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Variational methods to calculate the hydrostatic structure of rotating planets

S. Abad,[1] A.F. Pacheco[1] and J. Sañudo[2]

[1]*Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza,* 50009 *Zaragoza, Spain*
[2]*Departamento de Física, Facultad de Ciencias, Universidad de Extremadura,* 06071 *Badajoz, Spain*

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SUMMARY

Two variational strategies to calculate the internal flattening induced in the structure of slowly rotating hydrostatic planets are discussed. In the first procedure, the minimization of energy fixes the physical coefficients of a polynomial that describes the dependence of the flattening on depth. In the second, the planet is assumed to be divided into thin equidensity shells, and the condition of minimum energy leads to an algebraic method that can compete with the usual one based on Clairaut's equation. These methods are applied to the Earth. The differences between them and other previous variational strategies are discussed.

Key words: planetary structure, planetary rotation.

1 INTRODUCTION

The hydrostatic theory of slowly rotating self-gravitating bodies, and its application to the Earth, was developed by Clairaut and subsequently completed by other authors (see, for example, Jeffreys 1976). According to this theory, to the first order, the external flattening, $f(R) = (a - c)/a$, is given by

$$f(R) = (5m/2)\left\{1 + \left(\frac{25}{4}\right)\left[1 - \frac{3}{2}\left(\frac{C}{MR^2}\right)\right]^2\right\}^{-1}, \tag{1.1}$$

where m is the ratio of the centrifugal to the gravitational acceleration at the equator, C is the moment of inertia about the axis of rotation, M is the planetary mass, R is the mean spherical radius of the planet, and a and c are the equatorial and polar radii, respectively.

The value of the flattening at an arbitrary depth $f(r)$ ($0 \le r \le R$ is the spherically averaged radius of the spheroidal equidensity differential shells of the rotating planet) fulfils Clairaut's equation, i.e.

$$r^2\frac{d^2f}{dr^2} + 6\frac{\rho}{\rho_m}\left(f + r\frac{df}{dr}\right) - 6f = 0, \tag{1.2}$$

and reaches, at the external surface, the value given by eq. (1.1). $\rho(r)$ is the mass density, and $\rho_m(r)$ is its average within the sphere of radius r. Clairaut's equation is obtained simply by imposing the condition of hydrostatic equilibrium on the planetary structure.

As is well known, the condition of hydrostatic equilibrium can be derived from a variational principle (see, for example, Moritz 1990). From this perspective, one defines an energy functional, E, formed by the addition of the self-gravitating energy of the body to the centrifugal potential energy. No compressional energy term is included in E, because the radial density profile used in this method, $\rho(r)$, corresponds to that of a planet which is actually rotating (i.e. already expanded) and not to what it would be in the absence of rotation. The variational objective can be stated as the search for the eccentricity, existing at any depth, that, for a fixed radial density profile $\rho(r)$, minimizes the energy functional, E.

Having formulated the variational objective, one can devise different specific strategies to implement it, and compare their performances. In this paper we will develop two of these procedures. In the first, the squared eccentricity $\varepsilon^2(r)$ will be expressed as a polynomial, of various orders, in the r coordinate. In the limit, where ε is assumed to be independent of r, one obtains a mean eccentricity for the whole planet. Increasing the order of the polynomial, i.e. $\varepsilon = \bar{\varepsilon}_0 + \bar{\varepsilon}_1 r + \bar{\varepsilon}_2 r^2 + \cdots$, with $\bar{\varepsilon}_i$ constants, improves the accuracy obtained. In this formulation, the coefficients $\bar{\varepsilon}_i$ are the independent parameters that are to be varied in order to minimize E. In the second procedure, the planet will be conceptually divided into a number of thin shells, each with a constant density, and the eccentricities of the separating spheroidal surfaces between the equidensity shells will be the parameters that are varied to obtain a minimum energy. This strategy, which leads to a simple and elegant algebraic method, is quite efficient and can be refined to a second order of perturbation theory, or beyond, in a straightforward manner.

The second method initially resembles those developed by Macke *et al.* (1964) and Voss (1965, 1966), who, also using a thin-shell strategy, mimicked Clairaut's equation in a discrete form. Our method does not use the Euler–Lagrange equation that derives from the variational principle applied to E; on the contrary, it approaches the minimization of E in a direct way. In addition, we do not have to deal with Lagrange multipliers as these authors did, because our minimization process avoids the existence of constraints.

The organization of the paper is as follows. In Section 2, we will define the energy functional, E, applied in all cases. In Section 3, we develop the first method, calculating E as a function of the coefficients of the above-mentioned polynomial, and proceeding to its minimization. In Section 4, the thin-shell method is explained to a first order of approximation. In Section 5 we develop the second order of approximation, and check the size of the correction. In Section 6 we apply both methods to the Earth, using the density function given by the PREM model (Dziewonski & Anderson 1981), and state our conclusions. Finally, in two appendices, we detail some of the technical steps appearing in the calculation of the gravitational energy terms.

The content of the two appendices may appear somewhat redundant, bearing in mind that when the width of the thin shells of Appendix B tends to zero, one is dealing with spheroidal surfaces, which are essentially the objects studied in Appendix A. However, we decided to arrange them in this way to afford the maximum convenience to the reader.

2 ENERGY FUNCTIONAL

Let us first define the energy functional, E, which is the cornerstone of the method. It is given by

$$E = E_g + E_r, \tag{2.1}$$

where E_g is the self-gravitating energy, and E_r is the centrifugal potential energy due to the rotation. Hence,

$$E_g = -\frac{1}{2} G \int \frac{\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|} d\mathbf{x}_1\, d\mathbf{x}_2, \tag{2.2}$$

and

$$E_r = -\frac{1}{2}\omega^2 \int \rho(\mathbf{x})(|\mathbf{x}| \sin\theta)^2\, d\mathbf{x}, \tag{2.3}$$

where G is the gravitational constant, $\rho(\mathbf{x})$ the local mass density, ω is the angular velocity of uniform rotation, and $\mathbf{x}$ is a position vector defined throughout the body. As usual, $|\mathbf{x}|$ denotes the length of $\mathbf{x}$, and θ and φ are its polar and azimuthal angles, respectively.

