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VARIATIONAL PRINCIPLES FOR PARABOLIC EQUATIONS

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1. While the variational principles are well-known and applied since a long time in the boundary value problems for elliptic and in mixed problems for hyperbolic partial differential equations, in the problems, described by parabolic equations, their relation to variational principles were quite different. The effort to find a functional, the Euler's equation of which would be e.g.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

was failing. G. ADLER [1] even proved, that such functional of the form

$$\mathscr{I}(u) = \int_D F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \mathrm{d}x \, \mathrm{d}t \tag{1.2}$$

with F analytical in some domain R of the five-dimensional Euclidean space, where D is a subdomain of R, does not exist.

Nevertheless, P. ROSEN [2], M. A. BIOT [3], [4], [5], R. A. SCHAPERY [6] and M. E. GURTIN [7] suggested some new sorts of variational principles in the theory of heat conduction, to which the equation (1.1) (or its operational transcription) represents the corresponding Euler's equation. Rosen imposed certain restrictions on the variations of $\partial u/\partial t$, Biot used the operational calculus, Schapery and Gurtin used the convolutions with respect to time. The convolution functionals are the nearest to the functional (1.2), but they differ from (1.2) by involving products of "asynchronic" quantities of convolution type, e.g.

$$u(t_1 - t)\, u(t)\,, \quad \frac{\partial u}{\partial t}(t)\, u(t_1 - t)\,.$$

In the following sections we attempt to give a systematic survey of those principles, to complete and extend them onto a further kind of boundary conditions. Our approach will be purely mathematical, without use of any physical concept or law, so that the results are applicable to all analogous problems of mathematical physics. The physical interpretation, however, will be explained on the example of heat conduction only, in order to show the proper situation of the principles mentioned above [2], [3], [4], [5], [6], [7] in the complete group of variational principles.

We can divide the principles into two kinds. Let us call the principles of the first kind *primary principles*, because they characterize the original problem, expressed by means of the unknown function $u(X, t)$. The principles of the second kind will be referred to as *secondary principles*, as they characterize the same physical problem, but expressed by means of another unknown functions. In the theory of heat conduction the primary principles are those "for temperature" and secondary principles are those "for entropy displacement" or "heat flux" respectively and furthermore the generalized principles "for temperature, entropy displacement, heat forces a.s.o.".

All the variational principles for parabolic equations may be applied to complex fields of mathematical physics. as e.g. to coupled thermoelasticity and thermo-viscoelasticity. Some of the above-mentioned principles have been already established there by Biot [3], [4], [5], Schapery [6], Herrmann [8], Ben-Amoz [9] and others.

For the sake of clearness and simplicity, we shall not introduce here the assumptions providing the existence, uniqueness and the necessary regularity of the solution to the original problem (compare e.g. [15], [16]). Let us suppose, that such solution exists, which enables us to carry out all the steps and transformations required in the course of the following explanation.

2. Primary variational principles. Let us consider a bounded region Ω of the N-dimensional Euclidean space with a Cartesian coordinates frame $X \equiv (x_1, \ldots, x_N)$. Let the boundary Γ of the region Ω consists of four mutually disjoint parts Γ_u, Γ_h, Γ_v and Γ_0,[1]) the latest of which has the zero $(N - 1)$-dimensional measure, i.e.

$$\Gamma = \Gamma_u \cup \Gamma_h \cup \Gamma_v \cup \Gamma_0 .$$

Let the problem be given as follows:
to find such function $u(X, t)$, which satisfies the equation

$$\dot{u} = (K_{ij}u_{,j})_{,i} + f \quad \text{for} \quad X \in \Omega , \quad t > 0 , \tag{2.1}$$

the initial condition

$$u(X, 0) = u_0(X) \quad \text{for} \quad X \in \Omega \tag{2.2}$$

[1]) Each of Γ_u, Γ_h, Γ_v is either vacuous or has a positive $(N - 1)$-dimensional Hausdorf's measure.

and the boundary conditions

$$(2.3a) \qquad u = P \quad \text{for} \quad X \in \Gamma_u, \quad t > 0,$$

$$(2.3b) \qquad K_{ij} n_i u_{,j} = P \quad \text{for} \quad X \in \Gamma_h, \quad t > 0,$$

$$(2.3c) \qquad Au + K_{ij} n_i u_{,j} = P \quad \text{for} \quad X \in \Gamma_v, \quad t > 0.$$

Here we use the notation

$$\dot{u} \equiv \frac{\partial u}{\partial t}, \quad u_{,j} \equiv \frac{\partial u}{\partial x_j}, \quad j = 1, 2, \ldots, N$$

and summation over repeated subscripts $i, j = 1, \ldots, N$. $K_{ij}(X) = K_{ji}(X)$ are prescribed functions for $X \in \bar{\Omega} = \Omega \cup \Gamma$, $A(X) \geqq \alpha > 0$ for $X \in \Gamma_v$, $f(X, t)$ is given for $X \in \Omega$, $t \geqq 0$ and $P(X, t)$ for $X \in \Gamma \dot{-} \Gamma_0$, $t > 0$. n_i denote the components of unit outward normal to Γ. In heat conduction theory, $u(X, t)$ denotes the temperature distribution, f the internal sources of heat, the boundary condition (2.3b) prescribes the heat flux and (2.3c) the interchange of heat with the surrounding medium. The body, occupying the closed region $\bar{\Omega}$, is non-homogeneous in general, if the functions K_{ij} and A vary with the spatial coordinates. For simplicity, however, the specific heat is supposed to be constant.

In the following we shall use the concept of *convolution* of two functions. Therefore, let us recall its definition: assume that functions $f(X, t)$ and $g(X, t)$ are continuous in $t \in \langle 0, \infty)$ for each fixed $X \in M$. Then by the convolution $f * g$ of these two functions we understand the function, defined through the relation

$$[f * g](X, t) = \int_0^t f(X, t - \tau)\, g(X, \tau)\, d\tau$$

for $X \in M$, $t \in \langle 0, \infty)$.

Convolution has the following properties (see e.g. [13]):

$$f * g = g * f,$$

$$f * (g * h) = (f * g) * h = f * g * h,$$

$$\frac{\partial}{\partial t}(f * g) = \dot{f} * g + f(X, 0)\, g(X, t)$$

There exists a group of primary variational principles related to the problem (2.1)–(2.3c), namely *convolution principles, their Laplace transforms, operational principles and principles with special variations*. Moreover, each of those four types has two alternatives: *integral* and *differential*.[2]) Next we shall deal with all the eight types, starting with convolution principles.

[2]) Let us remark, that the terminology of those principles has not yet stabilized, so that our notation and classification is to be regarded as a suggestion only.

