^z - \hat{N}_{xx,xx} - 6\alpha M_{xz,x}\right) - \Psi_{zz}^s \left(\hat{N}_{xx,x}^z + 6\alpha M_{xz}^z\right) \\
& + \rho \left(I + \alpha^2 H - 2\alpha F\right) \ddot{\beta} + \rho \left(\alpha^2 H - \alpha F\right) \ddot{w}_{,x} \\
& + \rho \Psi_{xx}^d \left[-\left(I + \alpha^2 H - 2\alpha F\right) \ddot{\beta}_{,xx} - \left(\alpha^2 H - \alpha F\right) \ddot{w}_{,xxx}\right] \\
& + \rho \Psi_{zz}^d \left[\left(A + 9\alpha^2 F - 6\alpha I\right) \ddot{\beta} + \left(9\alpha^2 F - 3\alpha I\right) \ddot{w}_{,x}\right] - \left(F_x^1 - \alpha F_x^3\right) = 0,
\end{aligned} \tag{3.52}
$$

$$
\begin{aligned}
& -\alpha P_{xx,xx} - \hat{N}_{xz,x} + \Psi_{xx}^s \left(\alpha P_{xx,xxxx} + \hat{N}_{xz,xxx}\right) \\
& + \Psi_{xz}^s \left(\alpha P_{xx,xxx}^z + \hat{N}_{xz,xx}^z - 3\alpha R_{xx,xxx} + 6\alpha M_{xz,xx}\right) + \Psi_{zz}^s \left(-3\alpha R_{xx,xx}^z + 6\alpha M_{xz,x}^z\right) \\
& + \rho A \ddot{w} - \rho\alpha^2 H \ddot{w}_{,xx} - \rho \left(\alpha^2 H - \alpha F\right) \ddot{\beta}_{,x} \\
& + \rho \Psi_{xx}^d \left[\alpha^2 H \ddot{w}_{,xxxx} - A \ddot{w}_{,xx} + \left(\alpha^2 H - \alpha F\right) \ddot{\beta}_{,xxx}\right] \\
& + \rho \Psi_{zz}^d \left[-\left(9\alpha^2 F - 3\alpha I\right) \ddot{\beta}_{,x} - 9\alpha^2 F \ddot{w}_{,xx}\right] - \left(F_z + \alpha F_{x,x}^3\right) = 0
\end{aligned} \tag{3.53}
$$

where

$$
(I, F, H) = \int_A \left(z^2, z^4, z^6\right) \mathrm{d}A. \tag{3.54}
$$

The order of equation (3.52) with respect to β and of equation (3.53) with respect to w is four and six respectively. Therefore, two boundary conditions with respect to β and three boundary conditions with respect to w is expected at each end of the beam (at $x = 0$, $x = L$). These boundary conditions are

$$
\begin{aligned}
&\hat{M}_{xx} + \Psi^s_{xx}\left(-\hat{M}_{xx,xx} + \hat{N}_{xz,x}\right) \\
&+ \Psi^s_{xz}\left(-\hat{M}^z_{xx,x} + \hat{N}^z_{xz} + \hat{N}_{xx,x}\right) + \Psi^s_{zz}\hat{N}^z_{xx} \\
&+ \rho\Psi^d_{xx}\left[\left(I + \alpha^2 H - 2\alpha F\right)\ddot{\beta}_{,x} + \left(\alpha^2 H - \alpha F\right)\ddot{w}_{,xx}\right] - \bar{P}^S = 0 \quad \text{or } \delta\beta = 0 \\[1em]
&\Psi^s_{xx}\hat{M}_{xx,x} + \Psi^s_{xz}\hat{M}^z_{xx} - \bar{P}^S_h = 0 \quad \text{or} \quad \delta\beta_{,x} = 0 \\[1em]
&\alpha P_{xx,x} + \hat{N}_{xz} + \Psi^s_{xx}\left(-\alpha P_{xx,xxx} - \hat{N}_{xz,xx}\right) \\
&+ \Psi^s_{xz}\left(-\alpha P^z_{xx,xx} - \hat{N}^z_{xz,x} + 3\alpha R_{xx,xx} - 6\alpha M_{xz,x}\right) \\
&+ \Psi^s_{zz}\left(3\alpha R^z_{xx,x} - 6\alpha M^z_{xz}\right) \\
&+ \rho\alpha^2 H\ddot{w}_{,x} + \rho\left(\alpha^2 H - \alpha F\right)\ddot{\beta} \\
&+ \rho\Psi^d_{xx}\left[-\left(\alpha^2 H - \alpha F\right)\ddot{\beta}_{,xx} - \alpha^2 H\ddot{w}_{,xxx} + A\ddot{w}_{,x}\right] \\
&+ \rho\Psi^d_{zz}\left[\left(9\alpha^2 F - 3\alpha I\right)\ddot{\beta} + 9\alpha^2 F\ddot{w}_{,x}\right] - \bar{V}^S = 0 \qquad \text{or } \delta w = 0 \\[1em]
&- \alpha P_{xx} + \Psi^s_{xx}\left(\alpha P_{xx,xx} + \hat{N}_{xz,x}\right) \\
&+ \Psi^s_{xz}\left(\alpha P^z_{xx,x} + \hat{N}^z_{xz} - 3\alpha R_{xx,x}\right) - 3\alpha\Psi^s_{zz}R^z_{xx} \\
&+ \rho\Psi^d_{xx}\left[\left(\alpha^2 H - \alpha F\right)\ddot{\beta}_{,x} + \alpha^2 H\ddot{w}_{,xx}\right] - \bar{M}^S = 0 \quad \text{or } \delta w_{,x} = 0 \\[1em]
&-\alpha\Psi^s_{xx}P_{xx,x} - \alpha\Psi^s_{xz}P^z_{xx} - \bar{M}^S_h = 0 \quad \text{or} \quad \delta w_{,xx} = 0
\end{aligned}
\tag{3.55}
$$