We will assume that equidensity differential shells inside the rotating planet have a spheroidal form. The semi-axes of these shells are generically denoted by c' (or c_i) and a' (or a_i); these letters have primes (in the polynomial method) or a subscript (in the thin-shell method) in order to distinguish them from c and a, which correspond to the external values. As usual, we define the eccentricity of a shell characterized by c' and a' as $\varepsilon^2(c') = 1 - c'^2/a'^2$. For a given $\rho(r)$, $\varepsilon^2(c')$ for $0 \le c' \le c$ [or equivalently $\varepsilon^2(r)$] fixes the structure of the body. It is the calculation of $\varepsilon^2(c')$ that is our goal in this paper.

As was stated in Section 1, if one formally imposes the condition of a minimum for E, the corresponding Euler–Lagrange differential equation will be satisfied by $\varepsilon^2(r)$, and this will result in Clairaut's equation. Here, we will instead proceed to perform direct parametric minimizations.

3 POLYNOMIAL METHOD

Working with the coordinates c', θ, and φ, the volume element is $d\mathbf{x} = g\, dc'd\theta\, d\varphi$, with the Jacobian, g, given by

$$g = \frac{c'^2 \sin\theta}{[1 - \varepsilon^2(c')\sin^2\theta]^{3/2}} + \frac{c'^3 \varepsilon(c')\varepsilon'(c')\sin^3\theta}{[1 - \varepsilon^2(c')\sin^2\theta]^{5/2}}, \tag{3.1}$$

where $\varepsilon'(c') = \left.\frac{d\varepsilon}{dc''}\right|_{c''=c'}$.

Denoting by r' the specific value of r corresponding to the shell (a', c'), i.e. $r' = (a'^2c')^{1/3}$, the relationship between r', c' and $\varepsilon(c')$ is

$$r' = \frac{c'}{[1 - \varepsilon^2(c')]^{1/3}}. \tag{3.2}$$

For our purposes, we can now proceed to write E_g and E_r in a convenient form. Using eqs (2.2) and (3.1) we obtain (see Appendix A)

$$\begin{aligned} E_g = -4\pi^2 G \Bigg\{ & \int_0^c \rho(c') \frac{c'^2}{1-\varepsilon^2} dc' \int_{c'}^c \rho(c'') \\ & \times \left[\frac{4c''}{(1-\varepsilon^2)^{1/2}} \frac{\sin^{-1}\varepsilon}{\varepsilon} + \varepsilon' c''^2 \gamma \right] dc'' \\ & - \frac{2}{3} \int_0^c \rho(c') c'^4 \frac{\varepsilon^2}{(1-\varepsilon^2)^2} dc' \int_{c'}^c \rho(c'')\beta\varepsilon'\, dc'' \\ & - \frac{4}{15} \int_0^c \rho(c')\varepsilon\varepsilon' c'^5 \frac{1+\varepsilon^2}{(1-\varepsilon^2)^3} dc' \int_{c'}^c \rho(c'')\beta\varepsilon'\, dc'' \\ & + \frac{2}{3} \int_0^c \rho(c') c'^3 \frac{\varepsilon\varepsilon'}{(1-\varepsilon^2)^2} dc' \\ & \times \int_{c'}^c \rho(c'') \left[\frac{4c''}{(1-\varepsilon^2)^{1/2}} \frac{\sin^{-1}\varepsilon}{\varepsilon} + \varepsilon' c''^2 \gamma \right] dc'' \Bigg\}, \end{aligned} \tag{3.3}$$

where the primes on the c coordinates are self-explanatory, depending on the integral to which they belong. Likewise,

$$E_r = -\pi\omega^2 \left[\frac{4}{3} \int_0^c \rho(c') \frac{c'^4}{(1-\varepsilon^2)^2} dc' + \frac{16}{15} \int_0^c \rho(c') \frac{c'^5 \varepsilon\varepsilon'}{(1-\varepsilon^2)^3} dc' \right]. \tag{3.4}$$

Now we transform these integrals into others which exhibit the spherically averaged density profile, $\rho(r)$. This is achieved by inserting eq. (3.2) into eqs (3.3) and (3.4).

After expanding in powers of ε^2 and truncating to the smallest power, we obtain

$$E = -16\pi^2 G\left\{\int_0^R \rho(r')r'^2\,dr' \int_{r'}^R \rho(r'')\right.$$
$$\times\left[r''\left(1-\frac{\varepsilon^4}{45}\right)-\frac{2}{45}\varepsilon^3\varepsilon' r''^2\right]dr''$$
$$+\frac{2}{45}\int_0^R \rho(r')r'^4\varepsilon^2\,dr' \int_{r'}^R \rho(r'')\varepsilon\varepsilon'\,dr''$$
$$\left.+\frac{4}{225}\int_0^R \rho(r')\varepsilon\varepsilon' r'^5\,dr' \int_{r'}^R \rho(r'')\varepsilon\varepsilon'\,dr''\right\}$$
$$-\frac{4}{3}\pi\omega^2\left\{\int_0^R \rho(r')r'^4\left(1+\frac{\varepsilon^2}{3}\right)dr'\right.$$
$$\left.+\frac{2}{15}\int_0^R \rho(r')r'^5\varepsilon\varepsilon'\,dr'\right\}+O(\varepsilon^6). \tag{3.5}$$

It is important to realize that the factor ω^2 appearing in the E_r terms increases by one the order in the perturbative sense. This is the reason why the terms in E_g and terms in E_r differ by one order of the truncation process. This fact will also be taken into account in Sections 4 and 5.

