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Variational Principle and Diverse Wave Structures of the Modified Benjamin-Bona-Mahony Equation Arising in the Optical Illusions Field

Kang-Jia Wang

School of Physics and Electronic Information Engineering, Henan Polytechnic University, Jiaozuo 454003, China; kangjiaw@hpu.edu.cn

Abstract: This study focuses on investigating the modified Benjamin-Bona-Mahony equation that is used to model the long wave in nonlinear dispersive media of the optical illusion field. Two effective techniques, the variational direct method and He's frequency formulation method, are employed to seek the travelling wave solutions. Using these two techniques, abundant exact solutions such as the bright wave, bright-dark wave, bright-like wave, kinky-bright wave and periodic wave solutions, are obtained. The 3-D contours and 2-D curves are drawn to present the dynamic physical behaviors of the solutions by assigning the proper parameters. It shows that the proposed methods are effective but simple and only need one or two steps to construct the exact solutions, which are expected to provide some new insights to study the travelling wave solutions of the PDEs arising in physics.

Keywords: variational direct method; He's frequency formulation; travelling wave solutions; semi-inverse method

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1. Introduction

In the field of physics, the mathematical analysis of many physical phenomena is reduced to a special mathematical model [1–4]. This model is often a nonlinear evolution equation [5–7]. The soliton theory plays a very important role in the field of nonlinear science. Its application covers almost the entirety of natural science, among which the problem of finding the exact solution of nonlinear evolution equation is of the most concern. Therefore, the study of the exact solution of the nonlinear evolution equation has become an important task [8–10]. At present, there are many ideal methods to find the exact solution of the partial differential equations (PDEs), such as the trial equation method [11,12], sine-Gordon expansion method [13,14], generalized (G′/G)-expansion method [15,16], simplified extended tanh-function method [17,18], extended rational sine-cosine and extended rational sinh-cosh techniques [19,20], exp-function method [21–24], Sardar subequation method [25–28] and so on [29–32]. In this work, we focus on the modified Benjamin-Bona-Mahony equation (MBBME) that reads as follows [33]:

$$\psi_t + \psi_x + k\psi^2\psi_x + \psi_{xxt} = 0, \tag{1}$$

where k is an arbitrary constant. Thus far, many famous scholars have made outstanding contributions to solving Equation (1). In [34], the exp-function method is used to construct some new soliton solutions. In [35], the modified simple equation is employed to study the problem. In [36], the extended generalized Riccati equation mapping method is adopted to find the exact solutions. Three different effective methods, the extended simplest equation method, modified Kudryashov method and sech-tanh expansion method, are applied in [37]. In [38], the first integral method is adopted to solve Equation (1). The bifurcation method of dynamical systems is used to study Equation (1) in [39]. In [40], the generalized perturbation reduction method is employed. The exp-expansion method is presented to find the abundant solutions of Equation (1) in [41]. Moreover, the modified Khater method is utilized to find the solutions in [42]. In recent years, the variational direct method (VDM) and He's frequency formulation method (HFFM) have attracted extensive attention since they can construct the travelling wave solutions (TWSs) only in one or two steps. Thus, in this work, we will study the TWSs by applying these two methods. The rest of this paper is arranged as follows. In Section 2, brief introductions to the algorithms of the two effective methods are presented. In Section 3, the VDM and HFFM are employed to find the desired TWSs. In Section 4, the physical explanations are given. Finally, the conclusions and future recommendations are presented in Section 5.

2. The Two Methods

In this section, we will give a brief introduction to the algorithms of the VDM and HFFM.

Consider the PDE with the form as follows:

$$F(u, u_x, u_{xx}, u_{xxx}, \ldots, u_t, \ldots) = 0. \tag{2}$$

We introduce the following transformation:

$$u(x,t) = U(\xi),\ \xi = kx + rt, \tag{3}$$

where k, r are arbitrary nonzero constants. Applying the transformation, the PDE can be converted into the following ODE:

$$F(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, \ldots, \ldots) = 0. \tag{4}$$