Different combinations of classical boundary conditions (3.55-1), (3.55-3), (3.55-4) result in eight boundary types of TSD beam in the framework of classical elasticity. Their relations to four conventional boundary types of a beam are worth separate discussion. The selection of non-classical boundary conditions (3.55-2, 3.55-5), splits each of eight aforementioned types by four subtypes (from singly to fourthly) yielding in that way all together 32 combinations. We do not present the full classification of all possible boundary types and note that physical meaning for many of them needs a more accurate study on the behaviour of the beam.

For a beam with rectangular cross-section, the governing equations (3.52) and (3.53) can be written in terms of deflection and rotation as

$$
\begin{aligned}
&- \left(\tilde{D}_{xx}\beta_{,x} - \alpha\bar{F}_{xx}w_{,xx}\right)_{,x} + \tilde{A}_{xz}\left(w_{,x} + \beta\right) \\
&+ \Psi^s_{xx}\left[\left(\tilde{D}_{xx}\beta_{,x} - \alpha\bar{F}_{xx}w_{,xx}\right)_{,xxx} - \tilde{A}_{xz}\left(w_{,x} + \beta\right)_{,xx}\right] \\
&+ \Psi^s_{xz}\left[-\left(\tilde{A}_{zz} + 6\alpha\bar{D}_{zz}\right)\left(w_{,x} + 2\beta\right)_{,xx} + 3\alpha\bar{D}_{zz}w_{,xxx}\right] \\
&- \Psi^s_{zz}\left[\left(\tilde{A}_{xx}\beta_{,x} - 3\alpha\hat{D}_{xx}w_{,xx}\right)_{,x} - 36\alpha^2 D_{xz}\left(w_{,x} + \beta\right)\right] \\
&+ \rho\left(I + \alpha^2 H - 2\alpha F\right)\ddot{\beta} + \rho\left(\alpha^2 H - \alpha F\right)\ddot{w}_{,x} \\
&+ \rho\Psi^d_{xx}\left[-\left(I + \alpha^2 H - 2\alpha F\right)\ddot{\beta}_{,xx} - \left(\alpha^2 H - \alpha F\right)\ddot{w}_{,xxx}\right] \\
&+ \rho\Psi^d_{zz}\left[\left(A + 9\alpha^2 F - 6\alpha I\right)\ddot{\beta} + \left(9\alpha^2 F - 3\alpha I\right)\ddot{w}_{,x}\right] - \left(F^1_x - \alpha F^3_x\right) = 0,
\end{aligned}
\tag{3.56}
$$

and

$$
\begin{aligned}
&- \alpha \left(\bar{F}_{xx}\beta_{,x} - \alpha H_{xx} w_{,xx}\right)_{,xx} - \tilde{A}_{xz}\left(w_{,x} + \beta\right)_{,x} \\
&+ \Psi^s_{xx}\left[\alpha\left(\bar{F}_{xx}\beta_{,x} - \alpha H_{xx} w_{,xx}\right)_{,xxxx} + \tilde{A}_{xz}\left(w_{,x} + \beta\right)_{,xxx}\right] \\
&+ \Psi^s_{xz}\left[-6\alpha\bar{D}_{zz} w_{,xxxx} + \left(\tilde{A}_{zz} + 3\alpha\bar{D}_{zz}\right)\beta_{,xxx}\right] \\
&+ \Psi^s_{zz}\left[-3\alpha\left(\hat{D}_{xx}\beta_{,x} - 3\alpha F_{xx} w_{,xx}\right)_{,xx} - 36\alpha^2 D_{xz}\left(w_{,x} + \beta\right)_{,x}\right] \\
&+ \rho A\ddot{w} - \rho\alpha^2 H\ddot{w}_{,xx} - \rho\left(\alpha^2 H - \alpha F\right)\ddot{\beta}_{,x} \\
&+ \rho\Psi^d_{xx}\left[\alpha^2 H\ddot{w}_{,xxxx} - A\ddot{w}_{,xx} + \left(\alpha^2 H - \alpha F\right)\ddot{\beta}_{,xxx}\right] \\
&+ \rho\Psi^d_{zz}\left[-\left(9\alpha^2 F - 3\alpha I\right)\ddot{\beta}_{,x} - 9\alpha^2 F\ddot{w}_{,xx}\right] - \left(F_z + \alpha F^3_{x,x}\right) = 0.
\end{aligned}
\tag{3.57}
$$

where

$$
\begin{aligned}
&\tilde{D}_{xx} = \bar{D}_{xx} - \alpha\bar{F}_{xx},\ \tilde{A}_{xz} = \hat{A}_{xz} - 3\alpha\hat{D}_{xz}, \\
&\tilde{A}_{xx} = \hat{A}_{xx} - 3\alpha\hat{D}_{xx},\ \tilde{A}_{zz} = \hat{A}_{zz} - 3\alpha\hat{D}_{zz}.
\end{aligned}
\tag{3.58}
$$

