# VARIATIONAL ITERATION METHOD TO SOLVE MOVING BOUNDARY PROBLEM WITH TEMPERATURE DEPENDENT PHYSICAL PROPERTIES 

by

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#### Abstract

In this paper, variational iteration method is used to solve a moving boundary problem arising during melting or freezing of a semi infinite region when physical properties (thermal conductivity and specific heat) of the two regions are temperature dependent. The result is compared with result obtained by exact method (when thermal conductivity and specific heat in two regions are temperature independent) and semi analytical method (when thermal conductivity and specific heat are temperature dependent) and are in good agreement. We obtain the solution in the form of continuous functions. The method performs extremely well in terms of efficiency and simplicity and effective for solving the moving boundary problems.


Key words: moving boundary problem, phase change process, temperature
distribution, thermal conductivity and specific heat, freezing,
Stefan number, variational iteration method

## Introduction

Moving boundary problems involving heat phenomena (melting or freezing) occur in numerous important areas of science, engineering, and industry [1-3] and have been of special interest due to its non-linearity and unknown position of moving interface [4-7]. The exact solution of these problems is limited which are well documented in the literature [8, 9]. During a melting or freezing process, the special situations which arise are temperature dependent thermal conductivity and specific heat of materials in two regions. The resulting moving boundary problem can not be solved exactly. Thus, semi analytical [10-12], and numerical methods [13, 14] have been used to solve them. Semi analytical method such as heat balance integral method [15, 16], variable space grid method [17], Galerkin's method [18], non-integral method [19, 20], regular perturbation method [21, 22], and strained co--ordinate method [23] exist in the literature. Most of these methods have difficulty in accommodating time dependent applied surface condition, choice of a suitable approximate profile for the transfer potentials of heat and their applicability is restricted to short times.

[^0]$\mathrm{He}[24,25]$ first proposed the variational iteration method (VIM) to solve non-linear differential and integral equations. This method was successfully applied to solve initial spherical growth during equiaxed solidification by Yao [26], reliable treatment of heat equation with Ill-defined initial data by Chun [27]. In 2009, Cao et al. [28] solved the non-linear Ill-posed operator equations by homotopy perturbation technique. Recently, He et al. [29] discussed new algorithm of the VIM. This method is applicable wide range of non-linear problem. Many authors [30-34] applied this method in various different problems of non-linear differential and integral equations. Slota [35] used this method in direct and inverse onephase Stefan problem.

To the best of author knowledge solution of the two phase moving boundary problem with temperature dependent thermal conductivity and specific heat have not been solved yet using the VIM. In this paper, the proposed method is used to obtain a semi analytical solution to a two phase moving boundary problem when thermal conductivity and specific heat of the two regions are temperature dependent.

## Formulation of the problem

A semi-infinite medium consisting of a solid/melt is initially at a temperature $T_{\mathrm{i}}$ which is slightly below/above the melting/freezing temperature of the solid/melt, $T_{\mathrm{m}} / T_{\mathrm{f}}$. At time $t=0$, the surface $x=0$ is subjected to a temperature $T_{0}$. As a result, melting/freezing starts at the surface $x=0$ and the liquid solid/solid liquid interface $x=s(t)$ moves in the positive $x$-direction. Temperature in the two phases is unknown. Hence the problem is a two phase problem. In given problem, we take the freezing process only. The dynamics of freezing can be described by the following equations. The basic equation for phase 1 and phase 2 are, respectively:

$$
\begin{gather*}
{\left[\rho_{1} c_{1}\left(T_{1}\right)\right] \frac{\partial T_{1}}{\partial t}=\frac{\partial}{\partial x}\left[k_{1}\left(T_{1}\right) \frac{\partial T_{1}}{\partial x}\right], \quad 0<x<s(t), t>0}  \tag{1}\\
{\left[\rho_{2} c_{2}\left(T_{2}\right)\right] \frac{\partial T_{2}}{\partial t}+\left(\rho_{1}-\rho_{2}\right) c_{2}\left(T_{2}\right) \frac{\mathrm{d} s}{\mathrm{~d} t} \frac{\partial T_{2}}{\partial x}=\frac{\partial}{\partial x}\left[k_{2}\left(T_{2}\right) \frac{\partial T_{2}}{\partial x}\right], \quad x>s(t), t>0} \tag{2}
\end{gather*}
$$

The associated initial and boundary conditions are:

$$
\begin{align*}
T_{1}(x, 0) & =T_{i}  \tag{3}\\
T_{1}(0, t) & =T_{0} \tag{4}
\end{align*}
$$

The condition of temperature continuity and the Stefan condition on the moving interface:

$$
\begin{equation*}
T_{1}[s(t), t]=T_{2}[s(t), t]=T_{\mathrm{f}} \tag{5}
\end{equation*}
$$

and interface equation:

$$
\begin{equation*}
k_{1}\left(T_{1}\right) \frac{\partial T_{1}}{\partial x}-k_{2}\left(T_{2}\right) \frac{\partial T_{2}}{\partial x}=\rho_{1} H \frac{\mathrm{~d} s}{\mathrm{~d} t} \quad \text { at } \quad x=s(t) \tag{6}
\end{equation*}
$$

As $x \rightarrow \infty$ the material is at initial temperature i. e.:

$$
\begin{equation*}
s(0)=0 \tag{7}
\end{equation*}
$$

$$
\begin{equation*}
\lim _{x \rightarrow \infty} T_{2}(x, t)=T_{i} \tag{8}
\end{equation*}
$$

