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Variational-like inequalities for n-dimensional fuzzy-vector-valued functions and fuzzy optimization

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Abstract: The existing results on the variational inequality problems for fuzzy mappings and their applications were based on Zadeh's decomposition theorem and were formally characterized by the precise sets which are the fuzzy mappings' cut sets directly. That is, the existence of the fuzzy variational inequality problems in essence has not been solved. In this paper, the fuzzy variational-like inequality problems is incorporated into the framework of n-dimensional fuzzy number space by means of the new ordering of two n-dimensional fuzzy-number-valued functions we proposed [Fuzzy Sets and Systems 295 (2016) 19-36]. As a theoretical basis, the existence and the basic properties of the fuzzy variational inequality problems are discussed. Furthermore, the relationship between the variational-like inequality problems and the fuzzy optimization problems is discussed. Finally, we investigate the optimality conditions for the fuzzy multiobjective optimization problems.

Keywords: n-dimensional fuzzy-number-valued functions, generalized convexity, variational-like inequality, fuzzy optimization

1 Introduction

Variational inequality theory, where the function is a vector-valued mapping, known either in the form presented by Hartman and Stampacchia [1] or in the form introduced by Minty [2], has become an effective and powerful tool for studying a wide class of linear/nonlinear problems arising in diverse applied fields such as optimization and control, mechanics, economics and engineering sciences. Vector variational inequality, where the function is a matrix-valued mapping, was first introduced and studied by Giannessi [3] in finite-dimensional Euclidean spaces. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criteria considering. In the study of problems related to stochastic impulse control, Bensoussan and Lions [4] proposed quasi-variational inequality [5–7], where the function is a set-valued mapping. However, one frequently observes that there are objects that have an ambiguous status in the real world. The fuzzy set theory, introduced by Zadeh [8] in 1965, offers a wide variety of techniques for analyzing imprecise data and fuzzy numbers [9] have been investigated extensively. In order to deal with the variational inequalities derived from some fuzzy environments, in 1989, Chang and Zhu [10] introduced the concepts of variational inequalities for fuzzy mapping in abstract spaces and investigated the existence of some types

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of variational-like inequalities for fuzzy mappings. Since then, several types of variational inequalities and complementarity problems for fuzzy mappings have been studied by various researchers [11–18].

On the other hand, variational inequalities are efficient tool for the investigation of optimization problems because these inequalities ensure the existence of efficient solutions, under the condition of convexity or generalized convexity. Many works of these type of inequalities have been focused on looking for the relations between the solutions of various type of variational inequalities and optimization problems [19, 20]. While very few investigations have appeared to study the relationships between fuzzy variational inequalities and fuzzy optimization problems. Wu and Xu [21, 22] introduced the generalized monotonicity of fuzzy mappings and discussed the relationship between the fuzzy variational-like inequality and fuzzy optimization problems. Weir [23] and Noor [24, 25] have studied some basic properties of the preinvex functions and their role in optimization and variational-like inequality problems. In [24], Noor has pointed out that the concept of invexity plays exactly the same role in variational-like inequality problems as the classical convexity plays in variational inequality problems, and has shown that the variational-like inequality problems are well defined in the setting of invexity. Recently, Ruiz-Garzon et al. [26] established relationships between vector variational-like inequality and optimization problems, under the assumptions of pseudo-invexity. However, the exiting results on the variational inequalities for fuzzy mappings are focused on two methods. Since the cut set of a 1-dimensional fuzzy number is a close interval on R, one method is investigates the n-dimensional fuzzy-vector-valued function whose components are the 1-dimensional fuzzy numbers by means of the ordering of two fuzzy numbers proposed by Goetschel and Voxman [27] or by Nanda and Kar [28]; the other method is transformed into the classical set-valued variational inequalities, because the cut set of an n-dimensional fuzzy number is a nonempty compact convex subset of R^n. To the best of our knowledge, very few studies have investigated the variational inequalities for n-dimensional fuzzy number-valued functions directly in n-dimensional fuzzy number space. The main reason is that there is almost no related research about the ordering and the difference of n-dimensional fuzzy numbers. Until 2016, Gong and Hai [29] introduced the concept of a convex fuzzy-number-valued function based on a new ordering $\preceq_c$ of n-dimensional fuzzy numbers, and investigated some relations among the convexity and quasiconvex of n-dimensional fuzzy-number-valued functions, and also study the local-global minimum properties of the convex fuzzy number-valued functions. The present study is to incorporate the fuzzy variational-like inequality problems into the framework of n-dimensional fuzzy number space by the new ordering of two n-dimensional fuzzy numbers, which is a further study in theoretical research and more convenient in practical application.

The aim of this paper is to incorporate the fuzzy variational-like inequality problems into the framework of n-dimensional fuzzy number space. To make our analysis possible, we present the preliminary terminology used throughout this paper in Section 2. In Section 3, the concept of generalized monotonicity and invexity for n-dimensional fuzzy-number-valued functions are presented and some properties are discussed. In Section 4, we introduce the fuzzy variational-like inequality based on the order $\succeq_c$ and obtain the existence of a solution of the fuzzy variational-like inequality. The relationship between the variational-like inequality problems and fuzzy optimization problems is given in Section 5. We investigate the optimality conditions for the fuzzy multiobjective optimization problems in Section 6. Section 7 concludes this paper.

2 Preliminaries

Throughout this paper, R^n denotes the n-dimensional Euclidean space, $\mathcal{K}^n$ and $\mathcal{K}^n_C$ denote the spaces of nonempty compact and compact convex sets of R^n, respectively. Let $\mathcal{F}(R^n)$ be the set of all fuzzy subsets on R^n. A fuzzy set u on R^n is a mapping $u : R^n \to [0, 1]$, and $u(x)$ is the degree of membership of the element x in the fuzzy set u. For each fuzzy set u, we denote its r-level set as $[u]^r = \{x \in R^n : u(x) \geq r\}$ for any $r \in (0, 1]$, and in some references also denoted by u_r for short. The support of u we denote by supp$u$ where supp$u = \{x \in R^n : u(x) > 0\}$. The closure of supp$u$ defines the 0-level of u, i.e. $[u]^0 = cl(\text{supp}u)$. Here $cl(M)$ denotes the closure of set M. Fuzzy set $u \in \mathcal{F}(R^n)$ is called a fuzzy number if [30, 31]

(i) u is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(ii) u is a convex fuzzy set, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0, 1]$,
(iii) u is upper semicontinuous ,
(iv) $[u]^0 = cl(\mathrm{supp}u) = cl(\bigcup_{r\in(0,1]}[u]^r)$ is compact.
We use E^n to denote the fuzzy number space. Note that if $u : R \to [0, 1]$, then u is a 1-dimensional fuzzy number, denoted by $u \in E$, and $[u]^r = [u_-(r), u^+(r)]$ is a close interval on R.

It is clear that each $u \in R^n$ can be considered as a fuzzy number u defined by

$$u(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

In particular, the fuzzy number 0 is defined as $0(x) = 1$ if $x = 0$, and $0(x) = 0$ otherwise.

Example 2.1. *Let $u \in E^2$ is defined by*

$$u(x, y) = \begin{cases} \sqrt{1 - x^2 - y^2}, & x^2 + y^2 \leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

then $[u]^r = \{(x, y) : x^2 + y^2 \leq 1 - r^2\}, r \in [0, 1]$.

Theorem 2.2. *[32] If $u \in E^n$, then*
(i) $[u]^r$ is a nonempty compact convex subset of R^n for any $r \in (0, 1]$,
(ii) $[u]^{r_1} \subseteq [u]^{r_2}$, whenever $0 \leq r_2 \leq r_1 \leq 1$,
(iii) if $r_n > 0$ and r_n converging to $r \in [0, 1]$ is nondecreasing, then $\bigcap_{n=1}^{\infty}[u]^{r_n} = [u]^r$.

Conversely, suppose for any $r \in [0, 1]$, there exists an $A^r \subseteq R^n$ which satisfies the above (i)-(iii), then there exists a unique $u \in E^n$ such that $[u]^r = A^r$, $r \in (0, 1]$, $[u]^0 = \overline{\bigcup_{r\in(0,1]}[u]^r} \subseteq A^0$.

Let $u, v \in E^n$, $k \in R$. For any $x \in R^n$, the addition and scalar multiplication can be defined, respectively, as:

$$(u + v)(x) = \sup_{s+t=x} \min\{u(s), v(t)\}, \tag{2.3}$$

$$(ku)(x) = u(\frac{x}{k}), k \neq 0, \tag{2.4}$$

$$(0u)(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases} \tag{2.5}$$

It is well known that for any $u, v \in E^n$ and $k \in R$, the addition $u + v$ and the scalar multiplication ku have the level sets

$$[u + v]^r = [u]^r + [v]^r = \{x + y : x \in [u]^r, y \in [v]^r\}, \tag{2.6}$$

$$[ku]^r = k[u]^r = \{kx : x \in [u]^r\}. \tag{2.7}$$

Proposition 2.3. *[33] If $u, v \in E^n$, $k, k_1, k_2 \in R$, then*
(i) $k(u + v) = ku + kv$,
(ii) $k_1(k_2u) = (k_1k_2)u$,
(iii) $(k_1 + k_2)u = k_1u + k_2u$ when $k_1 \geq 0$ and $k_2 \geq 0$.