The governing equations and boundary conditions of a Timoshenko beam can be simply obtained by setting $\alpha = 0$ in equations (3.55), (3.56) and (3.57) giving:

$$
\begin{aligned}
&- D_{xx}\beta_{,xx} + A_{xz}\left(w_{,x} + \beta\right) + \Psi^s_{xx}\left[D_{xx}\beta_{,xxxx} - A_{xz}\left(w_{,x} + \beta\right)_{,xx}\right] \\
&- \Psi^s_{xz} A_{zz}\left(w_{,x} + 2\beta\right)_{,xx} - \Psi^s_{zz} A_{xx}\beta_{,xx} + \rho I\ddot{\beta} - \rho\Psi^d_{xx} I\ddot{\beta}_{,xx} + \rho\Psi^d_{zz} A\ddot{\beta} - F^1_x = 0,
\end{aligned}
\tag{3.59}
$$

$$
-A_{xz}\left(w_{,x} + \beta\right)_{,x} + \Psi^s_{xx} A_{xz}\left(w_{,x} + \beta\right)_{,xxx} + \Psi^s_{xz} A_{zz}\beta_{,xxx} + \rho A\ddot{w} - \rho\Psi^d_{xx} A\ddot{w}_{,xx} - F_z = 0. \tag{3.60}
$$

$$
\begin{aligned}
&M_{xx} + \Psi^s_{xx}\left(-M_{xx,xx} + N_{xz,x}\right) \\
&\ + \Psi^s_{xz}\left(-M^z_{xx,x} + N^z_{xz} + N_{xx,x}\right) + \Psi^s_{zz} N^z_{xx} + \rho\Psi^d_{xx} I\ddot{\beta}_{,x} - T^1_x = 0 \quad \text{or} \quad \delta\beta = 0 \\
&\Psi^s_{xx} M_{xx,x} + \Psi^s_{xz} M^z_{xx} - Q^1_x = 0 \quad \text{or} \quad \delta\beta_{,x} = 0 \\
&N_{xz} - \Psi^s_{xx} N_{xz,xx} - \Psi^s_{xz} N^z_{xz,x} + \rho\Psi^d_{xx} A\ddot{w}_{,x} - T_z = 0 \quad \text{or} \quad \delta w = 0 \\
&\Psi^s_{xx} N_{xz,x} + \Psi^s_{xz} N^z_{xz} - Q_z = 0 \quad \text{or} \quad \delta w_{,x} = 0
\end{aligned}
\tag{3.61}
$$

3.2.1. Simplifications for particular kinds of materials

The formulation can be readily simplified for materials of more practical use such as orthotropic and isotropic materials. For this purpose, Voigt notation is employed (Voigt, 1928):

$$
C_{ijkl} \to C_{st},\ s,t \to 1,2,...,6:\ 11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 13 \to 5,\ 12 \to 6. \tag{3.62}
$$

According to equation (5.93) of Appendix (A), the tensor of static anisotropic length scale for orthotropic materials (which are composed of orthorhombic crystals) is

$$
\Psi^s_{mn} = \begin{bmatrix} \Psi^s_{xx} & 0 & 0 \\ 0 & \Psi^s_{yy} & 0 \\ 0 & 0 & \Psi^s_{zz} \end{bmatrix} \quad \Psi^s_{xx} > 0, \quad \Psi^s_{yy} > 0, \quad \Psi^s_{zz} > 0. \tag{3.63}
$$

Similarly; the tensor of kinetic length scale for orthotropic materials is considered to be

$$\Psi^d_{mn} = \begin{bmatrix} \Psi^d_{xx} & 0 & 0 \\ 0 & \Psi^d_{yy} & 0 \\ 0 & 0 & \Psi^d_{zz} \end{bmatrix} \quad \Psi^d_{xx} > 0, \quad \Psi^d_{yy} > 0, \quad \Psi^d_{zz} > 0. \tag{3.64}$$

By using Voigt notation, the elastic modulus tensor for orthotropic material is given by

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{3.65}$$

Therefore, in order to obtain the equilibrium equations and boundary conditions of a beam made of orthotropic material, it is sufficient to simplify the equations (3.56), (3.57) and (3.55) by using equations (3.63), (3.64) and (3.65).

By setting $\Psi^s_{xx} = \Psi^s_{zz} = l_s^2$ and $\Psi^d_{xx} = \Psi^d_{zz} = l_d^2$ and considering the elastic modulus tensor of isotropic materials, the governing equations and boundary conditions of an isotropic gradient elastic TSD beam will be obtained (Yaghoubi et al., 2015).