We assume that $\varepsilon(r') = \sum_{i=0}^{P-1}\tilde{\varepsilon}_i r'^i$, and aim to find out the value of the coefficients $\tilde{\varepsilon}_i$ that make E a minimum. Substituting the polynomial into eq. (3.5) and calculating the radial integrals, we obtain E as an algebraic function in the unknowns, $\tilde{\varepsilon}_i$. If $P=1$ or 2, the conditions of minimum E can be worked out explicitly, and algebraically solved. For $P>2$, however, we have used numerical routines for minimizing functions that depend on several parameters. In Section 6, we will compare these results for the case of the Earth, but we mention here that, for $P=1$ and for homogeneous bodies, we obtain $f=5m/4$, i.e. the same result as for the Maclaurin spheroid in the slow-rotation regime.

4 THIN-SHELL METHOD

In this section we present an alternative parametric strategy for implementing the variational principle. We will assume the mass distribution of the planet under consideration to be an onion-like structure with successive constant-density shells. It is important to emphasize that there is no loss of generality in making this assumption: a continuously varying density profile can always be approximated by a stratified model, by assuming that the shells are infinitesimally thin. These shells are numbered from the centre outwards: $i=1,2,\dots N$. Thus an increase in N implies a greater accuracy of the method. The external radii of the spherically averaged shells, having a given density ρ_i, will be denoted by R_i, and the eccentricity of the intershell surfaces by ε_i^2. As pointed out before, in these spheroids the polar (equatorial) radii will be denoted by $c_i(a_i)$, and, as $\varepsilon_i^2 = 1 - c_i^2/a_i^2$, we have

$$c_i = (1-\varepsilon_i^2)^{1/3}R_i. \tag{4.1}$$

This relation is the parallel of eq. (3.2) in the previous section.

A convenient energy notation for this case is as follows: the potential energy of the shell i, due to its own gravity, will be denoted by $S(i)$; the gravitational potential energy existing between the shells i and j $(i<j)$ will be denoted by $U(i,j)$; finally, the centrifugal potential energy of the shell i, due to rotation, will be denoted by $R(i)$. Thus, the energy functional is defined as

$$E = \sum_{i=1}^{N}[S(i)+R(i)] + \sum_{\substack{i,j=1\\ i<j}}^{N} U(i,j). \tag{4.2}$$

When $S(i)$, $R(i)$ and $U(i,j)$ are calculated, one finds

$$S(i) = -16\pi^2 G\rho_i^2\left[(1-\varepsilon_i^2)^{-3/2}\frac{\sin^{-1}\varepsilon_i}{\varepsilon_i}\frac{c_i^5}{15}\right.$$
$$-(1-\varepsilon_{i-1}^2)^{-3/2}\frac{\sin^{-1}\varepsilon_{i-1}}{\varepsilon_{i-1}}\frac{c_{i-1}^5}{15}$$
$$\left.-\frac{\Delta A_2(i)}{12}\frac{c_{i-1}^3}{1-\varepsilon_{i-1}^2}+\Delta A_1(i)\frac{c_{i-1}^5}{30}\frac{\varepsilon_{i-1}^2}{(1-\varepsilon_{i-1}^2)^2}\right], \tag{4.3a}$$

$$R(i) = -\frac{4\pi\omega^2}{15}\rho_i\left[\frac{c_i^5}{(1-\varepsilon_i^2)^2}-\frac{c_{i-1}^5}{(1-\varepsilon_{i-1}^2)^2}\right], \tag{4.3b}$$

$$U_{i<j}(i,j) = -\frac{4\pi^2 G}{3}\rho_i\rho_j$$
$$\times\left\{\Delta A_2(j)\left(\frac{c_i^3}{1-\varepsilon_i^2}-\frac{c_{i-1}^3}{1-\varepsilon_{i-1}^2}\right)\right.$$
$$\left.-\frac{2}{5}\Delta A_1(j)\left[\frac{c_i^5\varepsilon_i^2}{(1-\varepsilon_i^2)^2}-\frac{c_{i-1}^5\varepsilon_{i-1}^2}{(1-\varepsilon_{i-1}^2)^2}\right]\right\}, \tag{4.3c}$$

where

$$\Delta A_1(i) \equiv \frac{(1-\varepsilon_i^2)^{1/2}}{\varepsilon_i^3}\sin^{-1}\varepsilon_i$$
$$-\frac{(1-\varepsilon_{i-1}^2)^{1/2}}{\varepsilon_{i-1}^3}\sin^{-1}\varepsilon_{i-1}-\frac{1}{\varepsilon_i^2}+\frac{1}{\varepsilon_{i-1}^2}, \tag{4.4a}$$

$$\Delta A_2(i) \equiv 2\left[\frac{c_i^2\sin^{-1}\varepsilon_i}{\varepsilon_i(1-\varepsilon_i^2)^{1/2}}-\frac{c_{i-1}^2\sin^{-1}\varepsilon_{i-1}}{\varepsilon_{i-1}(1-\varepsilon_{i-1}^2)^{1/2}}\right]. \tag{4.4b}$$

To obtain eqs (4.3a–c), one performs appropriate subtractions between the energies of constant-density solid spheroids. Technical details can be found in Appendix B.