α-integral convolution principle. *Let $\mathscr{K}$ be a set of functions $u(X, t)$, which satisfy the boundary condition* (2.3a) *and for which the functional*

$$\mathscr{F}_t(u) = \int_\Omega \{u * u + K_{ij} * u_{,i} * u_{,j} - 2 * f * u - 2u_0 * u\} \, dX - \tag{2.4}$$

$$- \int_{\Gamma_h} 2 * P * u \, d\Gamma + \int_{\Gamma_v} (A * u * u - 2 * P * u) \, d\Gamma .$$

may be defined.

Then

$$\delta \mathscr{F}_t(u) = 0 \tag{2.5}$$

holds on $\mathscr{K}$ for every $t > 0$, if and only if u satisfies the equation (2.1), *initial condition* (2.2) *and the boundary conditions* (2.3b), (2.3c).

Proof. Integrating by parts, using the properties of convolution and the symmetry of $K_{ij} = K_{ji}$, we derive

$$\tfrac{1}{2}\delta \mathscr{F}_t(u) = \int_\Omega \{u - (K_{ij} * u_{,i})_{,j} - 1 * f - u_0\} * \delta u +$$

$$+ \int_{\Gamma_h} [K_{ij} * u_{,i} n_j - 1 * P] * \delta u \, d\Gamma + \int_{\Gamma_v} [K_{ij} * u_{,i} n_j - 1 * P + A * u] * \delta u \, d\Gamma .$$

On the base of some lemmas, which are analogous to the fundamental lemmas of the calculus of variations (see Lemma 2.1 and 2.2 in [10]), (2.5) yields

$$u - u_0 = (K_{ij} * u_{,i})_{,j} + 1 * f \quad \text{for} \quad X \in \Omega , \quad t > 0 , \tag{2.6}$$

$$K_{ij} * u_{,i} n_j = 1 * P \quad \text{for} \quad X \in \Gamma_h , \quad t > 0 , \tag{2.7}$$

$$A * u + K_{ij} * u_{,i} n_j = 1 * P \quad \text{for} \quad X \in \Gamma_v , \quad t > 0 . \tag{2.8}$$

Differentiating (2.6) with respect to t, (2.1) can be obtained for $t > 0$. Similarly, from (2.7) and (2.8) the conditions (2.3b) and (2.3c) follow respectively. As $u(X, t)$ is continuous in t, the limit transition $t \to 0+$ in (2.6) yields the initial condition (2.2).

Remark 2.1. The α-principle was suggested by Schapery [6] and Gurtin [7] for the boundary conditions (2.3a) and (2.3b) in the theory of heat conduction.

Remark 2.2. If we restrict the definition of the problem (2.1)–(2.3c) onto a finite interval $0 \leqq t \leqq t_1 < \infty$, then α-principle may be modified as follows:

$$\delta \mathscr{F}_{t_1}(u) = 0$$

holds on $\mathscr{K}$, if and only if u is a solution to the problem. The proof goes through like previously.

Next we are going to show, that to every integral convolution principle (α-type) an equivalent "differential" principle exists, which we shall denote by β. In fact, there is

Lemma 2.1. *Let* $\mathbf{v}$ *be a vector-function, each component of which* $v_i(X, t)$ *is continuous in* $t \in \langle 0, \infty)$ *for every fixed* $X \in \bar{\Omega}$. *Let* $\mathscr{I}_t(\mathbf{v})$ *be such functional, that its variation takes the form*

$$\delta \mathscr{I}_t(\mathbf{v}) = \int_\Omega \sum_{i=1}^{J} F_i(\mathbf{v}) * \delta v_i \, \mathrm{d}X + \int_\Gamma \sum_{i=1}^{J} f_i(\mathbf{v}, X) * \delta v_i \, \mathrm{d}\Gamma ,$$

where F_i, f_i *are such functions, for which* $\delta \mathscr{I}_t(\mathbf{v})$ *can be defined and*

$$\frac{\partial}{\partial t} F_i(\mathbf{v}(X, t), X, t) \quad or \quad \frac{\partial}{\partial t} f_i(\mathbf{v}(X, t), X, t)$$

respectively are continuous in $t \in \langle 0, \infty)$ *for each fixed* $X \in \Omega$ *or* $X \in \Gamma$. *Let*

$$\delta \mathscr{I}_t(\mathbf{v}) = 0$$

holds for every $t > 0$. *Then also*

$$\delta \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{I}_t(\mathbf{v}) \right) = 0$$

holds for every $t > 0$ *and the latter equality is true, if and only if*

(a) $\quad \dfrac{\partial}{\partial t} F_i(\mathbf{v}(X, t), X, t) = 0 \quad for \quad X \in \Omega, \quad t > 0; \quad F_i(\mathbf{v})|_{t=0} = 0 \quad on \quad \Omega,$

(b) $\quad \dfrac{\partial}{\partial t} f_i(\mathbf{v}(X, t), X, t) = 0 \quad for \quad X \in \Gamma, \quad t > 0; \quad f_i(\mathbf{v})|_{t=0} = 0 \quad on \quad \Gamma.$

Proof. We may write

$$\delta \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{I}_t(\mathbf{v}) \right) = \frac{\mathrm{d}}{\mathrm{d}t} \delta \mathscr{I}_t(\mathbf{v}) = \int_\Omega \{ \sum_i \dot{F}_i(\mathbf{v}) * \delta v_i + \sum_i F_i(\mathbf{v})|_{t=0} \, . \, \delta v_i(t) \} \, \mathrm{d}X +$$

$$+ \int_\Gamma \{ \sum_i \dot{f}_i(\mathbf{v}) * \delta v_i + \sum_i f_i(\mathbf{v})|_{t=0} \, . \, \delta v_i(t) \} \, \mathrm{d}\Gamma$$

for every $t > 0$.

Let us denote by $\mathscr{D}(M, (0, t))$ and $\mathscr{D}(M)$ the linear manifolds of functions with compact support in $M \times (0, t)$ and M respectively (here M represents Ω or Γ), having continuous derivatives of all orders. Let i be an arbitrary subscript. Then choosing suitable functions

$$\delta v_i \in \mathscr{D}(\Omega, (0, t)), \quad \delta v_i(t) \in \mathscr{D}(\Omega), \quad \delta v_i \in \mathscr{D}(\Gamma, (0, t)), \quad \delta v_i(t) \in \mathscr{D}(\Gamma),$$

and $\delta v_j = 0$ for $j \neq i$, the assertions (a) and (b) follow gradually (compare e.g. the proof of lemma 2.1 in [10]).