3.2.2. Free vibration of a simply supported TSD beam

A doubly simply supported anisotropic TSD beam with a rectangular cross section is considered. In order to study the free vibration of the beam, the external loads are assumed to be zero. The boundary conditions for a doubly simply supported rectangular beam are

$$\begin{aligned}
&\hat{M}_{xx} + \Psi^s_{xx}\left(-\hat{M}_{xx,xx} + \hat{N}_{xz,x}\right) \\
&+ \Psi^s_{xz}\left(-\hat{M}^z_{xx,x} + \hat{N}^z_{xz} + \hat{N}_{xx,x}\right) + \Psi^s_{zz}\hat{N}^z_{xx} \\
&+ \rho\Psi^d_{xx}\left[\left(I + \alpha^2 H - 2\alpha F\right)\ddot{\beta}_{,x} + \left(\alpha^2 H - \alpha F\right)\ddot{w}_{,xx}\right] = 0 \\
&\beta_{,x} = 0 \\
&w = 0 \\
&-\alpha P_{xx} + \Psi^s_{xx}\left(\alpha P_{xx,xx} + \hat{N}_{xz,x}\right) \\
&+ \Psi^s_{xz}\left(\alpha P^z_{xx,x} + \hat{N}^z_{xz} - 3\alpha R_{xx,x}\right) - 3\alpha\Lambda_{zz}R^z_{xx} \\
&+ \rho\Psi^d_{xx}\left[\left(\alpha^2 H - \alpha F\right)\ddot{\beta}_{,x} + \alpha^2 H\ddot{w}_{,xx}\right] = 0 \\
&w_{,xx} = 0
\end{aligned} \tag{3.66}$$

The governing equations (3.56) and (3.57) together with the boundary conditions (3.66) when the external loads are zero have a serial solution of the form

$$w(x,t) = \sum_{n=1}^{\infty} w^d_n \sin(\frac{n\pi x}{L})e^{i\omega_n t}, \ \beta(x,t) = \sum_{n=1}^{\infty} \beta^d_n \cos(\frac{n\pi x}{L})e^{i\omega_n t} \tag{3.67}$$

Above, ω_n is the vibrational frequency and i is the imaginary number defined by $i^2 = -1$. Substitution of (3.67) into (3.56) and (3.57) results in

$$\begin{bmatrix} k_1 - k_4\omega_n^2 & k_2 - k_5\omega_n^2 \\ k_2 - k_5\omega_n^2 & k_3 - k_6\omega_n^2 \end{bmatrix} \begin{bmatrix} \beta^d_n \\ w^d_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.68}$$

where

$$
\begin{cases}
k_1 = (1 + \Psi_{xx}^s \gamma^2) a_1 + a_2 + a_3 + 2a_8, \\
k_2 = (1 + \Psi_{xx}^s \gamma^2) a_4 + \gamma a_3 + a_5 + \gamma a_8 + a_9, \\
k_3 = (1 + \Psi_{xx}^s \gamma^2) a_6 + \gamma^2 a_3 + a_7 + 2\gamma a_9 \\
k_4 = \left(1 + \Psi_{xx}^d \gamma^2\right) b_1 + \Psi_{zz}^d b_3, \\
k_5 = \left(1 + \Psi_{xx}^d \gamma^2\right) \gamma b_2 + \Psi_{zz}^d \gamma b_4, \\
k_6 = \left(1 + \Psi_{xx}^d \gamma^2\right) \gamma^2 b_5 + b_6 + b_7
\end{cases}
\tag{3.69}
$$

$$
\begin{cases}
a_1 = \gamma^2 \tilde{D}_{xx} + \tilde{A}_{xz}, & a_2 = \Psi_{zz}^s \gamma^2 \tilde{A}_{xx}, \\
a_3 = 36\alpha^2 \Psi_{zz}^s D_{xz}, & a_4 = -\alpha\gamma^3 \bar{F}_{xx} + \gamma \tilde{A}_{xz}, \\
a_5 = -3\alpha \Psi_{zz}^s \gamma^3 \hat{D}_{xx}, & a_6 = \alpha^2 H_{xx} \gamma^4 + \gamma^2 \tilde{A}_{xz}, \\
a_7 = 9\alpha^2 \Psi_{zz}^s \gamma^4 F_{xx}, & a_8 = \Psi_{xz}^s \gamma^2 (\tilde{A}_{zz} + 6\alpha \bar{D}_{zz}), \\
a_9 = -3\alpha \Psi_{xz}^s \gamma^3 \bar{D}_{zz}.
\end{cases}
\tag{3.70}
$$

$$
\begin{cases}
b_1 = \rho\left(I + \alpha^2 H - 2\alpha F\right), & b_2 = \rho\left(\alpha^2 H - \alpha F\right), \\
b_3 = \rho\left(A + 9\alpha^2 F - 6\alpha I\right), & b_4 = \rho\left(9\alpha^2 F - 3\alpha I\right), \\
b_5 = \rho \alpha^2 H, & b_6 = \rho A, \\
b_7 = \rho \gamma^2 \left(\Psi_{xx}^d A + 9\alpha^2 F \psi_{zz}^d\right)
\end{cases}
\tag{3.71}
$$

For the existence of a nontrivial solution, the determinant of the coefficient matrix of (3.68) has to vanish. This condition leads to

$$
R_1 \omega_n^4 + R_2 \omega_n^2 + R_3 = 0 \tag{3.72}
$$

while

$$
R_1 = k_4 k_6 - k_5^2, R_2 = 2k_2 k_5 - k_1 k_6 - k_3 k_4, R_3 = k_1 k_3 - k_2^2. \tag{3.73}
$$

Hence, by solving the bi-quadratic equation (3.72) one can obtain two branches of eigenspectrum:

$$
\omega_{n_{1,2}} = \sqrt{\frac{-R_2 \pm \sqrt{R_2^2 - 4R_1 R_3}}{2R_1}}. \tag{3.74}
$$

The second spectral brunch also exists in the numerical solution considered in subsequent section. Discussion about the physical meaning of this brunch (with "+" in front of discriminant in (3.74)) is out of scope of the present contribution.