Now, using eq. (4.1), we express c_i in terms of R_i and ε_i. In the limit of small ε_i, we will retain only the leading contribution to E, i.e. up to quadratic terms in ε_i^2. Using the dimensionless variables $\delta_i \equiv \rho_i/\rho_1$ and $y_i \equiv R_i/R$, we easily find

$$S(i) = \pi^2 G R^5 \rho_1^2 [s_{\rm I}(i) + s_{\rm II}(i)\varepsilon_{i-1}^4 + s_{\rm III}(i)\varepsilon_{i-1}^2\varepsilon_i^2 + s_{\rm IV}(i)\varepsilon_i^4] + O(\varepsilon^6), \tag{4.5a}$$

$$s_{\rm I}(i) = \left(\frac{8}{3} y_i^2 y_{i-1}^3 - \frac{16}{15} y_i^5 - \frac{8}{5} y_{i-1}^5\right)\delta_i^2, \tag{4.5b}$$

$$s_{\rm II}(i) = -\frac{8}{225} y_{i-1}^5 \delta_i^2, \tag{4.5c}$$

$$s_{\rm III}(i) = \frac{16}{225} y_{i-1}^5 \delta_i^2, \tag{4.5d}$$

$$s_{\rm IV}(i) = \left(\frac{16}{675} y_i^5 - \frac{8}{135} y_i^2 y_{i-1}^3\right)\delta_i^2, \tag{4.5e}$$

and

$$R(i) = \pi\omega^2 R^5 \rho_1 [r_{\rm I}(i) + r_{\rm II}(i)\varepsilon_{i-1}^2 + r_{\rm III}(i)\varepsilon_i^2] + O(\varepsilon^6), \tag{4.6a}$$

$$r_{\rm I}(i) = -\frac{4}{15}(y_i^5 - y_{i-1}^5)\delta_i, \tag{4.6b}$$

$$r_{\rm II}(i) = \frac{4}{45} y_{i-1}^5 \delta_i, \tag{4.6c}$$

$$r_{\rm III}(i) = -\frac{4}{45} y_i^5 \delta_i. \tag{4.6d}$$

Likewise, we obtain

$$U_{i<j}(i,j) = \pi^2 G R^5 \rho_1^2 [u_{\rm I}(i,j) + u_{\rm II}(i,j)\varepsilon_{i-1}^2\varepsilon_{j-1}^2 + u_{\rm III}(i,j)\varepsilon_{i-1}^2\varepsilon_j^2 + u_{\rm IV}(i,j)\varepsilon_i^2\varepsilon_{j-1}^2 + u_{\rm V}(i,j)\varepsilon_i^2\varepsilon_j^2 + u_{\rm VI}i(,j)\varepsilon_{j-1}^4 + u_{\rm VII}(i,j)\varepsilon_j^4] + O(\varepsilon^6), \tag{4.7a}$$

$$u_{\rm I}(i,j) = -\frac{8}{3}(y_j^2 - y_{j-1}^2)(y_i^3 - y_{i-1}^3)\delta_i\delta_j, \tag{4.7b}$$

$$u_{\rm II}(i,j) = -u_{\rm III}(i,j) = -\frac{16}{225} y_{i-1}^5 \delta_i\delta_j, \tag{4.7c}$$

$$u_{\rm IV}(i,j) = -u_{\rm V}(i,j) = \frac{16}{225} y_i^5 \delta_i\delta_j, \tag{4.7d}$$

$$u_{\rm VI}(i,j) = -\frac{8}{135} y_{j-1}^2 (y_i^3 - y_{i-1}^3)\delta_i\delta_j, \tag{4.7e}$$

$$u_{\rm VII}(i,j) = \frac{8}{135} y_j^2 (y_i^3 - y_{i-1}^3)\delta_i\delta_j, \tag{4.7f}$$

where we now explicitly observe the quadratic truncation in the energy terms.

Consequently, the N independent conditions of extremum of the energy,

$$\left.\frac{\partial E}{\partial \varepsilon_i^2}\right|_{\varepsilon_j^2 = \text{constant}} = 0, \tag{4.8}$$

$(j \neq i)$
$(i = 1, 2, \dots, N)$

can easily be expressed in a matrix form:

$$\sum_{\ell=1}^{N} M_{k,\ell}\varepsilon_\ell^2 = \kappa v_k, \tag{4.9}$$

where

$$M_{k,\ell} = \begin{cases} u_{\rm II}(\ell+1, k+1) + u_{\rm III}(\ell+1, k) \\ \quad + u_{\rm IV}(\ell, k+1) + u_{\rm V}(\ell, k) & (\ell \leq k-2) \\ u_{\rm II}(\ell+1, k+1) + u_{\rm IV}(\ell, k+1) \\ \quad + u_{\rm V}(\ell, k) + s_{\rm III}(k) & (\ell = k-1) \\ u_{\rm IV}(\ell, k+1) + u_{\rm IV}(k, \ell+1) \\ \quad + 2\sum_{k'=1}^{k} [u_{\rm VI}(k', k+1) + u_{\rm VII}(k'-1, k)] \\ \quad + 2[s_{\rm II}(k+1) + s_{\rm IV}(k)] & (\ell = k) \\ u_{\rm II}(k+1, \ell+1) + u_{\rm IV}(k, \ell+1) \\ \quad + u_{\rm V}(k, \ell) + s_{\rm III}(k+1) & (\ell = k+1) \\ u_{\rm II}(k+1, \ell+1) + u_{\rm III}(k+1, \ell) \\ \quad + u_{\rm IV}(k, \ell+1) + u_{\rm V}(k, \ell), & (\ell \geq k+2) \end{cases} \tag{4.10}$$

$$v_k = -r_{\rm II}(k+1) - r_{\rm III}(k), \tag{4.11}$$

and

$$\kappa = \omega^2(\pi G \rho_1)^{-1}. \tag{4.12}$$

κ in eq. (4.12) sets the scale of deformation for the successive shells of the body, and the ε_i^2 are directly calculated in terms of the numerical coefficients $M_{k,\ell}$ and κv_k.