Give two subsets A, $B \subseteq R^n$ and $k \in R$, the Minkowski difference is given by $A - B = A + (-1)B = \{a - b : a \in A,\ b \in B\}$. However, in general, $A + (-A) \neq 0$, i.e. the opposite of A is not the inverse of A in Minkowski addition (unless $A = \{a\}$ is a singleton). The spaces $\mathcal{K}^n$ and $\mathcal{K}_C^n$ are not linear spaces since they do not contain inverse elements and therefore subtraction is not defined. To partially overcome this situation, Hukuhara [36] introduced the following H-difference $A \ominus B = C \Longleftrightarrow A = B + C$ and an important property of $\ominus$ is that $A \ominus A = \{0\}$, $\forall A \in R^n$ and $(A+B)\ominus B = A$, $\forall A, B \in R^n$. The H-difference is unique, but a necessary condition for $A \ominus_H B$ to exist is that A contains a translation $\{c\} + B$ of B. In order to overcome this situation, Stefanini [37] defined the generalized Hukuhara difference of two sets $A,\ B \in \mathcal{K}^n$ as follows

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} \quad (1)\ \ A = B + C, \\ \text{or } (2)\ \ B = A + (-1)C. \end{cases} \tag{2.8}$$

The generalized Hukuhara difference has been extended to the fuzzy case in [38]. For any $u, v \in E^n$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} \quad (1)\ \ u = v + w, \\ \text{or } (2)\ \ v = u + (-1)w. \end{cases} \tag{2.9}$$

It is possible that the gH-difference of two fuzzy numbers does not exist. To solve this shortcoming, in [39] a new difference between fuzzy numbers was proposed. Using the convex hull (conv) the new difference was defined as follows.

Definition 2.4. *[39, 40] The generalized difference (g-difference for short) of two fuzzy numbers $u, v \in E^n$ is given by its level sets as*

$$[u \ominus_g v]^r = cl(\text{conv}\bigcup_{\beta \geq r}([u]^\beta \ominus_{gH} [v]^\beta)),\ \forall r \in [0,1], \tag{2.10}$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[u]^\beta$ and $[v]^\beta$.

A necessary condition for $u \ominus_g v$ to exist is that either $[u]^r$ contains a translation of $[v]^r$ or $[v]^r$ contains a translation of $[u]^r$ for any $r \in [0,1]$.

Proposition 2.5. *[41] Let $u, v \in E^n$. Then*
(i) if the g-difference exists, it is unique,
(ii) $u \ominus_g u = 0$,
(iii) $(u + v) \ominus_g v = u$, $(u + v) \ominus_g u = v$,
(iv) $u \ominus_g v = -(v \ominus_g u)$.

Given $u, v \in E^n$, the distance $D : E^n \times E^n \to [0, +\infty)$ between u and v is defined by the equation

$$D(u, v) = \sup_{r\in[0,1]} d([u]^r, [v]^r), \tag{2.11}$$

where d is the Hausdorff metric given by

$$\begin{aligned} d([u]^r, [v]^r) &= \inf\{\varepsilon : [u]^r \subset N([v]^r, \varepsilon), [v]^r \subset N([u]^r, \varepsilon)\} \\ &= \max\{\sup_{a\in[u]^r} \inf_{b\in[v]^r} \|a - b\|, \sup_{b\in[v]^r} \inf_{a\in[u]^r} \|a - b\|\}. \end{aligned}$$

$N([u]^r, \varepsilon) = \{x \in R^n : d(x, [u]^r) = \inf_{y\in[u]^r} d(x, y) \leq \varepsilon\}$ is the ε-neighborhood of $[u]^r$. Then, (E^n, D) is a complete metric space, and satisfies $D(u + w, v + w) = D(u, v)$, $D(ku, kv) = |k|D(u, v)$ for any $u, v, w \in E^n$ and $k \in R$.

Let $S^{n-1} = \{x \in R^n : \|x\| = 1\}$ be the unit sphere of R^n and $\langle\cdot,\cdot\rangle$ be the inner product in R^n, i.e. $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, where $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$. Suppose $u \in E^n$, $r \in [0, 1]$ and $x \in S^{n-1}$, the support function of u is defined by

$$u^*(r, x) = \sup_{a\in[u]^r} \langle a, x\rangle. \tag{2.12}$$

Theorem 2.6. *[42] Suppose $u \in E^n$, $r \in [0,1]$, then*

$$[u]^r = \{y \in R^n : \langle y, x\rangle \leq u^*(r, x),\ x \in S^{n-1}\}. \tag{2.13}$$

For $u \in E^n$, we denote the centroid of $[u]^r$, $r \in [0,1]$ as

$$\left(\frac{\int\cdots\int_{[u]^r} x_1 dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}, \frac{\int\cdots\int_{[u]^r} x_2 dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}, \cdots, \frac{\int\cdots\int_{[u]^r} x_n dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}\right),$$

where $\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n$ is the solidity of $[u]^r$, $r \in [0,1]$ and $\int\cdots\int_{[u]^r} x_i dx_1 dx_2\cdots dx_n$ $(i = 1, 2, \cdots, n)$ is the multiple integral of x_i on measurable sets $[u]^r$, $r \in [0,1]$. Next we define an order $\preceq_C$ for E^n.

Let $\tau : E^n \to R^n$ be a real vector-valued function defined by ([29])

$$\tau(u) = \Big(2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_1 dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}dr, 2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_2 dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}dr, \cdots, 2\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_n dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}dr\Big), \tag{2.14}$$

where $\int_0^1 r\frac{\int\cdots\int_{[u]^r} x_i dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}dr$ $(i = 1, 2, \cdots, n)$ is the Lebesgue integral of $r\frac{\int\cdots\int_{[u]^r} x_i dx_1 dx_2\cdots dx_n}{\int\cdots\int_{[u]^r} 1 dx_1 dx_2\cdots dx_n}$ $(i = 1, 2, \cdots, n)$ on $[0,1]$. The vector-valued function τ is called a ranking value function defined on E^n.

Definition 2.7. *[29] Let $u, v \in E^n$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. We say that $u \preceq_c v$ (u precedes v) if*

$$\tau(v) \in \tau(u) + C. \tag{2.15}$$

We say that $u \prec_c v$ if $u \preceq_c v$ and $\tau(u) \neq \tau(v)$. Sometimes we may write $v \succeq_c u$ (resp. $v \succ_c u$) instead of $u \preceq_c v$ (resp. $u \prec_c v$). In addition, $\widetilde{\varepsilon} \in E^n$ is said to be an arbitrary positive fuzzy-number if $\widetilde{\varepsilon} \succ_c 0$ $(0 \in R^n)$ and $D(\widetilde{\varepsilon}, 0) < \varepsilon$, where ε is an arbitrary positive real number.

Example 2.8. *If $u, v \in E^1$, then $\tau(u) = \int_0^1 r(u_r^- + u_r^+)dr$, $\tau(v) = \int_0^1 r(v_r^- + v_r^+)dr$. Suppose $C = R^+ = [0, +\infty) \subseteq R$, $u \preceq_c v$ if and only if $\tau(u) \leq \tau(v)$, i.e., $\tau(v) \in \tau(u) + [0, +\infty)$. Therefore, when $u, v \in E^1$, Definition 2.7 coincides with the definition of ordering of u, v proposed by Goetschel ([27]).*

If $u, v \in E^2$, in Definition 2.7, let C be the set of nonnegative orthant of R^2, i.e., $C = R^{2+} = \{(x_1, x_2) \in R^2 : x_1 \geqslant 0,\ x_2 \geqslant 0)\} \subseteq R^2$.

Example 2.9. *A special kind of n-dimension fuzzy numbers is the fuzzy n-cell numbers proposed in [43]. Let $u \in L(E^n)$, i.e., $[u]^r = \prod_{i=1}^n [u_i^-(r), u_i^+(r)] = [u_1^-(r), u_1^+(r)] \times [u_2^-(r), u_2^+(r)] \times \cdots \times [u_n^-(r), u_n^+(r)]$ for any $r \in [0,1]$, where the left endpoint function and the right endpoint function $u_i^-(r), u_i^+(r) \in R$ with $u_i^-(r) \leq u_i^+(r)$ $(i = 1, 2, \cdots, n)$, then we have*

$$\tau(u) = \Big(\int_0^1 r(u_1^-(r) + u_1^+(r))dr, \int_0^1 r(u_2^-(r) + u_2^+(r))dr, \cdots, \int_0^1 r(u_n^-(r) + u_n^+(r))dr\Big). \tag{2.16}$$

For $u, v \in L(E^n)$, suppose $C = R^{n+} = \{(x_1, x_2, \cdots, x_n) \in R^n : x_1 \geqslant 0,\ x_2 \geqslant 0, \cdots,\ x_n \geqslant 0)\} \subseteq R^n$, then we have $u \succeq_c v \Longleftrightarrow \tau(u) \in \tau(v) + c$. Furthermore, for $k_1, k_2 \in R$, we obtain

$$\tau(k_1 u + k_2 v) = k_1\tau(u) + k_2\tau(v). \tag{2.17}$$

Let M be a convex set of m-dimensional Euclidean space R^m and F be an n-dimensional fuzzy-number-valued function (fuzzy-number-valued function for short) from M into E^n.

