Aplikace matematiky

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Aplikace matematiky, Vol. 31 (1986), No. 4, 270–281

Persistent URL: http://dml.cz/dmlcz/104206

31 (1986) APLIKACE MATEMATIKY No. 4, 270—281

VARIATIONAL INEQUALITIES IN PLASTICITY WITH STRAIN-HARDENING — EQUILIBRIUM FINITE ELEMENT APPROACH

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(Received April 16, 1985)

Summary. The incremental finite element method is applied to find the numerical solution of the plasticity problem with strain-hardening. Following Watwood and Hartz, the stress field is approximated by equilibrium triangular elements with linear functions. The field of the strain-hardening parameter is considered to be piecewise linear. The resulting nonlinear optimization problem with constraints is solved by the Lagrange multipliers method with additional variables. A comparison of the results obtained with an experiment is given.

1. INTRODUCTION

The flow theory of plasticity with strain-hardening material has been studied recently by Johnson [1] and Hlaváček, Nečas [2] from a new point of view, pioneered by Nguyen Quoc Son [4] and Halphen-Nguyen Quoc Son [5]. The common idea of their existence proofs is to formulate the problem by means of the variational inequality of evolution and to use a penalty method.

In the present paper we propose an incremental finite element method, starting from the formulation of the quasi-static problem in terms of stresses and hardening parameters only [3]. Whereas in the mixed method of [6], [18], the stresses and hardening parameters are approximated by piecewise constant functions and the displacements by piecewise linear functions, we employ piecewise linear functions for both the stresses and the hardening parameters. The stress approximations consist of Watwood-Hartz equilibriated triangular elements [8]. The finite element method will produce approximations to the stresses successively at a finite number of time levels. At each time level one has to solve a constrained nonlinear optimization problem. We also discuss the Lagrange multipliers method with slack variables [9], [7], [21] for solving this problem. With a particular choice of finite element spaces, the optimization problem is solvable.

Numerical tests for the method proposed were performed for thin perforated strips of the strain-hardening material subjected to the uniform tension. The stress applied was increased monotonically from the elastic region of loading to values producing an impending plastic flow. The numerical results are in a good agreement with the experiment [10].

Let Ω be a polyhedral bounded domain in R^n, $n = 2, 3$, $x = (x_1, \ldots, x_n)$ a Cartesian coordinate system. Denote by $I = [0, T]$, $0 < T < \infty$, a fixed interval of time. Let R_σ be the space of symmetric $n \times n$ matrices (stress tensors). A repeated index implies summation over the range 1, ..., n.

Assume that a yield function $f: R_\sigma \to R$ is given, which is convex, continuous in R_σ, continuously differentiable in $R_\sigma - Q$, where Q is a subspace of dimension one, and

(2.1) $$f(\lambda\sigma) = |\lambda| f(\sigma), \quad \forall \lambda \in R, \quad \forall \sigma \in R_\sigma.$$

Note that such function satisfies also the condition

(2.2) $$\left|\frac{\partial f}{\partial \sigma_{ij}}\right| < C, \quad i, j = 1, \ldots, n, \quad C = \text{constant}, \quad \forall \sigma \in R_\sigma - Q.$$

For instance, we can employ the von Mises yield function $f(\sigma) = (\sigma^D_{ij}\sigma^D_{ij})^{1/2}$.

Let us introduce the following notations:

$$\|\tau\|_{R_\sigma} = (\tau_{ij}\tau_{ij})^{1/2},$$

$$S = \{\tau: \Omega \to R_\sigma, \ \tau_{ij} \in L^2(\Omega), \ \forall i, j\},$$

$$\|\tau\|_S = \left(\int_\Omega \|\tau\|_R^2 \, dx\right)^{1/2},$$

$$H = S \times L^2(\Omega).$$

Let

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_\sigma,$$

$$\Gamma_u \cap \Gamma_\sigma = \emptyset$$

where Γ_u and Γ_σ are either empty or open in $\partial\Omega$.