Now lemma 2.1 yields

β-differential convolution principle. *Let us define*

$$\text{(2.9)} \quad \mathscr{F}_t(u) = \int_\Omega \{\dot{u} * u + K_{ij} u_{,i} * u_{,j} - 2f * u - 2u_0 u + u(X, 0)\, u\} \, \mathrm{d}X - \\ - \int_{\Gamma_h} 2P * u \, \mathrm{d}\Gamma + \int_{\Gamma_v} (Au * u - 2P * u) \, \mathrm{d}\Gamma,$$

for every $t > 0$ and $u \in \mathscr{K}$. Then

$$\text{(2.10)} \qquad \delta \mathscr{F}_t(u) = 0$$

holds on $\mathscr{K}$ for every $t > 0$ [3]*, if and only if u satisfies the equation* (2.1), *the initial condition* (2.2) *and the boundary conditions* (2.3b), (2.3c).

Proof. Substituting $v = v_1 = u(X, t)$ in Lemma 2.1, then $F_1(u) = 0$ is expressed by (2.6), $f_1(u, X) = 0$ by (2.7) and (2.8). The remainder of the proof follows immediately from the Lemma 2.1.

Remark 2.3. The β-principle was suggested by Schapery [6] in the theory of heat conduction for $f = u_0 = 0$ on the subset $\mathscr{K}_0$ of functions from $\mathscr{K}$, satisfying also the initial condition (2.2) and for $\Gamma = \Gamma_h \cup \Gamma_u$.

It is well known, that the Laplace transform

$$\hat{u}(X, p) = \int_0^\infty e^{-pt} u(X, t) \, \mathrm{d}t$$

is often used to the solution of the problem. Then the product $\hat{f}\hat{g}$ corresponds with the convolution $f * g$ and u_0/p with the function $u_0(X)$ independent of t. We may establish

$\mathscr{L}\alpha$-Laplace transform of integral principle. *Let $\mathscr{L}\mathscr{K}$ be a set of all functions $\hat{u}(X, p)$, $X \in \Omega$, $p > 0$, which satisfy the transformed boundary condition*

$$\mathscr{L}\text{(2.3a)} \qquad \hat{u} = \hat{P} \quad \text{for} \quad X \in \Gamma_u, \quad p > 0,$$

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[3]) Restricting the definition of the problem onto a finite interval $t \in \langle 0, t_1 \rangle$, $t = t_1$ may be inserted in all the convolutions. (See also Remark 2.2).

and for which the functional

$$\mathscr{L}\mathscr{F}_p(\hat{u}) = \int_\Omega \left\{\hat{u}^2 + \frac{1}{p} K_{ij}\hat{u}_{,i}\hat{u}_{,j} - \frac{2}{p} u_0 \hat{u} - \frac{2}{p}\hat{f}\hat{u}\right\} \mathrm{d}X - \\ - \int_{\Gamma_h} \frac{2}{p} \hat{P}\hat{u} \,\mathrm{d}\Gamma + \int_{\Gamma_v} \left(\frac{A}{p} \hat{u}^2 - \frac{2}{p} \hat{P}\hat{u}\right) \mathrm{d}\Gamma$$

may be defined for each real $p > 0$. *Then*

$$\delta \mathscr{L}\mathscr{F}_p(\hat{u}) = 0$$

on $\mathscr{L}\mathscr{K}$ *for each* $p > 0$, *if and only if* $\hat{u}$ *satisfies the transformed equations*

$\mathscr{L}(2.6)$ $$\frac{1}{p} [\hat{f} + (K_{ij}\hat{u}_{,j})_{,i}] = \hat{u} - \frac{u_0}{p} \quad \textit{for} \quad X \in \Omega\,, \quad p > 0\,,$$

$\mathscr{L}(2.3\text{b})$ $$K_{ij}\hat{u}_{,j}n_i = \hat{P} \quad \textit{for} \quad X \in \Gamma_h\,, \quad p > 0\,,$$

$\mathscr{L}(2.3\text{c})$ $$A\hat{u} + K_{ij}\hat{u}_{,j}n_i = \hat{P} \quad \textit{for} \quad X \in \Gamma_v\,, \quad p > 0\,.$$

The proof follows from the relation

$$\tfrac{1}{2}\delta \mathscr{L}\mathscr{F}_p(\hat{u}) = \int_\Omega \left\{\hat{u} - \frac{1}{p} (K_{ij}\hat{u}_{,j})_{,i} - \frac{\hat{f}}{p} - \frac{u_0}{p}\right\} \delta\hat{u} \,\mathrm{d}X + \\ + \int_{\Gamma_h} \left(\frac{K_{ij}}{p} \hat{u}_{,i}n_j - \frac{\hat{P}}{p}\right) \delta\hat{u} \,\mathrm{d}\Gamma + \int_{\Gamma_v} \left(\frac{K_{ij}}{p} \hat{u}_{,j}n_i + \frac{A}{p} \hat{u} + \frac{\hat{P}}{p}\right) \delta\hat{u} \,\mathrm{d}\Gamma = 0\,.$$

Remark 2.4. Let $p = p_0$ be an arbitrary positive number and the matrix K_{ij} symmetric and positive definite in $\bar{\Omega}$, i.e. let such $\mu > 0$ exists, that

$$K_{ij}(X)\, \xi_i\xi_j \geqq \mu\xi_i\xi_i$$

holds for every real vector $(\xi_1, \xi_2, \ldots, \xi_N)$ and every $X \in \bar{\Omega}$. Denote by $W_2^{(1)}(\Omega)$ the Sobolev space of square-integrable functions (in the Lebesgue's sense), which have square-integrable generalized first derivatives. Assume that such function $\tilde{u}(X) \in \in W_2^{(1)}(\Omega)$ exists, that

$$\tilde{u}(X) = \hat{P}(X, p_0)$$

on Γ_u in the sense of traces. Furthermore, let K_{ij} be bounded and measurable functions of $X \in \bar{\Omega}$ for each $i, j, \hat{f}(X, p_0)$ and $u_0(X)$ square-integrable on Ω. Denote by $\mathscr{L}\mathscr{K}(p_0)$ the set of functions, resulting from the set $\mathscr{L}\mathscr{K}$ by fixing the parameter $p = p_0$ in each $\hat{u}(X, p) \in \mathscr{L}\mathscr{K}$.

Then $\mathscr{L}\mathscr{F}_{p_0}(\hat{u})$ attains its absolute minimum on $\mathscr{L}\mathscr{K}(p_0)$, if and only if $\delta\mathscr{L}\mathscr{F}_{p_0}(\hat{u}) = = 0$ on $\mathscr{L}\mathscr{K}(p_0)$.

The proof of this assertion is based on the inequality

$$\int_{\Omega}\left(K_{ij}\frac{1}{p_0}\,\hat{u}_{,i}\hat{u}_{,j}+\hat{u}^2\right)\mathrm{d}X+\int_{\Gamma_v}\frac{A}{p_0}\,\hat{u}^2\,\mathrm{d}\Gamma\geqq C\int_{\Omega}(\hat{u}_{,i}\hat{u}_{,i}+\hat{u}^2)\,\mathrm{d}X$$

and a similar approach may be used, as in [14] (Sections 2 and 3), where the principle of minimum potential energy for a general boundary-value problem of linear elasticity is proved.

