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Truong Q. Bao
Wayne State University

Boris S. Mordukhovich
Wayne State University, boris@math.wayne.edu

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TRUONG Q. BAO and BORIS. S. MORDUKHOVICH

WAYNE STATE UNIVERSITY

Detroit, MI 48202

Department of Mathematics

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VARIATIONAL PRINCIPLES FOR SET-VALUED MAPPINGS WITH APPLICATIONS TO MULTIOBJECTIVE OPTIMIZATION[1]

by

Truong Q. Bao and Boris S. Mordukhovich

Department of Mathematics, Wayne State University, Detroit, MI 48202
tqbao@math.wayne.edu, boris@math.wayne.edu

Dedicated to Stephan Rolewicz

Abstract: This paper primarily concerns the study of general classes of constrained multiobjective optimization problems (including those described via set-valued and vector-valued cost mappings) from the viewpoint of modern variational analysis and generalized differentiation. To proceed, we first establish two variational principles for set-valued mappings, which—being certainly of independent interest—are mainly motivated by applications to multiobjective optimization problems considered in this paper. The first variational principle is a set-valued counterpart of the seminal derivative-free Ekeland variational principle, while the second one is a set-valued extension of the subdifferential principle by Mordukhovich and Wang formulated via an appropriate subdifferential notion for set-valued mappings with values in partially ordered spaces. Based on these variational principles and corresponding tools of generalized differentiation, we derive new conditions of the coercivity and Palais-Smale types ensuring the existence of optimal solutions to set-valued optimization problems with noncompact feasible sets in infinite dimensions and then obtain necessary optimality and suboptimality conditions for nonsmooth multiobjective optimization problems with general constraints, which are new in both finite-dimensional and infinite-dimensional settings.

Keywords: multiobjective optimization, variational principles, generalized differentiation, existence of optimal solutions, necessary optimality and suboptimality conditions.

1 Introduction

The primary goal of this paper is to study constrained *multiobjective optimization* problems generally given by

$$\text{minimize } F(x) \text{ subject to } x \in \Omega \subset X \tag{1.1}$$

by using advanced tools of modern *variational analysis* and *generalized differentiation*. In (1.1), the cost mapping $F\colon X \rightrightarrows Z$ may be *set-valued*, and "minimization" is understood with respect to some *partial ordering* on Z. Thus (1.1) is a problem of *set-valued optimization*, while the term of *vector optimization* is usually used when $F = f\colon X \to Z$ is a single-valued mapping. In this paper we unify both set-valued and vector optimization problems under the name of multiobjective optimization.

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There is an abundant literature on various problems of multiobjective optimization. One of the first work in modern variational theory for such problems was done by Rolewicz; see [23] and [20, Chapter 10]. We refer the reader to the books [5, 8, 14, 16, 18, 20] and the bibliographies therein for more information on history, results, and methods in multiobjective optimization and related problems.

A characteristic feature of the current stage of variational analysis is the broad usage of modern *variational principles* started with the seminal work by Ekeland [6]. The fundamental *Ekeland variational principle* asserts that, given a proper and lower semicontinuous function $\varphi\colon X \to \overline{I\!R} := (-\infty, \infty]$ bounded from below on the complete metric space (X, d), for every $\varepsilon > 0$, $\lambda > 0$, and $x_0 \in X$ with $\varphi(x_0) < \inf_X \varphi(x) + \varepsilon$ there is $\bar{x} \in X$ satisfying the conditions $\varphi(\bar{x}) \le \varphi(x_0)$, $d(\bar{x}, x_0) \le \lambda$, and

$$\varphi(x) - \varphi(\bar{x}) + \frac{\varepsilon}{\lambda} d(x, \bar{x}) > 0 \ \text{ whenever } \ x \in X \ \text{ with } \ x \neq \bar{x}. \tag{1.2}$$

Note that (1.2) means that the *perturbed* function $\varphi(x) + (\varepsilon/\lambda) d(x, \bar{x})$ attains its *strict global minimum* over X at $\bar{x}$. If X is Banach and f is Gâteaux *differentiable*, then (1.2) easily implies the *perturbed stationary* condition

$$\big\|\nabla\varphi(\bar{x})\big\| \le \frac{\varepsilon}{\lambda}, \tag{1.3}$$

which can be treated as a *suboptimality* condition to the problem of minimizing $\varphi(x)$—with *no* assumption on the *existence* of optimal solutions to this problem over X particularly restrictive in infinite dimensions—and which was among the strongest original motivations for developing Ekeland's variational principle in [6] and its subsequent applications.

When φ is *nonsmooth*—just extended-real-valued, lower semicontinuous, and bounded from below as in the afore-mentioned Ekeland general result—another variational principle is established by Mordukhovich and Wang [19] under the name of *subdifferential variational principle.* It gives the same conclusions as Ekeland's principle with replacing the minimization condition (1.2) by the subdifferential one:

$$\|x^*\| \le \varepsilon/\lambda \ \text{ for some } \ x^* \in \widehat{\partial}\varphi(\bar{x}), \tag{1.4}$$

where $\widehat{\partial}\varphi(\bar{x})$ stands for the so-called *Fréchet subdifferential* of φ at $\bar{x}$ defined by

$$\widehat{\partial}\varphi(\bar{x}) := \Big\{x^* \in X^* \Big|\ \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \ge 0\Big\}, \tag{1.5}$$

and where the space X is assumed to be *Asplund*, i.e., a Banach space whose separable subspaces have separable duals; see, e.g., [21] for more information and references on the broad class of Asplund spaces that includes, in particular, all reflexive Banach spaces.

The subdifferential variational principle is established in [19] (see also [17, Theorem 2.28]) as a consequence of (actually an *equivalence* to) the *extremal principle*, which is a *variational* counterpart of local *separation* for *nonconvex* sets being a variational principle of the *geometric* type independent of the analytic Ekeland variational principle; see the books [17, 18] for a comprehensive variational theory and numerous applications of the extremal principle. Observe that the subdifferential condition (1.4) is a *nonsmooth* counterpart of the *almost stationary* condition (1.3); furthermore, it implies certain *enhanced* versions of (analytic) *smooth variational principles* under additional smoothness assumptions of the space X in question; see [17, Subsection 2.3.3].