2.1. The VDM

Applying the semi-inverse method [43–48], the variational principle of Equation (4) can be constructed as

$$J(U) = \int L(U, U', U'', \ldots) d\xi \tag{5}$$

We obtain the solutions of Equation (4) with the following different forms:

$$\begin{aligned}
&\textbf{\textit{Formone}}: && U(\xi) = H_1 \operatorname{sech}(\xi). \\
&\textbf{\textit{Formtwo}}: && U(\xi) = \frac{H_2 \sinh(\xi)}{[\cosh(\xi)]^{\frac{3}{2}}} \\
&\textbf{\textit{Formthree}}: && U(\xi) = \frac{H_3}{1+\cosh(\xi)}. \\
&\textbf{\textit{Formfour}}: && U(\xi) = H_4 \operatorname{sech}^2(\xi).
\end{aligned}$$

Substituting the above forms into Equation (5), respectively, and applying the Ritz-like method [49] with the stationary conditions, we have

$$\begin{aligned}
&\textbf{\textit{Resultsone}}: && \frac{dJ}{dH_1} = 0. \\
&\textbf{\textit{Resultstwo}}: && \frac{dJ}{dH_2} = 0. \\
&\textbf{\textit{Resultsthree}}: && \frac{dJ}{dH_3} = 0. \\
&\textbf{\textit{Resultsfour}}: && \frac{dJ}{dH_4} = 0.
\end{aligned}$$

On solving them, we can determine the expressions of H_1, H_2, H_3 and H_4.

Note: It can be seen that the VDM is based on the stationary condition, so we can obtain the optimal solution. Moreover, the VDM is very simple and can construct the different solutions in two steps.

2.2. The HFFM

Consider Equation (4) with the form as follows:

$$U'' + N(U) = 0, \tag{6}$$

where $U'' = \frac{d^2U}{d\xi^2}$, $N(U)$ is a function about U. Here, we can assume the periodic solution of Equation (6) as

$$U = \Lambda \cos(\varpi \xi). \tag{7}$$

Based on the HFFM [50–53], the amplitude frequency relationship can be determined in one step as

$$\varpi = \sqrt{\frac{dN}{dU}}\Bigg|_{U=\frac{\Lambda}{2}}. \tag{8}$$

Note: It can be easily found that this method is very simple and can construct the periodic solution in one step.

3. Applications

To find the TWSs, we can adopt the wave variable $\gamma = x - ct$ for Equation (1) and obtain an ODE as follows:

$$-c\psi' + \psi' + k\psi^2\psi' - c\psi''' = 0, \tag{9}$$

where $\psi' = \frac{d\psi}{d\gamma}$, $\psi'' = \frac{d^2\psi}{d\gamma^2}$. Integrating Equation (9) once and setting the integration constant to zero yields

$$-c\psi + \psi + \frac{1}{3}k\psi^3 - c\psi'' = 0, \tag{10}$$

which is

$$c\psi'' + (c-1)\psi - \frac{1}{3}k\psi^3 = 0. \tag{11}$$

3.1. The VDM

On the basis of the VDM, we first set up the variational principle of Equation (11) as follows:

$$J(\psi) = \int_0^\infty \left\{ -\frac{1}{2}c(\psi')^2 + \frac{1}{2}(c-1)\psi^2 - \frac{1}{12}k\psi^4 \right\} d\gamma. \tag{12}$$

The bright wave solution

We assume that Equation (11) has the solution as follows:

$$\psi_1(\gamma) = A_1 \mathrm{sech}(\gamma). \tag{13}$$

Taking this into Equation (12) yields

$$\begin{aligned} J(A_1) &= \int_0^\infty \left\{ -\tfrac{1}{2}c(\psi')^2 + \tfrac{1}{2}(c-1)\psi^2 - \tfrac{1}{12}k\psi^4 \right\} d\gamma \\ &= \int_0^\infty \left\{ -\tfrac{1}{2}c[-A_1\mathrm{sech}(\gamma)\tanh(\gamma)]^2 + \tfrac{1}{2}(c-1)[A_1\mathrm{sech}(\gamma)]^2 - \tfrac{1}{12}k[A_1\mathrm{sech}(\gamma)]^4 \right\} d\gamma. \\ &= -\tfrac{1}{18}A_1^2\left(9 + A_1^2 k - 6c\right) \end{aligned} \tag{14}$$