Assume that a (reference) body force vector $F^0 \in [C(\Omega)]^n$ and a (reference) surface traction vector $g^0 \in [L^2(\Gamma_\sigma)]^n$ are given. If $\Gamma_u = \emptyset$, the total equilibrium conditions for F^0, g^0 are satisfied.

Let the actual body forces and surface tractions be

$$F(t, x) = \gamma(t)\, F^0(x) \quad \text{in} \quad I \times \Omega,$$

$$g(t, x) = \gamma(t)\, g^0(x) \quad \text{on} \quad I \times \Gamma_\sigma.$$

Here $\gamma: I \to R$ is a non-negative function from $C^2(I)$ such that

(2.3) $$\exists t_1 > 0, \quad \gamma(t) = 0, \quad \forall t \in [0, t_1],$$

(2.4) $$\gamma(t) \in [\gamma(t_{n-1}), \gamma[t_n)] \quad \text{if} \quad \gamma(t_{n-1}) \leqq \gamma(t_n),$$

$$\gamma(t) \in [\gamma(t_n), \gamma(t_{n-1})] \quad \text{if} \quad \gamma(t_n) \leqq \gamma(t_{n-1})$$

holds in any subinterval $I_n = [t_{n-1}, t_n]$ of all time discretizations which will be considered in the following.

For any $t \in I$ we introduce the set of statically admissible stress tensors

$$E(t) = E(F(t), g(t)) = \left\{\sigma \in S, \int_\Omega \sigma_{ij} e_{ij}(v)\,\mathrm{d}x = \int_\Omega F_i(t)\, v_i\,\mathrm{d}x + \int_{\Gamma_\sigma} g_i(t)\, v_i\,\mathrm{d}s\,, \quad \forall v \in V\right\}$$

where

$$V = \{v \in [H^1(\Omega)]^n\,, \quad v = 0 \text{ on } \Gamma_u\}\,,$$

$$e_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Let us define

$$\mathscr{F}(\tau, \alpha) = f(\tau) - \alpha\,,$$

$$B = \{(\tau, \alpha) \in R_\sigma \times R,\ \mathscr{F}(\tau, \alpha) \leqq 0\}\,,$$

$$P = \{(\tau, \alpha) \in H,\ (\tau(x), \alpha(x)) \in B \text{ a.e. in } \Omega\}\,,$$

$$K(t) = (E(t) \times L^2(\Omega)) \cap P\,, \quad t \in I\,.$$

Let the elasticity coefficients $A_{ijkl} \in L^\infty(\Omega)$ be given $(i, j, k, l = 1, \ldots, n)$ such that

$$A_{ijkl} = A_{jikl} = A_{klij} \quad \text{a.e. in } \Omega$$

and $\exists c_0 > 0$ such that

$$A_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqq c_0 \varepsilon_{ij}\varepsilon_{ij}\,, \quad \forall \varepsilon \in R_\sigma$$

holds a.e. in Ω.

Moreover, let positive constants $\varkappa \in R$ and $\alpha_0 \in R$ be given.

We introduce the following bilinear forms for $\hat\sigma, \hat\tau \in H$, $\hat\sigma = (\sigma, \alpha)$, $\hat\tau = (\tau, \beta)$:

$$\langle\hat\sigma, \hat\tau\rangle = \sigma_{ij}\tau_{ij} + \alpha\beta\,, \quad |\hat\tau| = \langle\hat\tau, \hat\tau\rangle^{1/2}\,,$$

$$(\hat\sigma, \hat\tau)_0 = \int_\Omega \langle\hat\sigma, \hat\tau\rangle\,\mathrm{d}x\,, \quad \|\hat\sigma\| = (\hat\sigma, \hat\sigma)_0^{1/2}\,,$$

$$\{\hat\sigma, \hat\tau\} = \int_\Omega A_{ijkl}\sigma_{ij}\tau_{kl}\,\mathrm{d}x + \varkappa \int_\Omega \alpha\beta\,\mathrm{d}x\,,$$

$$|||\hat\sigma||| = \{\hat\sigma, \hat\sigma\}^{1/2}\,.$$

Notice that the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent. Denote by $\|\cdot\|_{0,\Omega}$ the norm in $L^2(\Omega)$.