The counterpart of the differential convolution principle is

$\mathscr{L}\beta$**-Laplace transform of differential principle.** *Let us define the functional*

$$\mathscr{L}\mathscr{F}^{\cdot}_p(\hat{u})=\int_{\Omega}\{p\hat{u}^2+K_{ij}\hat{u}_{,i}\hat{u}_{,j}-2\hat{f}\hat{u}-2u_0\hat{u}\}\,\mathrm{d}X-\int_{\Gamma_h}2\hat{P}\hat{u}\,\mathrm{d}\Gamma+$$
$$+\int_{\Gamma_v}(A\hat{u}^2-2\hat{P}\hat{u})\,\mathrm{d}\Gamma\,,$$

for $\hat{u}\in\mathscr{L}\mathscr{K}$ *and real* $p>0$.

Then

$$\delta\mathscr{L}\mathscr{F}^{\cdot}_p(\hat{u})=0$$

on $\mathscr{L}\mathscr{K}$ *for every real* $p>0$, *if and only if* $\hat{u}$ *satisfies the transformed equation* $\mathscr{L}$(2.6) *and the transformed boundary conditions* $\mathscr{L}$(2.3b), $\mathscr{L}$(2.3c).

Proof. It suffices to write

$$\mathscr{L}\mathscr{F}^{\cdot}_p(\hat{u})=p\mathscr{L}\mathscr{F}_p(\hat{u})$$

and to use the proof of $\mathscr{L}\alpha$-principle.

Remark 2.5. The assertion of Remark 2.4, that the stationary value of $\mathscr{L}\mathscr{F}_{p_0}(\hat{u})$ is a minimum value, remains in validity also for $\mathscr{L}\beta$-principle.

Next we are coming to the operational principles, which correspond closely with the Laplace transforms of convolution principles.

$\mathscr{B}\alpha$**-integral operational principle.** *Let* $p\neq 0$ *be a fixed real parameter. Define*

$$\mathscr{B}\mathscr{F}_{pt}(u)=\int_{\Omega}\left\{u^2+\frac{K_{ij}}{p}\,u_{,i}u_{,j}-\frac{2}{p}fu-2u_0u\right\}\mathrm{d}X-$$
$$-\int_{\Gamma_h}\frac{2}{p}\,Pu\,\mathrm{d}\Gamma+\int_{\Gamma_v}\left(\frac{A}{p}\,u^2-\frac{2}{p}\,Pu\right)\mathrm{d}\Gamma\,,$$

for every $u\in\mathscr{K}$ *and* $t>0$. *Then*

$$\delta\mathscr{B}\mathscr{F}_{pt}(u)=0$$

on $\mathscr{K}$ for every $t > 0$, if and only if u satisfies the equation

$$u - u_0 = \frac{1}{p}\left[(K_{ij}u_{,i})_{,j} + f\right] \tag{2.11}$$

and the boundary conditions (2.3b), (2.3c), *for $t > 0$.*

The proof runs just like that for $\mathscr{L}\alpha$-principle.

If we interpret the parameter p in (2.11) as the differential operator and $1/p$ as the integral operator, i.e. if

$$p \equiv \frac{\partial}{\partial t}, \quad \frac{1}{p} \equiv \int_0^t (\quad)\,d\tau\,,$$

then from (2.11) both the equation (2.1) and the initial condition (2.2) follow. Hence $\mathscr{B}\alpha$-principle is equivalent to the original problem in the sense mentioned above.

Remark 2.6. $\mathscr{B}\alpha$-principle was suggested by Biot [3] in the theory of heat conduction for $f = 0$, $\Gamma = \Gamma_u \cup \Gamma_h$.

Remark 2.7. Again, the functional $\mathscr{BF}_{p_0t_0}(u)$ attains its minimum on $\mathscr{K}(t_0)$ for any fixed $p = p_0 > 0$, $t = t_0 > 0$, if and only if $\delta\mathscr{BF}_{p_0t_0}(u) = 0$, provided the same assumptions as in Remark 2.4 are valid. Here $\mathscr{K}(t_0)$ is the set, which results from the set of functions $u \in \mathscr{K}$ by fixing $t = t_0$.

$\mathscr{B}\beta$ – differential operational principle. *Let $p \neq 0$ be a fixed real parameter. Define*

$$\mathscr{BF}^{\cdot}_{pt}(u) = \int_\Omega \{pu^2 + K_{ij}u_{,i}u_{,j} - 2fu - 2pu_0u\}\,dX -$$
$$- \int_{\Gamma_h} 2Pu\,d\Gamma + \int_{\Gamma_v} (Au^2 - 2Pu)\,d\Gamma\,.\,[4]$$

for $u \in \mathscr{K}$ and $t > 0$. Then

$$\delta\mathscr{BF}^{\cdot}_{pt}(u) = 0$$

on $\mathscr{K}$ for every $t > 0$, if and only if u satisfies the equation (2.11) *and the boundary conditions* (2.3b), (2.3c).

Proof. The variation being carried out, we apply the fundamental lemmas of the calculus of variation for $u(X, t)$ with a fixed t and divide by p.

[4]) If we restrict the definition of $\mathscr{BF}^{\cdot}_{pt}(u)$ onto a subset $\mathscr{K}_0 \subset \mathscr{K}$ of functions, which satisfy also the initial condition, then the term $-2pu_0u$ may be omitted. Consequently, instead of (2.11) we have the equation

$$pu = (K_{ij}u_{,j})_{,i} + f\,.$$

Interpreting p again as the differential operator like in $\mathscr{B}\alpha$-principle, we obtain the equivalence of $\mathscr{B}\beta$-principle with the original problem (2.1)–(2.3c).

Remark 2.8. $\mathscr{B}\beta$-principle was suggested by Biot [5] in the theory of heat conduction for $f = 0$, $\Gamma = \Gamma_u \cup \Gamma_h$.

Remark 2.9. An assertion analogous to that of Remark 2.6 holds.

The survey of primary variational principles for parabolic equations may be completed by a couple of principles with special variations.

$\tilde{\alpha}$-integral principle with special variations. *Define*

$$\tilde{\mathscr{F}}_t(u) = \int_\Omega \{u^2 + K_{ij} * u,_i u,_j - 2 * fu - 2u_0 u\} \,\mathrm{d}X - \\ - \int_{\Gamma_h} 2 * Pu \,\mathrm{d}\Gamma + \int_{\Gamma_v} (A * u^2 - 2 * Pu) \,\mathrm{d}\Gamma$$

for $u \in \mathscr{K}$ and $t > 0$.