In this paper we derive an appropriate analog of the afore-mentioned subdifferential variational principle for *set-valued* (in particular, *vector-valued*) mappings with values in partially ordered spaces. We need such a result for the subsequent applications to constrained *multiobjective optimization* problems of type (1.1). The proof of the *set-valued subdifferential variational principle* (SVSVP) obtained in this paper is based on the *extremal principle* and a new version of the *set-valued Ekeland variational principle* (SVEVP) established below. The required version of the latter needed for our purposes (while certainly of independent interest) is different from various vector and set-valued extensions of Ekeland's seminal result known in the literature; see, e.g., [2, 3, 5, 8, 9, 10, 12, 13, 15] and the references therein as well as further comments in Section 3.

The rest of the paper is organized as follows. In Section 2 we briefly review (for the reader's convenience) certain basic tools of variational analysis and generalized differentiation widely used in the paper. Then we introduce new *subdifferential* notions for *set-valued mappings* (in particular, for vector-valued mappings) with values in *partially ordered spaces* and establish some of their important properties needed in the sequel.

In Section 3 we first derive a new version of the *SVEVP* and then use it in the proof of the new *SVSVP* via the *extremal principle.* The formulation of the SVSVP result, which plays a crucial role in the subsequent applications in this paper, involves the subdifferentials of set-valued mappings introduced in Section 2. We discuss relationships of the results obtained with those known in the literature.

Section 4 contains applications of the *variational techniques and principles* developed in Section 3 to deriving efficient conditions for the *existence of optimal solutions* to set-valued constrained optimization problems. In particular, we establish new conditions of the *coercivity type* and of the *subdifferential Palais-Smale type* for set-valued and nonsmooth single-valued mappings ensuring the existence of weak minimizers to the multiobjective optimization problems under consideration.

The concluding Section 5 is devoted to applications of the variational principles established in Section 3 and some basic calculus rules of generalized differentiation from [17] to deriving *necessary optimality conditions* for multiobjective optimization problems with general geometric constraints as well as their specifications for multiobjective problems of mathematical programming with equality and inequality constraints given by nonsmooth functions. In this section we also obtain *suboptimality conditions* for the afore-mentioned multiobjective problems, which do not assume the existence of optimal solutions and are important for both theoretical and numerical aspects of multiobjective optimization.

Throughout the paper we use standard notation from variational analysis and set-valued optimization; cf. the books [14, 17, 22]. Some special symbols are described in the text when they are introduced. Recall that $I\!N = \{1, 2, \ldots\}$, that $I\!B$ and $I\!B^*$ stand for the closed unit ball of the space in question and its topologically dual, and that $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. Unless otherwise stated, the norm on the product $X \times Y$ of Banach spaces we define by

$$\|(x,y)\| := \|x\| + \|y\|, \quad (x,y) \in X \times Y.$$

Given a set-valued mapping $G\colon X \rightrightarrows X^*$ between a Banach space X and its dual space X^*, the *Painlevé-Kuratowski sequential outer/upper limit* of F as $x \to \bar{x}$ is defined by

$$\mathop{\rm Lim\,sup}_{x \to \bar{x}} G(x) = \Big\{ x^* \in X^* \Big| \ \exists \text{ sequences } x_k \to \bar{x}, \ x_k^* \xrightarrow{w^*} x^* \text{ such that } x_k^* \in G(x_k) \text{ for all } k \in I\!N \Big\}, \tag{1.6}$$

where $\xrightarrow{w^*}$ signifies the *weak** convergence on X^*.

2 Subdifferentials of set-valued mappings

The primary goal of this section is to introduce and discuss new notions of *subdifferentials* for *set-valued* and *vector-valued* mappings with values in partially ordered spaces. To proceed, we first need to recall some well-recognized generalized differential constructions of variational analysis widely used in this paper. We mainly follow the recent books by Mordukhovich [17, 18], where the reader can find more details, references, and discussions. We also recommend the book by Rockafellar and Wets [22] for related and additional material in finite dimensions and the one by Borwein and Zhu [4] for that in Fréchet smooth spaces.

Given a nonempty subset $\Omega \subset X$ of a Banach space X, define the collection of ε-*normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{x^* \in X^* \Big|\ \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le \varepsilon\Big\}, \quad \varepsilon \ge 0, \tag{2.1}$$

with $\widehat{N}_\varepsilon(\bar{x};\Omega) := \emptyset$ if $\bar{x} \notin \Omega$. For $\varepsilon = 0$ in (2.1), the construction $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ is known as the *Fréchet normal cone* (or *prenormal cone*) to Ω at $\bar{x}$. When $X = I\!R^n$, the dual/polar cone to $\widehat{N}(\bar{x};\Omega)$ agrees with the (Bouligand-Severi) *contingent cone* to Ω at $\bar{x}$. Note that the Fréchet subdifferential $\widehat{\partial}\varphi(\bar{x})$ defined in (1.5) for an extended-real-valued function $\varphi\colon X \to \overline{I\!R}$ finite at $\bar{x}$ admits the following equivalent geometric representation:

$$\widehat{\partial}\varphi(\bar{x}) = \big\{x^* \in X^* \big|\ (x^*,-1) \in \widehat{N}\big((\bar{x},\varphi(x)); \operatorname{epi}\varphi\big)\big\} \tag{2.2}$$

via Fréchet normals to the epigraph $\operatorname{epi}\varphi := \{(x,\mu) \in X \times I\!R |\ \mu \ge \varphi(x)\}$. The basic (limiting, Mordukhovich) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \mathop{\rm Lim\,sup}_{\substack{x \to \bar{x}\\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega) \tag{2.3}$$

via the Painlevé-Kuratowski sequential outer limit (1.6). If the space X is *Asplund* and the set Ω is locally closed around $\bar{x}$, we can equivalently put $\varepsilon = 0$ in (2.3). Note that both cones $\widehat{N}(\bar{x};\Omega)$ and $N(\bar{x};\Omega)$ reduce to the normal cone of convex analysis for convex sets Ω.