Let $C_0^1(I, S)$ be the space of continuously differentiable mappings $\tau: I \to S$ such that $\tau(0) = 0$. Let $H_0^1(I, S)$ be the closure of $C_0^1(I, S)$ with respect to the norm

$$\left(\int_0^T \left\|\frac{\mathrm{d}\tau}{\mathrm{d}t}\right\|_S^2 \mathrm{d}t\right)^{1/2}.$$

Similarly, let $H^1(I, L^2)$ be the closure of $C^1(I, L^2(\Omega))$ with respect to the norm

$$\left(\int_0^T \left(\|\beta\|_{0,\Omega}^2 + \left\|\frac{\mathrm{d}\beta}{\mathrm{d}t}\right\|_{0,\Omega}^2\right) \mathrm{d}t\right)^{1/2}.$$

Definition 2.1. *A weak solution of the plasticity problem with strain-hardening is a pair of functions*

$$\hat{\sigma} \equiv (\sigma, \alpha) \in H_0^1(I, S) \times H^1(I, L^2)$$

such that

$$\alpha(0) = \alpha_0 \,, \quad \hat{\sigma}(t) \in K(t)$$

and

$$\left\{\frac{\mathrm{d}\hat{\sigma}(t)}{\mathrm{d}t}, \hat{\tau} - \hat{\sigma}(t)\right\} \geqq 0 \,, \quad \forall \hat{\tau} \equiv (\tau, \alpha) \in K(t) \tag{2.5}$$

holds for a.e. $t \in I$.

Remark 2.1. The existence and uniqueness of a weak solution has been discussed in [2] for $\partial\Omega = \Gamma_\sigma$ and in [1] for $\partial\Omega = \Gamma_u$.

3. FINITE ELEMENT APPROXIMATIONS

In the present section, we will extend some results of Johnson [11] to the case of plasticity with strain-hardening, using also several procedures published in [1] and [6] and following the paper [3].

We shall use the following approximations of the set $E(t)$:

$$E_h(t) = \chi(t) + E_h^0 \,, \quad 0 < h \leqq h_0 < \infty \tag{3.1}$$

where $\chi \in H_0^1(I, S)$ is a fixed stress field such that $\chi(t) \in E(t)$ a.e. in I and $E_h^0 \subset E(0, 0)$ is a finite-dimensional subspace of self-equilibriated stress fields. Then $E_h(t) \subset E(t)$.

Let $V_h \subset L^2(\Omega)$ be a finite-dimensional subspace, an approximation of $L^2(\Omega)$. Assume that V_h contains constant functions.

Define

$$K_h(t) = (E_h(t) \times V_h) \cap P$$

so that $K_h(t) \subset K(t)$.

We introduce a discretization of the time interval as follows: Let N be a positive integer, $k = T/N$, $t_n = nk$, $n = 0, 1, \ldots, N$, $I_n = [t_{n-1}, t_n]$, $\hat{\tau}^n = \hat{\tau}(t_n)$, $\partial\hat{\tau}^n = (\hat{\tau}^n - \hat{\tau}^{n-1})/k$.

We define the following approximate problem of (2.5):

Definition 3.1. *Find a* $\hat{\sigma}_{hk}^n \in K_h(t_n)$ *such that*

$$\{\partial\hat{\sigma}_{hk}^n, \hat{\tau} - \hat{\sigma}_{hk}^n\} \geqq 0 \,, \quad \forall \hat{\tau} \in K_h(t_n) \,, \quad n = 1, \ldots, N \,, \tag{3.2}$$

$$\hat{\sigma}_{hk}^0 = (0, \alpha_0) \,.$$

Remark 3.1. Since $\hat{\sigma}_{hk}^n$ minimizes the strictly convex functional

$$\tfrac{1}{2}|||\hat{\sigma}|||^2 - \{\hat{\sigma}, \hat{\sigma}^{n-1}\} \tag{3.3}$$

over the closed convex set $K_h(t)$, there exists a unique $\hat{\sigma}_{hk}^n$ provided $K_h(t_n) \neq \emptyset$.