Let the variations $\tilde{\delta} u$ do not depend on t, i.e.

$$\tilde{\delta} u = \tilde{\delta} u(X)\,, \quad \frac{\partial}{\partial t} \tilde{\delta} u = \tilde{\delta} \dot{u} = 0\,.$$

Then

$$\tilde{\delta} \tilde{\mathscr{F}}_t(u) = 0$$

on $\mathscr{K}$ for each $t > 0$ if and only if u satisfies the equation (2.1), *the initial condition* (2.2) *and the boundary conditions* (2.3b), (2.3c).

The proof runs likewise that of α-principle, replacing only the convolution products with δu by the usual multiplication and using the fundamental lemmas of the calculus of variations.

Remark 2.10. The functional $\tilde{\mathscr{F}}_t(u)$ coincides with $\mathscr{B}\mathscr{F}_{pt}(u)$, $1/p$ being interpreted as the integral operator. Nevertheless, variations in $\mathscr{B}\alpha$-principle may depend on t.

$\tilde{\beta}$-differential principle with special variations. *Let $\mathscr{K}_0 \subset \mathscr{K}$ be the subset of such functions from $\mathscr{K}$, which satisfy also the initial condition* (2.2). *Define*

$$\tilde{\mathscr{F}}^{\cdot}_t(u) = \int_\Omega \{2\dot{u}u + K_{ij} u,_i u,_j - 2fu\} \,\mathrm{d}X - \int_{\Gamma_h} 2Pu \,\mathrm{d}\Gamma + \int_{\Gamma_v} (Au^2 - 2Pu) \,\mathrm{d}\Gamma$$

for each $u \in \mathscr{K}_0$ and $t > 0$. Let the variations $\tilde{\delta} u$ do not depend on t, i.e.

$$\tilde{\delta} u = \tilde{\delta} u(X)\,, \quad \frac{\partial}{\partial t} \tilde{\delta} u = \tilde{\delta} \dot{u} = 0\,.$$

Then

$$\tilde{\delta} \tilde{\mathscr{F}}^{\cdot}_t(u) = 0$$

on $\mathscr{K}_0$ for every $t > 0$, if and only if u satisfies the equation (2.1), *and the boundary conditions* (2.3b), (2.3c).

The proof follows easily from the relation

$$\tfrac{1}{2}\delta\tilde{\mathscr{F}}_t^{\cdot}(u) = \int_\Omega \{\dot{u} - (K_{ij}u_{,i})_{,j} - f\}\, \delta u \, \mathrm{d}X + \int_{\Gamma_h} (K_{ij}n_j u_{,i} - P)\, \delta u \, \mathrm{d}\Gamma +$$
$$+ \int_{\Gamma_v} (Au + K_{ij}u_{,i}n_j - P)\, \delta u \, \mathrm{d}\Gamma = 0 \, .$$

Remark 2.11. The functional $\tilde{\mathscr{F}}_t^{\cdot}(u)$ coincides with $\mathscr{B}\mathscr{F}_{pt}^{\cdot}(u)$, if we restrict $\mathscr{B}\beta$-principle onto the subset $\mathscr{K}_0$ and interpret the parameter p as the differential operator with respect to t.

Remark 2.12. A similar principle was suggested by Rosen [2] in the theory of heat conduction.

3. Secondary variational principles. In the present section we shall derive a group of variational principles, which characterize the original problem, but expressed by means of some other unknown functions. The well – known Friedrichs' method of inverting the minimum problem into a maximum problem [11] will be applied in a way similar to that used for the derivation of the principle of minimum complementary energy from the principle of minimum potential energy in the theory of elasticity (see e.g. [12]). Furthermore, some generalized variational principles, counterparts of Hellinger-Reissner principle and Hu-Washizu principle in elasticity, will be established.

Let us start with some of those primary principles, which imply the minimum property in the sense of Remarks 2.4 (or 2.5, 2.7, 2.9 respectively). For simplicity, let us consider the $\mathscr{B}\beta$-principle and take the assumptions of Remark 2.4 (where $\hat{P}$ and $\hat{f}$ will be replaced by P, f) for granted. Then fixing $p_0 > 0$ and $t_0 > 0$, the condition of stationary value implies the minimum value for the functional $\mathscr{B}\mathscr{F}_{p_0t_0}^{\cdot}(u)$ on the set $\mathscr{K}(t_0)$. According to the Friedrichs' method, we rewrite this minimum condition in the form

$$\mathscr{B}'\mathscr{F}_{p_0t_0}^{\cdot}(u, g_i) = \min$$

for $u \in \mathscr{K}(t_0)$ with subsidiary conditions

(3.1) $$u_{,i} = g_i \quad \text{in} \quad \Omega \, ,$$

where

(3.2) $$\mathscr{B}'\mathscr{F}_{p_0t_0}^{\cdot}(u, g_i) = \int_\Omega \{p_0u^2 + K_{ij}g_ig_j - 2fu - 2p_0u_0u\} \, \mathrm{d}X -$$
$$- \int_{\Gamma_h \cup \Gamma_v} 2Pu \, \mathrm{d}\Gamma + \int_{\Gamma_v} Au^2 \, \mathrm{d}\Gamma \, .$$

First we shall restrict ourselves onto a subset $\overline{\mathscr{K}}(t_0)$ of functions from $\mathscr{K}(t_0)$, which satisfy also the boundary condition (2.3c), and express the latter by means of u and g_i, i.e.

$$Au + K_{ij}n_ig_j = P \quad \text{for} \quad X \in \Gamma_v, \quad t = t_0. \tag{3.3}$$

Inserting (3.3) into (3.2) and joining the side conditions (3.1) and (2.3a) by means of Lagrange multipliers $2\lambda_i$ and $2\mu_0$ respectively, we obtain the functional

$$\mathscr{B}''\mathscr{F}^{\cdot}_{p_0t_0}(u, g_i, \lambda_i, \mu_0) = \int_\Omega \{p_0u^2 + K_{ij}g_ig_j - 2fu - 2p_0u_0u + 2\lambda_i(g_i - u_{,i})\} \, dX - $$
$$- \int_{\Gamma_h} 2Pu \, d\Gamma + \int_{\Gamma_v} \left\{\frac{1}{A}(P - K_{ij}n_ig_j)^2 - 2\frac{P}{A}(P - K_{ij}n_ig_j)\right\} d\Gamma +$$
$$+ \int_{\Gamma_u} 2\mu_0(u - P) \, d\Gamma .$$

Integrating the term $\lambda_i u_{,i}$ by parts and substituting again from (3.3), we may write

$$\mathscr{B}''\mathscr{F}^{\cdot}_{p_0t_0}(u, g_i, \lambda_i, \mu_0) = \int_\Omega \{p_0u^2 + K_{ij}g_ig_j - 2fu - 2p_0u_0u + \tag{3.4}$$
$$+ 2\lambda_ig_i + 2\lambda_{i,i}u\} \, dX - 2\int_{\Gamma_h} (Pu + \lambda_in_iu) \, d\Gamma +$$
$$+ 2\int_{\Gamma_u} [\mu_0(u - P) - \lambda_in_iu] \, d\Gamma +$$
$$+ \int_{\Gamma_v} \frac{1}{A} [(K_{ij}n_ig_j)^2 - P^2 - 2\lambda_in_i(P - K_{ij}n_ig_j)] \, d\Gamma .$$