Example 2.10. *The following function is a 2-dimensional fuzzy-number-valued function. For constants $s, t \in R$, $F : [-\sqrt{-\ln\frac{1}{5}}, \sqrt{-\ln\frac{1}{5}}]^2 \to E^2$ is defined as*

$$F(s,t)(x,y) = \begin{cases} \frac{5}{4}e^{-[(x-s)^2+(y-t)^2]} - \frac{1}{4}, & -\sqrt{-\ln\frac{1}{5}} \leq x, y \leq \sqrt{-\ln\frac{1}{5}} \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

Example 2.11. *The following function is a fuzzy* 1*-cell number function. Furthermore, for constants* $s \in R$, $F(s)(x) = f(s)u(x)$.

$$F(s)(x) = \begin{cases} \frac{x+e^s}{2e^s}, & -e^s \le x \le e^s, \\ \sqrt{\frac{2e^s-x}{e^s}}, & e^s \le x \le 2e^s, \\ 0, & \text{otherwise}, \end{cases} \tag{2.19}$$

where $f(s) = e^s$, *and*

$$u(x) = \begin{cases} \frac{x+1}{2}, & -1 \le x \le 1, \\ \sqrt{2-x}, & 1 < x < 2, \\ 0, & \text{otherwise}. \end{cases}$$

The epigraph of F, denoted by epi(F), is defined as

$$\text{epi}(F) = \{(t,u) : t \in M, u \in E^n, F(t) \preceq_c u\}. \tag{2.20}$$

For $u, v \in E^n$, we say that u and v are comparable, if either $u \preceq_c v$ or $v \preceq_c u$,; otherwise, they are non-comparable. F is said to be a comparable fuzzy-number-valued function if for each pair $t_1, t_2 \in M$ and $t_1 \neq t_2$, $F(t_1)$ and $F(t_2)$ are comparable; otherwise, F is said to be a non-comparable fuzzy-number-valued function.

F is said to be lower semicontinuous (l.c.) at $t_0 \in M$, if for any $\widetilde{\varepsilon} \succ_c 0$, there exists a neighborhood U of t_0, when $t \in U$, we have $F(t_0) \prec_c F(t) + \widetilde{\varepsilon}$; F is said to be upper semicontinuous (u.c.) at $t_0 \in M$, if for any $\widetilde{\varepsilon} \succ_c 0$, there exists a neighborhood U of t_0, when $t \in U$, we have $F(t) \prec_c F(t_0) + \widetilde{\varepsilon}$. F is continuous at $t_0 \in M$, if it is both l.c. and u.c. at t_0, and that it is continuous if and only if it is continuous at every point of M ([29]).

Definition 2.12. *([29]) Let* $F : M \to E^n$ *be a fuzzy-number-valued function.*
(1) An element $t_0 \in M$ *is called a local minimum point of F if there exists a neighborhood U of* t_0, $F(t_0) \preceq_c F(t)$ *for any* $t \in U$.
(2) An element $t_0 \in M$ *is called a global minimum point of F if* $F(t_0) \preceq_c F(t)$ *for any* $t \in M$.
(3) An element $t_0 \in M$ *is called a strictly local minimum point of F if there exists a neighborhood U of* t_0, $F(t_0) \prec_c F(t)$ *for any* $t \in U$ *and* $t \neq t_0$.
(4) An element $t_0 \in M$ *is called a strictly global minimum point of F if* $F(t_0) \prec_c F(t)$ *for any* $t \in M$ *and* $t \neq t_0$.

Definition 2.13. *Let* $\mathbf{A} = (u_1, u_2, \cdots, u_n) \in (E^n)^n$, $u_i \in E^n$, $i = 1, 2, \cdots, n$, *and* $T = (t_1, t_2, \cdots, t_n) \in R^n$ *be an n-dimensional fuzzy vector and an n-dimensional real vector, respectively. We define the product of a fuzzy vector with a real vector as* $T\mathbf{A} = \sum_{i=1}^n t_i u_i$, *which is an n-dimensional fuzzy number. In addition, if* $T\mathbf{A} = 0$, *then we say* $\mathbf{A}$ *is fuzzy orthogonal to* T.

We denote the fuzzy vector $\mathbf{0}$ by $\mathbf{0} = \{\underbrace{0, 0, \cdots, 0}_{n}\}$, where $0 \in E^n$. If $\mathbf{A} = (u_1, u_2, \cdots, u_n) \in (E)^n$, $u_i \in E^1$, $i = 1, 2, \cdots, n$, then Definition 2.13 coincides with Definition 2.4 proposed in [34]. It is not difficult to obtain

$$[T\mathbf{A}]^r = \bigcup_{i=1}^n t_i [u_i]^r = \bigcup_{i=1}^n \{t_i x_i : x \in [u_i]^r\}. \tag{2.21}$$

For any n-dimensional fuzzy vectors $\mathbf{X}$ and $\mathbf{Y}$, let $\mathbf{X} = \{X_1, X_2, \cdots, X_n\}$, $\mathbf{Y} = \{Y_1, Y_2, \cdots, Y_n\}$, we use the following convention for equalities and inequalities throughout the paper:

(a) $\mathbf{X} \le \mathbf{Y} \Longleftrightarrow x_i \preceq_c y_i$, $i = 1, 2, \cdots n$, with strict inequality holding for at least one i;
(b) $\mathbf{X} \leqq \mathbf{Y} \Longleftrightarrow x_i \preceq_c y_i$, $i = 1, 2, \cdots n$;
(c) $\mathbf{X} = \mathbf{Y} \Longleftrightarrow x_i =_c y_i$, $i = 1, 2, \cdots n$;
(d) $\mathbf{X} < \mathbf{Y} \Longleftrightarrow x_i \prec_c y_i$, $i = 1, 2, \cdots n$.

In the following, we assume that the fuzzy-number-valued function $F : M \to E^n$ and fuzzy-vector-valued function $\mathbf{F} : M \to (E^n)^n$ are comparable, respectively.

Definition 2.14. *Let* $\mathbf{F} : M \to (E^n)^n$ *be an n-dimensional fuzzy-vector-valued function, denoted by* $\mathbf{F}(t) = (u_1(t), u_2(t), \cdots, u_n(t))$*, where* $u_i(t)$ $(i = 1, 2, \cdots, n)$ *is a fuzzy-number-valued function on* M*. For the sake of brevity, F is called a fuzzy-vector-valued function.*
(1) $\mathbf{F}$ *is said to be a comparable fuzzy-vector-valued function if any* $u_i(t)$ $(i = 1, 2, \cdots, n)$ *is a comparable fuzzy-number-valued function.*
(2) For $s, t \in M$*, we define g-difference of fuzzy-vector-valued functions as*

$$\mathbf{F}(s) \ominus_g \mathbf{F}(t) = (u_1(s) \ominus_g u_1(t), u_2(s) \ominus_g u_2(t) \cdots, u_n(s) \ominus_g u_n(t)). \tag{2.22}$$

Example 2.15. *Let* $f(t) = (t_1, t_2, t_3) = (e^t, \frac{e^t(e^t-1)}{3}, \frac{e^t}{3}) \in R^3,\ t \in R$*, be a 3-dimensional real-vector-valued function, and* $\mathbf{u} = (u_1, u_2, u_3) \in (E)^3$ $(u_i \in E,\ i = 1, 2, 3)$ *be a 3-dimensional fuzzy-vector-valued function, where*

$$u_1 = \begin{cases} x + 2, & -2 \le x \le -1, \\ 1, & -1 \le x \le 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$u_2 = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$u_3 = \begin{cases} 1 - x, & 0 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then according to Definition 2.13, we have the following 1*-dimensional fuzzy-number-valued function* $F : (0, \infty)^2 \to E$ *and*

$$F(t)(x) = f\mathbf{u} = f_1u_1 + f_2u_2 + f_3u_3 \begin{cases} \frac{2e^t+x}{e^t}, & -2e^t \le x \le -e^t, \\ 1, & -e^t \le x \le \frac{e^{2t}-e^t}{3}, \\ \frac{2e^t-3x}{e^t}, & \frac{e^{2t}-e^t}{3} \le x \le \frac{e^{2t}}{3}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\begin{aligned} F_r(t) &= [e^t(r-2), \frac{e^t(e^t-r)}{3}] \\ &= e^t[r-2, 0] + \frac{e^t(e^t-1)}{3}[0, 1] + \frac{e^t}{3}[0, 1-r],\ \forall r \in [0, 1]. \end{aligned}$$

Theorem 2.16. *Let* $\mathbf{F}, \mathbf{G} \in (E^n)^n$*. Then*
(i) if the g-difference exists, it is unique,
(ii) $\mathbf{F} \ominus_g \mathbf{F} = \mathbf{0}$,
(iii) $(\mathbf{F} + \mathbf{G}) \ominus_g \mathbf{G} = \mathbf{F}$, $(\mathbf{F} + \mathbf{G}) \ominus_g \mathbf{F} = \mathbf{G}$,
(iv) $\mathbf{F} \ominus_g \mathbf{G} = -(\mathbf{G} \ominus_g \mathbf{F})$.

Proof. It is not difficult to obtain from Proposition 2.5 and Definition 2.14.