The convergence of the finite element approximations is proved in the following theorem:

Theorem 3.1. *Let us denote*

$$\varepsilon(h, k) = \inf_{\hat{\tau} \in \mathscr{K}} \|\hat{\sigma} - \hat{\tau}\|_{l^2(H)}$$

where

$$\mathscr{K} = \{\hat{\tau} = (\hat{\tau}^1, \ldots, \hat{\tau}^N), \quad \hat{\tau}^n \in K_h(t_n), \quad n = 1, \ldots, N\},$$

$$\|q\|_{l^2(H)} = (\sum_{n=1}^{N} k\|q^n\|^2)^{1/2}, \quad q = (q^1, \ldots, q^N), \quad q^n \in H.$$

Assume that if $\Gamma_\sigma \neq \emptyset$, then there exists

$$\chi^0 \in [L^\infty(\Omega)]^{n^2} \cap E(F^0, g^0).$$

Then there exist such positive constants C and k_0 that for $k \leqq k_0$,

$$\max_{n=1,\ldots,N} \|\hat{\sigma}^n - \hat{\sigma}^n_{hk}\| \leqq C(\sqrt{(\varepsilon(h, k))} + \sqrt{k}). \tag{3.4}$$

Proof. See [3], Th. 2.1.

Remark 3.2. Construction of a fixed stress field χ^0: Let F^0 be continuous in $\bar{\Omega}$. Then there exists

$$\chi^1 \in S \cap [L^\infty(\Omega)]^{n^2}$$

such that

$$\operatorname{div} \chi^1 = -F^0 \text{ in } \Omega$$

(χ^1 can be obtained by integration).

Let the vector-function $g^0 - \chi^1 . \nu$, where ν denotes the unit outward normal, be piecewise linear on Γ_σ with respect to a simplicial partition of Γ_σ. Then there exists a simplicial partition of Ω and $\chi^2 \in E^0_h$, where E^0_h consists of piecewise linear stress fields such that

$$\chi^2 . \nu = g^0 - \chi^1 . \nu .$$

Setting $\chi^0 = \chi^1 + \chi^2$, we obtain

$$\chi^0 \in [L^\infty(\Omega)]^{n^2},$$
$$\operatorname{div} \chi^0 = -F^0 \quad \text{in} \quad \Omega,$$
$$\chi^0 . \nu = g^0 \quad \text{on} \quad \Gamma_\sigma,$$

which implies $\chi^0 \in E(F^0, g^0)$.

4. EQUILIBRIUM FINITE ELEMENT MODEL IN TWO-DIMENSIONAL PROBLEMS

In the following we shall consider the problems in R^2 and evaluate the quantity $\varepsilon(h, k)$ introduced in Theorem 3.1 for a piecewise linear finite element model assuming a certain regularity of the exact solution $\hat{\sigma}$.

We assume that the reference body forces F^0 are constant and the reference surface tractions g^0 are piecewise linear on Γ_σ.

Let us consider a regular family $\{\mathcal{T}_h\}$, $0 < h \leqq h_0$, of triangulations of the domain Ω (i.e., there exists a positive ϑ_0 such that all angles in all triangulations are not less than ϑ_0). Let h denote the maximal length of all sides in $\mathcal{T}_h$.

We employ the self-equilibriated triangular block-elements of Watwood and Hartz [8]. The model consists of triangular block-elements, each of them being generated by connecting the vertices of the triangle $\mathbf{K}$ with its centre of gravity. On each subtriangle $\mathbf{K}_i$ three linear functions – components of a self-equilibriated stress tensor – are defined. The stress vector has to be continuous when crossing any common boundary between the subtriangles.