Having now a set-valued mapping $F\colon X \rightrightarrows Z$ between Banach spaces with the graph

$$\operatorname{gph} F := \big\{(x,z) \in X \times Z \big|\ z \in F(x)\big\},$$

define its ε-*coderivative* $\widehat{D}^*_\varepsilon F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ at $(\bar{x},\bar{z}) \in \operatorname{gph} F$ by

$$\widehat{D}^*_\varepsilon F(\bar{x},\bar{z})(z^*) := \big\{x^* \in X^* \big|\ (x^*,-z^*) \in \widehat{N}_\varepsilon\big((\bar{x},\bar{z}); \operatorname{gph} F\big)\big\}, \quad \varepsilon \ge 0, \tag{2.4}$$

where $\widehat{D}^*_\varepsilon F(\bar{x},\bar{z})(z^*) = \emptyset$ for $(\bar{x},\bar{y}) \notin \operatorname{gph} F$, and where $\widehat{D}^* F(\bar{x},\bar{z}) := \widehat{D}^*_0 F(\bar{x},\bar{z})$ is a positively homogeneous set-valued mapping called the *Fréchet coderivative* of F at $(\bar{x},\bar{z})$. Based on (2.4) considered at points nearby the reference one, construct as in [17] two (sequential) *limiting coderivatives* of F at $(\bar{x},\bar{z})$ called, respectively, the *normal coderivative*

$$D^*_N F(\bar{x},\bar{z})(\bar{z}^*) := \mathop{\rm Lim\,sup}_{\substack{(x,z)\to(\bar{x},\bar{z})\\ \varepsilon\downarrow 0\\ z^* \xrightarrow{w^*} \bar{z}^*}} \widehat{D}^*_\varepsilon F(x,z)(z^*) \tag{2.5}$$

and the *mixed coderivative* of F at $(\bar{x}, \bar{z})$ that is given by

$$D^*_M F(\bar{x}, \bar{z})(z^*) := \mathop{\mathrm{Lim\,sup}}_{\substack{(x,z)\to(\bar{x},\bar{z}) \\ \varepsilon\downarrow 0 \\ \|z^*-\bar{z}^*\|\to 0}} \widehat{D}^*_\varepsilon F(x, z)(z^*) \tag{2.6}$$

We can equivalently put $\varepsilon = 0$ in (2.5) and (2.6) if both spaces X and Z are Asplund and if the mapping F is locally closed-graph around $(\bar{x}, \bar{z})$.

Note that, by definition (1.6) of the sequential outer limit, the only difference between (2.5) and (2.6) is that the *weak** convergence w^* is used in (2.5) on both dual spaces X^* and Z^*, while in (2.6) the *strong/norm* convergence is employed on Z^* versus the weak* convergence on X^*. Thus these limiting coderivatives agree when $\dim Z < \infty$ (they both reduce to the original construction by Mordukhovich; see [17] with the references and commentaries therein), while $D^*_M F(\bar{x}, \bar{z})$ may be essentially smaller than $D^*_N F(\bar{x}, \bar{z})$ even for single-valued Lipschitzian mappings $f\colon I\!R \to H$ to an arbitrary Hilbert space H as [17, Example 1.35]. Note that the normal coderivative (2.5) can be equivalently defined by

$$D^*_N F(\bar{x}, \bar{z})(z^*) = \big\{x^* \in X^* \big|\ (x^*, -z^*) \in N\big((\bar{x}, \bar{z}); \operatorname{gph} F\big)\big\}$$

via the basic normal cone (2.3) to the graph of F.

Now let us consider a set-valued mapping $F\colon X \rightrightarrows Z$ between Banach spaces, where Z is *partially ordered* by a convex and closed cone $\Theta \subset Z$. Denoting the ordering relation on Z under consideration by "$\leq$", we therefore have its description:

$$z_1 \leq z_2 \ \text{ iff } \ z_1 - z_2 \in \Theta. \tag{2.7}$$

Given $F\colon X \rightrightarrows Z$, define its (generalized) *epigraph* with respect to the above order by

$$\operatorname{epi} F := \big\{(x, z) \in X \times Z \big|\ z \in F(x) - \Theta\big\}$$

and associate with F the *epigraphical multifunction* $\mathcal{E}_F\colon X \rightrightarrows Z$ defined by

$$\mathcal{E}_F(x) := \big\{z \in Z \big|\ z \in F(x) - \Theta\big\} \ \text{ with } \ \operatorname{gph} \mathcal{E}_F = \operatorname{epi} F, \tag{2.8}$$

where the ordering cone Θ is not mentioned in the epigraphical notation for simplicity.

Our goal is to introduce appropriate *subdifferentials of set-valued mappings* with values in partially ordered spaces by using the corresponding *coderivatives* of the associated epigraphical multifunctions. Although there are many various definitions of subdifferentials for (single-valued) *vector* functions with values in partially ordered spaces, our coderivative approach and the subdifferential constructions below are different from those known in the literature (see, e.g., a very good survey on vector subdifferentials by Stamate [24]). Furthermore, our constructions apply to *set-valued* mappings/multifunctions with values in partially ordered spaces, which is important for the main results of this paper.

The following definition contains only those subdifferential constructions, which are used in this paper. Based on the coderivative approach and employing various limiting procedures on dual spaces, the reader can construct other subdifferentials that may be different from the ones given below in infinite dimensions.

Definition 2.1 (subdifferentials of set-valued mappings). *Let $F\colon X \rightrightarrows Z$ be a mapping between Banach spaces, let $\Theta \subset Z$ be a closed, convex, and pointed cone that generates* *a partial order on* Z *by* (2.7), *and let* $(\bar{x}, \bar{z}) \in \operatorname{epi} F$. *We define the following subdifferentials of* F *at* $(\bar{x}, \bar{z})$ *via the corresponding coderivatives of the epigraphical multifunction* (2.8):

—*the* ε-SUBDIFFERENTIAL *of* F *at* $(\bar{x}, \bar{z})$ *by*

$$\widehat{\partial}_\varepsilon F(\bar{x}, \bar{z}) := \big\{x^* \in X^* \big|\, x^* \in \widehat{D}^*_\varepsilon \mathcal{E}_F(\bar{x}, \bar{z})(z^*), \quad z^* \in N(0; \Theta), \quad \|z^*\| = 1\big\}, \quad \varepsilon \geq 0, \tag{2.9}$$

where $\widehat{\partial} F(\bar{x}, \bar{z}) := \widehat{\partial}_0 F(\bar{x}, \bar{z})$ *is the* FRÉCHET SUBDIFFERENTIAL *of* F *at this point;*

—*the* LIMITING SUBDIFFERENTIAL *of* F *at* $(\bar{x}, \bar{z})$ *by*

$$\partial_L F(\bar{x}, \bar{z}) := \operatorname*{Lim\,sup}_{\substack{(x,z) \to (\bar{x}, \bar{z}) \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon F(x, z), \tag{2.10}$$

where $\widehat{\partial}_\varepsilon F(x, z) := \emptyset$ *if* $(x, z) \notin \operatorname{epi} F$, *and where one can equivalently put* $\varepsilon = 0$ *if both* X *and* Z *are Asplund and if* $\operatorname{epi} F$ *is locally closed around* $(\bar{x}, \bar{z})$;