Introducing the dimensionless variable and similarity criteria:

$$
\begin{equation*}
\theta_{\mathrm{k}}=\frac{T_{\mathrm{k}}-T_{0}}{T_{i}-T_{0}} \quad k=1,2, f \tag{9}
\end{equation*}
$$

$$
\left.\begin{array}{l}
a_{01}=\frac{k_{01}}{c_{01} \rho_{1}} \\
a_{02}=\frac{k_{02}}{c_{02} \rho_{2}} \\
a_{12}=\frac{a_{02}}{a_{01}}, \rho_{12}=\frac{\rho_{2}}{\rho_{1}} \text { and } k_{12}=\frac{k_{02}}{k_{01}}  \tag{10}\\
\lambda=\frac{s(t)}{2 \sqrt{a_{01} t}},(\lambda \text { is a calculated parameter }) \\
\operatorname{Ste}(t)=\frac{g_{1}\left(\theta_{\mathrm{f}}\right) c_{01}\left|T_{\mathrm{i}}-T_{0}\right|}{H}
\end{array}\right\}
$$

and using transformation:

$$
\begin{equation*}
y=\frac{x}{2 \sqrt{a_{01} t}} \tag{11}
\end{equation*}
$$

the system of eqs. (1) to (8) reduces to the following non-dimensional form:

$$
\begin{gather*}
\frac{\mathrm{d}}{\mathrm{~d} y}\left[g_{1}\left(\theta_{1}\right) \frac{\mathrm{d} \theta_{1}}{\mathrm{~d} y}\right]+2 y f_{1}\left(\theta_{1}\right) \frac{\mathrm{d} \theta_{1}}{\mathrm{~d} y}=0  \tag{12}\\
a_{12} \frac{\mathrm{~d}}{\mathrm{~d} y}\left[g_{2}\left(\theta_{2}\right) \frac{\mathrm{d} \theta_{2}}{\mathrm{~d} y}\right]+2\left[y+\left(\frac{1}{\rho_{12}}-1\right) \lambda\right] f_{2}\left(\theta_{2}\right) \frac{\mathrm{d} \theta_{2}}{\mathrm{~d} y}=0  \tag{13}\\
\theta_{1}(0)=0  \tag{14}\\
\theta_{1}(\lambda)=\theta_{2}(\lambda)=\theta_{\mathrm{f}} \tag{15}
\end{gather*}
$$

Interface equation is:

$$
\begin{gather*}
\frac{\mathrm{d} \theta_{1}}{\mathrm{~d} y}-k_{12} \frac{g_{2}\left(\theta_{\mathrm{f}}\right)}{g_{1}\left(\theta_{\mathrm{f}}\right)} \frac{\mathrm{d} \theta_{2}}{\mathrm{~d} y}=\frac{2 \lambda}{\text { Ste }} \text { at } y=\lambda  \tag{16}\\
\lim _{y \rightarrow \infty} \theta_{2}(y)=1 \tag{17}
\end{gather*}
$$

The thermal conductivity and specific heat in two regions varies with temperature and is assumed to be:

$$
\left.\begin{array}{rl}
k_{1} & =k_{01} g_{1}\left(\theta_{1}\right) \\
k_{2} & =k_{02} g_{2}\left(\theta_{2}\right) \\
c_{1} & =c_{01} f_{1}\left(\theta_{1}\right)  \tag{18}\\
c_{2} & =c_{02} f_{2}\left(\theta_{2}\right)
\end{array}\right\}
$$

## Variational iteration method

To clarify the basic ideas of He's VIM, we consider the differential equation:

$$
\begin{equation*}
L u+N u=g(t) \tag{19}
\end{equation*}
$$

where $L$ and $N$ are linear and non-linear operators, respectively, and $g(t)$ is the source of inhomogeneous term. A correction functional for eq. (19) can be written as:

$$
\begin{equation*}
u_{n+1}(t)=u_{n}(t)+\int_{0}^{t} \lambda\left[L u_{n}(\xi)+N \tilde{u}_{n}(\xi)-g(\xi)\right] \mathrm{d} \xi, \quad n \geq 0 \tag{20}
\end{equation*}
$$

where $\lambda$ is a general Lagrange multiplier, which can be identified optimally via the variational theory. The subscript $n$ indicate the $n^{\text {th }}$ approximation and $\tilde{u}_{n}$ is a restricted variation, which means $\delta \tilde{u}_{n}=0$. Therefore, we first determine the Lagrange multiplier $\lambda$ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(t), n \square 0$ of the solution $u(t)$ will be readily obtained upon using the Lagrange multiplier obtained and by using selective function $u_{0}(t)$.

Consequently, the exact solution may be obtained by using:

$$
\begin{equation*}
u(t)=\lim _{n \rightarrow \infty} u_{n}(t) \tag{21}
\end{equation*}
$$

## Solution of the problem

In order to simplify the solution of the problem we have to consider two cases:
Case 1: When thermal conductivity and specific heat varies exponentially i.e.

$$
\left.\begin{array}{l}
f_{1}\left(\theta_{1}\right)=\mathrm{e}^{\alpha_{1} \theta_{1}}  \tag{22}\\
f_{2}\left(\theta_{2}\right)=\mathrm{e}^{\alpha_{2} \theta_{2}} \\
g_{1}\left(\theta_{1}\right)=\mathrm{e}^{\beta_{1} \theta_{1}} \\
g_{2}\left(\theta_{2}\right)=\mathrm{e}^{\beta_{2} \theta_{2}}
\end{array}\right\}
$$

We construct a correction functional, for eq. (12) restrict as follows:

$$
\begin{equation*}
\theta_{1, n+1}(y)=\theta_{1, n}(y)+\int_{0}^{y} \lambda(\xi)\left[\mathrm{e}^{\beta_{1} \theta_{1, n}(\xi)} \frac{\mathrm{d}^{2} \theta_{1, n}(\xi)}{\mathrm{d} \xi^{2}}+\mathrm{e}^{\beta_{1} \theta_{1, n}(\xi)} \beta_{1}\left(\frac{\mathrm{~d} \tilde{\theta}_{1, n}(\xi)}{\mathrm{d} \xi}\right)^{2}+2 \xi \mathrm{e}^{\alpha_{1} \theta_{1, n}(\xi)} \frac{\mathrm{d} \tilde{\theta}_{1, n}(\xi)}{\mathrm{d} \xi}\right] \mathrm{d} \xi \tag{23}
\end{equation*}
$$

taking variation both sides and using:

$$
\begin{equation*}
\delta \tilde{\theta}_{1, n}=0 \tag{24}
\end{equation*}
$$

$$
\begin{equation*}
\delta \theta_{1, n+1}(y)=\delta \theta_{1, n}(y)+\delta \int_{0}^{y} \lambda(\xi) \mathrm{e}^{\beta_{1} \theta_{1, n}(\xi)} \frac{\mathrm{d}^{2} \theta_{1, n}}{\mathrm{~d} \xi^{2}} \mathrm{~d} \xi \tag{25}
\end{equation*}
$$