Note that under the assumptions on F^0 and g^0, the auxiliary function χ^0 can be chosen piecewise linear with respect to the triangulation $\mathcal{T}_{h_0}$. Then $\chi(t_n) = \gamma(t_n)\,\chi^0$ is piecewise linear as well. In the sequel, we assume that each $\mathcal{T}_h$ of the family $\{\mathcal{T}_h\}$ of the triangulations is generated by a regular refinement of $\mathcal{T}_{h_0}$.

Let us define

$$\begin{aligned}
M(\mathbf{K}) &= \{\tau_{11} = \beta_1 + \beta_2 x_1 + \beta_3 x_2,\ \tau_{22} = \beta_4 + \beta_5 x_1 + \beta_6 x_2,\\
&\qquad \tau_{12} = \tau_{21} = \beta_7 - \beta_6 x_1 - \beta_2 x_2,\ \beta \in R^7\}\ ;\\
N(\mathbf{K}) &= \{\tau = (\tau^1, \tau^2, \tau^3),\ \tau^i = \tau|_{K_i} \in M(\mathbf{K}),\ T(\tau^i) + T(\tau^{i+1}) = 0\,,\\
&\qquad T(\tau) = \{T_1(\tau), T_2(\tau)\}\,,\quad T_i(\tau) = \tau_{ij}\nu_j\}\ ;\\
N_h(\Omega) &= \{\tau \in S,\ \tau|_K \in N(\mathbf{K}),\ \forall \mathbf{K} \in \mathcal{T}_h,\ T(\tau)|_K + T(\tau)|_{K'} = 0 \text{ on } \bar{K} \cap \bar{K}'\}\ ;\\
E_h^0 &= N_h(\Omega) \cap E(0,0) = \{\tau \in N_h(\Omega),\ \tau\,.\,\nu = 0 \text{ on } \Gamma_\sigma\}\,.
\end{aligned}$$

An a priori error estimate is presented in the following theorem.

Theorem 4.1. *Let the solution $\hat{\sigma} = (\sigma, \alpha)$ be such that for $\sigma_0 = \sigma - \chi$ and α and for any $\mathbf{K}^0 \in \mathcal{T}_{h_0}$,*

$$\sup_{t \in I} \|\sigma_0(t)\|_{[C^2(K^0)]^4} \equiv \|\sigma_0\|_{L^\infty(I,[C^2(K^0)]^4)} < \infty\,,$$

$$\sup_{t \in I} \|\alpha(t)\|_{H^2(K_i{}^0)} \equiv \|\alpha\|_{L^\infty(I,H^2(K_i{}^0))} < \infty\,,\quad i = 1, 2, 3\,.$$

Then there exist constants C and k_0 such that

$$\max_{n=1,\ldots,N} \|\hat{\sigma}^n - \hat{\sigma}^n_{hk}\| \leqq C(h + \sqrt{k}) \tag{4.1}$$

holds for $k \leqq k_0$, $h \leqq h_0$.

Proof. see [3], Th. 3.2.

Remark 4.1. For $\tau_0^n \in E_h^0$ we obtain

$$(\chi^n + \tau_0^n, \beta^n) \in P \Leftrightarrow \beta^n(a_j) \geqq f(\chi^n + \tau_0^n)(a_j) \tag{4.2}$$

at all vertices $a_j \in \mathbf{K}_i \subset \mathbf{K}$ of all triangles $\mathbf{K} \in \mathcal{T}_h$.

Thus we have also nonlinear constraints for the parametrs of β^n and τ_0^n. In the case of von Mises' yield function f these constraints are quadratic.

Remark 4.2. (An algorithm for solving the approximate problem (3.2).) At each time level we have to minimize the quadratic functional (3.3) with the nonlinear constraints (4.2) and with linear constraints (equations) which guarantee the continuity of the stress vectors across the interelement boundaries. We employ the initial values

$$\sigma(0) = 0\,,$$
$$\alpha(0) = \alpha_0$$

and for the initial values of the next time step we take the values calculated at the previous one.
The choice of a suitable algorithm of nonlinear programming is discussed in the next chapter.

Remark 4.3. In three-dimensional problems, Theorem 4.1 can be proved if we use tetrahedral block-elements [12].