4. Variational formulations and Isogeometric analysis

This section is devoted to variational formulations of eigenvalue problems for the Euler-Bernoulli and TSD beam models with a short description of the isogeometric Galerkin method used for obtaining numerical solutions for a benchmark problem.

In what follows, notation $L^2(\Omega)$ is used for a set of square-integrable real-valued functions defined on $\Omega = (0, L)$ and $H^s(\Omega)$ for a real Sobolev space of order s.

4.1. Euler-Bernoulli beam

In order to formulate the eigenvalue problem for the Euler-Bernoulli model, we assume the following form of the particular variable-separable solution for equation (3.33):

$$
w(x,t) = w(x) e^{-i\omega t}, \tag{4.75}
$$

where $i = \sqrt{-1}$ stands for the imaginary unit.

Substitution of assumption (4.75) into the strain (3.26), kinetic (3.27), and external (3.30) energies with subsequent utilisation of Hamilton's principle (2.14) and integration by parts results in the weak formulation of the eigenvalue problem (the variation of deflection δw is replaced by test function $\hat{w}$):

Problem 1. *Find all eigenpars $\{w, \lambda\}$, $w \in \mathcal{W}$, $\lambda = \omega^2 \in \mathbb{R}$, such that*

$$a(w;\hat{w}) - \omega^2 b(w;\hat{w}) = 0 \ \forall \hat{w} \in \hat{\mathcal{W}}, \tag{4.76}$$

where the components of the bilinear forms $a : \mathcal{W} \times \hat{\mathcal{W}} \to \mathbb{R}$ and $b : \mathcal{W} \times \hat{\mathcal{W}} \to \mathbb{R}$, are defined as

$$a(w;\hat{w}) = \int_0^L \Big[D_{xx} w_{,xx} \hat{w}_{,xx} + \Psi^s_{xx} \left(D_{xx} w_{,xx}\right)_{,x} \hat{w}_{,xxx} + \Psi^s_{zz} A_{xx} w_{,xx} \hat{w}_{,xx} \Big] \, \mathrm{d}x; \tag{4.77}$$

$$b(w;\hat{w}) = \int_0^L \rho \left[I w_{,x} \hat{w}_{,x} + A w \hat{w} + \Psi^d_{xx} \left(I w_{,xx} \hat{w}_{,xx} + A w_{,x} \hat{w}_{,x} \right) + \Psi^d_{zz} A w_{,x} \hat{w}_{,x} \right] \mathrm{d}x. \tag{4.78}$$

The trial function set

$$W = \{ v \in H^3(\Omega) \ | \ v_{|\Gamma_{\mathrm{SS}} \cup \Gamma_{\mathrm{C}}} = \bar{w}, \ v'_{|\Gamma_{\mathrm{ES}} \cup \Gamma_{\mathrm{C}}} = \bar{\beta}, \ v''_{|\Gamma_{\mathrm{C_d}} \cup \Gamma_{\mathrm{SS_d}} \cup \Gamma_{\mathrm{S_d}}} = \bar{\kappa} \} \tag{4.79}$$

consists of functions satisfying the essential boundary conditions, with given Dirichlet data $\bar{w}, \bar{\beta}, \bar{\kappa}$, while the test function space $\hat{w}$ consists of H^3 functions satisfying the corresponding homogeneous Dirichlet type boundary conditions.

An analogue to Problem 1 for the isotropic beam can be derived by making the following substitutions into the bilinear forms (4.77) and (4.78):

$$D_{xx} \to EI, \ A_{xx} = EA, \ \psi^s_{xx} \to l_s^2, \ \psi^s_{zz} \to l_s^2, \ \psi^d_{xx} \to l_d^2, \ \psi^d_{zz} \to l_d^2, \tag{4.80}$$

which leads to

$$a(w;\hat{w}) = \int_0^L \Big[EI w_{,xx} \hat{w}_{,xx} + l_s^2 \left(EI w_{,xx}\right)_{,x} \hat{w}_{,xxx} + l_s^2 EA w_{,xx} \hat{w}_{,xx} \Big] \, \mathrm{d}x; \tag{4.81}$$

$$b(w;\hat{w}) = \int_0^L \rho \left[I w_{,x} \hat{w}_{,x} + A w \hat{w} + l_d^2 I w_{,xx} \hat{w}_{,xx} + 2 l_d^2 A w_{,x} \hat{w}_{,x} \right] \mathrm{d}x. \tag{4.82}$$

As it can be seen, the formed structures of the weak formulations for isotropic and anisotropic cases are identical and therefore one can use the same numerical method for them. Detailed analysis of isogeometric Galerkin methods for isotropic case of gradient-elastic Euler-Bernoulli beam model is carried out in Niiranen et al. (2017) and is not repeated in the present contribution.