We compute its stationary condition as

$$\frac{dJ(A_1)}{dA_1} = 0, \tag{15}$$

which gives

$$-\frac{1}{9}A_1\left(9 + 2A_1^2k - 6c\right) = 0. \tag{16}$$

On solving Equation (16), there is

$$A_1 = \pm\sqrt{\frac{6c-9}{2k}}. \tag{17}$$

Thus, we have

$$\psi_1^{\pm}(x,t) = \pm\sqrt{\frac{6c-9}{2k}}\mathrm{sech}(x-ct). \tag{18}$$

The bright-dark wave solution

We can also assume the solution of Equation (11) as follows:

$$\psi_2(\gamma) = \frac{A_2\sinh(\gamma)}{[\cosh(\gamma)]^{\frac{3}{2}}}. \tag{19}$$

Inserting Equation (19) into Equation (12) gives

$$\begin{aligned} J(A_2) &= \int_0^\infty \left\{-\tfrac{1}{2}c(\psi')^2 + \tfrac{1}{2}(c-1)\psi^2 - \tfrac{1}{12}k\psi^4\right\}d\gamma \\ &= \int_0^\infty \left\{-\tfrac{1}{2}c\left[\tfrac{A_2}{[\cosh(\gamma)]^{\frac{1}{2}}} - \tfrac{3A_2\sinh^2(\gamma)}{2[\cosh(\gamma)]^{\frac{5}{2}}}\right]^2 + \tfrac{1}{2}(c-1)\left[\tfrac{A_2\sinh(\gamma)}{[\cosh(\gamma)]^{\frac{3}{2}}}\right]^2 - \tfrac{1}{12}k\left[\tfrac{A_2\sinh(\gamma)}{[\cosh(\gamma)]^{\frac{3}{2}}}\right]^4\right\}d\gamma. \\ &= -\tfrac{A_2^2\left[15(16-5c)\pi + 32A_2^2k\right]}{1920} \end{aligned} \tag{20}$$

whose stationary condition is

$$\frac{dJ(A_2)}{dA_2} = 0, \tag{21}$$

which yields

$$-\frac{A_2^3k}{15} + \frac{A_2(-16+5c)\pi}{64} = 0. \tag{22}$$

On solving it, we obtain

$$A_2 = \pm\frac{1}{8}\sqrt{\frac{15\pi(5c-16)}{k}}. \tag{23}$$

Then, we have the solution of Equation (1) as follows:

$$\psi_2^{\pm}(x,t) = \pm\frac{1}{8}\sqrt{\frac{15\pi(5c-16)}{k}}\frac{\sinh(x-ct)}{[\cosh(x-ct)]^{\frac{3}{2}}}. \tag{24}$$

The bright-like wave solution

The solution of Equation (11) can be supposed as follows:

$$\psi_3(\gamma) = \frac{A_3}{1+\cosh(\gamma)}. \tag{25}$$

Inserting it into Equation (12), we have

$$
\begin{aligned}
J(A_3) &= \int_0^\infty \left\{-\tfrac{1}{2}c(\psi')^2 + \tfrac{1}{2}(c-1)\psi^2 - \tfrac{1}{12}k\psi^4\right\}d\gamma \\
&= \int_0^\infty \left\{-\tfrac{1}{2}c\left[-\tfrac{A_3\sinh(\gamma)}{[1+\cosh(\gamma)]^2}\right]^2 + \tfrac{1}{2}(c-1)\left[\tfrac{A_3}{1+\cosh(\gamma)}\right]^2 - \tfrac{1}{12}k\left[\tfrac{A_3}{1+\cosh(\gamma)}\right]^4\right\}d\gamma. \\
&= -\tfrac{A_3^2\left(35+A_3^2k-28c\right)}{210}
\end{aligned}
\tag{26}
$$

We take its stationary condition as

$$\frac{dJ(A_3)}{dA_3} = 0. \tag{27}$$

We have

$$-\frac{A_3\left(35+2A_3^2k-28c\right)}{105} = 0. \tag{28}$$

Form Equation (28), there is

$$A_3 = \pm\sqrt{\frac{7(4c-5)}{2k}}. \tag{29}$$

Thus, the third solution of Equation (1) is found as follows:

$$\psi_3^{\pm}(\gamma) = \pm\sqrt{\frac{7(4c-5)}{2k}}\frac{1}{1+\cosh(x-ct)}. \tag{30}$$