To find the optimal value of $\lambda(\xi)$ we integrate the equation by parts and obtain stationary condition (expanding exponential series and restrict all $\theta_{1, n}(\xi)$ :

$$
\begin{equation*}
\left.\lambda^{\prime \prime}(\xi)\right|_{\xi=y}=0, \quad 1-\left.\lambda^{\prime}(\xi)\right|_{\xi=y}=0, \quad \lambda(\xi)_{\xi=y}=0 \tag{26}
\end{equation*}
$$

The Lagrangian multiplier can identified as solving above equation:

$$
\begin{equation*}
\lambda(\xi)=\xi-y \tag{27}
\end{equation*}
$$

Putting this value in eq (23), as a result, we obtain the iterative formula:

$$
\begin{gather*}
\theta_{1, n+1}(y)=\theta_{1, n}(y)+\int_{0}^{y}(\xi-y)\left[\mathrm{e}^{\beta_{1} \theta_{1, n}(\xi)} \frac{\mathrm{d}^{2} \theta_{1, n}(\xi)}{\mathrm{d} \xi^{2}}+\beta_{1} \mathrm{e}^{\beta_{1} \theta_{1, n}(\xi)}\left(\frac{\mathrm{d} \theta_{1, n}(\xi)}{\mathrm{d} \xi}\right)^{2}+\right. \\
\left.+2 \xi \mathrm{e}^{\alpha_{1} \theta_{1, n}(\xi)} \frac{\mathrm{d} \theta_{1, n}(\xi)}{\mathrm{d} \xi}\right] \mathrm{d} \xi \tag{28}
\end{gather*}
$$

Taking the initial approximation according to initial and boundary condition:

$$
\begin{equation*}
\theta_{1,0}(y)=\frac{\theta_{\mathrm{f}}}{\lambda} y \tag{29}
\end{equation*}
$$

using the variation iteration formula (28) we have:

$$
\begin{equation*}
\theta_{1,1}(y)=\frac{1}{\beta_{1}}-\frac{\mathrm{e}^{\frac{\beta_{1} y \theta_{\mathrm{f}}}{\lambda}}}{\beta_{1}}+\frac{2 y \theta_{\mathrm{f}}}{\lambda}-\frac{2 y \lambda}{\alpha_{1}^{2} \theta_{\mathrm{f}}}-\frac{2 \mathrm{e}^{\frac{\alpha_{1} y \theta_{\mathrm{f}}}{\lambda}} y \lambda}{\alpha_{1}^{2} \theta_{\mathrm{f}}}-\frac{4 \lambda^{2}}{\alpha_{1}^{3} \theta_{\mathrm{f}}^{2}}+\frac{4 \mathrm{e}^{\frac{\alpha_{1} y \theta_{\mathrm{f}}}{\lambda}} \lambda^{2}}{\alpha_{\mathrm{l}}^{3} \theta_{\mathrm{f}}^{2}} \tag{30}
\end{equation*}
$$

and so on.
Similarly we can find $\theta_{2}(y)$ for differential eq. (13), now we construct correctional functional:

$$
\begin{align*}
& \theta_{2, n+1}(y)=\theta_{2, n}(y)+\int_{0}^{y} \lambda(\xi)\left\{a_{12}{ }^{\beta \theta_{2} \theta_{n}(\xi)} \frac{\mathrm{d}^{2} \theta_{2, n}(\xi)}{\mathrm{d} \xi^{2}}+\beta_{2} a_{12} \mathrm{e}^{\beta_{2} \theta_{2 n}(\xi)}\left[\frac{\mathrm{d} \tilde{\theta}_{2, n}(\xi)}{\mathrm{d} \xi}\right]^{2}+\right. \\
&\left.+\left[2 \xi+2 \lambda\left(\frac{1}{\rho_{12}}-1\right)\right] \mathrm{e}^{\alpha_{2} \theta_{2 n}(\xi)} \frac{\mathrm{d} \tilde{\theta}_{2, n}(\xi)}{\mathrm{d} \xi}\right\} \mathrm{d} \xi \tag{31}
\end{align*}
$$

Its stationary condition (expanding exponential series and restrict all $\theta_{2, n}(\xi)$ ):

$$
\begin{equation*}
\lambda^{\prime \prime}(\xi)_{\xi=y}=0, \quad 1-\left.a_{12} \lambda^{\prime}(\xi)\right|_{\xi=y}=0,\left.\quad \lambda(\xi)\right|_{\xi=y}=0 \tag{32}
\end{equation*}
$$

The Langrangian multiplier obtained:

$$
\begin{equation*}
\lambda(\xi)=\frac{1}{a_{12}}(\xi-y) \tag{33}
\end{equation*}
$$

The following iteration formula becomes:

$$
\begin{align*}
& \theta_{2, n+1}(y)=\theta_{2, n}(y)+ \int_{0}^{y} \\
& \frac{1}{a_{12}}(\xi-y)\left\{a_{12} \mathrm{e}^{\beta_{2} \theta_{2 n}(\xi)} \frac{\mathrm{d}^{2} \theta_{2, n}(\xi)}{\mathrm{d} \xi^{2}}+\beta_{2} a_{12} \mathrm{e}^{\beta_{2} \theta_{2 n}(\xi)}\left(\frac{\mathrm{d} \theta_{2, n}(\xi)}{\mathrm{d} \xi}\right)^{2}+\right.  \tag{34}\\
&\left.+\left[2 \xi+2 \lambda\left(\frac{1}{\rho_{12}}-1\right)\right] \mathrm{e}^{\alpha_{2} \theta_{2 n}(\xi)} \frac{\mathrm{d} \theta_{2, n}}{\mathrm{~d} \xi}\right\} \mathrm{d} \xi
\end{align*}
$$