—*the* NORMAL SUBDIFFERENTIAL *of* F *at* $(\bar{x}, \bar{z})$ *by*

$$\partial_N F(\bar{x}, \bar{z}) := \big\{x^* \in X^* \big|\, x^* \in D^*_N \mathcal{E}_F(\bar{x}, \bar{z})(z^*), \quad z^* \in N(0; \Theta), \quad \|z^*\| = 1\big\}; \tag{2.11}$$

—*the* SINGULAR SUBDIFFERENTIALS *of* F *at* $(\bar{x}, \bar{z})$ *by*

$$\partial^\infty F(\bar{x}, \bar{z}) := D^*_M \mathcal{E}_F(\bar{x}, \bar{z})(0). \tag{2.12}$$

As usual, we drop $\bar{z} = f(\bar{x})$ in the subdifferential notation (2.9)–(2.12) if $F = f\colon X \to Z$ is single-valued. When $\varphi\colon X \to \overline{I\!R}$ is an *extended-real-valued function* finite at $\bar{x}$ with the standard order $\Theta = I\!R_-$ on $I\!R$, the epigraphical multifunction (2.8) agrees with the standard one $E_\varphi(x) = \{\mu \in I\!R|\ \mu \geq \varphi(x)\}$ and the subdifferentials (2.9)–(2.12) reduce to their well-known prototypes, namely:

—construction (2.9) with $\varepsilon = 0$ reduces to the Fréchet subdifferential $\widehat{\partial}\varphi(\bar{x})$ in (1.5)—due to the geometric representation (2.2) of the latter;

—both limiting (2.10) and normal (2.11) subdifferentials reduce to the *basic subdifferential* $\partial\varphi(\bar{x})$ by Mordukhovich [17];

—the singular subdifferential in (2.12) reduces to the one $\partial^\infty\varphi(\bar{x})$ in [17].

Among the strongest advantages of the coderivative approach to subdifferentials of set-valued and vector-valued mappings is a *full coderivative calculus* [17], which induces a variety of calculus rules for the subdifferential constructions defined in (2.9)–(2.12). Other major advantages include *complete coderivative characterizations* of fundamental properties in nonlinear analysis related to *metric regularity*, *linear openness*, and *robust Lipschitzian stability* of set-valued mappings; see [17, 22]. These characterizations generate the corresponding results for mappings with values in partial ordered spaces via the subdifferentials (2.9)–(2.12) introduced in this paper.

In *infinite-dimensional* spaces, the afore-mentioned calculus and characterizations require certain additional "sequential normal compactness" properties of sets and mappings, which are automatic in finite dimensions, while turn out to be a crucial ingredient of variational analysis in infinite dimensions; see the books [17, 18] for a comprehensive theory and numerous applications of various properties of this type.

Let us recall some of these properties needed in the paper. Considering generally a set $\Omega \subset X \times Z$ in the product of Banach spaces, we say that it is *sequentially normally compact* at $\bar{v} = (\bar{x}, \bar{z}) \in \Omega$ if for any sequences

$$\varepsilon_k \downarrow 0, \quad v_k \xrightarrow{\Omega} \bar{v}, \quad \text{and} \quad (x_k^*, z_k^*) \in \widehat{N}_{\varepsilon_k}(v_k; \Omega), \quad k \in I\!N, \tag{2.13}$$

one has the implication $(x_k^*, z_k^*) \xrightarrow{w^*} 0 \Longrightarrow \|(x_k^*, z_k^*)\| \to 0$ as $k \to \infty$. The product structure of the space in question plays no role in this property (we can put $Z = \{0\}$ without loss of generality) in contrast to its following partial modifications. We say that Ω is *partially sequentially normally compact* (PSNC) with respect to X at $\bar{v} \in \Omega$ if for any sequences $(\varepsilon_k, v_k, x_k^*, z_k^*)$ satisfying (2.13) one has the implication

$$\big[x_k^* \xrightarrow{w^*} 0, \quad \|z_k^*\| \to 0\big] \Longrightarrow \|x_k^*\| \to 0 \quad \text{as} \quad k \to \infty.$$

Finally, a set Ω is *strongly PSNC* with respect to X at $\bar{v}$ if for any sequences $(\varepsilon_k, v_k, x_k^*, z_k^*)$ satisfying (2.13) one has $(x_k^*, z_k^*) \xrightarrow{w^*} 0 \Longrightarrow \|x_k^*\| \to 0$ as $k \to \infty$. We can equivalently put $\varepsilon_k = 0$ in (2.13) for all the above properties if both spaces X and Z are Asplund and if the set Ω is locally closed around $\bar{v}$.

Given a *set-valued mapping* $F\colon X \rightrightarrows Z$ between Banach spaces, its SNC/PSNC properties at $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ induce by the corresponding properties of its graph. In particular, we say that F is *PSNC* at $(\bar{x}, \bar{z})$ if its graph is PSNC with respect to X at this point. The reader can find in [17] a number of efficient conditions for the fulfillment of SNC/PSNC properties of sets and mappings, which often relate to their Lipschitzian behavior of some kind. Furthermore, there is a well-developed *SNC calculus* in [17] ensuring the preservation of SNC and PSNC properties under natural operations performed on sets and mappings.

For mappings $F\colon X \rightrightarrows Z$ with values in Banach spaces Z *partially ordered* by convex cones $\Theta \subset Z$ as in (2.7), the above SNC and PSNC properties induce the corresponding *epigraphical* counterparts by applying to their epigraphical multifunctions (2.8). Following this way, we say that such a mapping F is *sequentially normally epicompact* (SNEC) or, respectively, *partially SNEC* at $(\bar{x}, \bar{z}) \in \operatorname{epi} F$ if the epigraphical multifunction $\mathcal{E}_F$ is SNC (resp. PSNC) at this point.

Employing the SNC property of the ordering convex cone $\Theta \subset Z$ and the above definitions, we now establish relationships between the limiting subdifferential (2.10) and the normal subdifferential (2.11) of an arbitrary set-valued mapping $F\colon X \rightrightarrows Z$ in the Banach space setting with the range space Z ordered by Θ.