The kinky-bright wave solution

We can also assume the solution of Equation (11) as follows:

$$\psi_4(\gamma) = A_4\mathrm{sech}^2(\gamma). \tag{31}$$

Substituting Equation (31) into Equation (32) leads to

$$
\begin{aligned}
J(A_4) &= \int_0^\infty \left\{-\tfrac{1}{2}c(\psi')^2 + \tfrac{1}{2}(c-1)\psi^2 - \tfrac{1}{12}\psi^4\right\}d\gamma \\
&= \int_0^\infty \left\{-\tfrac{1}{2}c\left[-2A_4\mathrm{sech}^2(\gamma)\tanh(\gamma)\right]^2 + \tfrac{1}{2}(c-1)\left[A_4\mathrm{sech}^2(\gamma)\right]^2 - \tfrac{1}{12}k\left[A_4\mathrm{sech}^2(\gamma)\right]^4\right\}d\gamma. \\
&= -\tfrac{1}{105}A_4^2\left(35+4A_4^2k-7c\right)
\end{aligned}
\tag{32}
$$

Using its stationary condition,

$$\frac{dJ(A_4)}{dA_4} = 0. \tag{33}$$

In other words,

$$-\frac{2}{105}A_4\left(35+8A_4^2k-7c\right) = 0. \tag{34}$$

Solving the above equation gives

$$A_4 = \pm\frac{1}{2}\sqrt{\frac{7(c-5)}{2k}}. \tag{35}$$

Then, we attain the fourth solution:

$$\psi_4^{\pm}(x,t) = \pm\frac{1}{2}\sqrt{\frac{7(c-5)}{2k}}\mathrm{sech}^2(x-ct). \tag{36}$$

3.2. The HFFM

To apply the HFFM, we first rewrite Equation (11) as

$$\psi'' + \frac{c-1}{c}\psi - \frac{k}{3c}\psi^3 = 0, \tag{37}$$

which is

$$\psi'' + \Re(\psi) = 0, \tag{38}$$

where

$$\Re(\psi) = \frac{c-1}{c}\psi - \frac{k}{3c}\psi^3 \tag{39}$$

The periodic wave solution

We assume that the periodic solution of Equation (37) has the following form:

$$\psi_5(\gamma) = A_5 \cos(\omega\gamma),\ \omega > 0. \tag{40}$$

In view of the HFFM, the amplitude–frequency relationship can be determined at once as

$$\omega = \sqrt{\frac{d\Re}{d\psi}}\Bigg|_{\psi=\frac{A_5}{2}}, \tag{41}$$

which leads to

$$\omega = \sqrt{\frac{c-1}{c} - \frac{kA_5^2}{4c}} > 0. \tag{42}$$

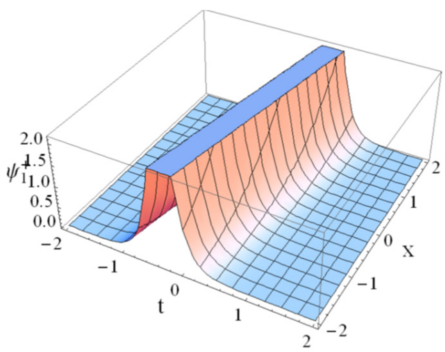
Then, the periodic wave solution of Equation (1) is obtained as follows:

$$\psi_5(x,t) = A_5 \cos\left[\sqrt{\frac{c-1}{c} - \frac{kA_5^2}{4c}}(x-ct)\right]. \tag{43}$$

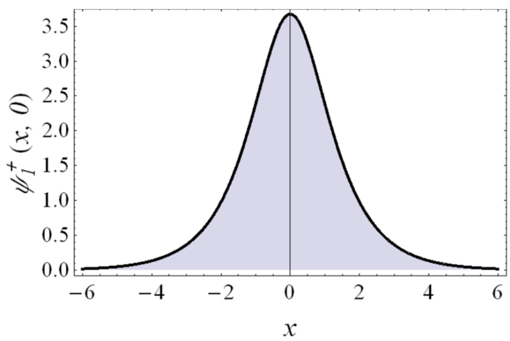
4. Results and Discussion

The main purpose of this section is to elaborate the physical interpretation of the solutions by assigning proper parameters.