In this method, the ideal case—a planet composed of two major constant-density shells—is easily derived, and here we will merely give the results:

$$\varepsilon_1^2 = \kappa\frac{15}{4}[5y_1^3 + 5\delta_2(1 - y_1^3)]D^{-1}, \tag{4.13a}$$

$$\varepsilon_2^2 = \kappa\frac{15}{4}[(2 + 3y_1^5) + 3\delta_2(1 - y_1^5)]D^{-1}, \tag{4.13b}$$

$$D = (10y_1^3) + \delta_2(4 + 5y_1^3 - 9y_1^5) + \delta_2^2(6 - 15y_1^3 + 9y_1^5). \tag{4.13c}$$

These equations provide a simple estimate for a planet formed by a well-contrasted mantle and core.

5 THIN-SHELL METHOD: SECOND-ORDER CORRECTION

The results of Section 4 were obtained by truncating the terms of E to the smallest order in perturbation theory. Obviously, this can be improved by retaining the additional terms of these expansions. Thus, here we will retain up to cubic terms in ε^2 for E_g, and up to quadratic terms in ε^2 for E_r. Eqs 4.5a, 4.6a and 4.7a thus become

$$S(i) = \pi^2 G R^5 \rho_1^2 [s_{\rm I}(i) + s_{\rm II}(i)\varepsilon_{i-1}^4 + s_{\rm III}(i)\varepsilon_{i-1}^2\varepsilon_i^2 + s_{\rm IV}(i)\varepsilon_i^4 + s'_{\rm I}(i)\varepsilon_{i-1}^6 + s'_{\rm II}(i)\varepsilon_{i-1}^4\varepsilon_i^2 + s'_{\rm III}(i)\varepsilon_{i-1}^2\varepsilon_i^4 + s'_{\rm IV}(i)\varepsilon_i^6] + O(\varepsilon^8), \tag{5.1a}$$

$$R(i) = \pi\omega^2 R^5 \rho_1 [r_{\rm I}(i) + r_{\rm II}(i)\varepsilon_{i-1}^2 + r_{\rm III}(i)\varepsilon_i^2 + r'_{\rm I}(i)\varepsilon_{i-1}^4 + r'_{\rm II}(i)\varepsilon_i^4] + O(\varepsilon^8), \tag{5.1b}$$

$$U_{i<j}(i,j) = \pi^2 G R^5 \rho_1^2 [u_{\rm I}(i,j) + u_{\rm II}(i,j)\varepsilon_{i-1}^2\varepsilon_{j-1}^2 + u_{\rm III}(i,j)\varepsilon_{i-1}^2\varepsilon_j^2 + u_{\rm IV}(i,j)\varepsilon_i^2\varepsilon_{j-1}^2 + u_{\rm V}(i,j)\varepsilon_i^2\varepsilon_j^2 + u_{\rm VI}(i,j)\varepsilon_{j-1}^4 + u_{\rm VII}(i,j)\varepsilon_j^4 + u'_{\rm I}(i,j)\varepsilon_{i-1}^4\varepsilon_{j-1}^2 + u'_{\rm II}(i,j)\varepsilon_{i-1}^4\varepsilon_j^2 + u'_{\rm III}(i,j)\varepsilon_{i-1}^2\varepsilon_{j-1}^4 + u'_{\rm IV}(i,j)\varepsilon_{i-1}^2\varepsilon_j^4 + u'_{\rm V}(i,j)\varepsilon_i^4\varepsilon_{j-1}^2 + u'_{\rm VI}(i,j)\varepsilon_i^4\varepsilon_j^2 + u'_{\rm VII}(i,j)\varepsilon_i^2\varepsilon_{j-1}^4 + u'_{\rm VIII}(i,j)\varepsilon_i^2\varepsilon_j^4 + u'_{\rm IX}(i,j)\varepsilon_{j-1}^6 + u'_{\rm X}(i,j)\varepsilon_j^6] + O(\varepsilon^8), \tag{5.1c}$$

where the new terms, identified by primes, are

$$s'_{\rm I}(i) = -\frac{416}{14\,175} y_{i-1}^5 \delta_i^2, \tag{5.2a}$$

$$s'_{\rm II}(i) = \frac{16}{675} y_{i-1}^5 \delta_i^2, \tag{5.2b}$$

$$s'_{\rm III}(i) = \frac{64}{1575} y_{i-1}^5 \delta_i^2, \tag{5.2c}$$

$$s'_{\rm IV}(i) = \frac{496}{42\,525} y_i^2 (2y_i^3 - 5y_{i-1}^3)\delta_i^2, \tag{5.2d}$$

$$r'_{\rm I}(i) = \frac{8}{135} y_{i-1}^5 \delta_i, \tag{5.3a}$$

$$r'_{\rm II}(i) = -\frac{8}{135} y_i^5 \delta_i, \tag{5.3b}$$

and

$$u'_{\rm I}(i,j) = -u'_{\rm II}(i,j) = -\frac{16}{675} y_{i-1}^5 \delta_i \delta_j, \tag{5.4a}$$

$$u'_{\rm III}(i,j) = -u'_{\rm IV}(i,j) = -\frac{64}{1575} y_{i-1}^5 \delta_i \delta_j, \tag{5.4b}$$

$$u'_{\rm V}(i,j) = -u'_{\rm VI}(i,j) = \frac{16}{675} y_i^5 \delta_i \delta_j, \tag{5.4c}$$

$$u'_{\rm VII}(i,j) = -u'_{\rm VIII}(i,j) = \frac{64}{1575} y_i^5 \delta_i \delta_j, \tag{5.4d}$$

$$u'_{\rm IX}(i,j) = -\frac{496}{8505} y_{j-1}^2 (y_i^3 - y_{i-1}^3)\delta_i \delta_j, \tag{5.4e}$$

$$u'_{\rm X}(i,j) = \frac{496}{8505} y_j^2 (y_i^3 - y_{i-1}^3)\delta_i \delta_j. \tag{5.4f}$$