Theorem 2.2 (relationships between limiting and normal subdifferentials of set-valued mappings). *Let $F\colon X \rightrightarrows Z$ be a mapping between Banach spaces with Z ordered by a convex cone Θ in (2.7), and let $(\bar{x}, \bar{z}) \in \operatorname{epi} F$. Assume that the closed unit ball $I\!B^*$ of Z^* is sequentially compact in the weak* topology (this is surely the case of Asplund spaces, in particular) and that the ordering cone Θ is SNC at the origin. Then*

$$\partial_L F(\bar{x}, \bar{z}) \subset \partial_N F(\bar{x}, \bar{z}). \tag{2.14}$$

If furthermore $\dim Z < \infty$, *then* (2.14) *holds as equality.*

Proof. First we justify (2.14). Taking any $x^* \in \partial_L F(\bar{x}, \bar{z})$ and using subsequently definitions (2.10), (1.6) and (2.9), find sequences $\varepsilon_k \downarrow 0$, $(x_k, z_k) \to (\bar{x}, \bar{z})$ with $z_k \in F(x_k) - \Theta$, and $z_k^* \in N(0; \Theta)$ with $\|z_k^*\| = 1$ such that

$$x_k^* \in \widehat{D}^*_{\varepsilon_k} \mathcal{E}_F(x_k, z_k)(z_k^*) \quad \text{for all} \quad k \in I\!N. \tag{2.15}$$

Since the unit ball $I\!B^*$ of Z^* is sequentially compact, we select a subsequence of $\{z_k^*\}$ that weak* converges to some $z^* \in I\!B^*$. Note that $z^* \neq 0$, because the converse property implies that $z_k^* \stackrel{w^*}{\to} 0$ and hence $\|z_k^*\| \to 0$ by the assumed SNC property of Θ, which is impossible due to $\|z_k^*\| = 1$ for all $k \in I\!N$. Supposing with no loss of generality that $\|z^*\| = 1$ and passing to the limit in (2.15), we get by definition (2.5) of the normal coderivative that $x^* \in D_N^*\mathcal{E}_F(\bar{x},\bar{z})(z^*)$. Since $z^* \in N(0;\Theta)$ by the closed-graph property of the normal cone to convex sets, we get $x^* \in \partial_N F(\bar{x},\bar{z})$ and complete the proof of inclusion (2.14).

Next let us show that the opposite inclusion holds in (2.14) provided that Z is *finite-dimensional.* Picking any $x^* \in \partial_N F(\bar{x},\bar{z})$, we have $z^* \in N(0;\Theta)$ with $\|z^*\| = 1$ and find by (2.11) and (2.5) sequences $\varepsilon_k \downarrow 0$, $(x_k,z_k) \to (\bar{x},\bar{z})$, and $(x_k^*,z_k^*) \stackrel{w^*}{\to} (x^*,z^*)$ with

$$(x_k,z_k) \in \operatorname{epi} F \quad\text{and}\quad (x_k^*,-z_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k,z_k);\operatorname{epi} F\big) \quad\text{for all } k \in I\!N. \tag{2.16}$$

Since Z is finite-dimensional, we have $\|z_k^*\| \to 1$ as $k \to \infty$. It follows from the second inclusion in (2.16) that for any $\gamma > 0$ there is $\eta > 0$ such that

$$\big\langle (x_k^*,-z_k^*),(x,z)-(x_k,z_k)\big\rangle \le (\gamma+\varepsilon_k)\big\|(x,z)-(x_k,z_k)\big\| \tag{2.17}$$

for all $(x,z) \in \operatorname{epi} F$ with $x \in x_k + \eta I\!B$ and $z \in z_k + \eta I\!B$ and for all $k \in I\!N$. By the definition of $\operatorname{epi} F$ with respect to the ordering cone Θ we have

$$z_k = v_k - \theta_k \quad\text{for some}\quad v_k \in F(x_k) \quad\text{and}\quad \theta_k \in \Theta, \quad k \in I\!N. \tag{2.18}$$

Taking further an arbitrary vector $(u,v) \in \operatorname{epi} F$ with $u \in x_k + \eta I\!B$ and $v \in v_k + \eta I\!B$, observe by the above ordering that

$$v = \widetilde{v}_k - \widetilde{\theta}_k \quad\text{for some}\quad \widetilde{v}_k \in F(u) \quad\text{and}\quad \widetilde{\theta}_k \in \Theta, \quad k \in I\!N. \tag{2.19}$$

Now we define $\widetilde{z}_k \in Z$ by

$$\widetilde{z}_k := v + (z_k - v_k)$$

and get due to (2.18), (2.19), and the *convexity* of the cone Θ that

$$\widetilde{z}_k = \widetilde{v}_k - \widetilde{\theta}_k - \theta_k \in F(u) - \Theta, \quad k \in I\!N.$$

Since $\|\widetilde{z}_k - z_k\| = \|v - v_k\| \le \eta$ by the choice of $\widetilde{z}_k$, we have $(u,\widetilde{z}_k) \in \operatorname{epi} F$ with $u \in x_k + \eta I\!B$ and $\widetilde{z}_k \in z_k + \eta I\!B$. Substituting $(u,\widetilde{z}_k)$ into (2.17) gives

$$\big\langle (x_k^*,-z_k^*),(u,\widetilde{z}_k)-(x_k,z_k)\big\rangle \le (\gamma+\varepsilon_k)\big\|(u,\widetilde{z}_k)-(x_k,z_k)\big\|, \quad k \in I\!N,$$

and hence, by $\widetilde{z}_k - z_k = v - v_k$, we get

$$\big\langle (x_k^*,-z_k^*),(u,v)-(x_k,v_k)\big\rangle \le (\gamma+\varepsilon_k)\big\|(u,v)-(x_k,v_k)\big\|, \quad k \in I\!N. \tag{2.20}$$

Remember that points (u,v) were chosen *arbitrarily* in $\operatorname{epi} F$ and in the η-neighborhood of (x_k,z_k) and that (x_k,v) is one of such points. Putting (x_k,v) into (2.20), we arrive at

$$\langle -z_k^*, v - v_k\rangle \le (\gamma+\varepsilon_k)\|v - v_k\| \quad\text{whenever}\quad v \in (v_k - \Theta) \cap (v_k + \eta I\!B),$$

which implies by (2.1) and the convexity of Θ that

$$-z_k^* \in \widehat{N}_{\varepsilon_k}(v_k, v_k - \Theta) = \widehat{N}_{\varepsilon_k}(0;-\Theta) = -N(0,\Theta) + \varepsilon_k I\!B^*, \quad k \in I\!N;$$

see [17, Proposition 1.3] for the latter conclusion for convex sets. Thus there is $\widetilde{z}_k^* \in N(0;\Theta)$ satisfying the relationships

$$\widetilde{z}_k^* \in N(0;\Theta) \text{ and } \|\widetilde{z}_k^* - z_k^*\| \le \varepsilon_k, \quad k \in I\!N. \tag{2.21}$$