Let us consider the variational equation

$$\delta\mathscr{B}''\mathscr{F}^{\cdot}_{p_0t_0} = 0$$

with respect to independent variables u, g_i, λ_i in Ω, u, λ_i on $\Gamma_h \cup \Gamma_u$, g_i, λ_i on Γ_v and μ_0, u on Γ_u. Among all Euler's equations and natural boundary conditions let us choose only the relations complementary to the subsidiary conditions (3.1), (2.3a) of the original problem, i.e.

$$p_0(u - u_0) = f - \lambda_{i,i} \quad \text{in} \quad \Omega, \tag{3.5}$$
$$K_{ij}g_j + \lambda_i = 0 \quad \text{in} \quad \Omega, \tag{3.6}$$
$$P + \lambda_in_i = 0 \quad \text{on} \quad \Gamma_h, \tag{3.7}$$
$$\mu_0 - \lambda_in_i = 0 \quad \text{on} \quad \Gamma_u, \tag{3.8}$$
$$K_{ij}n_j(K_{ij}n_ig_j + n_i\lambda_i) = 0 \quad \text{on} \quad \Gamma_v. \tag{3.9}$$

Let us insert the results from (3.5)–(3.9) into the functional (3.4), i.e.

$$(3.5') \qquad u - u_0 = \frac{1}{p_0}(f - \lambda_{i,i}) \quad \text{in} \quad \Omega\,,$$

$$(3.6') \qquad g_i = -K_{ij}^{-1}\lambda_j \quad \text{in} \quad \Omega\,,^{5)}$$

$$\lambda_i n_i = -P \quad \text{on} \quad \Gamma_h\,, \quad \mu_0 = \lambda_i n_i \quad \text{on} \quad \Gamma_u\,,$$

$$(3.9') \qquad K_{ij} n_i g_j = -\lambda_i n_i \quad \text{on} \quad \Gamma_v\,.^{6)}$$

Thus we are led to the functional

$$-\mathscr{B}'\mathscr{G}^{\cdot}_{p_0 t_0}(\lambda_i) = -\int_\Omega \left\{\frac{1}{p_0}(f + p_0 u_0 - \lambda_{i,i})^2 + K_{ij}^{-1}\lambda_i\lambda_j\right\} \mathrm{d}X -$$

$$-2\int_{\Gamma_h} \lambda_i n_i P \,\mathrm{d}\Gamma - \int_{\Gamma_v} \frac{1}{A}\left[(\lambda_i n_i)^2 + 2P\lambda_i n_i + P^2\right] \mathrm{d}\Gamma\,.$$

Now we are already able to formulate **the secondary $\mathscr{B}\beta$-differential operational principle:**

Let Λ be the set of vector-functions $\lambda_i(X, t)$, which satisfy the boundary condition

$$(3.10) \qquad \lambda_i n_i + P = 0$$

for $X \in \Gamma_h$, $t > 0$, and for which the functional

$$(3.11) \qquad \mathscr{B}\mathscr{G}^{\cdot}_{tp}(\lambda_i) = \int_\Omega \left\{K_{ij}^{-1}\lambda_i\lambda_j + \frac{1}{p}(f + pu_0 - \lambda_{i,i})^2\right\} \mathrm{d}X +$$

$$+\int_{\Gamma_u} 2P\lambda_i n_i \,\mathrm{d}\Gamma + \int_{\Gamma_v} \frac{1}{A}\left[(\lambda_i n_i)^2 + 2P\lambda_i n_i\right] \mathrm{d}\Gamma$$

may be defined for any $p \neq 0$, $t \geqq 0$. Then

$$(3.12) \qquad \delta\mathscr{B}\mathscr{G}^{\cdot}_{tp}(\lambda_i) = 0$$

holds on Λ for all $t \geqq 0$, if and only if λ_i satisfy also the equations

$$(3.13) \qquad K_{ij}^{-1}\lambda_j + \frac{1}{p}(f + pu_0 - \lambda_{j,j})_{,i} = 0 \quad \textit{for} \quad X \in \Omega\,, \quad t \geqq 0$$

---

[5]) K_{ij}^{-1} denotes the inverse matrix to K_{ij}.

[6]) By virtue of the positive definiteness of K_{ij} on $\bar{\Omega}$, at least one of the components $K_{ij}n_j$ does not vanish.

and the boundary conditions

$$P - \frac{1}{p}(f + pu_0 - \lambda_{j,j}) = 0 \quad \textit{for} \quad X \in \Gamma_u, \quad t > 0, \tag{3.14}$$

$$\frac{A}{p}(f + pu_0 - \lambda_{j,j}) - \lambda_i n_i = P \quad \textit{for} \quad X \in \Gamma_v, \quad t > 0. \tag{3.15}$$

The proof follows easily by carrying out the variation of (3.12), integrating the term with $\delta\lambda_{i,i}$ by parts and using the relation

$$n_i \delta\lambda_i = 0 \quad \text{on} \quad \Gamma_h,$$

which is a consequence of (3.10).

Remark 3.1. Taking (3.5′) and (3.6′) for the definitions of $u(X, t)$ and $g_i(X, t)$ respectively (and replacing at the same time p_0, t_0 by parameters $p \neq 0$, $t \geqq 0$), defining $K_{ij}n_i g_j$ through (3.9′) and finally interpreting p, $1/p$ as

$$p \equiv \frac{\partial}{\partial t}, \quad \frac{1}{p} \equiv \int_0^t (\quad) \, d\tau,$$

then (3.13) may be understood as the relation

$$g_i = u_{,i} \quad \text{for} \quad X \in \Omega, \quad t \geqq 0,$$

(3.14) as

$$u = P \quad \text{for} \quad X \in \Gamma_u, \quad t > 0$$

and (3.15) as the condition

$$Au + K_{ij}n_i g_j = P \quad \text{for} \quad X \in \Gamma_v, \quad t > 0.$$

Altogether (3.12) characterizes the dual problem, expressed by means of a vector-function λ_i by means of the relations (3.13), (3.14), (3.15) and (3.10).