Definition 2.17. *Let* $\mathbf{F} : M \to (E^n)^n$ *be a fuzzy-vector-valued function, denoted by* $\mathbf{F}(t) = (u_1(t), u_2(t), \cdots, u_n(t))$*, where* $u_i(t)$ $(i = 1, 2, \cdots, n)$ *is a fuzzy-number-valued function on* M*.*
(1) $\mathbf{F}$ *is said to be lower semicontinuous (l.c.) at* $t_0 \in M$ *if there exists a neighborhood* U *of* t_0*, any* $u_i(t)$ $(i = 1, 2, \cdots, n)$ *is l.c. at* t_0*.*
(2) $\mathbf{F}$ *is said to be upper semicontinuous (u.c.) at* $t_0 \in M$ *if there exists a neighborhood* U *of* t_0*, any* $u_i(t)$ $(i = 1, 2, \cdots, n)$ *is u.c. at* t_0*.*
A fuzzy-vector-valued function $\mathbf{F} : M \to (E^n)^n$ *is continuous at* $t_0 \in M$*, if it is both l.c. and u.c. at* t_0*, and that it is continuous if and only if it is continuous at every point of* M*.*

Definition 2.18. *Let $F : M \to E^n$ be a fuzzy-number-valued function, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$. If g-difference $F(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g F(t_1^0, \cdots, t_j^0, \cdots, t_m^0)$ exists and there exists $u_j \in E^n$ ($j = 1, 2, \cdots, m$), such that*

$$\lim_{h \to 0} \frac{F(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g F(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h} = u_j,$$

then we say that F has the jth partial generalized derivative (g-derivative for short) at t_0, denoted by $u_j = \partial F / \partial t_j^0$. Here the limit is taken in the metric space (E^n, D). If all the partial g-derivatives at t_0 exist, then we say F is said to be generalized differentiable (g-differentiable for short) on t_0. If F is g-differentiable at any interior point of M, then F is said to be g-differentiable on M. The fuzzy vector $(u_1, u_2, \cdots, u_m) \in (E^n)^m$ is said to be the gradient of F at t_0, denoted by $\nabla F(t_0)$, that is,

$$\nabla F(t_0) = (u_1, u_2, \cdots, u_m) = (\partial F/\partial t_1^0, \partial F/\partial t_2^0, \cdots, \partial F/\partial t_m^0).$$

In addition, $t_0 \in M$ is said to be a stationary point of F if $\nabla F(t_0) = \mathbf{0}$.

Note that if $M = [a, b]$, then Definition 2.18 coincides with the definition of F is g-differentiable on $[a, b]$ proposed by Gong and Hai ([41]).

We call $\mathbf{F} : [a, b] \to (L(E^n))^n$, denoted by $\mathbf{F} = (F_1, F_2, \cdots, F_n)$, is an n-dimensional fuzzy n-cell vector-valued function (fuzzy n-cell vector-valued function for short). If $\mathbf{F} = (f_1(t)u_1, f_2(t)u_2, \cdots, f_n(t)u_n)$, where $f_i : [a, b] \to R,\ u_i \in L(E^n),\ i = 1, 2, \cdots n$, then the gradient of $\mathbf{F}$ at t_0 is defined as

$$\nabla \mathbf{F}(t_0) = (\nabla F_1, \nabla F_2, \cdots, \nabla F_n),$$

and it is not difficult to obtain $\nabla F_i = (u_i \partial f_i/\partial t_1^0, u_i \partial f_i/\partial t_2^0, \cdots, u_i \partial f_i/\partial t_m^0),\ i = 1, 2, \cdots n$.

Definition 2.19. *The function $\eta : M \times M \to R^n$ is said to be a skew function if*

$$\eta(x, y) = -\eta(y, x),\ \ \forall x, y \in M. \tag{2.23}$$

Definition 2.20. *[35] An n-dimensional fuzzy set u is a fuzzy cone if $u(\gamma x) = u(x)$ for all $\gamma > 0$ and $x \in R^n$.*

Definition 2.21. *Let $\mathbf{A} = (u_1, u_2, \cdots, u_n) \in (E^n)^n$ ($u_i \in E^n,\ i = 1, 2, \cdots, n$) be an n-dimensional fuzzy vector. A fuzzy dual cone of $\mathbf{A}$ is the n-dimensional fuzzy vector $\mathbf{A}^\star$ given by*

$$\mathbf{A}^\star(y) = \Big(\inf_{x \in R^n : xy < 0}(1 - u_1(x)), \inf_{x \in R^n : xy < 0}(1 - u_2(x)), \cdots, \inf_{x \in R^n : xy < 0}(1 - u_n(x))\Big) \tag{2.24}$$

for nonzero $y \in R^n$, and $\mathbf{A}^\star(0) = (\underbrace{1, 1, \cdots, 1}_{n})$.

Notice that if $\mathbf{A} = (u) \in E^n$ is a 1-dimensional fuzzy vector, i.e., an n-dimensional fuzzy number, then Definition 2.21 reduces to Definition 8 proposed in [35].

3 Generalized convex fuzzy-number-valued functions

It is well known that the role of generalized monotonicity of the operator in vector variational inequality problems corresponds to the role of generalized convexity of the objective function in the optimization problem. In this section, we generalize convexity from vector-valued maps to fuzzy number-valued functions. The concepts of invexity and generalized monotonicity for n-dimensional fuzzy-number-valued functions are presented and some relative properties are discussed. In the following, suppose $M \subseteq R^n$ be a convex set.

Definition 3.1. *The mapping $\mathbf{F} : M \to (E^n)^n$ is said to be*
(1) fuzzy monotone over M if

$$(y - x)(\mathbf{F}(y) \ominus_g \mathbf{F}(x)) \succeq_c 0,\ \forall x, y \in M. \tag{3.1}$$

(2) fuzzy invex monotone over M if there exists a continuous map $\eta : M \times M \to R^n$ such that

$$\eta(y,x)(\mathbf{F}(y) \ominus_g \mathbf{F}(x)) \succeq_c 0, \ \forall x, y \in M. \tag{3.2}$$

Note that this definition reduces to the definition of monotone functions if $\eta(y,x) = y - x$.
(3) fuzzy strictly invex monotone over M if there exists a continuous map $\eta : M \times M \to R^n$ such that

$$\eta(y,x)(\mathbf{F}(y) \ominus_g \mathbf{F}(x)) \succeq_c 0, \ \forall x, y \in M, \ x \neq y. \tag{3.3}$$

Definition 3.2. *A g-differentiable fuzzy mapping $F : M \to E^n$ is called*
(1) fuzzy invex (FIX) with respect to a function $\eta : M \times M \to R^n$, if for all $x, y \in M$

$$F(x) \ominus_g F(y) \succeq_c \eta(x,y)\nabla F(y). \tag{3.4}$$

(2) fuzzy strictly invex (FSIX) with respect to a function $\eta : M \times M \to R^n$, if for all $x, y \in M$

$$F(x) \ominus_g F(y) \succ_c \eta(x,y)\nabla F(y), \ \ \forall x \neq y. \tag{3.5}$$

(3) fuzzy incave (FIC) with respect to a function $\eta : M \times M \to R^n$, if for all $x, y \in M$

$$F(x) \ominus_g F(y) \preceq_c \eta(x,y)\nabla F(y). \tag{3.6}$$

(4) fuzzy strictly incave (FSIC) with respect to a function $\eta : M \times M \to R^n$, if for all $x, y \in M$

$$F(x) \ominus_g F(y) \prec_c \eta(x,y)\nabla F(y), \ \ \forall x \neq y. \tag{3.7}$$

Theorem 3.3. *The function $F : M \to E^n$ will be a fuzzy invex fuzzy-number-valued function with respect to some function η if and only if each stationary point of F is a global minimum point.*

Proof. Necessity. Let F be a fuzzy invex fuzzy-number-valued function with respect to some function η. If x_0 is a stationary point of F, then $\nabla F(x_0) = \mathbf{0}$. Since F is fuzzy invex, using (3.4), we have $F(x) \ominus_g F(x_0) \succeq_c \eta(x,y)\nabla F(x_0) = 0, \ \forall x \in M$. Thus, we obtain $F(x_0) \preceq_c F(x), \ \forall x \in M$. Therefore, x_0 is a global minimum point.

Sufficiency. If y is a stationary point of F, i.e., $\nabla F = \mathbf{0}$, and also a global minimum point of F, then for a function $\eta : M \times M \to R^n$, we have

$$F(x) \ominus_g F(y) \succeq_c 0 = \eta(x,y)\nabla F, \ \ \forall x \in M.$$

Therefore, F is a fuzzy invex fuzzy-number-valued function. □

Theorem 3.4. *If a g-differentiable fuzzy mapping $F : M \to E^n$ is fuzzy invex on M with respect to $\eta : M \times M \to R^n$ and η is a skew function. Then, $\nabla F : M \to (E^n)^n$ is fuzzy invex monotone with respect to the same η.*

Proof. By the fuzzy invexity of F, there exists $\eta(x,y) \in R^n$, such that

$$F(x) \ominus_g F(y) \succeq_c \eta(x,y)\nabla F(y), \ \ \forall x, y \in M.$$

By changing x for y,

$$F(y) \ominus_g F(x) \succeq_c \eta(y,x)\nabla F(x).$$

Adding the above two formulas, we obtain

$$0 \succeq_c \eta(x,y)\nabla F(y) + \eta(y,x)\nabla F(x).$$

Since η is a skew function, $\eta(y,x) = -\eta(x,y)$, thus, we have

$$\eta(y,x)\nabla F(y) \ominus_g \eta(y,x)\nabla F(x) \succeq_c 0,$$

that is,

$$\eta(y,x)(\nabla F(y) \ominus_g \nabla F(x)) \succeq_c 0.$$

Therefore, ∇F is fuzzy invex monotone. □

Theorem 3.5. *If a g-differentiable fuzzy mapping $F : M \to E^n$ is fuzzy strictly invex on M with respect to $\eta : M \times M \to R^n$ and η is a skew function. Then, $\nabla F : M \to (E^n)^n$ is fuzzy strictly invex monotone with respect to the same η.*