The N new conditions of minimization obtained in the process of differentiation with respect to ε_i^2 can be expressed in an analogous form to those of Section 4, by defining a set of quantities Q_k, in the form

$$\sum_{\ell=1}^{N} M_{k,\ell}\varepsilon_\ell^2 = \kappa v_k + Q_k(\varepsilon^4, \varepsilon^2\kappa), \tag{5.5}$$

with

$$\begin{aligned} Q_k(\varepsilon^4, \varepsilon^2\kappa) = -\Big\{ & s'_{\rm II}(k)\varepsilon_{k-1}^4 + 2s'_{\rm II}(k+1)\varepsilon_k^2\varepsilon_{k+1}^2 \\ & + 2s'_{\rm III}(k)\varepsilon_k^2\varepsilon_{k-1}^2 + 3[s'_{\rm I}(k+1) + s'_{\rm IV}(k)]\varepsilon_k^4 \\ & + s'_{\rm III}(k+1)\varepsilon_{k+1}^4 + \sum_{j>k+1}^{N} [2u'_{\rm I}(k+1,j)\varepsilon_k^2\varepsilon_{j-1}^2 \\ & + 2u'_{\rm II}(k+1,j)\varepsilon_k^2\varepsilon_j^2 + u'_{\rm III}(k+1,j)\varepsilon_{j-1}^4 \\ & + u'_{\rm IV}(k+1,j)\varepsilon_j^4] \\ & + \sum_{j>k}^{N} [2u'_{\rm V}(k,j)\varepsilon_k^2\varepsilon_{j-1}^2 + 2u'_{\rm VI}(k,j)\varepsilon_k^2\varepsilon_j^2 \\ & + u'_{\rm VII}(k,j)\varepsilon_{j-1}^4 + u'_{\rm VIII}(k,j)\varepsilon_j^4] \\ & + \sum_{i<k+1} [u'_{\rm I}(i,k+1)\varepsilon_{i-1}^4 \\ & + 2u'_{\rm III}(i,k+1)\varepsilon_k^2\varepsilon_{i-1}^2 \\ & + u'_{\rm V}(i,k+1)\varepsilon_i^4 + 2u'_{\rm VII}(i,k+1)\varepsilon_i^2\varepsilon_k^2 \\ & + 3u'_{\rm IX}(i,k+1)\varepsilon_k^4] \\ & + \sum_{i<k} [u'_{\rm II}(i,k)\varepsilon_{i-1}^4 + 2u'_{\rm IV}(i,k)\varepsilon_{i-1}^2\varepsilon_k^2 \\ & + u'_{\rm VI}(i,k)\varepsilon_i^4 + 2u'_{\rm VIII}(i,k)\varepsilon_i^2\varepsilon_k^2 + 3u'_{\rm X}(i,k)\varepsilon_k^4] \Big\} \\ & - 2\kappa\varepsilon_k^2[r'_{\rm I}(k+1) + r'_{\rm II}(k)]. \end{aligned} \tag{5.6}$$

The Q_k terms are one order of magnitude lower than the previous independent terms κv_k, and hence the correction they introduce to the previously calculated ε_i^2 is small. Thus, we can evaluate Q_k terms using the first-order solution, and they will appear as the independent terms of the new system of N linear equations,

$$\sum_{\ell=1}^{N} M_{k,\ell}\Gamma_\ell = Q_k, \tag{5.7}$$

$M_{k,\ell}$ being the same array of coefficients as used in the system of equations, to minimal order, in Section 4. The Γ_ℓ thus are the corrections to the eccentricities.

In the particular case of a Maclaurin spheroid, the values of the coefficients introduced here are $M_{11} = 32/675$; $v_1 = 4/45$; $\varepsilon_1^2 = 5m/2$; $Q_1 = -992\varepsilon_1^4/14175 + 16\kappa\varepsilon_1^2/135$; and $\kappa = 4m/3$. As a result, this second-order correction to the flattening is $\Gamma_1 = -25m^2/28$. This coincides with what one would obtain from the exact development up to that order.

6 APPLICATION TO THE EARTH, AND CONCLUSIONS

In this work, the density profile of the Earth has been taken from the PREM model (Dziewonski & Anderson 1981). The PREM model divides the Earth into eight major zones, with new, finer subdivisions. This density profile allows a very fast calculation of all the radial integrals appearing in the polynomial method, so that we can proceed to the

Table 1. The set of parameters and results of the polynomial method.

P	1	2	3	4	5
$\tilde{\varepsilon}_0$	0.080167	0.059115	0.063731	0.084111	0.075718
$\tilde{\varepsilon}_1$	-	0.022525	0.009923	-.075383	-.009281
$\tilde{\varepsilon}_2$	-	-	0.008031	0.122415	-.051028
$\tilde{\varepsilon}_3$	-	-	-	-.049487	0.136152
$\tilde{\varepsilon}_4$	-	-	-	-	-.069917
$f^{-1}(R)$	310.702	299.573	299.228	299.447	299.540
$\Delta E/GM^2R^{-1}$ (10^{-7})	-6.11171	-6.16909	-6.16921	-6.16936	-6.16939

minimization of E. The numerical values of the physical coefficients calculated in this process are listed in Table 1, depending on the number of parameters, P. Note that in the first column the physical coefficients have been multiplied by suitable powers of R, in order to deal with dimensionless quantities. In this table the prediction of the method for the external flattening is shown, together with the energy increment with respect to the strictly spherical rotating situation. As expected, as P grows, ΔE also grows.

The results of this method are plotted in Fig. 1 and compared with those from Clairaut's equation (1.2), in which the adopted boundary condition at the surface is the actual value of the Earth's flattening. The value chosen for m, which is necessary to implement our method, is $m = 3.449786 \times 10^{-3}$, in agreement with Moritz (1990).