4.2. TSD beam

Let us consider an eigenvalue problem of a beam with *doubly simply supported* boundaries (3.66). Similarly to subsection 4.1, we assume the following form of the particular variable-separable solution for equations (3.56) and (3.57):

$$w(x,t) = w(x) e^{-i\omega t}; \quad \beta(x,t) = \beta(x) e^{-i\omega t}, \tag{4.83}$$

and substitute it into the energy expressions (3.39), (3.48), (3.50). Then by utilising integration by parts and Hamilton's principle (2.14) one obtains the weak formulation:

Problem 2. *Find all eigenpairs $\{(w,\beta), \lambda\}$, $w \in \mathcal{W}, \beta \in \mathcal{V}$, $\lambda = \omega^2 \in \mathbb{R}$, such that*

$$a(w,\beta;\hat{w},\hat{\beta}) - \omega^2 b(w,\beta;\hat{w},\hat{\beta}) = 0 \ \forall \ \hat{w} \in \hat{\mathcal{W}}, \forall \ \hat{\beta} \in \hat{\mathcal{V}}, \tag{4.84}$$

where the components of the bilinear forms $a : (\mathcal{W} \times \mathcal{V}) \times (\hat{\mathcal{W}} \times \hat{\mathcal{V}}) \to \mathbb{R}$, *and* $b : (\mathcal{W} \times \mathcal{V}) \times (\hat{\mathcal{W}} \times \hat{\mathcal{V}}) \to \mathbb{R}$, *and function spaces are defined as*

$$\begin{aligned}
a(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L \Big[& \tilde{D}_{xx}\beta_{,x}\hat{\beta}_{,x} - \alpha\bar{F}_{xx}(w_{,xx}\hat{\beta}_{,x} + \beta_{,x}\hat{w}_{,xx}) + \alpha^2 H_{xx}w_{,xx}\hat{w}_{,xx} + \tilde{A}_{xz}(w_{,x} + \beta)(\hat{w}_{,x} + \hat{\beta}) \\
& + \Psi^s_{xx}\left[\tilde{D}_{xx}\beta_{,xx}\hat{\beta}_{,xx} - \alpha\bar{F}_{xx}(w_{,xxx}\hat{\beta}_{,xx} + \beta_{,xx}\hat{w}_{,xxx}) + \alpha^2 H_{xx}w_{,xxx}\hat{w}_{,xxx} + \tilde{A}_{xz}(w_{,xx} + \beta_{,x})(\hat{w}_{,xx} + \hat{\beta}_{,x})\right] \\
& + \Psi^s_{xz}\left[\tilde{A}_{zz}\Big((w_{,xx} + 2\beta_{,x})\hat{\beta}_{,x} + \beta_{,x}\hat{w}_{,xx}\Big) + 3\alpha\bar{D}_{zz}\Big((w_{,xx} + 4\beta_{,x})\hat{\beta}_{,x} - (2w_{,xx} - \beta_{,x})\hat{w}_{,xx}\Big)\right] \\
& - \Psi^s_{zz}\left[(\tilde{A}_{xx}\beta_{,x} - 3\alpha\hat{D}_{xx}w_{,xx})\hat{\beta}_{,x} + 36\alpha^2 D_{xz}(w_{,x} + \beta)(\hat{w}_{,x} + \hat{\beta}) - 3\alpha(\hat{D}_{xx}\beta_{,x} - 3\alpha F_{xx}w_{,xx})\hat{w}_{,xx}\right]\Big]\,\mathrm{d}x;
\end{aligned} \tag{4.85}$$

$$\begin{aligned}
b(w, \beta; \hat{w}, \hat{\beta}) = \int_0^L \rho\Big[& Aw\hat{w} + b_1\beta\hat{\beta} + \alpha^2 H w_{,x}\hat{w}_{,x} + b_2(w_{,x}\hat{\beta} + \beta\hat{w}_{,x}) \\
& + \Psi^d_{xx}\left[Aw_{,x}\hat{w}_{,x} + b_1\beta_{,x}\hat{\beta}_{,x} + \alpha^2 H w_{,xx}\hat{w}_{,xx} + b_2(w_{,xx}\hat{\beta}_{,x} + \beta_{,x}\hat{w}_{,xx})\right] \\
& + \Psi^d_{zz}\left[b_3\beta\hat{\beta} + 9\alpha^2 F w_{,x}\hat{w}_{,x} + b_4(w_{,x}\hat{\beta} + \beta\hat{w}_{,x})\right]\Big]\,\mathrm{d}x;
\end{aligned} \tag{4.86}$$

$$\begin{aligned}
\mathcal{W} &= \{u \in H^3(\Omega)|\ u|_{0,L} = 0,\ u''|_{0,L} = 0\}; \\
\mathcal{V} &= \{v \in H^2(\Omega)|\ v'|_{0,L} = 0\}.
\end{aligned} \tag{4.87}$$

4.3. Basics of Isogeometric analysis

Let us recall the main definitions concerning isogeometric discretizations without going into deep details (de Falco et al., 2011). For unknown functions of deflection and rotation, we use the following approximations

$$w(x) = \sum_{i=1}^{n} N_{i,p}(x)d_i^w;\ \beta(x) = \sum_{i=1}^{n} N_{i,p}(x)d_i^\beta, \tag{4.88}$$

where d_i^w and d_i^β denote the control variables and act as problem unknowns. B-spline basis functions $N_{i,p}$ of order p are used. They can be defined with the aid of an open knot vector $\{0 = x_1, ..., x_i, ..., x_{n+p+1} = L\}$ by the use of Cox-de Boor recursion formula:

$$\begin{aligned}
N_{i,p}(x) &= \frac{x - x_i}{x_{i+p} - x_i}N_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}}N_{i+1,p-1}(x) \quad for \quad p = 1, 2, 3, ... \\
N_{i,0}(x) &= \begin{cases} 1 & \text{if } x_i \le x \le x_{i+1}; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.89}$$

After the substitution of the approximations (4.88) into the weak form (Problem 2), we use the standard Galerkin approach and calculate the stiffness and mass matrices.