The inequality in (2.21) implies that $\widetilde{z}_k^* \to z^*$ by $z_k^* \to z^*$. Since $\|\widetilde{z}_k^*\| \to 1$ as $k \to \infty$, we assume without loss of generality that $\|\widetilde{z}_k^*\| = 1$ for all $k \in I\!N$. It follows from (2.16) and (2.21) that

$$(x_k^*, -\widetilde{z}_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k, z_k); \operatorname{epi} F\big) + \varepsilon_k I\!B^*, \quad k \in I\!N,$$

and thus, by taking (2.21) into account, we have

$$x_k^* \in \widetilde{D}^*_{2\varepsilon_k}\mathcal{E}_F(x_k, z_k)(\widetilde{z}_k^*) \text{ and so } x_k^* \in \widehat{\partial}_{2\varepsilon_k} F(x_k, z_k), \quad k \in I\!N.$$

The latter gives by (2.10) and (1.6) that $x^* \in \partial_L F(\bar{x}, \bar{z})$, which completes the proof of the equality in (2.14) and of the whole theorem. △

Next we formulate a robust Lipschitzian property of set-valued mappings with values in *ordered* Banach spaces, which ensures simultaneously the partial SNEC property and the triviality of the singular subdifferential (2.12), which are both important in what follows. We say that a set-valued mapping $F: X \rightrightarrows Z$ is *epi-Lipschitz-like* (ELL) around a point $(\bar{x}, \bar{z}) \in \operatorname{epi} F$ with respect to the ordering cone $\Theta \subset Z$ if the associated epigraphical multifunction (2.8) is Lipschitz-like (or enjoys Aubin's "pseudo-Lipschitzian" property; see [17, 22]) around this point, i.e., there are neighborhoods U of $\bar{x}$ and V of $\bar{z}$ and a number $\ell \ge 0$ such that one has the inclusion

$$\mathcal{E}_F(x) \cap V \subset \mathcal{E}_F(u) + \ell\|x - u\| I\!B \text{ for all } x, u \in U.$$

This robust Lipschitzian property of $\mathcal{E}_F$ is known to be equivalent to both *metric regularity* and *linear openness* properties of the *inverse* multifunction.

Proposition 2.3 (singular subdifferential and partial SNEC property of ELL mappings). *Let $F: X \rightrightarrows Z$ be a mapping between Banach spaces, where Z is ordered by a cone Θ. Assume that F is ELL around $(\bar{x}, \bar{z}) \in \operatorname{epi} F$. Then F is partially SNEC at $(\bar{x}, \bar{z})$, and one has the singular subdifferential condition*

$$\partial^\infty F(\bar{x}, \bar{z}) = \{0\}. \tag{2.22}$$

Proof. The partial SNEC property of F follows from [17, Proposition 1.68] due to the above definitions of this and ELL properties, while the subdifferential condition (2.22) as a consequence of definition (2.12) and [17, Theorem 1.44]. △

Finally in this section, we formulate the (approximate) *extremal principle* from [17, Chapter 2], which is the main driving force for the development of the afore-mentioned calculus results and characterizations (including the SNC calculus in infinite dimensions) and plays a crucial role in this paper. Given two sets $\Omega_1, \Omega_2 \subset X$ locally closed around $\bar{x} \in \Omega_1 \cap \Omega_2$, we say that $\bar{x}$ is *local extremal* point of the set system $\{\Omega_1, \Omega_2\}$ if there is a neighborhood U of $\bar{x}$ such that for any $\varepsilon > 0$ there is $a \in \varepsilon I\!B$ with

$$(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset.$$

The Extremal Principle. *Let $\bar{x}$ be a local extremal point of the set system $\{\Omega_1, \Omega_2\}$ in the Asplund space X, and let both Ω_1 and Ω_2 be locally closed around $\bar{x}$. Then for any $\varepsilon > 0$ there are $x_i \in \Omega_1 \cap (\bar{x} + \varepsilon I\!B)$ and $x_i^* \in \widehat{N}(x_i; \Omega_i) + \varepsilon I\!B^*$, $i = 1, 2$, such that*

$$\|x_1^*\| + \|x_2^*\| = 1, \quad x_1^* + x_2^* = 0.$$

3 Variational principles for set-valued mappings

In this section we derive two major *variational principles* that are extensions of the variational principles discussed in Section 1 from scalar functions to *vector-valued* and *set-valued* mappings. Let us start with an appropriate extension of the *Ekeland variational principle*, which is of undoubted interest for its own sake and is used in what follows for deriving a required extension of the subdifferential variational principle to set-valued mappings in terms of the subdifferential constructions introduced in Section 2.

It is well understood that the conventional proof of the classical Ekeland variational principle for extended-real-valued functions (see [6, 7] and also [17, Theorem 2.26]) cannot be directly extended to the vector and set-valued mappings with merely *partially ordered* (while not totally ordered) range spaces. Several approaches to vector/set-valued extensions of this fundamental result and its proof are suggested in the literature (based on certain vector metrics, scalarization techniques, etc.—compare [2, 5, 8, 10, 12, 13, 15] for more details, discussions, and references), but unfortunately they do not allow us to arrive at all the conclusions needed for our purposes; see below. Our proof is based on a *new iterative procedure*, which does not involve any scalarization technique and deals directly with the set/vector-valued setting under consideration.

To formulate a set-valued extension of the Ekeland variational principle, we first recall some relevant notions from set-valued optimization mainly following the book by Jahn [14].

Let (X,d) be a complete metric space, and let Z be a *partially ordered* linear topological space, where the partial order "$\le$" is generated by a closed and convex cone Θ via (2.7). In what follows we always assume that the ordering cone Θ is *pointed*, i.e., $\Theta\cap(-\Theta)=\{0\}$.