Proof. By the fuzzy invexity of F, there exists $\eta(x, y) \in R^n$, such that

$$F(x) \ominus_g F(y) \succ_c \eta(x, y)\nabla F(y), \quad \forall x, y \in M.$$

By changing x for y,

$$F(y) \ominus_g F(x) \succ_c \eta(y, x)\nabla F(x).$$

Adding the above two formulas, we obtain

$$\eta(x, y)\nabla F(y) + \eta(y, x)\nabla F(x) \prec_c 0.$$

Since η is a skew function, $\eta(y, x) = -\eta(x, y)$, thus, we have

$$\eta(y, x)\nabla F(y) \ominus_g \eta(y, x)\nabla F(x) \succ_c 0,$$

that is,

$$\eta(y, x)(\nabla F(y) \ominus_g \nabla F(x)) \succ_c 0.$$

Therefore, ∇F is fuzzy strictly invex monotone. □

4 Variational-like inequalities for fuzzy-vector-valued functions

The existing results on the variational inequality problems for fuzzy mappings and their applications were based on Zadeh's decomposition theorem and were formally characterized by the precise sets which are the fuzzy mappings' cut sets directly. In this section, the fuzzy variational-like inequality problems is incorporated into the framework of n-dimensional fuzzy number space and proposed by means of the new ordering of two n-dimensional fuzzy-number-valued functions we proposed in [29]. In addition, we give the extension principle of the fuzzy variational inequality problems.

Theorem 4.1. *(Decomposition theorem)[39] If $u \in E^n$, then*

$$u = \bigcup_{\lambda\in[0,1]} (\lambda \cdot [u]^\lambda). \tag{4.1}$$

Let $f : M \to E^n$ be a fuzzy-number-valued function, then $\forall x \in M$, $[f(x)]^\alpha = f_\alpha(x) = f(x)(\alpha) = f(x, \alpha) = \{x \in R^n : f(x) \geqslant \alpha\}$, $\alpha \in [0, 1]$, denotes the α-cut set of f. According to Theorem 2.2, $\forall x \in M$, $f_\alpha(x) \subseteq \mathcal{K}_C^n \subseteq 2^{R^n}$, where 2^{R^n} is the family of all nonempty subsets of R^n.

Definition 4.2. *Let M be a closed and convex set in R^m. Given a continuous mapping $\eta : M \times M \to R^n$.*

(1) The variational-like inequality problem for n-dimensional fuzzy mappings (fuzzy variational-like inequality problem for short), denoted by FVLIP(M, $\mathbf{F}$, η), is to find $x \in M$ such that

$$\eta(x, y)\mathbf{F}(x) \succeq_c 0, \quad \forall y \in M, \tag{4.2}$$

where $\mathbf{F} : M \to (E^n)^n$ is a continuous fuzzy-vector-valued mapping.

(2) The variational inequality problem for n-dimensional fuzzy mappings (fuzzy variational inequality problem for short), denoted by FVIP(M, $\mathbf{F}$), is to find $x \in M$ such that

$$(y - x)\mathbf{F}(x) \succeq_c 0, \quad \forall y \in M, \tag{4.3}$$

where $\mathbf{F} : M \to (E^n)^n$ is a continuous fuzzy-vector-valued mapping.

(3) The generalized variational-like inequality problem for n-dimensional fuzzy mappings (generalized fuzzy variational-like inequality problem for short), denoted by GFVLIP(M, $\mathbf{F}$, η), is to find $x \in M$ with $\mathbf{x}^* \in \mathbf{F}(x)$ such that

$$\eta(x, y)\mathbf{x}^* \succeq_C 0, \quad \forall y \in M. \tag{4.4}$$

where $\mathbf{F} : M \to 2^{(E^n)^n}$ is a continuous set-valued fuzzy vector mapping, and $2^{(E^n)^n}$ is the family of all nonempty subsets of $(E^n)^n$.

(4) The generalized variational inequality problem for n-dimensional fuzzy mappings (generalized fuzzy variational inequality problem for short), denoted by GFVIP(M, $\mathbf{F}$), is to find $x \in M$ with $\mathbf{x}^* \in \mathbf{F}(x)$ such that

$$(y - x)\mathbf{x}^* \succeq_C 0, \quad \forall y \in M. \tag{4.5}$$

where $\mathbf{F} : M \to 2^{(E^n)^n}$ is a continuous set-valued fuzzy vector mapping, and $2^{(E^n)^n}$ is the family of all nonempty subsets of $(E^n)^n$.

Here we would like to point out that (FVLIP) and (FVIP) include many kinds of variational inequality problems as their special cases. For example,

(i) If $\mathbf{F} : M \to R^n$ is a continuous real-vector-valued mapping, and $C = R^+ = [0, \infty)$, then (4.3) reduces to the classical variational inequality problem: to finding $x \in K$ such that

$$(y - x)F(x) \geq 0, \quad \forall y \in M, \tag{4.6}$$

which was considered by Stampacchia [1].

(ii) If $\mathbf{F} : M \to 2^{R^n}$ is a continuous set-valued real-vector mapping, then (4.5) reduces to the classical generalized variational inequality problem: to finding $x \in M$ with $x^* \in \mathbf{F}(x)$ such that

$$(y - x)x^* \geq 0, \quad \forall y \in M. \tag{4.7}$$

This problem was considered and studied by Noor [24].

Suppose for any $r \in [0, 1]$, there exists an $A^r \subseteq R^n$ which satisfies the conditions (i)-(iii) in Theorem 2.2, then there exists a unique $F \in E^n$ such that $[F]^r = A^r$, $r \in (0, 1]$, $[F]^0 = \overline{\bigcup_{r\in(0,1]}[F]^r} \subseteq A^0$. We denote $A = \{A^r : r \in [0, 1]\}$, then $A \subseteq \mathcal{K}_C^n \subseteq 2^{R^n}$.

(iii) Let $G : M \to A$ be a set-valued mapping. Now we define a 1-dimensional fuzzy-vector-valued mapping $\mathbf{F}$ by

$$\mathbf{F} : M \to \mathcal{F}(A), \quad x \mapsto r \cdot \chi_A(x) \subseteq (E^n)^1,$$

where $\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$ is the characteristic function of the set A. Then (4.3) is equivalent to the variational inequality for fuzzy mapping, which was considered and studied by Noor [17], i.e., is to find $x \in M$ with $x^* \in G(x)$ such that

$$(y - x)x^* \geq 0, \quad \forall y \in M. \tag{4.8}$$

(iv) Let $G : M \to A$ be a set-valued mapping. Now we define a 1-dimensional fuzzy-vector-valued mapping $\mathbf{F}$ by

$$\mathbf{F} : M \to \mathcal{F}(A), \quad x \mapsto r \cdot \chi_A(x) \subseteq (E^n)^1,$$

where $\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$ is the characteristic function of the set A. Then (4.2) is equivalent to the variational-like inequality for fuzzy mapping,which was considered and studied by Rufián-Lizana [44], i.e., is to find $x \in M$ with $x^* \in G(x)$ such that

$$\eta(x, y)x^* \geq 0, \quad \forall y \in M. \tag{4.9}$$

Let $a : R^n \to [0,1]$ be a function, we have $f_{a(x)} = \{x \in R^n : f(x) \geqslant a(x)\}$. $\forall x \in R^n$, suppose for any $a(x) \in [0,1]$, there exists an $A^{a(x)} \subseteq R^n$ which satisfies the conditions (i)-(iii) in Theorem 2.2, then there exists a unique $F \in E^n$ such that $[F]^{a(x)} = A^{a(x)}$, $a(x) \in (0,1]$, and $[F]^0 = \overline{\bigcup_{a(x)\in(0,1]}[F]^{a(x)}} \subseteq A^0$. We denote $A = \{A^{a(x)} : a(x) \in [0,1]\}$, then $A \subseteq \mathcal{K}_C^n \subseteq 2^{R^n}$.

(v) Let $G : M \to A$ be a set-valued mapping. Now we define a 1-dimensional fuzzy-vector-valued mapping $\mathbf{F}$ by

$$\mathbf{F} : M \to \mathcal{F}(A),\;\; x \mapsto a(x) \cdot \chi_A(x) \subseteq (E^n)^1,$$

where $\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$ is the characteristic function of the set A. Then (4.2) is equivalent to the variational inequality for fuzzy mapping: is to find $x \in M$ with $\mathbf{x}^\star \in \mathbf{F}_{a(x)}(x)$ such that

$$(y - x)\mathbf{x}^\star \succeq_C 0,\;\; \forall y \in M. \tag{4.10}$$

This problem was studied by Huang [16], where the cut $a(x)$ depends on x. It is slightly different from that of Noor [17], where the cut is a constant. The advantages are that the cuts have more freedom than those of Noor, and the model includes that of Noor as a special case in the viewpoint of mathematics.