By using $k = 1$, $c = 6$, the behavior of $\psi_1^+(x,t)$ is as presented in Figure 1, where Figure 1a is the 3-D plot and Figure 1b is the 2-D curve for $t = 0$. Obviously, we can find that the profile of $\psi_1^+(x,t)$ is the bright wave, which represents the bright soliton propagating along the x-axis.



(**a**)



(**b**)

Figure 1. The dynamic characteristics of $\psi_1^+(x,t)$ with $k = 1$, $c = 6$. (**a**) 3-D plot, (**b**) 2-D curve for $t = 0$.

Assigning the same parameters, we illustrate the dynamic characteristics of $\psi_2^+(x,t)$ through the 3-D plot and 2-D curve in Figure 2. By observation, it can be seen that the outline of $\psi_2^+(x,t)$ is the bright-dark wave.

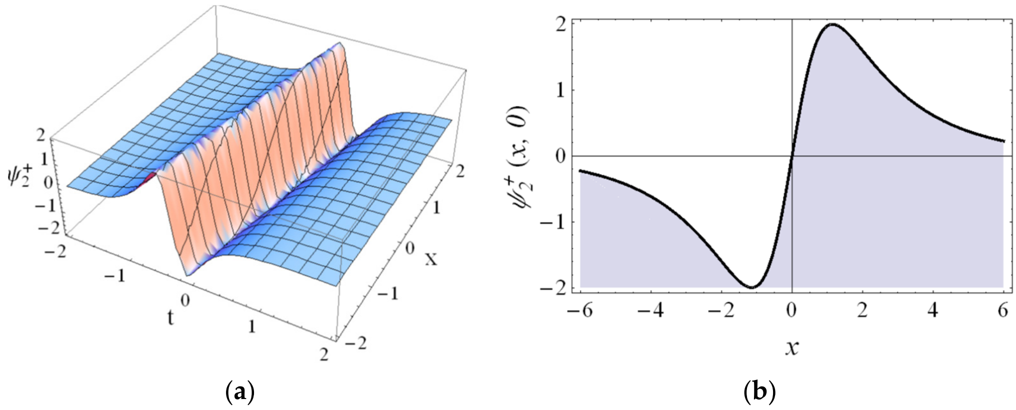


Figure 2. The dynamic characteristics of $\psi_2^+(x,t)$ with $k=1$, $c=6$. (**a**) 3-D plot, (**b**) 2-D curve for $t=0$.

Figures 3 and 4 plot the behaviors of $\psi_3^+(x,t)$ and $\psi_4^+(x,t)$, respectively, with $k=1$, $c=6$. It can be noticed that they are kinky-bright and bright-like waves, respectively, which both have the characteristics of bright solitary waves.

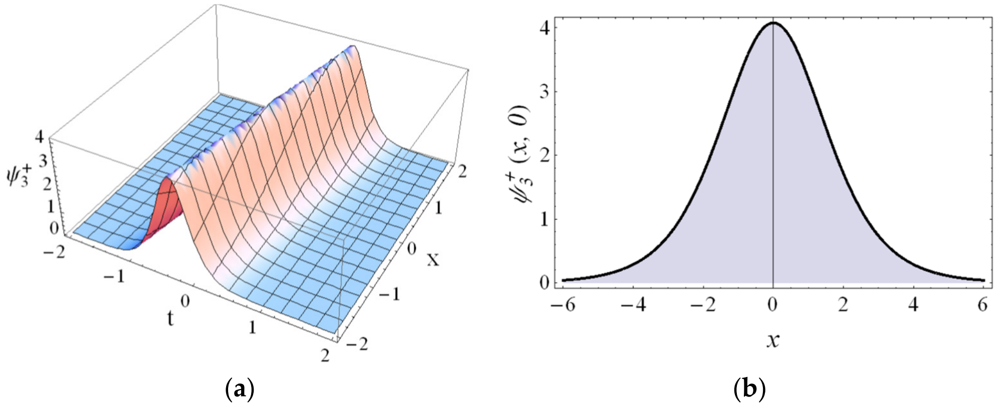


Figure 3. The dynamic characteristics of $\psi_3^+(x,t)$. with $k=1$, $c=6$. (**a**) 3-D plot, (**b**) 2-D curve for $t=0$.