Now we start with initial approximation $\theta_{2,0}(y)=1+\left(\theta_{\mathrm{f}}-1\right) \mathrm{e}^{(\lambda-y)}$ and we get the first approximation:

$$
\begin{align*}
& \theta_{2,1}(y)=1-\mathrm{e}^{(\lambda-y)}+\mathrm{e}^{\beta_{2}+e^{\lambda} \beta_{2}\left(\theta_{\mathrm{f}}-1\right)+\lambda} y+\mathrm{e}^{(\lambda-y)} \theta_{\mathrm{f}}-\mathrm{e}^{\beta_{2}+e^{\lambda} \beta_{2}\left(\theta_{\mathrm{f}}-1\right)+\lambda} y \theta_{\mathrm{f}}- \\
&-\frac{1}{a_{12}} \int_{0}^{y}\left\{2 \mathrm{e}^{\alpha_{2}\left[1+\mathrm{e}^{(\lambda-\xi)}\left(\theta_{\mathrm{f}}-1\right)\right]+(\lambda-\xi)}\left(\theta_{\mathrm{f}}-1\right)(\xi-y)\left[\left(\frac{1}{\rho_{12}}-1\right) \lambda+\xi\right]\right\} \mathrm{d} \xi \tag{35}
\end{align*}
$$

Similarly, we find other approximation. After getting expression of $\theta_{1, n}$ and $\theta_{2, n}$ we have:

$$
\begin{align*}
& \theta_{1}(y)=\lim _{n \rightarrow \infty} \theta_{1, n}(y)  \tag{36}\\
& \theta_{2}(y)=\lim _{n \rightarrow \infty} \theta_{2, n}(y) \tag{37}
\end{align*}
$$

Next we consider,
Case 2: When thermal conductivity and specific heat varies linearly with temperature [36]:

$$
\left.\begin{array}{l}
f_{1}\left(\theta_{1}\right)=1+\alpha_{1} \theta_{1}  \tag{38}\\
f_{2}\left(\theta_{2}\right)=1+\alpha_{2} \theta_{2} \\
g_{1}\left(\theta_{1}\right)=1+\beta_{1} \theta_{1} \\
g_{2}\left(\theta_{2}\right)=1+\beta_{2} \theta_{2}
\end{array}\right\}
$$

Now we construct a correction functional, for eq. (12) restrict as:

$$
\begin{equation*}
\theta_{1, n+1}(y)=\theta_{1, n}(y)+\int_{0}^{y} \lambda(\xi)\left\{\left[1+\beta_{1} \tilde{\theta}_{1, n}(\xi)\right] \frac{\mathrm{d}^{2} \theta_{1, n}(\xi)}{\mathrm{d} \xi^{2}}+\beta_{1}\left(\frac{\mathrm{~d} \tilde{\theta}_{1, n}(\xi)}{\mathrm{d} \xi}\right)^{2}+2 \xi\left[1+\alpha_{1} \tilde{\theta}_{1}(\xi)\right] \frac{\mathrm{d} \tilde{\theta}_{1, n}(\xi)}{\mathrm{d} \xi}\right\} \mathrm{d} \xi \tag{39}
\end{equation*}
$$

Taking variation both sides and using $\delta \tilde{\theta}_{1, n}=0$ :

$$
\begin{equation*}
\delta \theta_{1, n+1}(y)=\delta \theta_{1, n}(y)+\delta \int_{0}^{y} \lambda(\xi) \frac{\mathrm{d}^{2} \theta_{1, n}}{\mathrm{~d} \xi^{2}} \mathrm{~d} \xi \tag{40}
\end{equation*}
$$

Its stationary condition can be obtained as:

$$
\begin{equation*}
\left.\lambda^{\prime \prime}(\xi)\right|_{\xi=y}=0, \quad 1-\left.\lambda^{\prime}(\xi)\right|_{\xi=y}=0,\left.\quad \lambda(\xi)\right|_{\xi=y}=0 \tag{41}
\end{equation*}
$$

The Lagrangian multiplier can identified as solving eq. (41):

$$
\begin{equation*}
\lambda(\xi)=\xi-y \tag{42}
\end{equation*}
$$

Putting this value in eq. (39), as a result, we obtain:

$$
\begin{align*}
\theta_{1, n+1}(y)=\theta_{1, n}(y)+ & \int_{0}^{y}(\xi-y)\left\{\left[1+\beta_{1} \theta_{1, n}(\xi)\right] \frac{\mathrm{d}^{2} \theta_{1, n}(\xi)}{\mathrm{d} \xi^{2}}+\beta_{1}\left(\frac{\mathrm{~d} \theta_{1, n}(\xi)}{\mathrm{d} \xi}\right)^{2}+\right. \\
& \left.+2 \xi\left[1+\alpha_{1} \theta_{1, n}(\xi)\right] \frac{\mathrm{d} \theta_{1, n}(\xi)}{\mathrm{d} \xi}\right\} \mathrm{d} \xi \tag{43}
\end{align*}
$$

Taking the initial approximation:

$$
\begin{equation*}
\theta_{1,0}(y)=\frac{\theta_{\mathrm{f}}}{\lambda} y \tag{44}
\end{equation*}
$$

Using the variational correctional eq. (43) we have:

$$
\begin{equation*}
\theta_{1,1}(y)=\frac{y \theta_{\mathrm{f}}}{\lambda}-\frac{\beta_{1} y^{2} \theta_{f}^{2}}{2 \lambda^{2}}-\frac{\alpha_{1} y^{4} \theta_{f}^{2}}{6 \lambda^{2}}-\frac{y^{3} \theta_{\mathrm{f}}}{3 \lambda} \tag{45}
\end{equation*}
$$