Remark 4.3. *Let M be a convex cone in R^m and $\mathbf{F} : M \to (E^n)^n$. The fuzzy variational-like inequality problem is called complementarity-like problem, denoted by NCLP($\mathbf{F}$). The NCLP($\mathbf{F}$) is an important special case of FVILP($M, \mathbf{F}, \eta$). That is, the NCLP($\mathbf{F}$) is to find $x^\star \in M$ such that*

$$\mathbf{F}(x^\star) \in \mathbf{F}^\star,\;\; \eta(x^\star)\mathbf{F}(x^\star) = 0, \tag{4.11}$$

where $\mathbf{F}^\star$ denotes the fuzzy dual cone of $\mathbf{F}$, i.e.,

$$\mathbf{F}^\star(y) = \Big(\inf_{x\in D:xy<0}(1 - u_1(x)),\; \inf_{x\in M:xy<0}(1 - u_2(x)), \cdots, \inf_{x\in M:xy<0}(1 - u_n(x))\Big),\;\; y \in M.$$

Remark 4.4. *If $\mathbf{F} : M \to (L(E^n))^n$, the fuzzy variational-like inequality problem is called the fuzzy box constrained variational-like inequality problem, denoted by FBVLIP($M, \mathbf{F}, \eta$).*

Example 4.5. *If $\mathbf{F} : M \to (L(E^n))^n$ be fuzzy n-cell vector-valued function, then (FBVLIP) is to find $x \in M$ such that*

$$\eta(x, y)F(x) \succeq_C 0,\;\; \forall y \in M, \tag{4.12}$$

which is equivalent to

$$\tau(\eta(x, y)F(x)) \in \tau(0) + C,\;\; \forall y \in M, \tag{4.13}$$

where $C = R^{n+} = \{(x_1, x_2, \cdots, x_n) \in R^n : x_1 \geqslant 0,\; x_2 \geqslant 0, \cdots,\; x_n \geqslant 0\} \subseteq R^n$. Suppose that $\mathbf{F} = (F_1, F_2, \cdots, F_n)$, $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$, then (FVIP) is to find $x \in M$ such that

$$\tau(\eta(x, y)\mathbf{F}(x)) = \tau\Big(\sum_{i=1}^{n} \eta_i F_i(x)\Big) = \sum_{i=1}^{n} \eta_i \tau(F_i(x)) \geq 0, \tag{4.14}$$

where $\tau(F_i(x)) = \big(\int_0^1 r(F_{i1}^-(x)(r) + F_{i1}^+(x)(r))dr,\; \int_0^1 r(F_{i2}^-(x)(r) + F_{i2}^+(x)(r))dr, \cdots, \int_0^1 r(F_{in}^-(x)(r) + F_{in}^+(x)(r))dr\big)$, $i = 1, 2, \cdots n$.

Theorem 4.6. *Let M be a nonempty, compact and convex subset of R^m and let $\mathbf{F}$ be a continuous mapping from X into $(E^n)^n$. Then there exists a solution to the problem FVLIP($M, \mathbf{F}, \eta$), that is, there exists $x_0 \in M$ such that*

$$\eta(y, x_0)\mathbf{F}(x_0) \succeq_C 0,\;\; \forall y \in M. \tag{4.15}$$

Proof. If M is a point, the theorem is trivial. If M is not a point, then it can be supposed that M has interior points for otherwise, without loss of generality, R^n is replaced by a suitable subspace of R^n containing M. Since a translation of the space R^n dose not affect the assumption or assertion, it can be supposed that $x = 0$ is an interior point of M. We denote a half-space by $\partial M = \{x \in R^m : (x - p)n \le 0\}$, where p is a point in R^m and n is an nonzero vector in R^m.

Let $x_0 \in \partial M$. Then (4.15) holds if and only if there is a hyperplane π through x_0, that is, $\pi = \{x \in R^m : (x - p)n = 0\}$, supporting M such that if $\mathbf{N} \ne 0$ is a fuzzy vector which is fuzzy orthogonal to π and pointing into the half-space not containing M, then $\mathbf{F}(x_0) = -t\mathbf{N}$ for some $t \ge 0$.

*Case*1. ∂M is of class C^1. Assume that (4.15) fails to hold for all $x_0 \in M$. We shall show that

$$\mathbf{F}(x) = \mathbf{0} \tag{4.16}$$

has a solution $x_0 \in M$, which satisfies (4.15) trivially.

Let $\mathbf{N}(x_0)$ be the outward, unit normal vector at $x_0 \in \partial M$. Then

$$\mathbf{F}(x_0, t) = (1 - t)\mathbf{F}(x_0) + t\mathbf{N}(x_0), \; 0 \le t \le 1,$$

is a deformation of the vector field $\mathbf{F}(x_0)$, $x_0 \in \partial M$, into the vector field $\mathbf{N}(x_0)$. The assumption that (4.15) dose not hold for $x_0 \in \partial M$ implies that $\mathbf{F}(x_0) \ne \mathbf{0}$ for $x_0 \in \partial M$, $0 \le t \le 1$. Hence the indices of the vector fields $\mathbf{F}(x_0)$, $\mathbf{N}(x_0)$ with respect to $x = 0$ are identical.

There is a deformation $\mathbf{D}(x_0, s) = (1 - s)\mathbf{N}(x_0) + sx_0$, $0 \le s \le 1$, of $\mathbf{N}(x_0)$ into x_0 and $\mathbf{D}(x_0, s) \ne \mathbf{0}$ since $x = 0$ is an interior point of M. Since the vector field x_0, $x_0 \in \partial M$, has index 1 with respect to $x = 0$, the index of $\mathbf{N}(x_0)$ and, hence, of $\mathbf{F}(x_0)$ is 1. This proves that (4.16) has a solution in M.

*Case*2. ∂M is not of class C^1. By a theorem of Minkowski (see [45], pp. 36-37), there exists a sequence of compact convex sets $M_1 \subseteq M_2 \subseteq \cdots$ such that M is the closure of the union $M_1 \cup M_2 \cup \cdots$ and ∂M_m, $m = 1, 2, \cdots$, is of class C^1. By case 1, there exists $x_m \in M$ satisfying

$$\eta(y, x_m)\mathbf{F}(x_m) \succeq_c 0, \;\; \forall y \in M_m.$$

After a selection of a subsequence, it can be supposed that $x_0 = \lim x_m$ exists. Then, by continuity of $\mathbf{F}$, it follows that

$$\eta(y, x_0)\mathbf{F}(x_0) \succeq_c 0, \;\; \forall y \in M_m.$$

This implies (4.15) and completes the proof. □

Corollary 4.7. *Let M be a nonempty, closed and fuzzy invex subset of R^m and let $\mathbf{F} : R^n \to (E^n)^n$ be continuous. If there exists a nonempty bounded subset B of M such that for every $x \in M \setminus B$ there is a $y \in B$ with*

$$\eta(x, y)\mathbf{F}(x) \ge 0,$$

then the problem FVLIP($M, \mathbf{F}, \eta$) has a solution.

5 Relationship between fuzzy variational-like inequality problems and fuzzy optimization problems

In this section, we investigate the relationships between fuzzy variational-like inequality problems and fuzzy optimization problems.

The Fuzzy Optimization Problem (FOP) is defined as

$$\begin{array}{ll} \min & f(t) \\ \text{subject to} & t \in M, \end{array} \tag{5.1}$$

where M is closed and convex set and in R^n and $f : M \to E^n$ is continuously g-differentiable.

Theorem 5.1. *Suppose that $f : M \to E^n$ is fuzzy invex with respect to some continuous map $\eta : M \times M \to R^n$. If $t^\star \in M$ is a solution to FVLIP(M, F, η), where $\mathbf{F}(t) = \nabla f$, then $t^\star$ is a solution to the (FOP).*

Proof. By the fuzzy invexity of f, we have

$$f(t) \ominus_g f(t^\star) \succeq_c \eta(t, t^\star)\nabla f(t^\star), \ \forall t \in M.$$

Since $t^\star \in M$ is a solution to FVLIP(M, F, η), we have

$$\eta(t, t^\star)\mathbf{F}(t^\star) \succeq_c 0, \ \forall t \in M.$$

Now, setting $\mathbf{F}(t^\star) = \nabla f(t^\star)$, we obtain

$$f(t) \ominus_g f(t^\star) \succeq_c 0, \ \forall t \in M,$$

that is,

$$f(t) \succeq_c f(t^\star), \ \forall t \in M.$$

Thus, we have

$$f(t^\star) = \min_{t \in M} f(t).$$

Therefore, $t^\star$ is a solution to the (FOP). □

Theorem 5.2. *Let $K \subseteq R^n$ be an invex set with respect to η, $x^\star \in K$, and $F : K \to E^n$ be a g-differentiable incave fuzzy mapping (FIC) with respect to η. If $x^\star$ is a strictly local optimal solution to (FOP), then $(x^\star, \nabla F(x^\star))$ is a solution to (FVLIP).*

Proof. Let $x^\star$ be a strictly local optimal solution to (FOP). By contradiction, suppose that there exists an $\overline{x} \in K$ such that

$$\eta(\overline{x}, x^\star)\nabla F(x^\star) \preceq_c 0.$$

Since F is a g-differentiable incave fuzzy mapping,

$$F(\overline{x}) \ominus_g F(x^\star) \preceq_c \nabla\eta(\overline{x}, x^\star)F(x^\star).$$

Thus, we have

$$F(\overline{x}) \preceq_c F(x^\star).$$

This contradicts the fact that $x^\star$ is a strictly local optimal solution of (FOP). □

Theorem 5.3. *Let $K \subseteq R^n$ be an open invex set with respect to η, $x^\star \in K$, and $F : K \to E^n$ be a g-differentiable strictly incave fuzzy mapping (FSIC) with respect to η. If $x^\star$ is an optimal solution of (FOP), then $(x^\star, \nabla F(x^\star)$ is a solution to (FVLIP).*

Proof. Let $x^\star$ be an optimal solution to (FOP). By contradiction, suppose that there exists an $\overline{x} \in K$ such that

$$\eta(\overline{x}, x^\star)\nabla F(x^\star) \preceq_c 0.$$

Since F is a g-differentiable strictly incave fuzzy mapping,

$$F(\overline{x}) \ominus_g F(x^\star) \prec_c \nabla\eta(\overline{x}, x^\star)F(x^\star).$$

Therefore,

$$F(\overline{x}) \prec_c F(x^\star).$$

This contradicts the fact that $x^\star$ is an optimal solution to (FOP). □

6 Fuzzy multiobjective optimization

In this section, we investigate the optimality conditions for the multiobjective optimization problems.