Remark 3.2. If we restrict ourselves again onto fixed parameters $p_0 > 0$, $t_0 > 0$, then using Friedrichs' method, the following assertion can be proved (compare [11], [12]):

if a function $\mathring{u}$ exists, which minimizes the functional $\mathscr{B}\mathscr{F}^{\cdot}_{p_0t_0}(u)$ on the set $\overline{\mathscr{K}}(t_0)$, then the dual problem is a maximum problem

$$-\mathscr{B}'\mathscr{G}^{\cdot}_{p_0t_0}(\lambda_i) = \max., \quad \lambda_i \in \Lambda(t_0),$$

which has a solution $\mathring{\lambda}_i$ and it holds

$$-\mathscr{B}'\mathscr{G}^{\cdot}_{p_0t_0}(\mathring{\lambda}_i) = \mathscr{B}\mathscr{F}^{\cdot}_{p_0t_0}(\mathring{u}), \quad \mathring{\lambda}_i = -K_{ij}\mathring{u}_{,j}.$$

In order to show some examples of the secondary principles in the theory of heat conduction and to compare our results with the existing principles there, first let us derive the integral $\mathscr{B}\alpha$-type of the principle, related to (3.11), (3.12). Denote

$$\frac{1}{p}\,\mathscr{B}\mathscr{G}^{\cdot}_{tp}(\lambda_i) = \mathscr{B}\mathscr{G}'_{tp}(\lambda_i)$$

and substitute

$$\lambda_i = ph_i\,.$$

Then

$$\text{(3.16)}\quad \mathscr{B}\mathscr{G}_{tp}(h_i) \equiv \mathscr{B}\mathscr{G}'_{tp}(ph_i) = \int_\Omega \left\{K_{ij}^{-1} ph_i h_j + \left(\frac{f}{p} + u_0 - h_{i,i}\right)^2\right\} \mathrm{d}X + \\ + 2\int_{\Gamma_u} Ph_i n_i \,\mathrm{d}\Gamma + \int_{\Gamma_v} \frac{1}{A}\left[p(h_i n_i)^2 + 2Ph_i n_i\right] \mathrm{d}\Gamma$$

and we may formulate **the secondary $\mathscr{B}\alpha$-integral operational principle:**

Let $p \neq 0$ be a fixed real number. Denote by $\mathscr{H}(p)$ the set of vector-functions $h_i(X, t)$, which satisfy the boundary conditions

$$ph_i n_i + P = 0 \quad for \quad X \in \Gamma_h\,, \quad t > 0\,,$$

and for which the functional $\mathscr{B}\mathscr{G}_{tp}(h_i)$ may be defined through (3.16). *Then*

$$\text{(3.17)}\qquad \delta\mathscr{B}\mathscr{G}_{tp}(h_i) = 0$$

on $H(p)$ for every $t \geqq 0$, if and only if h_i satisfy also the equations

$$\text{(3.13')}\qquad pK_{ij}^{-1} h_j + \left(\frac{f}{p} + u_0 - h_{j,j}\right)_{,i} \quad for \quad X \in \Omega\,, \quad t \geqq 0$$

and the boundary conditions

$$\text{(3.14')}\qquad P - \left(\frac{f}{p} + u_0 - h_{j,j}\right) = 0 \qquad for \quad X \in \Gamma_u\,, \quad t > 0$$

$$\text{(3.15')}\qquad A\left(\frac{f}{p} + u_0 - h_{j,j}\right) - ph_i n_i = P \quad for \quad X \in \Gamma_v\,, \quad t > 0\,.$$

The proof follows easily carrying out the variation, integrating the term with $\delta h_{i,i}$ by parts and using $n_i \delta h_i = 0$ on Γ_h.

Remark 3.3. An assertion analogous to that fo Remark 3.1 holds again.

Remark 3.4. Let us interpret p as the differential operator $\partial/\partial t$,

$$\lambda_i = ph_i = \dot{h}_i = \dot{H}_i/c$$

as the reduced vector of heat flux and

$$h_i = \frac{1}{p} \dot{h}_i = \int_0^t \dot{h}_i(\tau)\, d\tau = T_r S_i / c \,,$$

where S_i denotes the vector of entropy displacement, T_r the (constant) relative temperature and c the (constant) coefficient of specific heat. Then the variational equation (3.17) represents an extension of Biot's equation [3], [4].

Secondary α-integral convolution principle. *Let us define $h_i = 1 * \dot{h}_i$ and*

$$\mathscr{G}_t(\dot{h}_i) = \int_\Omega \{K_{ij}^{-1} h_i * h_j + (1 * f + u_0 - h_{j,j}) * (1 * f + u_0 - h_{i,i})\}\, dX +$$

$$+ 2 \int_{\Gamma_u} P * h_i n_i \, d\Gamma + \int_{\Gamma_v} \frac{1}{A} [h_i n_i + 2P] * h_j n_j \, d\Gamma$$

for every $\dot{h}_i \in \bar{\Lambda}$ [7]) *and $t \geqq 0$. Then*

$$\delta \mathscr{G}_t(\dot{h}_i) = 0$$

on $\bar{\Lambda}$ for every $t > 0$,[8]) *if and only if $\dot{h}_i$ satisfy also the equations*

(3.13″) $$K_{ij}^{-1} h_j + (1 * f + u_0 - h_{j,j})_{,i} = 0 \quad \text{for} \quad X \in \Omega \,, \quad t \geqq 0$$

and the boundary conditions

(3.14″) $$1 * f + u_0 - h_{j,j} = P \quad \text{for} \quad X \in \Gamma_u \,, \quad t > 0 \,,$$

(3.15″) $$A(1 * f + u_0 - h_{j,j}) - h_i n_i = P \quad \text{for} \quad X \in \Gamma_v \,, \quad t > 0 \,.$$

The proof follows easily, if we carry out the variation, integrate the term with $\delta h_{j,j}$ by parts, use the relations

$$n_i \delta \dot{h}_i = 0 \quad \text{on} \quad \Gamma_h \,, \qquad K_{ij}^{-1} = K_{ji}^{-1} \,,$$

$$\dot{h}_i * \delta h_j = \dot{h}_i * (1 * \delta \dot{h}_j) = (1 * \dot{h}_i) * \delta \dot{h}_j = h_i * \delta \dot{h}_j$$

and the counterparts of the fundamental lemmas of the calculus of variations (see [10]). Finally the differentiation with respect to t yields (3.13″)–(3.15″).

Remark 3.5. If we define

$$S_i = c h_i / T_r$$

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[7]) The set $\bar{\Lambda}$ contains those $\dot{h}_i \in \Lambda$, which may be inserted into convolution, i.e. for example $\dot{h}_i(X, t)$ continuous in t.

[8]) A modified principle in the sense of Remark 2.2 is true, too.

as the entropy displacement, then $\mathscr{G}_t(\dot{S}_i T_r/c)$ represents an extension of Schapery's functional in [6], (2.47), the latter being completed by a non-homogeneous initial condition and the boundary condition on Γ_v. Moreover, here we use the reduced heat flux vector $\dot{h}_i \equiv \lambda_i$, proportional to the "entropy velocities" $\dot{S}_i$, as independent variables, while Schapery employed the entropy displacements S_i.