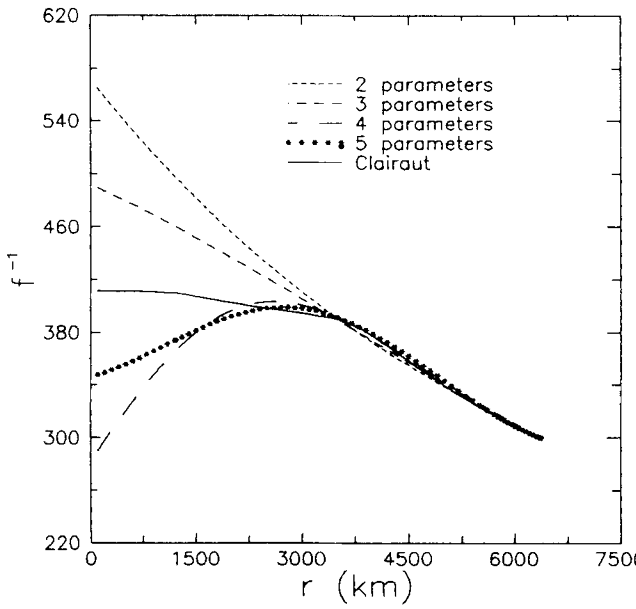

Figure 1. Performance of the polynomial method, to various orders, when used to describe the flattening of the Earth.

From Fig. 1 one can easily make three observations: (i) as the order of the polynomial grows, $\varepsilon^2(r)$ moves closer to Clairaut's solution; (ii) a good description at the external surface demands very few parameters; (iii) an accurate description of ε^2 near the origin, however, is awkward because variations of ε^2 at small r/R values involve very tiny energy differences. As a result, the method is rather insensitive there.

To implement the second method, we have considered the Earth as being divided into 80 shells of equal depth, calculating their mean density according to the PREM model. The linear system of equations described in Sections 4 and 5 have been solved, and the results (to the second order) are shown in Fig. 2, together with Clairaut's results [for an accurate application of Clairaut's equation to the Earth, see, for example, Denis & Ibrahim (1981)]. The two curves are practically coincident. The slight discrepancies can be attributed to the fact that Clairaut's solution has been

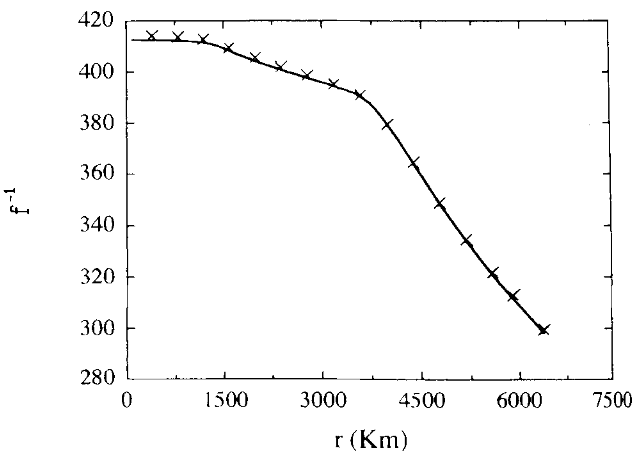

Figure 2. Results of the thin-shell method when used to describe the flattening of the Earth. The crosses correspond to the variational method, and the continuous line has been obtained from Clairaut's equation.

forced to fulfil the real external flattening of the Earth, while in our method the only input is the $\rho(r)$ taken from PREM. In our second variational method, the inner parts are described with great accuracy. This is a clear advantage over the polynomial method. For the external figure, this second method gives, to the first order, $f^{-1}(R) = 299.4$, and, to the second order, $f^{-1}(R) = 299.7$. With respect to the energies, using the same notation as before, we obtain $\Delta E/GM^2R^{-1} = -0.61709 \times 10^{-6}$, which is a bigger difference than that obtained with the polynomial method. This is a good check that the thin-shell method is indeed more powerful.

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APPENDIX A: CALCULATION OF E_g IN THE POLYNOMIAL METHOD

We start with the gravitational potential created by a homogeneous solid spheroid, characterized by ρ, ε and c, at a point $\mathbf{x} \equiv (x_1, x_2, x_3)$ in its interior (Chandrasekhar 1969):

$$\phi = -\pi G\rho[A - A_1(x_1^2 + x_1^2) - A_3x_3^2], \tag{A1}$$

where

$$A = \frac{2c^2}{(1-\varepsilon^2)^{1/2}}\frac{\sin^{-1}\varepsilon}{\varepsilon},$$
$$A_1 = \frac{(1-\varepsilon^2)^{1/2}}{\varepsilon^3}\sin^{-1}\varepsilon - \frac{1-\varepsilon^2}{\varepsilon^2}, \tag{A2}$$
$$A_3 = \frac{2}{\varepsilon^2} - 2\frac{(1-\varepsilon^2)^{1/2}}{\varepsilon^3}\sin^{-1}\varepsilon.$$

From this expression, we can obtain the potential created by a differential homogeneous spheroidal shell as the difference between those created by a solid homogeneous spheroid characterized by $c'' + dc''$ and $\varepsilon + \varepsilon'\,d\varepsilon$, and another similar spheroid characterized by c'' and ε. As the shell is infinitesimally thin, we can substitute the difference by the differentiation,

$$d\phi = -\pi G\rho\left[\frac{dA}{dc''} - \frac{dA_1}{dc''}(x_1^2 + x_2^2) - \frac{dA_3}{dc''}x_3^2\right]dc'', \tag{A3}$$

and hence

$$d\phi(x_1, x_2, x_3) = -\pi G\rho(c'')\left[\frac{4c''\sin^{-1}\varepsilon}{\varepsilon(1-\varepsilon^2)^{1/2}} + \varepsilon' c''^2\gamma + \beta\varepsilon'(2x_3^2 - x_1^2 - x_2^2)\right]dc'', \tag{A4}$$

where

$$\gamma = \frac{2}{\varepsilon(1-\varepsilon^2)} - 2\frac{(1-2\varepsilon^2)}{\varepsilon(1-\varepsilon^2)^{3/2}}\frac{\sin^{-1}\varepsilon}{\varepsilon}, \tag{A5a}$$

$$\beta = \frac{3}{\varepsilon^3} - \frac{(3-2\varepsilon^2)}{\varepsilon^3(1-\varepsilon^2)^{1/2}}\frac{\sin^{-1}\varepsilon}{\varepsilon}. \tag{A5b}$$