Remark 4.4. The convergence of the approximations can be proved without any regularity assumptions (see [3], Th. 4.1).

5. NONLINEAR OPTIMIZATION PROBLEM

Using Remark 3.1, for each time level we can define the following nonlinear optimization problem for the plasticity with strain-hardening:

Definition 5.1. For each time level t_n find a minimum of the functional

$$J(S) = \sum_{p=1}^{N_e} J_p(S_1', S^\alpha) \tag{5.1}$$

with linear constraints (equations)

$$\underset{N_A \times N_S}{[A]}\ \underset{N_S \times 1}{\{S\}} = \underset{N_A \times 1}{\{R\}} \tag{5.2}$$

and with nonlinear constraints (inequalities, cf. (4.2), $\alpha(S^\alpha) \geqq 0$)

$$\underset{9N_e \times 1}{\{\alpha(S^\alpha)\}^2} - \underset{9N_e \times 1}{\{f(S_1')\}^2} \geqq 0 \tag{5.3}$$

where

$S = (S_1', S^\alpha)^{\mathrm{T}}$ are parameters,

$$J_p(S_1', S^\alpha) = \frac{1}{2}\left\{\int_K A_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}x + \varkappa\int_K \alpha^2\,\mathrm{d}x\right\} - \left\{\int_K A_{ijkl}\sigma_{ij}\sigma_{kl}^{n-1}\,\mathrm{d}x + \varkappa\int_K \alpha\alpha^{n-1}\,\mathrm{d}x\right\}$$

— the functional defined on one block-element,

$[A]$ — the matrix of linear constraints,

$\{R\}$ – the vector of external applied stresses,
$\alpha(S^{\alpha})$ – the linear approximation of the hardening parameter,
$f(S'_1)$ – the approximation of the yield function,
N_S – the dimension of the problem,
N_A – the number of the linear constraints,
N_e – the number of the elements in the model.

The constrained nonlinear programming problem (5.1)–(5.3) is for each t_n equivalent to the problem of finding the saddle-point of the Lagrangian with slack variables v and with multipliers λ, $\bar{\lambda}$, [7], [13]–[15],

$$L(S, v, \lambda, \bar{\lambda}) = J(S) + \{\lambda\}^{\mathrm{T}} \{[A]\{S'_1\} - \{R\}\} + \\ + \{\bar{\lambda}\}^{\mathrm{T}} \{\{\alpha^2\} - \{f^2\} - \{v^2\}\} . \tag{5.4}$$

In this form the constrained problem (5.1)–(5.3) is formulated as an unconstrained one with S, v, λ, $\bar{\lambda}$ variables. Many algorithms have been frequently investigated in the mathematical programming theory, and their global convergence theorem was proved [9], [13]–[15], [17], [19], [20].

In this paper we have used the Newton-like type algorithm [9], [13]–[15], [19]. Consequently, the stationary point of (5.4) can be found by the nonlinear equations

$$\frac{\partial L}{\partial S} = 0 , \quad \frac{\partial L}{\partial v} = 0 , \quad \frac{\partial L}{\partial \lambda} = 0 , \quad \frac{\partial L}{\partial \bar{\lambda}} = 0 . \tag{5.5}$$

Using the Newton-Raphson method we have

$$x^{k+1} = x^k - [\nabla^2 L(x^k)]^{-1} \nabla L(x^k) \tag{5.6}$$

where

$x^{k+1} = (S^{k+1}, v^{k+1}, \lambda^{k+1}, \bar{\lambda}^{k+1})^{\mathrm{T}}$ is a solution for the $(k+1)$st step,

$\nabla^2 L(x^k)$ – the Hessian of the Lagrangian at the point x^k.

The problem (5.6) can be written in the form of the linear equations

$$\nabla L(x^k) + \nabla^2 L(x^k)(x^{k+1} - x^k) = 0 . \tag{5.7}$$

The choice of an efficient algorithm for the solution of large linear systems (5.7) is of great practical importance. We have used the frontal algorithm solver for the semidefinite matrix [16]. Approximately four to seven iterations were required for finding the solution of (5.5).