The described method provides C^{p-1} global regularity over the mesh. Consequently, in order to guarantee the desired $H^3(\Omega)$-conforming discretization for Problem 2 we have to use functions of order $p \ge 3$.

4.4. Numerical results and error estimations

We consider again a beam made of a hypothetical? anisotropic material with *doubly simply supported* boundary conditions, described in section 3.2.2. Values of all required geometrical and mechanical parameters can be found in Table 1.

Recently, Admal et al. (2017) presented a new method for obtaining atomistic definitions for elastic tensors appearing in the first strain-gradient elasticity theory for an arbitrary multi-lattice. Their method is based on the condition of energetic equivalence between continuum and atomic representations of a crystal, when the kinematics of the latter is governed by Cauchy-Born rule. The tensor of elastic moduli as well as the components of the strain gradient elastic tensor (D_{ijmkln}) are computed for a large class of materials and the results are available in OpenKIM Repository at https://www.openkim.org (Admal et al., 2017). Furthermore, in a recent paper by Po et al. (2017), methods for obtaining the tensor of anisotropic length scale parameters (Ψ^s_{mn}) from the tensors C_{ijkl} and D_{ijmkln} are proposed. In this study, the values of the static and kinetic length scale parameters of Table 1 are assumed in the order of ....scale in order to demonstrate the size effect and verify the accuracy of the IGA method.

Table 1: Problem parameters.

Parameter	Value
beam length L	200 μm
cross section height L_Z	4 μm
cross section width L_Y	10 μm
mass density ρ	4020 kg/m^3
C_{xxxx}	317.5 GPa
C_{xxxz}	100 GPa
C_{xzxz}	75.8 GPa
Ψ^s_{xx}	2 μm^2
Ψ^s_{xz}	1.2 μm^2
Ψ^s_{zz}	0.4 μm^2
Ψ^d_{xx}	0.8 μm^2
Ψ^d_{zz}	0.2 μm^2

For a verification of the numerical method, we compare results obtained with the aid of IGA and the analytical solution.

Figure 1 illustrates how the accuracy of the numerical results changes with the increase of the frequency number. The ratio of the numerically obtained eigen frequencies ω^h_n to the analytically obtained ones ω_n (see (3.74)) is plotted along the vertical axis. Along the horizontal axis, one can see the frequency number n divided by the total number of calculated frequencies N (equal to the number of degrees of freedom, DoFs). Shape and behaviour of the spectral curves do not depend on the number of DoFs.

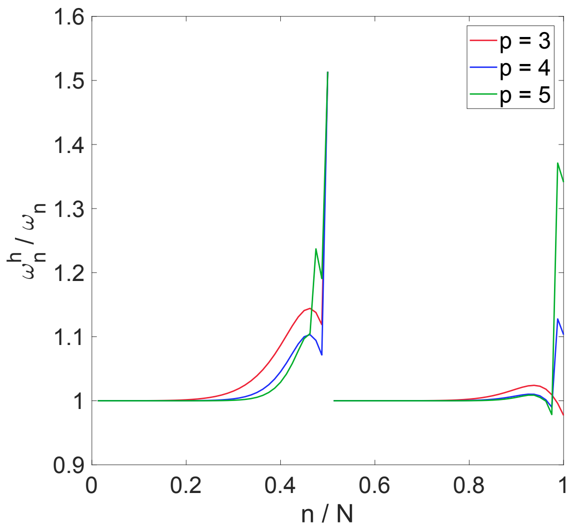

Figure 1: Normalized discrete spectra for p=3,4,5

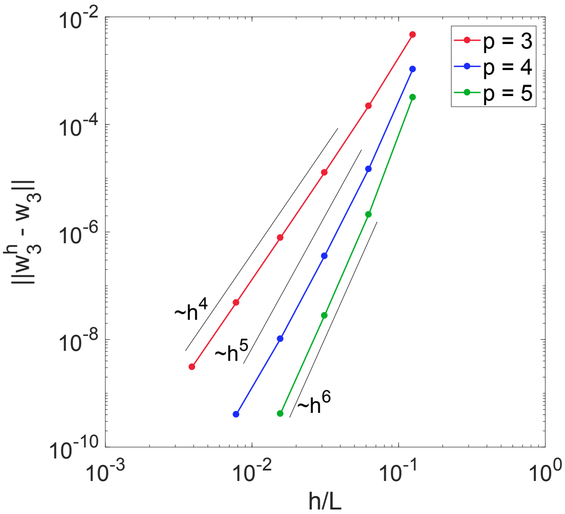

Figure 2: Error in L^2 norm for 3^{rd} eigen mode of deflection versus element size for p=3,4,5

The problem is solved for three different basis function orders $p = 3, 4, 5$. There are two spectral branches for the considered model and both demonstrate separately the typical behaviour for isogeometric Galerkin methods in full accordance with the results which can be found in literature (e.g. for longitudinal classical bars see Cottrell et al. (2009).

One should keep in mind the fact of the existence of the second branch and distinguish it carefully. The easiest way to do it is to choose the number of degrees of freedom in such a manner that the last eigen frequency from the first spectral branch is less than the first one from the second branch. For the considered set of problem parameters it is assumed that $N = 40$.