Once the potential created by an external differential shell is known we calculate, by integration, the contributions of all the external shells:

$$\phi(c', \theta) = \pi G\int_{c'}^{c}\rho(c'')\left[\frac{4c''\sin^{-1}\varepsilon}{\varepsilon(1-\varepsilon^2)^{1/2}} + \varepsilon c''^2\gamma + \beta\varepsilon'\frac{c'^2(3\cos^2\theta - 1)}{1 - \varepsilon^2(c')\sin^2\theta}\right]dc''. \tag{A6}$$

The spheroidal surface that contains the point under focus will be characterized by c', which is the lower limit of integration. The upper limit is c. We refer to the eccentricity at the point at which the potential is being calculated as $\varepsilon^2(c')$; otherwise it is $\varepsilon^2(c'')$.

From the previous considerations, we have

$$E_g = \int_0^{2\pi}\int_0^{\pi}\int_0^{c}\rho(c')\phi(c', \theta)g\,dc'\,d\theta\,d\varphi, \tag{A7}$$

where the Jacobian, the eccentricity and its derivative are all functions of c'. After substituting and performing the angular integrations, we obtain eq. (3.3).

APPENDIX B: CALCULATION OF E_g IN THE THIN-SHELL METHOD

The gravitational potential energy of a heterogeneous body stratified in homogeneous homeoidal shells is the sum of the $S(i)$ and the $U(i, j)$, using the notation described in Section 4. Let us start with $U(i, j)$.

Once again, we start with the value of the gravitational potential created by a homogeneous spheroid of polar semi-axis c, density ρ and eccentricity ε^2, at a point $\mathbf{x}(x_1, x_2, x_3)$ located in its interior (see eqs A1 and A2).

From these formulae we can compute the potential created by a shell j at a point in its interior. The potential will be the difference between the potentials created by two homogeneous spheroids whose external surfaces constitute the borders of our shell. Denoting these by c_j, ε_j and c_{j-1}, ε_{j-1}, and writing ρ_j for the density of the shell, we have

$$\phi_j(x_1, x_2, x_3) = -\pi G\rho_j[\Delta A_2(j) - \Delta A_1(j)(x_1^2 + x_2^2) - \Delta A_3(j)x_3^2], \tag{B1}$$

where

$$\Delta A_2(j) = 2\left[\frac{c_j^2}{(1-\varepsilon_j^2)^{1/2}}\frac{\sin^{-1}\varepsilon_j}{\varepsilon_j} - \frac{c_{j-1}^2}{(1-\varepsilon_{j-1}^2)^{1/2}}\frac{\sin^{-1}\varepsilon_{j-1}}{\varepsilon_{j-1}}\right]$$

$$\Delta A_1(j) = \frac{(1-\varepsilon_j^2)^{1/2}}{\varepsilon_j^3}\sin^{-1}\varepsilon_j - \frac{1}{\varepsilon_j^2} - \frac{(1-\varepsilon_{j-1}^2)^{1/2}}{\varepsilon_{j-1}^3}\sin^{-1}\varepsilon_{j-1} + \frac{1}{\varepsilon_{j-1}^2}, \tag{B2}$$

$$\Delta A_3(j) = -2\Delta A_1(j).$$

Having calculated this potential, we can now compute the gravitational energy of an inner shell i caused by the field created by an exterior shell j. Let us denote this by $U_{i<j}(i-j)$. With the convention that $V(0\cdots a)$ is the volume contained by the external border of shell a and that $V(b)$ is the volume of the individual shell b,

$$U_{i<j}(i-j) = \frac{1}{2}\int_{V(i)} \rho_i\phi_j(\mathbf{x})\,d\mathbf{x} = \frac{1}{2}\int_{V(0\cdots i)} \rho_i\phi_j(\mathbf{x})\,d\mathbf{x} - \frac{1}{2}\int_{V(0\cdots i-1)} \rho_i\phi_j(\mathbf{x})\,d\mathbf{x} \tag{B3}$$

Using the function $\phi_j(\mathbf{x})$ previously calculated and the volume element

$$d\mathbf{x} = \frac{c^2\sin\theta\,dc\,d\theta\,d\varphi}{(1-\varepsilon^2\sin^2\theta)^{3/2}}, \tag{B4}$$

these integrals can be computed, and from them the energy $U_{i<j}(i-j)$ extracted.

By symmetry, this energy coincides with that of the external shell j under the field of the internal shell i. We thus find eq. (4.3c).

To calculate $S(i)$, we start with the gravitational potential energy of a homogeneous solid spheroid, characterized by ρ_i, c_i, and ε_i (Chandrasekhar 1969):

$$S_{\text{solid}}(i) = -\frac{16}{15}\pi^2 G\rho_i^2\frac{c_i^5}{(1-\varepsilon_i^2)^{3/2}}\frac{\sin^{-1}\varepsilon_i}{\varepsilon_i}. \tag{B5}$$

In this spheroid one can always assume the existence of an intermediate spheroidal surface, characterized by c_{i-1} and ε_{i-1}, which divides the body into two parts: the internal part will be another solid homogeneous spheroid, and the external one will be a shell of the type considered here. The total energy of the spheroid is then

$$S_{\text{solid}}(i) = S_{\text{solid}}(i-1) + U(i-1, i) + S(i), \tag{B6}$$

and from this one can deduce $S(i)$, because the other terms are known. Now, with the appropriate substitution of index labels, we obtain eq. (4.3a).