The Fuzzy Multiobjective Optimization Problem (FMOP1) is defined as

$$\begin{array}{ll} \min & \mathbf{F}(x) = (f_1(x), f_2(x), \cdots, f_p(x)) \\ \text{subject to} & \mathbf{G}(x) \leq \mathbf{0}, \\ & \mathbf{H}(x) = \mathbf{0}, \\ & x \in M, \end{array} \tag{6.1}$$

where $M \subseteq R^m$ is closed and convex set, the objective function $\mathbf{F}(x) : M \to (L(E^n))^p$ is a fuzzy-vector-valued function, $\mathbf{G}(x) : M \to (L(E^n))^l$ and $\mathbf{H}(x) : M \to (L(E^n))^t$ in constraint conditions are fuzzy-vector-valued functions, denoted by $\mathbf{G}(x) = (g_1(x), g_2(x), \cdots, g_l(x))$, $\mathbf{H}(x) = (h_1(x), h_2(x), \cdots, h_t(x))$, where $f_i(x), g_k(x), h_s(x) : M \to E^n,\ i = 1, 2, \cdots, p,\ k = 1, 2, \cdots, l,\ s = 1, 2, \cdots, t$.

$X = \{x \in M : \mathbf{G}(x) \leq \mathbf{0},\ \mathbf{H}(x) = \mathbf{0}\}$ is said to be the feasible set of (FMOP1). Let $x_0 \in X$, if there does not exist $x \in M$ such that $\mathbf{F}(x) \leq \mathbf{F}(x_0)$, then x_0 is said to be an optimal solution to (FMOP1).

Let $C = R^{n+} = \{(x_1, x_2, \cdots, x_n) \in R^n : x_1 \geqslant 0,\ x_2 \geqslant 0, \cdots,\ x_n \geqslant 0\} \subseteq R^n$. Then we have

$$\mathbf{G}(x) \leq \mathbf{0} \Longleftrightarrow \tau(g_k(x)) \leq 0\ (0 \in R^n),\ k = 1, 2, \cdots, l. \tag{6.2}$$

where $\tau(g_k(x)) = (\int_0^1 r(g_{k1}^-(x)(r) + g_{k1}^+(x)(r))dr, \int_0^1 r(g_{k2}^-(x)(r) + g_{k2}^+(x)(r))dr, \cdots, \int_0^1 r(g_{kn}^-(x)(r) + g_{kn}^+(x)(r))dr)$, $k = 1, 2, \cdots l$, thus, we obtain

$$\mathbf{G}(x) \leq \mathbf{0} \Longleftrightarrow \int_0^1 r(g_{kj}^-(x)(r) + g_{kj}^+(x)(r))dr \leq 0\ (0 \in R),\ k = 1, 2, \cdots, l,\ j = 1, 2, \cdots, n. \tag{6.3}$$

Similarly, we have

$$\mathbf{H}(x) = \mathbf{0} \Longleftrightarrow \int_0^1 r(h_{sj}^-(x)(r) + h_{sj}^+(x)(r))dr = 0\ (0 \in R),\ s = 1, 2, \cdots, t,\ j = 1, 2, \cdots, n. \tag{6.4}$$

We denote $G_{k'}(x) = \int_0^1 r(g_{kj}^-(x)(r) + g_{kj}^+(x)(r))dr$, $H_{s'}(x) = \int_0^1 r(h_{sj}^-(x)(r) + h_{sj}^+(x)(r))dr$, $k' = 1, 2, \cdots, l \times n,\ s' = 1, 2, \cdots, t \times n$, then the fuzzy multiobjective optimization problem (FMOP1) can be transformed into the following fuzzy multiobjective optimization problem (FMOP2)

$$\begin{array}{ll} \min & \mathbf{F}(x) = (f_1(x), f_2(x), \cdots, f_p(x)) \\ \text{subject to} & G_{k'}(x) \leq 0, \\ & H_{s'}(x) = 0, \\ & x \in M, \end{array} \tag{6.5}$$

where $G_{k'}, H_{s'} : M \to R$.

Obviously, the feasible set of (FMOP2) is equivalent to the feasible set of (FMOP1).

In the following, suppose that the feasible set of (FMOP2) $X = \{x \in \mathrm{int}M : G_{k'}(x) \leq 0,\ H_{s'}(x) = 0,\ k' = 1, 2, \cdots, l \times n,\ s' = 1, 2, \cdots, t \times n\} \subseteq \mathcal{K}_C^n$, the real-valued functions $G_{k'}(x),\ k' = 1, 2, \cdots, l \times n$, are convex on M, continuous and differentiable at $x_0 \in X$.

Definition 6.1. *Let $\mathbf{F} : M \to (L(E^n))^p$, denoted by $\mathbf{F}(x) = (f_1(x), f_2(x), \cdots, f_p(x))$. If for any $x_1, x_2 \in \mathrm{int}M$ and $\lambda \in [0, 1]$, the inequalities*

$$f_{ij}^-(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f_{ij}^-(x_1, r) + (1 - \lambda)f_{ij}^-(x_2, r),\ i = 1, 2, \cdots, p,\ j = 1, 2, \cdots, n, \tag{6.6}$$

and

$$f_{ij}^+(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f_{ij}^+(x_1, r) + (1 - \lambda)f_{ij}^+(x_2, r),\ i = 1, 2, \cdots, p,\ j = 1, 2, \cdots, n, \tag{6.7}$$

uniformly hold for all $r \in [0, 1]$, then $\mathbf{F}(x)$ is said to be endpoint-wise fuzzy convex.

Definition 6.2. *Let* $\mathbf{F} : M \to (L(E^n))^p$, *denoted by* $\mathbf{F}(x) = (f_1(x), f_2(x), \cdots, f_p(x))$. *Then we say F is endpoint-wise differentiable at* x_0, *that is, if there exists* $u^-_{kij}, u^+_{kij} \in R$, $k = 1, 2, \cdots, p$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, *such that*

$$\lim_{h\to 0} \frac{f^-_{ki}(x^0_1, \cdots, x^0_j + h, \cdots, x^0_m, r) - f^-_{ki}(x^0_1, \cdots, x^0_j, \cdots, x^0_m, r)}{h} = u^-_{kij}, \ k = 1, 2, \cdots, p,$$

and

$$\lim_{h\to 0} \frac{f^+_{ki}(x^0_1, \cdots, x^0_j + h, \cdots, x^0_m, r) - f^+_{ki}(x^0_1, \cdots, x^0_j, \cdots, x^0_m, r)}{h} = u^+_{kij}, \ k = 1, 2, \cdots, p,$$

uniformly for $r \in [0, 1]$, *then we say* $\mathbf{F}$ *has jth partial endpoint-wise differentiable at* x_0, *and we denote* $\frac{\partial F^-_i(x_0,r)}{\partial x^0_j} = u^-_{ij}$, $\frac{\partial F^+_i(x_0,r)}{\partial x^0_j} = u^+_{ij}$. *If all the partial endpoint-wise derivatives at* x_0 *exist, then we say* $\mathbf{F}$ *is endpoint-wise differentiable at* x_0.