Given a set $\Lambda\subset Z$ and a point $\bar z\in\Lambda$, we say that $\bar z$ is a *minimal point* of Λ if

$$\Lambda\cap(\bar z+\Theta)=\{\bar z\}.$$

The collection of minimum points to Λ can be equivalently described by

$$\operatorname{Min}\Lambda:=\big\{\bar z\in\Lambda\big|\ z-\bar z\notin\Theta\ \text{ whenever }\ z\in\Lambda\setminus\{\bar z\}\big\}.$$

If $\operatorname{int}\Theta\ne\emptyset$, we similarly consider *weak minimal points* $\bar z$ of Λ defined by

$$\Lambda\cap(\bar z+\operatorname{int}\Theta)=\emptyset.$$

A set-valued mapping $F\colon X\rightrightarrows Z$ is *epiclosed* if its epigraph with respect to the ordering cone Θ is closed in $X\times Z$. This mapping is *level-closed* if for all $z\in Z$ its z-level set

$$\mathcal{L}(z):=\big\{x\in X\big|\ \exists\,\zeta\in F(x)\ \text{ with }\ \zeta\le z\big\}$$

is closed in X. It is clear that every epiclosed mapping is level-closed but not vice versa. We say that F is Θ-*bounded from below* if there exists a bounded subset $M\subset Z$ such that

$$F(X)\subset M-\Theta,$$

and that F is *bounded from below* if the set M above can be chosen as a singleton.

Now having a mapping $F\colon X\rightrightarrows Z$ from a complete metric space (X,d) to a partially ordered linear topological space Z with the ordering cone Θ, we consider the *set-valued optimization problem*

$$\text{minimize }\ F(x)\ \text{ subject to }\ x\in X \tag{3.1}$$

with no explicit constraints on x, although they are hidden by

$$x \in \operatorname{dom} F := \{x \in Z \mid F(x) \neq \emptyset\}.$$

In this paper we study the following notions of exact and approximate minimizers to set-valued and vector-valued mappings.

Definition 3.1 (minimizers and approximate minimizers in set-valued optimization). *Given a mapping $F\colon X \rightrightarrows Z$ taking values in a partially ordered space with the ordering cone Θ, we consider the set-valued minimization problem* (3.1) *and say that:*

(i) *$(\bar{x},\bar{z}) \in \operatorname{gph} F$ is a* MINIMIZER *to* (3.1)*—or just to the mapping F—if $\bar{z} \in F(\bar{x})$ is a minimal point of the image set $F(X) := \cup_{x\in X} F(x)$, i.e.,*

$$(\bar{z} + \Theta) \cap F(X) = \{\bar{z}\}. \tag{3.2}$$

(ii) *$(\bar{x},\bar{z})$ is a* WEAK MINIMIZER *to* (3.1) *if $\bar{z} \in F(\bar{x})$ is a weak minimum point of the set $F(X)$, i.e.,* (3.2) *holds with the replacement of Θ by $\operatorname{int}\Theta \neq \emptyset$ and $\{\bar{z}\}$ by $\emptyset$.*

(iii) *Given $\varepsilon > 0$ and $\xi \in -\Theta \setminus \{0\}$, we say that $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is an* APPROXIMATE $\varepsilon\xi$-MINIMIZER *to* (3.1) *if*

$$z + \varepsilon\xi \not\leq \bar{z} \ \text{ for all } \ z \in F(x) \ \text{ with } \ x \neq \bar{x}.$$

(iv) *$(\bar{x},\bar{z}) \in \operatorname{gph} F$ is a* STRICT APPROXIMATE $\varepsilon\xi$-MINIMIZER *to* (3.1) *if there is a positive number $\tilde{\varepsilon} < \varepsilon$ such that $(\bar{x},\bar{z})$ is an approximate $\tilde{\varepsilon}\xi$-minimizer to this problem.*

Now we are ready to formulate and prove our set-valued extension of the Ekeland variational principle.

Theorem 3.2 (Ekeland variational principle for set-valued mappings). *Let (X,d) be a complete metric space, and let Z be a partially ordered linear topological space with order* (2.7) *generated by a convex, closed, and pointed cone $\Theta \neq \{0\}$. Consider a set-valued mapping $F\colon X \rightrightarrows Z$ and assume that F is Θ-bounded from below, level-closed, and that*

$$\text{for every } \ x \in X \ \text{ and } \ z \in F(x) \ \text{ there is } \ \bar{z} \in \operatorname{Min} F(x) \ \text{ with } \ \bar{z} \leq z, \tag{3.3}$$

where the minimum set $\operatorname{Min} F(x)$ is closed. Then for any $\varepsilon > 0$, $\lambda > 0$, $\xi \in -\Theta$ with $\|\xi\| = 1$, and $(x_0,z_0) \in \operatorname{gph} F$ there is a point $(\bar{x},\bar{z}) \in \operatorname{gph} F$ satisfying the relationships

$$\bar{z} - z_0 + \frac{\varepsilon}{\lambda} d(\bar{x},x_0)\xi \leq 0, \tag{3.4}$$

$$z - \bar{z} + \frac{\varepsilon}{\lambda} d(\bar{x},x)\xi \not\leq 0 \ \text{ for all } \ (x,z) \in \operatorname{gph} F \ \text{ with } \ (x,z) \neq (\bar{x},\bar{z}). \tag{3.5}$$

If (x_0,z_0) is an approximate $\varepsilon\xi$-minimizer to F, then $\bar{x}$ can be chosen such that in addition to (3.4) *and* (3.5) *we have*

$$d(\bar{x},x_0) \leq \lambda. \tag{3.6}$$

Proof. Note first that it is sufficient to prove the theorem in the case of $\varepsilon = \lambda = 1$. Indeed, the general case can be easily reduced to this special case by applying the latter to the mapping $\widetilde{F}(x) := \varepsilon^{-1}F(x)$ on the metric space $(X, \widetilde{d})$ with $\widetilde{d}(x,y) := \lambda^{-1}d(x,y)$.

Having this in mind, introduce a set-valued mapping $T : X \times Z \rightrightarrows X$ by

$$T(x,z) := \{y \in X \,|\, \exists\, v \in F(y) \ \text{ with } \ v - z + d(x,y)\xi \le 0\} \tag{3.7}$$

and observe that T has the following properties:

- The sets $T(x,z)$ are *nonempty* for all $z \in F(x)$, since $x \in T(x,z)$.
- The sets $T(x,z)$ are *closed* for all $z \in F(x)$, since the mapping F is level-closed.
- The sets $T(x,z)$ are *uniformly bounded* for all $z \in F(x)$, since the mapping F is Θ-bounded from below. Indeed, one has

  $$T(x,z) \subset \{y \in X \,|\, d(x,y)\xi \in z - M + \Theta\},$$

  where the bounded set M is taken from the above definition of Θ-boundedness of F from below.
- One has the inclusion $T(y,v) \subset T(x,z)$ for all $y \in T(x,z)$ and $v \in F(y)$ with

  $$v - z + d(x,y)\xi \le 0.$$

  Indeed, pick $u \in T(y,v)$ and by construction of T in (3.7) find $w \in F(u)$ satisfying the inequality $w - v + d(y,u)\xi \le 0$. Summing the last two inequalities and taking into account that $d(x,y) + d(y,u) \ge d(x,u)$, $\xi \in -\Theta$, and $\Theta + \Theta \subset \Theta$, we have

  $$\begin{aligned} w - z + d(x,u)\xi &= \big(w - v + d(y,u)\xi\big) + \big(v - z + d(x,y)\xi\big) + \big(d(x,u) - d(y,u) \\ &\quad -d(x,y)\big)\xi \in \Theta + \Theta + \Theta \subset \Theta, \end{aligned}$$

  which implies that $u \in T(x,z)$.