Secondary β-differential convolution principle. *Define* $h_i = 1 * \dot{h}_i$ *and*

$$\dot{\mathscr{G}}_t(\dot{h}_i) = \int_\Omega \{K_{ij}^{-1} h_i * \dot{h}_j + (1 * f + u_0 - h_{j,j}) * (f - \dot{h}_{j,j}) - u_0 h_{j,j}\} \, dX + + 2 \int_{\Gamma_u} P * \dot{h}_i n_i \, d\Gamma + \int_{\Gamma_v} \frac{1}{A} [h_i n_i + 2P] * \dot{h}_j n_j \, d\Gamma$$

for $\dot{h}_i \in \bar{\Lambda}$ *(see the footnote belonging to α-principle) and for every* $t \geqq 0$. *Then*

$$\delta \dot{\mathscr{G}}_t(\dot{h}_i) = 0$$

on $\bar{\Lambda}$ *for every* $t > 0$ [9]*, if and only if* $\dot{h}_i$ *satisfy also the equations* (3.13″), (3.14″), (3.15″).

The proof is based on the secondary α-principle and the lemma 2.1. In fact, equations (3.13″)–(3.15″) are derivatives with respect to t of Euler's equations and natural boundary conditions, following from

$$\delta \mathscr{G}_t(h_i) = 0 \, .$$

The initial conditions

$$F_i(\mathbf{v})|_{t=0} = 0 \, , \quad f_i(\mathbf{v})|_{t=0} = 0$$

of the lemma 2.1 are fulfilled already by virtue of the definition.

Remark 3.6. It is obvious, that in the above-mentioned principles, the independent variables $\dot{h}_i$ may be replaced by h_i or S_i respectively. We shall not discuss these possibilites, but notice only, that then the initial conditions $h_i(0) = 0$ or $S_i(0) = 0$ have to be satisfied a priori.

Remark 3.7. Also the secondary principles may be completed by a couple of Laplace transforms ($\mathscr{L}\alpha$, $\mathscr{L}\beta$) and a couple of principles with special variations ($\tilde{\alpha}, \tilde{\beta}$). Moreover, there is an optional choice of independent variables in the sense of Remarks 3.5, 3.6.

At the end we shall establish two generalized secondary principles, analogous to that of HELLINGER-REISSNER and HU-WASHIZU in the theory of elasticity. They may be derived e.g. like in [12] from the functional (3.2) or (3.11) respectively.

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[9]) A modified principle in the sense of Remark 2.2 holds again.

1. generalized $\mathscr{B}\beta$-principle. *Let $p \neq 0$ be a fixed chosen number. The functional*

$$\mathscr{B}\mathscr{H}^{\cdot}_{pt}(u, g_i, \lambda_i) = \int_\Omega \{pu^2 + K_{ij}g_ig_j - 2fu - 2pu_0u + 2\lambda_i(g_i - u_{,i})\} \, dX +$$

$$+ 2\int_{\Gamma_u} \lambda_i n_i(u - P) \, d\Gamma - 2\int_{\Gamma_h} Pu \, d\Gamma + \int_{\Gamma_v} (Au^2 - 2Pu) \, d\Gamma$$

attains its stationary value with respect to independent functions u, g_i, λ_i (without any side condition) for every $t \geqq 0$, if and only if u, g_i, λ_i satisfy the equations (3.5)[10]), (3.6), (3.1), (3.7), (2.3a), (2.3c).

By virtue of (3.1) and (3.6), (3.7) is equivalent to (2.3b). The parameter p being interpreted as the operator $\partial/\partial t$, the principle characterizes the solution of the problem, expressed in terms of functions u, g_i, λ_i.

Remark 3.8. The corresponding 1. generalized $\mathscr{B}\alpha$-principle (for $f = 0$, $\Gamma = \Gamma_u \cup \Gamma_h$ and regardless of the initial condition (2.2)) is involved in Herrmann's extension [8] of the Reissner-Hellinger principle from elasticity onto coupled thermo-elasticity.

2. generalized $\mathscr{B}\beta$-principle. *Let $p \neq 0$ be a fixed chosen number. The functional*

$$\mathscr{B}\mathscr{I}^{\cdot}_{pt}(u, g_i, \lambda_i, L) = \int_\Omega \Big\{-K_{ij}g_ig_j + \frac{1}{p}(f + pu_0 + L)^2 - 2u(L + \lambda_{i,i}) -$$

$$- 2g_i\lambda_i\Big\} \, dX + 2\int_{\Gamma_u} Pn_i\lambda_i \, d\Gamma + 2\int_{\Gamma_h} u(\lambda_i n_i + P) \, d\Gamma +$$

$$+ \int_{\Gamma_v} \frac{1}{A}[(\lambda_i n_i)^2 + 2P\lambda_i n_i] \, d\Gamma$$

attains its stationary value on the set of independent functions u, g_i, λ_i, L (without any side conditions) for every $t \geqq 0$, if and only if u, g_i, λ_i, L satisfy the equations

$$u - u_0 = \frac{1}{p}(f + L), \tag{3.18}$$

$$L + \lambda_{i,i} = 0 \tag{3.19}$$

and (3.1), (3.6) *in Ω for every $t \geqq 0$ and the boundary conditions*

$$Au - \lambda_i n_i = P \quad \text{for} \quad X \in \Gamma_v, \quad t \geqq 0, \tag{3.20}$$

(2.3a) *and* (3.7).

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[10]) We write p instead of p_0 in (3.5).

If we interpret the parameter p as the operator $\partial/\partial t$, the principle characterizes the solution to the problem, expressed in terms of u, g_i, λ_i, L.

Remark 3.9. The corresponding 2. generalized $\mathscr{B}\alpha$-principle (for $f = 0$, $\Gamma = \Gamma_u \cup \Gamma_h$ and regardless of the initial condition) is involved in the Ben-Amoz's extension [9] of Hu-Washizu principle from elasticity onto coupled thermoelasticity. In the theory of heat conduction the newly introduced variable L is proportional to the rate $\dot{S}$ of entropy change, as we may write

$$L = -\frac{T_r}{c} pS_{i,i} = \frac{T_r}{c} pS .$$

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Výtah

VARIAČNÍ PRINCIPY PRO PARABOLICKÉ ROVNICE

IVAN HLAVÁČEK

V článku je dán systematický přehled nových typů variačních principů ekvivalentních smíšené úloze pro parabolickou rovnici s počátečními a okrajovými podmínkami, které byly předloženy fyziky. Ačkoliv metoda je čistě matematická a jako taková může být použita k vyšetřování všech analogických úloh matematické fyziky, fyzikální interpretace se týká pouze teorie vedení tepla.

Principy jsou dvojího druhu: jedna třída dává variační charakteristiku počáteční úlohy, vyjádřenou jednou skalární funkcí (teploty); druhá třída charakterizuje tutéž úlohu za pomoci jiných proměnných (např. toku tepla nebo rozdělení entropie).

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