Theorem 6.3. *Let the objective function* $\mathbf{F} : M \to (L(E^n))^p$, *denoted by* $\mathbf{F}(x) = (f_1(x), f_2(x), \cdots, f_p(x))$, *be endpoint-wise fuzzy convex, let* $\mathbf{F}$ *be continuous and endpoint-wise differentiable at* $x_0 = \{x^0_1, x^0_2, \cdots, x^0_m\} \in \mathrm{int}M$. *If for any* $r \in [0, 1]$, *there exist* $\omega(r) = \{\omega_1(r), \omega_2(r), \cdots, \omega_p(r)\} \in R^{p+}$, $\alpha(r) = \{\alpha_1(r), \alpha_2(r), \cdots, \alpha_{l\times n}(r)\} \in R^{(l\times n)+}$ *and* $\beta(r) = \{\beta_1(r), \beta_2(r), \cdots, \beta_{t\times n}(r)\} \in R^{t\times n}$ *such that*

(1) $\sum\limits_{i=1}^{p} \omega_i(r) \frac{\partial f^-_{ij}(x,r)}{\partial x_{j'}}\big|_{x_0} + \sum\limits_{i=1}^{p} \omega_i(r) \frac{\partial f^+_{ij}(x,r)}{\partial x_{j'}}\big|_{x_0} + \sum\limits_{k'=1}^{l\times n} \alpha_{k'}(r) \frac{\partial G_{k'}(x)}{\partial x_{j'}}\big|_{x_0} + \sum\limits_{s'=1}^{t\times n} \beta_{s'}(r) \frac{\partial H_{s'}(x)}{\partial x_{j'}}\big|_{x_0} = 0$, $j' = 1, 2, \cdots, m$,

(2) $\alpha_{k'}(r) G_{k'}(x_0) = 0$, $k' = 1, 2, \cdots l \times n$,

then x_0 *is an optimal solution to (FMOP2).*

Note that $\omega(r), \alpha(r), \beta(r)$ *are called Lagrange multiplier vectors containing parameters, the condition (1) and (2) are called the Karush-Kuhn-Tucker (KKT for short) conditions for (FMOP2).*

Proof. $\forall r \in [0, 1]$, we denote $\overline{f}(x, r) = \{\overline{f}_1(x, r), \overline{f}_2(x, r), \cdots, \overline{f}_p(x, r)\}$, and

$$\overline{f}_{ij}(x, r) = f^-_i(x, r) + f^+_{ij}(x, r), \ i = 1, 2, \cdots, p, \ j = 1, 2, \cdots, n.$$

Since $\mathbf{F}$ is endpoint-wise fuzzy convex on M, and continuous and endpoint-wise differentiable at x_0, then the real-valued function $f^-_{ij}(x, r)$ and $f^+_{ij}(x, r)$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, n$, is convex on M, and continuous and differentiable at x_0. Therefore, for all $r \in [0, 1]$, $\overline{f}_{ij}(x, r)$ is convex, and continuous and differentiable at x_0, furthermore, we have

$$\frac{\partial \overline{f}_{ij}(x, r)}{\partial x_{j'}}\Big|_{x_0} = \frac{\partial f^-_{ij}(x, r)}{\partial x_{j'}}\Big|_{x_0} + \frac{\partial f^+_{ij}(x, r)}{\partial x_{j'}}\Big|_{x_0}, \ i = 1, 2, \cdots, p, \ j = 1, 2, \cdots, n, \ j' = 1, 2, \cdots, m.$$

Since $\forall r \in [0, 1]$, the KKT conditions are equivalent to

(1) $\sum\limits_{i=1}^{p} \omega_i(r) \frac{\partial \overline{f}_{ij}(x,r)}{\partial x_{j'}}\big|_{x_0} + \sum\limits_{k'=1}^{l\times n} \alpha_{k'}(r) \frac{\partial G_{k'}(x)}{\partial x_{j'}}\big|_{x_0} + \sum\limits_{s'=1}^{t\times n} \beta_{s'}(r) \frac{\partial H_{s'}(x)}{\partial x_{j'}}\big|_{x_0} = 0$,

(2) $\alpha_{k'}(r) G_{k'}(x_0) = 0$, $k' = 1, 2, \cdots l \times n$,

thus, x_0 is an optimal solution to the multiobjective optimization problem under this constraint conditions (1) and (2), where the mutiobjective function $\overline{f}(x, r) = \{\overline{f}_1(x, r), \overline{f}_2(x, r), \cdots, \overline{f}_p(x, r)\}$, that is $\forall x \in \mathrm{int}M$, we have $\overline{f}(x_0, r) \le \overline{f}(x, r)$, that is,

$$\overline{f}_{ij}(x_0, r) \le \overline{f}_{ij}(x, r), \ i = 1, 2, \cdots, p, \ j = 1, 2, \cdots, n, \tag{6.8}$$

By reductio ad absurdum, suppose that x_0 is not an optimal solution of (FMOP2), then there exists $x' \in \mathrm{int}M$ such that $\mathbf{F}(x') < \mathbf{F}(x_0)$.

Let $C = R^{n+} \subseteq R^n$, according to Definition 2.7, we have

$$\int_0^1 r(f^-_{ij}(x')(r) + f^+_{i1}(x')(r))dr < \int_0^1 r(f^-_{ij}(x_0)(r) + f^+_{i1}(x_0)(r))dr, \ i = 1, 2, \cdots, p, \ j = 1, 2, \cdots, n,$$

that is,

$$\int_0^1 r\bar{f}_{ij}(x', r)dr < \int_0^1 r\bar{f}_{ij}(x_0, r)dr,\ i = 1, 2, \cdots, p,\ j = 1, 2, \cdots, n,$$

which is in contradiction to (6.8). Therefore, x_0 is an optimal solution to (FMOP2). □

Theorem 6.4 Let the objective function $\mathbf{F} : M \to (L(E^n))^p$ be denoted by $\mathbf{F}(x) = (f_1(x)u_1, f_2(x)u_2, \cdots, f_p(x)u_p)$, where $f_i : [a, b] \to R,\ u_i \in L(E^n),\ i = 1, 2, \cdots p$, and $u_i \succeq_c 0$. Let $\mathbf{F}$ be endpoint-wise fuzzy convex, continuous and endpoint-wise differentiable at $x_0 = \{x_1^0, x_2^0, \cdots, x_m^0\} \in \mathrm{int}M$. If there exist $\omega = \{\omega_1, \omega_2, \cdots, \omega_p\} \in R^{p+}$, $\alpha(r) = \{\alpha_1(r), \alpha_2(r), \cdots, \alpha_{l\times n}(r)\} \in R^{(l\times n)+}$ and $\beta(r) = \{\beta_1(r), \beta_2(r), \cdots, \beta_{t\times n}(r)\} \in R^{t\times n}$ such that

(1) $\sum_{\omega_i=1}^{p} \omega_i \nabla f_i(x_0) + \sum_{k'=1}^{l\times n} \alpha_{k'}(r)\frac{\partial G_{k'}(x)}{\partial x_{j'}}\Big|_{x_0} + \sum_{s'=1}^{t\times n} \beta_{s'}(r)\frac{\partial H_{s'}(x)}{\partial x_{j'}}\Big|_{x_0} = 0,\ j' = 1, 2, \cdots, m,$

(2) $\alpha_{k'}(r)G_{k'}(x_0) = 0,\ k' = 1, 2, \cdots l \times n,$

then x_0 is an optimal solution to (FMOP2).

Note that ω, α, β are called Lagrange multiplier vectors.

Proof. By Definition 2.18, $\forall x_0 \in M$, we have $\nabla \mathbf{F}(x_0) = (\nabla(f_1(x_0)u_1), \nabla(f_2(x_0)u_2), \cdots, \nabla(f_p(x_0)u_p))$, and

$$\nabla(f_i(x_0)u_i) = (u_i\frac{\partial f_i}{\partial x_1^0}, u_i\frac{\partial f_i}{\partial x_2^0}, \cdots, u_i\frac{\partial f_i}{\partial x_m^0}),\ i = 1, 2, \cdots, p. \tag{6.9}$$

Since $\mathbf{F}$ is endpoint-wise fuzzy convex M, and continuous and endpoint-wise differentiable at x_0, then $f_i(x),\ i = 1, 2, \cdots, p$, is convex on M, continuous and differential at $x_0 \in \mathrm{int}M$, that is, the real-vector-valued function $f = (f_1(x), f_2(x), \cdots, f_p(x))$ is convex on M, continuous and differential at $x_0 \in \mathrm{int}M$. Consider the following multiobjective optimization problem

$$\begin{array}{ll} \min & f(x) = (f_1(x), f_2(x), \cdots, f_p(x)) \\ \text{subject to} & G_{k'}(x) \le 0, \\ & H_{s'}(x) = 0, \\ & x \in M. \end{array} \tag{6.10}$$

Obviously, the conditions (1) and (2) are the KKT conditions for this problem. Therefore, x_0 is an optimal solution to this problem, that is, $\forall x \in \mathrm{int}M$, we have $f(x_0) \le f(x)$, that is,

$$f_i(x_0) \le f_i(x),\ i = 1, 2, \cdots, p. \tag{6.11}$$

By reductio ad absurdum, suppose that x_0 is not an optimal solution to (FMOP2), then there exists $x' \in \mathrm{int}M$ such that $\mathbf{F}(x') < \mathbf{F}(x_0)$, that is, $f_i(x')u \prec_c f_i(x_0)u,\ i = 1, , 2, \cdots, p$.

Let $C = R^{n+} \subseteq R^n$, according to Definition 2.7, we have

$$\tau(f_i(x')u) \in \tau(f_i(x_0)u) + C,$$

thus, we obtain $f_i(x')\tau(u) \in f_i(x_0)\tau(u) + C$ and $f_i(x')\tau(u) \neq f_i(x_0)\tau(u)$. Since $u_i \succeq_c 0$, we have $f_i(x') < f_i(x_0)$, which is in contradiction to (6.11). Therefore, x_0 is an optimal solution to (FMOP2). □

7 Conclusions

We define the fuzzy variational-like inequality problems by using the new ordering of two n-dimensional fuzzy-number-valued functions, and the existence and the basic properties of the fuzzy variational inequality problems are also investigated. We examine the relationship between the variational-like inequality problems and fuzzy optimization problems. Additionally, we discuss the optimality conditions for fuzzy multiobjective optimization. The next step for the continuation of the research direction proposed here is to investigate ill-posedness and regularization methods of the fuzzy variational-like inequality problems, and the duality for the fuzzy multiobjective optimization problems.

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