$$
\begin{gather*}
\theta_{1,2}(y)=\frac{y \theta_{\mathrm{f}}}{\lambda}-\frac{\beta_{1} y^{2} \theta_{\mathrm{f}}^{2}}{2 \lambda^{2}}-\frac{y^{3} \theta_{\mathrm{f}}}{3 \lambda}+\frac{\beta_{1}^{2} y^{3} \theta_{\mathrm{f}}^{3}}{2 \lambda^{3}}+\frac{\beta_{1} y^{4} \theta_{\mathrm{f}}^{2}}{2 \lambda^{2}}-\frac{\alpha_{1} y^{4} \theta_{\mathrm{f}}^{2}}{6 \lambda^{2}}-\frac{\beta_{1}^{3} y^{4} \theta_{\mathrm{f}}^{4}}{8 \lambda^{4}}+\frac{y^{5} \theta_{\mathrm{f}}}{10 \lambda}- \\
-\frac{\beta_{1}^{2} y^{5} \theta_{\mathrm{f}}^{3}}{6 \lambda^{3}}+\frac{19 \alpha_{1} \beta_{1} y^{5} \theta_{\mathrm{f}}^{3}}{60 \lambda^{3}}-\frac{\beta_{1} y^{6} \theta_{\mathrm{f}}^{2}}{18 \lambda^{2}}+\frac{2 \alpha_{1} y^{6} \theta_{\mathrm{f}}^{2}}{15 \lambda^{2}}-\frac{7 \alpha_{1} \beta_{1}^{2} y^{6} \theta_{\mathrm{f}}^{4}}{60 \lambda^{4}}+\frac{5 \alpha_{1}^{2} y^{7} \theta_{\mathrm{f}}^{3}}{126 \lambda^{3}}- \\
-\frac{2 \alpha_{1} \beta_{1} y^{7} \theta_{\mathrm{f}}^{3}}{21 \lambda^{3}}-\frac{\alpha_{1} y^{8} \theta_{\mathrm{f}}^{2}}{84 \lambda^{2}}-\frac{2 \alpha_{1}^{2} \beta_{1} y^{8} \theta_{\mathrm{f}}^{4}}{63 \lambda^{4}}-\frac{7 \alpha_{1}^{2} y^{9} \theta_{\mathrm{f}}^{3}}{648 \lambda^{3}}-\frac{\alpha_{1}^{3} y^{10} \theta_{\mathrm{f}}^{4}}{405 \lambda^{4}} \tag{46}
\end{gather*}
$$

and so on.
For differential eq. (13) we construct correctional functional:

$$
\begin{gather*}
\theta_{2, n+1}(y)=\theta_{2, n}(y)+\int_{0}^{y} \lambda(\xi)\left\{a_{12}\left[1+\beta_{2} \tilde{\theta}_{2, n}(\xi)\right] \frac{\mathrm{d}^{2} \theta_{2, n}(\xi)}{\mathrm{d} \xi^{2}}+a_{12} \beta_{2}\left(\frac{\mathrm{~d} \tilde{\theta}_{2, n}(\xi)}{\mathrm{d} \xi}\right)^{2}+\right. \\
\left.+\left[2 \xi+2 \lambda\left(\frac{1}{\rho_{12}}-1\right)\right]\left[1+\alpha_{2} \tilde{\theta}_{2, n}(\xi)\right] \frac{\mathrm{d} \tilde{\theta}_{2, n}}{\mathrm{~d} \xi}\right\} \mathrm{d} \xi \tag{47}
\end{gather*}
$$

Its stationary condition:

$$
\begin{equation*}
\left.\lambda^{\prime \prime}(\xi)\right|_{\xi=y}=0, \quad 1-\left.a_{12} \lambda^{\prime}(\xi)\right|_{\xi=y}=0,\left.\quad \lambda(\xi)\right|_{\xi=y}=0 \tag{48}
\end{equation*}
$$

The Langrangian multiplier obtained:

$$
\begin{equation*}
\lambda(\xi)=\frac{1}{a_{12}}(\xi-y) \tag{49}
\end{equation*}
$$

The following iteration eq. (47) becomes:

$$
\begin{align*}
\theta_{2, n+1}(y)=\theta_{2, n}(y) & +\int_{0}^{y} \frac{1}{a_{12}}(\xi-y)\left\{a_{12}\left[1+\beta_{2} \theta_{2, n}(\xi)\right] \frac{\mathrm{d}^{2} \theta_{2, n}(\xi)}{\mathrm{d} \xi^{2}}+a_{12} \beta_{2}\left(\frac{\mathrm{~d} \theta_{2, n}(\xi)}{\mathrm{d} \xi}\right)^{2}+\right. \\
& \left.+\left[2 \xi+2 \lambda\left(\frac{1}{\rho_{12}}-1\right)\right]\left[1+\alpha_{2} \theta_{2, n}(\xi)\right] \frac{\mathrm{d} \theta_{2, n}}{\mathrm{~d} \xi}\right\} \mathrm{d} \xi \tag{50}
\end{align*}
$$

Now we start with initial approximation:

$$
\begin{align*}
& \theta_{2,0}(y)=1+\left(\theta_{\mathrm{f}}-1\right) \mathrm{e}^{(\lambda-y)} \\
& \theta_{2,1}(y)=1-\mathrm{e}^{\lambda}+\frac{4 \mathrm{e}^{\lambda}}{a_{12}}-\frac{4 \mathrm{e}^{(\lambda-y)}}{a_{12}}+\frac{4 \mathrm{e}^{\lambda} \alpha_{2}}{a_{12}}-\frac{\mathrm{e}^{2 \lambda} \alpha_{2}}{2 a_{12}}-\frac{4 \mathrm{e}^{(\lambda-y)} \alpha_{2}}{a_{12}}+\frac{\mathrm{e}^{2(\lambda-y)} \alpha_{2}}{2 a_{12}}-\mathrm{e}^{\lambda} \beta_{2}+ \\
& +\frac{1}{2} \mathrm{e}^{2 \lambda} \beta_{2}+\beta_{2} \mathrm{e}^{(\lambda-y)}-\frac{1}{2} \mathrm{e}^{2(\lambda-y)} \beta_{2}+\mathrm{e}^{\lambda} y-\frac{2 \mathrm{e}^{\lambda} y}{a_{12}}-\frac{2 \mathrm{e}^{(\lambda-y)} y}{a_{12}}-\frac{2 \mathrm{e}^{\lambda} \alpha_{2} y}{a_{12}}+ \\
& +\frac{\mathrm{e}^{2 \lambda} \alpha_{2} y}{2 a_{12}}-\frac{2 \mathrm{e}^{(\lambda-y)} \alpha_{2} y}{a_{12}}+\frac{\mathrm{e}^{2(\lambda-y)} \alpha_{2} y}{2 a_{12}}+\mathrm{e}^{\lambda} \beta_{2} y-\mathrm{e}^{2 \lambda} \beta_{2} y+\mathrm{e}^{\lambda} \theta_{f}-\frac{4 \theta_{\mathrm{f}} \mathrm{e}^{\lambda}}{a_{12}}+\frac{4}{a_{12}} \mathrm{e}^{(\lambda-y)} \theta_{\mathrm{f}}- \\
& -\frac{4 \mathrm{e}^{\lambda} \alpha_{2} \theta_{\mathrm{f}}}{a_{12}}+\frac{\mathrm{e}^{2 \lambda} \alpha_{2} \theta_{\mathrm{f}}}{a_{12}}+\frac{4 \mathrm{e}^{(\lambda-y)} \alpha_{2} \theta_{\mathrm{f}}}{a_{12}}-\frac{\mathrm{e}^{2(\lambda-y)} \alpha_{2} \theta_{\mathrm{f}}}{a_{12}}+\mathrm{e}^{\lambda} \beta_{2} \theta_{\mathrm{f}}-\mathrm{e}^{2 \lambda} \beta_{2} \theta_{\mathrm{f}}-\mathrm{e}^{(\lambda-y)} \beta_{2} \theta_{\mathrm{f}} \\
& +\mathrm{e}^{2(\lambda-y)} \beta_{2} \theta_{\mathrm{f}}-\mathrm{e}^{\lambda} y \theta_{\mathrm{f}}+\frac{2 \mathrm{e}^{\lambda} y \theta_{\mathrm{f}}}{a_{12}}+\frac{2 \mathrm{e}^{(\lambda-y)} y \theta_{\mathrm{f}}}{a_{12}}+\frac{2 \mathrm{e}^{\lambda} \alpha_{2} y \theta_{\mathrm{f}}}{a_{12}}-\frac{\mathrm{e}^{2 \lambda} \alpha_{2} y \theta_{\mathrm{f}}}{a_{12}}+ \\
& +\frac{2 \mathrm{e}^{(\lambda-y)} \alpha_{2} y \theta_{\mathrm{f}}}{a_{12}}-\frac{\mathrm{e}^{2(\lambda-y)} \alpha_{2} y \theta_{\mathrm{f}}}{a_{12}}-\mathrm{e}^{\lambda} \beta_{2} y \theta_{\mathrm{f}}+2 \mathrm{e}^{2 \lambda} \beta_{2} y \theta_{\mathrm{f}}-\frac{\mathrm{e}^{2 \lambda} \alpha_{2} \theta_{\mathrm{f}}^{2}}{2 a_{12}}+\frac{\mathrm{e}^{2(\lambda-y)} \alpha_{2} \theta_{\mathrm{f}}^{2}}{2 a_{12}}+ \\
& +\frac{1}{2} \mathrm{e}^{2 \lambda} \beta_{2} \theta_{\mathrm{f}}^{2}-\frac{1}{2} \mathrm{e}^{2(\lambda-y)} \beta_{2} \theta_{\mathrm{f}}^{2}+\frac{\mathrm{e}^{2 \lambda} \alpha_{2} y \theta_{\mathrm{f}}^{2}}{2 a_{12}}+\frac{\mathrm{e}^{2(\lambda-y)} \alpha_{2} y \theta_{\mathrm{f}}^{2}}{2 a_{12}}-\mathrm{e}^{2 \lambda} \beta_{2} y \theta_{\mathrm{f}}^{2}+\frac{2 \mathrm{e}^{\lambda} k \lambda}{a_{12}}- \\
& -\frac{2 \mathrm{e}^{(\lambda-y)} k \lambda}{a_{12}}+\frac{2 \mathrm{e}^{\lambda} k \alpha_{2} \lambda}{a_{12}}-\frac{\mathrm{e}^{2 \lambda} k \alpha_{2} \lambda}{2 a_{12}}-\frac{2 \mathrm{e}^{(\lambda-y)} k \alpha_{2} \lambda}{a_{12}}+\frac{\mathrm{e}^{2(\lambda-y)} k \alpha_{2} \lambda}{2 a_{12}}-\frac{2 \mathrm{e}^{\lambda} k y \lambda}{a_{12}}- \\
& -\frac{2 \mathrm{e}^{\lambda} k \alpha_{2} y \lambda}{a_{12}}+\frac{\mathrm{e}^{2 \lambda} k \alpha_{2} y \lambda}{a_{12}}-\frac{2 \mathrm{e}^{\lambda} k \theta_{\mathrm{f}} \lambda}{a_{12}}+\frac{2 \mathrm{e}^{(\lambda-y)} k \theta_{\mathrm{f}} \lambda}{a_{12}}-\frac{2 \mathrm{e}^{\lambda} k \alpha_{2} \theta_{\mathrm{f}} \lambda}{a_{12}}+\frac{\mathrm{e}^{2 \lambda} k \alpha_{2} \theta_{\mathrm{f}} \lambda}{a_{12}}+ \\
& +\frac{2 \mathrm{e}^{(\lambda-y)} k \alpha_{2} \theta_{\mathrm{f}} \lambda}{a_{12}}-\frac{\mathrm{e}^{2(\lambda-y)} k \alpha_{2} \theta_{\mathrm{f}} \lambda}{a_{12}}+\frac{2 \mathrm{e}^{\lambda} k y \theta_{\mathrm{f}} \lambda}{a_{12}}+\frac{2 \mathrm{e}^{\lambda} k \alpha_{2} y \theta_{\mathrm{f}} \lambda}{a_{12}}-\frac{2 \mathrm{e}^{2 \lambda} k \alpha_{2} y \theta_{\mathrm{f}} \lambda}{a_{12}} \\
& -\frac{\mathrm{e}^{2 \lambda} k \alpha_{2} \theta_{\mathrm{f}}^{2} \lambda}{2 a_{12}}+\frac{\mathrm{e}^{2(\lambda-y)} k \alpha_{2} \theta_{\mathrm{f}}^{2} \lambda}{2 a_{12}}+\frac{\mathrm{e}^{2 \lambda} k \alpha_{2} y \theta_{\mathrm{f}}^{2} \lambda}{2 a_{12}} \tag{51}
\end{align*}
$$

We get the first approximation, in this expression $k=\left(1 / \rho_{12}\right)-1$, and so on.
In same manner we find the other approximation. After find out the expression of $\theta_{1, n}$ and $\theta_{2, n}$ we have:

$$
\begin{align*}
& \theta_{1}(y)=\lim _{n \rightarrow \infty} \theta_{1, n}(y)  \tag{52}\\
& \theta_{2}(y)=\lim _{n \rightarrow \infty} \theta_{2, n}(y) \tag{53}
\end{align*}
$$

## Numerical computation and discussion

Substituting, $\theta_{1}(y)$ and $\theta_{2}(y)$ for both cases in the interface eq. (16), we obtain a transcendental equation in the term of $\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2}$, and the Stefan number (Ste). The computation has been made and the results are presented in five figures. On the figures presented in this study, only the parameters whose values different from the reference values are indicated.