Solution errors in L^2-norm for 3^{rd} vibrational mode of deflection w versus the dimensionless size of the finite element h/L in logarithmic scales is represented in Figure 2. The study shows that the convergence rates follow order $\mathcal{O}(h^{p+1})$ for the considered orders of B-spline basis functions.

5. Conclusions

In the current paper, the derivation of the dynamic equations for anisotropic centrosymmetric gradient-elastic beams is presented. Three widespread beam models, namely Euler-Bernoulli, Timoshenko and third order shear-deformable are considered. The strain energy is generalized by strain gradients and a tensor of static length scale parameters. Moreover, the classical kinetic energy is enriched by the velocity gradients with the aid of introduction of a tensor of anisotropic kinetic length scale parameters which is usually missing in the papers devoted to gradient elastic theory. The resulting model enables one to study the size effect on the statical and dynamical behaviour of centrosymmetric anisotropic beams.

In addition to strong formulations, the weak variational formulations are presented. It is shown that the dynamic equation of a gradient-elastic anisotropic Euler-Bernoulli beam has the same structure as the one for the isotropic case.

The numerical C^2-continuous method based on isogeometric Galerkin discretization is implemented for the free-vibration problem of anisotropic gradient-elastic TSD beams.

Variational formulation and numerical solutions for the Timoshenko model are not presented in the present contribution. However, it is noteworthy that they can be easily obtained by a simplification of the TSD model and separate derivations and considerations are not needed. For TSD and Timoshenko beams, one can use the same numerical method (for the latter one, the minimal order of the basis functions is $p = 2$, though).

The numerical solution presented works for any kind of boundary conditions although it is tested for one case of boundary conditions for which an analytical solution can be found. Comparisons between analytical and numerical solutions show that the numerical method works properly in the sense of convergence.

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Appendix A

For different classes of crystal symmetry (Nye, 1957), the tensors of static and kinetic anisotropic length scale (Ψ^g_{mn}, $g \in \{s, d\}$) and the corresponding conditions for positive definiteness are listed (Lazar and Po, 2015a). The tensor of kinetic anisotropic length scale Ψ^d_{mn} is assumed to have the same structure as Ψ^s_{mn} for different classes of crystal symmetry.
Triclinic crystal:

$$\begin{aligned} &\Psi^g_{mn} = \begin{bmatrix} \Psi^g_{11} & \Psi^g_{12} & \Psi^g_{13} \\ \Psi^g_{12} & \Psi^g_{22} & \Psi^g_{23} \\ \Psi^g_{13} & \Psi^g_{23} & \Psi^g_{33} \end{bmatrix} \\ &\Psi^g_{11} > 0, \quad \begin{vmatrix} \Psi^g_{11} & \Psi^g_{12} \\ \Psi^g_{12} & \Psi^g_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} \Psi^g_{11} & \Psi^g_{12} & \Psi^g_{13} \\ \Psi^g_{12} & \Psi^g_{22} & \Psi^g_{23} \\ \Psi^g_{13} & \Psi^g_{23} & \Psi^g_{33} \end{vmatrix} > 0. \end{aligned} \tag{5.90}$$

Monoclinic crystal (standard orientation 2||b):

$$\begin{aligned} &\Psi^g_{mn} = \begin{bmatrix} \Psi^s_{11} & 0 & \Psi^g_{13} \\ 0 & \Psi^g_{22} & 0 \\ \Psi^g_{13} & 0 & \Psi^g_{33} \end{bmatrix} \\ &\Psi^g_{11} > 0, \quad \Psi^g_{22} > 0, \quad \Psi^g_{33} > 0, \quad \Psi^g_{11}\Psi^g_{33} - (\Psi^g_{13})^2 > 0. \end{aligned} \tag{5.91}$$

Monoclinic crystal (orientation 2||c):

$$\begin{aligned} &\Psi^g_{mn} = \begin{bmatrix} \Psi^g_{11} & \Psi^g_{12} & 0 \\ \Psi^g_{12} & \Psi^g_{22} & 0 \\ 0 & 0 & \Psi^g_{33} \end{bmatrix} \\ &\Psi^g_{11} > 0, \quad \Psi^g_{22} > 0, \quad \Psi^g_{33} > 0, \quad \Psi^g_{11}\Psi^g_{22} - (\Psi^g_{12})^2 > 0. \end{aligned} \tag{5.92}$$

Orthorhombic crystal:

$$\Psi^g_{mn} = \begin{bmatrix} \Psi^g_{11} & 0 & 0 \\ 0 & \Psi^g_{22} & 0 \\ 0 & 0 & \Psi^g_{33} \end{bmatrix} \quad \Psi^g_{11} > 0, \quad \Psi^g_{22} > 0, \quad \Psi^g_{33} > 0. \tag{5.93}$$

Tetragonal, hexagonal, and trigonal crystal:

$$\Psi^g_{mn} = \begin{bmatrix} \Psi^g_{11} & 0 & 0 \\ 0 & \Psi^g_{11} & 0 \\ 0 & 0 & \Psi^g_{33} \end{bmatrix} \quad \Psi^g_{11} > 0, \quad \Psi^g_{33} > 0. \tag{5.94}$$

Cubic crystal:

$$\Psi^g_{mn} = \begin{bmatrix} \Psi^g_{11} & 0 & 0 \\ 0 & \Psi^g_{11} & 0 \\ 0 & 0 & \Psi^g_{11} \end{bmatrix} \quad \Psi^g_{11} > 0. \tag{5.95}$$