Remark 5.1. From the physical point of view and using (5.5) we see that the solution $\bar{\lambda} = 0$, $v \neq 0$ satisfies in the elastic region and $\bar{\lambda} \neq 0$, $v = 0$ in the plastic region.

6. NUMERICAL RESULTS

Numerical tests of the method proposed were performed for thin perforated strips of a strain-hardening material subjected to a uniform tension [10] (Fig. 1).

The results have been compared with the experiment. The material and geometrical parameters considered were $E = 68\,700$ [MPa], Poisson's ratio $\nu = 0{\cdot}2$, $\varkappa =$

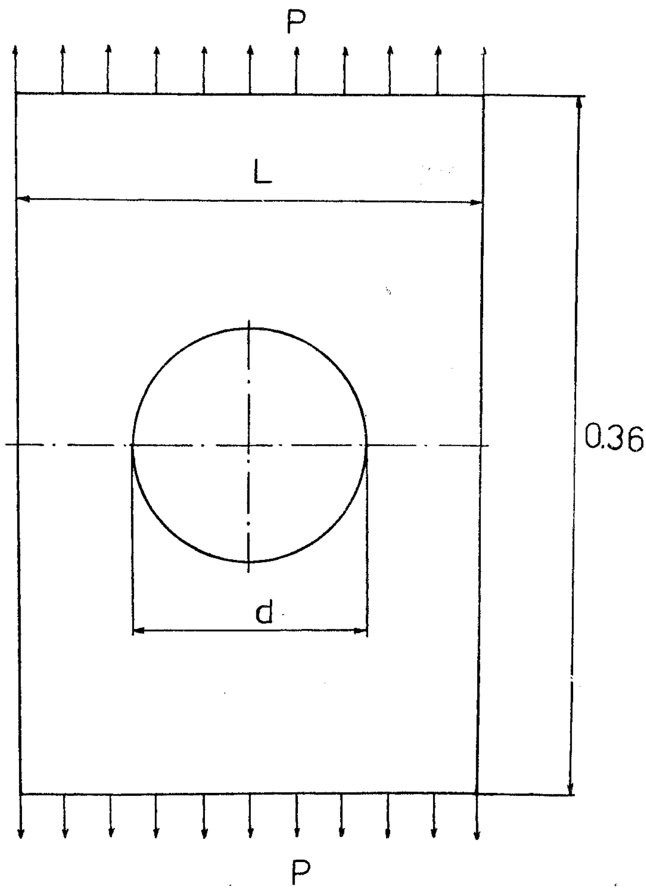

Fig. 1. Perforated tension strip.

$= 0{\cdot}555_{10}{-}3$ [MPa]$^{-1}$, $\alpha_0 = 238$ [MPa], $t = 0{\cdot}003$ [m], $L = 0{\cdot}2$ [m], $d = 0{\cdot}1$ [m]. The applied stress was increased monotonically from the elastic region of loading ($P = 104$ [MPa]) to values producing an impending plastic flow ($P = 117, 130, 144, 158, 172$ [MPa]). Due to the symmetry of the problem we can restrict the solution to a quarter of the strip only. The mesh is displayed in Fig. 2. It was represented by 28 triangular block-elements. The total number of parameters used for nonlinear optimization problem was 1204 with the frontwidth 84.

The average computer time required for the solution of the system of linear equations was 2·43 min. CPU (ICL 2958 computer). The total time was about 60 min. The ultimate elastic stress was calculated to 117 [MPa]. The measured value of this stress was 109 [MPa]. The first yielding appeared in the element at the root of the notch (node number 3). The propagation of the elastic-plastic points is shown in Fig.3. The figure includes the experimental results due to Theocaris and Marketos [10]. The results obtained by the present method are in good agreement with the experiment. It may be seen that a slight difference results from the coarse mesh used and the difference of load and boundary conditions between the calculation and the experiment.