In fig. 1, the dimensionless phase change location is plotted as a function of Ste. It has been observed that the result obtained by the VIM is exactly the same as that obtained by exact method. The dimensionless temperature $\theta_{1}$ and $\theta_{2}$ as a function of space co-ordinate are shown in fig. 2.


Figure 1. Graph between exact solution of linear equation and VIM; $\alpha_{1}=\alpha_{2}=0, \beta_{1}=\beta_{2}=0$, $\theta_{\mathrm{f}}=0.5, \rho_{12}=1.0, a_{12}=1.0$

In fig. 3, the dimensionless phase change location is plotted as a function of Stefan number, when thermal conductivity and specific heat in two regions varies linearly with temperature. It has been observed that as $\alpha_{1}$ and $\alpha_{2}$ increase, dimensionless phase change location decreases. This result is same as that obtained by Oliver et al. [36].

It is clear from fig. 4. that as $\beta_{1}$ and $\beta_{2}$ increases, dimensionless phase change location decreases. It is clear from figs. 3 and 4 that at higher values of heat of fusion (low values of Ste), the variation in thermal conductivity are important, but the effect of variable specific heat diminssishes. On fig. 5 the dimensionless phase change location is plotted as a function of Stefan number, when thermal conductivity and specific heat in two regions varies exponentially with temperature


Figure 4. Dimensionless phase change location vs.
Stefan number (Case 2); $\alpha_{1}=\alpha_{2}=1.0$,
$\theta_{\mathrm{f}}=0.5, \rho_{12}=1.0, a_{12}=1.0$


Figure 2. Dimensionless temperature distribution of phase 1 and phase 2 (Case 2); $\alpha_{1}=\alpha_{2}=1.0$, $\beta_{1}=\beta_{2}=1.0, \theta_{\mathrm{f}}=0.5, \rho_{12}=1.0, a_{12}=1.0, \lambda=0.13$


Figure 3. Dimensionless phase change location $v s$. Stefan number (Case 2); $\beta_{1}=\beta_{2}=1.0$, $\theta_{\mathrm{f}}=0.5, \rho_{12}=1.0, a_{12}=1.0$


Figure 5. Dimensionless phase change location vs. Stefan number (Case 1); $\beta_{1}=\beta_{2}=1.0, \theta_{\mathrm{f}}=0.5$, $\rho_{12}=1.0, a_{12}=1.0$
(Case 1). It has been observed that as $\alpha_{1}$ and $\alpha_{2}$ increases, dimensionless phase change location decreases.

## Conclusions

The variational iteration method is very powerful in finding the solution of moving boundary problem in freezing process. Sharing its application to moving boundary problem of temperature dependent thermal conductivity and specific heat of two regions, we may conclude that this method will be very much useful for solving moving and other many physical problems. The advantage of this method consists in obtaining the interface position and temperature distribution in the form of continuous function, instead of discrete form. Moreover, no linearization is needed and it avoids the accuracy of finding the temperature distribution by the numerical techniques.

## Nomenclature

Greek symbols
$\alpha \quad-$ specific heat coefficient
$\beta \quad-$ thermal conductivity coefficient, [-]

```
a - thermal diffusivity, [m2 }\mp@subsup{\textrm{m}}{}{-1}
```

a - thermal diffusivity, [m2 }\mp@subsup{\textrm{m}}{}{-1}
c - specific heat, [ }\mp@subsup{\textrm{kJkg}}{}{-1}\mp@subsup{\textrm{K}}{}{-1}
c - specific heat, [ }\mp@subsup{\textrm{kJkg}}{}{-1}\mp@subsup{\textrm{K}}{}{-1}
H - latent heat of fusion, [kJ kg}\mp@subsup{}{-1}{-1
H - latent heat of fusion, [kJ kg}\mp@subsup{}{-1}{-1
k - thermal conductivity, [Wm}\mp@subsup{}{}{-1}\mp@subsup{\textrm{K}}{}{-1}
k - thermal conductivity, [Wm}\mp@subsup{}{}{-1}\mp@subsup{\textrm{K}}{}{-1}
s(t) - moving interface, [m]
s(t) - moving interface, [m]
Ste - Stefan number, [-]
Ste - Stefan number, [-]
T - temperature, [ }\mp@subsup{}{}{\circ}\textrm{C}
T - temperature, [ }\mp@subsup{}{}{\circ}\textrm{C}
t - time, [s]
t - time, [s]
x - spatial co-ordinates, [m]
x - spatial co-ordinates, [m]
y - dimensionless co-ordinates, [-]

```
y - dimensionless co-ordinates, [-]
```

$\theta \quad-$ dimensionless temperature, [-]
$\lambda \quad-$ dimensionless phase change front, $[-]$
$\rho \quad-$ density, $\left[\mathrm{kgm}^{-3}\right]$
Subscripts
1,2 - phase 1 and 2, respectively
i - initial
f - freezing
m - melting
$0 \quad-$ at the surface, $x=0$
Acronym
VIM - variational iteration method