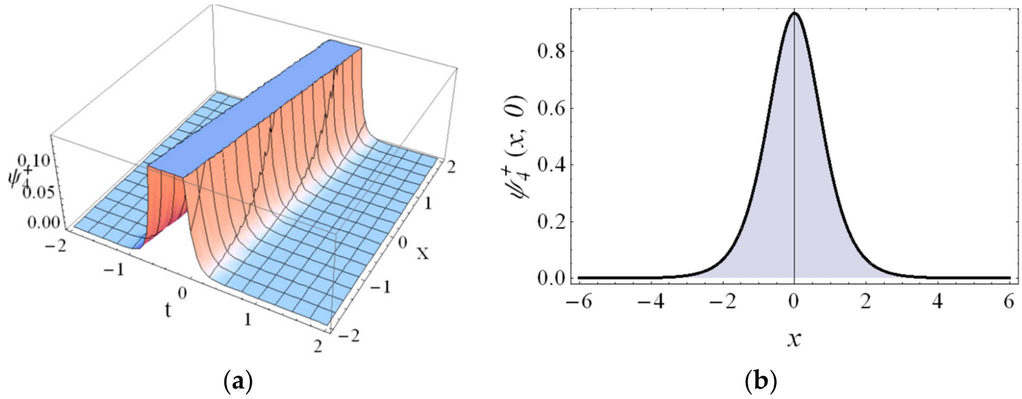
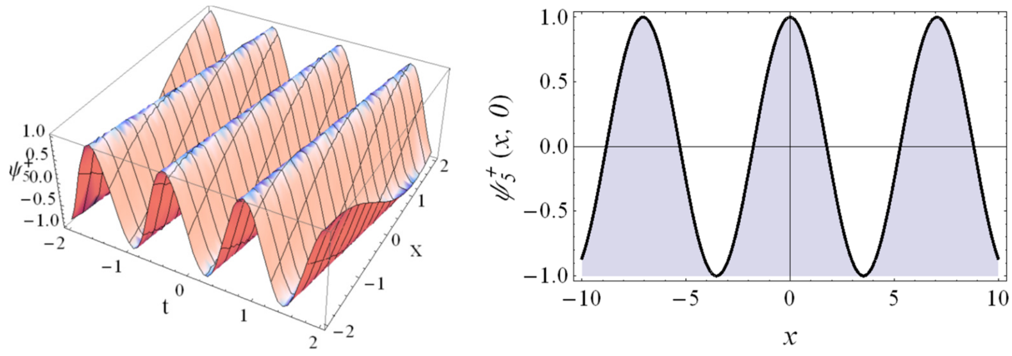


Figure 4. The dynamic characteristics of $\psi_4^+(x,t)$ with $k=1$, $c=6$. (**a**) 3-D plot, (**b**) 2-D curve for $t=0$.

If we choose the parameters as $A_5=1, k=1, c=6$, the performance of $\psi_5^+(x,t)$ is as described in Figure 5. As predicted, the contour of $\psi_5^+(x,t)$ is a perfect periodic wave.



(**a**) (**b**)

Figure 5. The dynamic characteristics of $\psi_5^+(x,t)$ with $A_5 = 1$, $k = 1$, $c = 6$. (**a**) 3-D plot, (**b**) 2-D curve for $t = 0$.

Note: The obtained different wave structures can enable us to better understand the physical characteristics and essential characteristics of the problem studied.

5. Conclusions and Future Recommendations

This work describes a detailed study of the modified Benjamin–Bona–Mahony equation. Two effective techniques, namely the variational direct method and He's frequency formulation method, are adopted to find the travelling wave solutions, and different forms of the solutions, such as the bright wave, bright-dark wave, bright-like wave, kinky-bright wave and periodic wave solutions, are constructed. It shows that the proposed methods are simple and straightforward, and they can obtain the travelling wave solutions through one or two steps. The results in this paper have potential to open some new horizons in the study of the variational theory and travelling wave solutions of the PDEs in physics.

In recent years, fractal and fractional calculus have attracted extensive attention in various fields [54–62]. The question of how to use the proposed methods to solve fractional PDEs will be our future research direction.

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