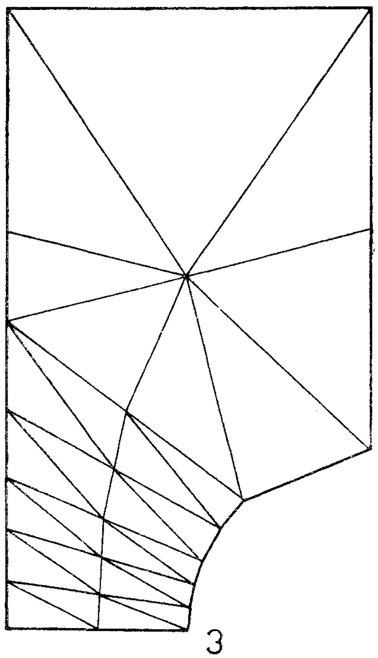

Fig. 2. Finite element mesh.

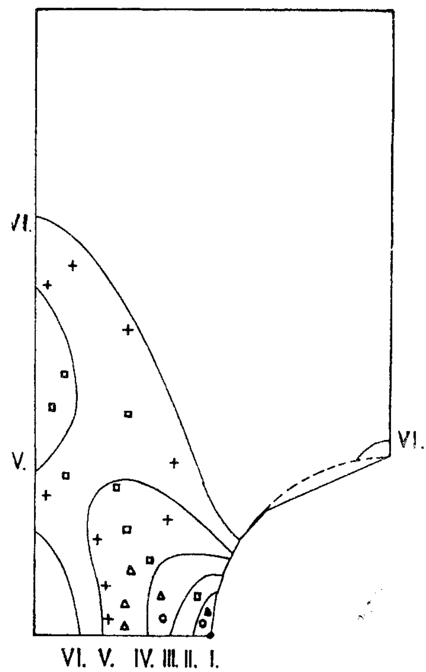

Fig. 3. Progressive yielding of perforated strip (numerical and experimental). Values at the centroid of elements: ● – 117 MPa, ○ – 130 MPa, △ – 144 MPa, □ – 158 MPa, + – 172 MPa, — – experimental.

7. CONCLUSION

The theory of variational inequalities applied to plasticity provides a firm basis for the theory and for the numerical algorithms. In the present paper the equilibrium finite element model has been used.

This new analysis has opened up great possibilities for better understanding of nonlinear models of the plastic bodies used in practice.

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Souhrn

VARIAČNÍ NEROVNICE V PLASTICITĚ SE ZPEVNĚNÍM DEFORMACÍ – UŽITÍ ROVNOVÁŽNÉHO MODELU METODY KONEČNÝCH PRVKŮ

ZDENĚK KESTŘÁNEK

V článku je aplikována přírůstková metoda konečných prvků k nalezení numerického řešení problému plasticity se zpevněním deformací. K aproximaci pole napětí je užito rovnovážných trojúhelníkových prvků s lineárními funkcemi podle Watwooda a Hartze. Pole parametru zpevnění deformací je uvažováno rovněž po částech lineární. K řešení výsledného nelineárního optimalizačního problému s vazbami je užito metody Lagrangeových multiplikátorů s přídatnými proměnnými. Získané numerické výsledky jsou porovnány s experimentem.

Резюме

ВАРИАЦИОННЫЕ НЕРАВЕНСТВА В ПЛАСТИЧНОСТИ С МЕХАНИЧЕСКИМ УПРОЧНЕНИЕМ – ПРИЛОЖЕНИЕ РАВНОВЕСНОЙ МОДЕЛИ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ

ZDENĚK KESTŘÁNEK

В статье применяется метод приращения конечных элементов к определению численного решения проблемы пластичности с механическим упрочнением. Для аппроксимации поля напряжений используются равновесные треугольные элементы с линейными функциями по Вотвуду и Харцу. Поле параметра механического упрочнения также считается кусочно линейным. Для решения результирующей нелинейной проблемы оптимизации с органичениями использован метод множителей Лагранжа с дополнительными переменными. Проведено сравнение полученных численных результатов с экспериментом.

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