## References

[1] Alexiades, V., Solomon, A. D., Mathematical Modeling of Melting and Freezing Processes, Hemisphere Publishing Corrporation, Taylor and Francis Group, Washington, DC, 1993
[2] Crank, J., Free and Moving Boundary Problems, Oxford University Press, Oxford, UK, 1984
[3] Lunardini, V. J., Heat Transfer with Freezing and Thawing, Elsevier, Amsterdam, The Netherlands, 1991
[4] Barber, J. R., An Asymptotic Solution for Short Time Transient Heat Conduction between Two Similar Contacting Bodies, Int. J. Heat Mass Transfer, 32 (1989), 5, pp. 943-949
[5] Briozzo, A. C., Natale, M. F., Tarzia, D. A., Determination of Unknown Thermal Coefficients for Storm's-Type Materials through a Phase-Change Process, Int. J. Nonlinear Mech., 34 (1999), 2, pp. 329-340
[6] Grzymkowski, R., Slota, D., Multi-Phase Inverse Stefan Problems Solved by Approximation Method (Eds. R., Wyrykowski, et al.), in: Parallel Processing and Applied Mathematics, LNCS 2328, Springer--Verlag, Berlin, 2002, pp. 679-686
[7] Ang, D. D., et al., Regularization of an Inverse Two Phase Stefan Problem, Nonlinear Anal., 34 (1998), pp. 719-731
[8] Carslaw, H. S., Jaeger, J. C., Conduction of Heat in Solids, $2^{\text {nd }}$ ed., Oxford University Press, London, 1956
[9] Ozisik, M. N., Heat Conduction, John Wiley and Sons, New York, USA, 1980
[10] H** $^{\text {C }}$, Computational Methods for Free and Moving Boundary Problems in Heat and Fluid Flow (Eds. L. C. Wrobel, C. A. Brebbia), Computational Mechanics Publications, Southampton, UK, 1993
[11] Fredrick, D., Greif, R., A Method for the Solution of Heat Transfer Problems with a Change of Phase, ASME J. Heat transfer, 107 (1985), 3, pp. 520-526
[12] Natale, M. F., Tarzia, D. A., Explicit Solutions to the One Phase Stefan Problem with TemperatureDependent Thermal Conductivity and a Convective Term, Int. J. Engi. Sci., 41 (2003), 15, pp. 16851698
[13] Gasiorski, A. K., Application of the Finite Element Method for the Analysis of Nonsteady-State Induction Device Problems with Rotational Symmetry, Compt. Elect. Eng., 13 (1987), 2, pp. 117-128
[14] Chen, H. T., Liu, J. Y., Effect of the Potential Field on Non-Fickian Diffusion Problems in a Sphere, Int. J. Heat Mass Transfer, 46 (2003), 15, pp.2809-2818
[15] Goodman, T. R., Shea, J. J., The Melting of Finite Slabs, ASME, J. Appl. Mech., 27 (1960), 1, pp. 16241632
[16] Goodman, T. R., The Heat-Balance Integral Method and its Application to Problems Involving a Change of Phase, Trans. ASME, 80 (1958), 2, pp. 335-342
[17] Savović, S., Caldwell, J., Numerical Solution of Stefan Problem with Time-Dependent Boundary Conditions by Variable Space Grid Method, Thermal Science, 13 (2009), 4, pp. 165-174
[18] Rai, K. N., Rai, S., Approximate Closed Form Analytical Solution of the Desublimation Problem in a Porous Medium, Int. J. Energy Research, 19 (1995), 4, pp. 279-288
[19] Annamalai, K., Lau, S. C., Kondepudi, S. N., A Non-Integral Technique for the Approximate Solution of Transport Problems, Int. Comm. Heat Mass Transfer, 13 (1986), 5, pp. 523-534
[20] Rai, K. N., Rai, S., An Analytical Study of the Solidification in a Semi-Infinite Porous Medium International J. Engg. Sci., 30 (1992), 2 , pp. 247-256
[21] Nayfeh, A. H., Perturbation Methods, John Wiley and Sons, New York, USA, 1973
[22] Liu, X. L., Oliveira, C. S., Iterative Modal Perturbation and Reanalysis of Eigen Value Problem, Comm. in Num. Meth in Engg., 19 (2003), 4, pp. 263-274
[23] Prud Homme, M., Hung Nguyen T., Long Nguyen, D., A Heat Transfer Analysis for Solidification of Slabs, Cylinders, and Spheres, Journal of Heat Transfer, 111, (1989), 3, pp. 699-705
[24] He, J. H., Variational Iteration Method - a Kind of Non Linear Analytical Technique: Some Example, Int. J. Nonlin. Mech., 34 (1999), 4, pp. 699-708
[25] He, J. H., Variational Iteration Method for Autonomous Ordinary Differential Systems, Appl. Math. Comput., 114 (2000), 2-3, pp.115-123
[26] Yao, X., A Model for Initial Spherical Growth during Equiaxed Solidification, Int. J. Nonlin. Sci. Num. Sim., 9 (2008), 3, pp. 283-288
[27] Chun, C., Variational Iteration Method for a Reliable Treatment of Heat Equations with Ill-Defined Initial Data, Int. J. Nonlin. Sci. Num. Sim., 9 (2008), 4, pp. 435-440
[28] Cao, L., Han, B., Wang, W., Homotopy Perturbation Method for Nonlinear Ill-Posed Operator Equations, Int. J. Nonlin. Sci. Num., 10 (2009), pp. 1319-1322
[29] He, J. H., Wu, G. C., Austin, F., The Variational Iteration Method Which Should be Followed, Nonlinear Science Letters A, 1 (2010), 1, pp. 1-30
[30] Ganji, D. D., Sadighi, A., Application of Homotopy-Perturbation and Variational Iteration Methods to Nonlinear Heat Transfer and Porous Media Equations, J. of Computational and Applied Mathematics, 207 (2007), 1, pp. 24 -34
[31] Momani, S., Abuasad, S., Application of He's Variational Iteration Method to Helmholtz Equation, Chaos, Solitons Fractals, 27 (2006), 5, pp. 1119-1123
[32] Abdou, M. A., Soliman, A. A., New Applications of Variational Iteration Method, Physica D 211, (2005), 1-2, pp. 1-8
[33] Abdou, M. A., Soliman, A. A., Variational Iteration Method for Solving Burgers and Coupled Burgers Equations, J. Comput. Appl. Math., 181 (2005), 2, pp. 245-251
[34] Rajeev, Rai, K. N., Das, S., Solution of One-Dimensional Moving Boundary Problem with Periodic Boundary Conditions by Variational Iteration Method, Thermal Science, 13 (2009), 2, pp. 199-204
[35] Slota, D., Direct and Inverse One-Phase Stefan Problem Solved by the Variational Iteration Method, Comp. and Math. with Applications, 54 (2007), 7-8, pp. 1139-1146
[36] Oliver, D. L. R., Sunderland, J. E., A Phase Change Problem with Temperature Dependent Thermal Conductivity and Specific Heat, Int. J. Heat Mass Transfer, 30 (1987), 12, pp. 2657-2661

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