Let us inductively construct a sequence of pairs $\{(x_k, z_k)\} \subset \operatorname{gph} F$ by the following *iterative procedure*: starting with (x_0, z_0) given in theorem and having the k-iteration (x_k, z_k), we select the next one (x_{k+1}, z_{k+1}) by

$$\begin{cases} x_{k+1} \in T(x_k, z_k), \\ d(x_k, x_{k+1}) \ge \displaystyle\sup_{x \in T(x_k, z_k)} d(x_k, x) - \frac{1}{k+1}, \\ z_{k+1} \in F(x_{k+1}), \quad z_{k+1} - z_k + d(x_k, x_{k+1})\xi \le 0. \end{cases} \tag{3.8}$$

It is clear from the construction and properties of $T(x,z)$ in (3.7) that the iterative procedure (3.8) is *well defined*. Summing up the last inequality in (3.8) from $k = 0$ to m, we get

$$\Big(\sum_{k=0}^{m} d(x_k, x_{k+1})\Big)\xi \in z_0 - z_{k+1} + \Theta \subset z_0 - M + \Theta$$

and, by passing to the limit as $k \to \infty$ and using the Θ-boundedness of the mapping F from below and the *pointedness* of the ordering cone Θ with $0 \ne \xi \notin \Theta$, arrive at the conclusions

$$\Big(\sum_{k=0}^{\infty} d(x_k, x_{k+1})\Big)\xi \in z_0 - M + \Theta \quad \text{and} \quad \sum_{k=0}^{\infty} d(x_k, x_{k+1}) < \infty.$$

Taking then into account that $\operatorname{diam} T(x_{k+1}, z_{k+1}) \le \operatorname{diam} T(x_k, z_k)$ and the choice of x_{k+1}, we have the estimate

$$\operatorname{diam} T(x_k, z_k) \le 2 \sup_{x \in T(x_k, z_k)} d(x_k, x) \le 2\Big(d(x_k, x_{k+1}) + \frac{1}{k+1}\Big),$$

and hence $\operatorname{diam} T(x_k, z_k) \downarrow 0$ as $k \to \infty$. Due to the *completeness* of X we conclude that the sets $T(x_k, z_k)$ shrink to a *singleton*:

$$\bigcap_{k=0}^{\infty} T(x_k, z_k) = \{\bar{x}\} \text{ with some } \bar{x} \in X. \tag{3.9}$$

Let us next justify the existence of $\bar{z} \in F(\bar{x})$ such that $(\bar{x}, \bar{z})$ satisfies relationships (3.4) and (3.5). For each $z_k \in Z$ from (3.8) define the set

$$R(x_k, z_k) := \big\{z \in \operatorname{Min} F(\bar{x}) \big|\ z - z_k + d(x_k, \bar{x})\xi \le 0\big\}, \quad k = 0, 1, \ldots. \tag{3.10}$$

Then we have the following properties:

- The set $R(x_k, z_k)$ is *nonempty* and *closed* for any $k = 0, 1, \ldots$ by the assumptions made in the theorem. Indeed, it is easily implied by the last line in (3.8) that whenever $m \ge 1$ one has $x_{k+m} \in \mathcal{L}(z_k - d(x_k, \bar{x})\xi)$ for the level set of F, which is assumed to be closed. Hence $\bar{x} \in \mathcal{L}(x_k - d(x_k, \bar{x})\xi)$, i.e., there is $\widetilde{z} \in F(\bar{x})$ satisfying

  $$\widetilde{z} - z_k + d(x_k, \bar{x})\xi \le 0.$$

  Furthermore, by condition (3.3) there is $\widehat{z} \in \operatorname{Min} F(\bar{x})$ with $\widehat{z} \le \widetilde{z}$. Taking into account the previous inequality, we get $\widehat{z} \in R(x_k, z_k)$, i.e., $R(x_k, z_k) \ne \emptyset$. The closedness of $R(x_k, z_k)$ follows directly from that of $\operatorname{Min} F(\bar{x})$ by construction (3.10).

- The set sequence $\{R(x_k, z_k)\}$ is *nonincreasing*, i.e., $R(x_{k+1}, z_{k+1}) \subset R(x_k, z_k)$ for all $k = 0, 1, \ldots$. To justify this, pick any $z \in R(x_{k+1}, z_{k+1})$ and observe that

  $$z - z_{k+1} + d(x_{k+1}, \bar{x})\xi \le 0.$$

  Adding the latter inequality to the one in (3.8), we have $z - z_k + d(x_k, \bar{x})\xi \le 0$, i.e., $z \in R(x_k, z_k)$ for all $k = 0, 1, \ldots$.

It follows from the above properties that

$$\emptyset \ne \bigcap_{k=0}^{\infty} R(x_k, z_k) \subset \operatorname{Min} F(\bar{x}).$$

Take an arbitrary vector $\bar{z}$ from the above intersection and show that the pair $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ satisfies relationships (3.4) and (3.5). Indeed, the one in (3.4) immediately follows from $\bar{z} \in R(x_0, z_0)$ and the construction of $R(\cdot, \cdot)$ in (3.10). To justify (3.5), suppose that it does not hold and then find a point

$$(x, z) \in \operatorname{gph} F \text{ with } (x, z) \ne (\bar{x}, \bar{z}) \text{ and } z - \bar{z} + d(x, \bar{x})\xi \le 0. \tag{3.11}$$

If $x = \bar{x}$ in (3.11), then we obviously have $z \le \bar{z}$, which contradicts the minimality of $\bar{z}$ on the set $F(\bar{x})$. If $x \ne \bar{x}$, then it follows from (3.11) and the construction of $\bar{z}$ that

$$z - \bar{z} + d(\bar{x}, x)\xi \le 0, \quad \bar{z} - z_k + d(\bar{x}, x_k)\xi \le 0, \text{ and thus } \bar{z} \in R(x_k, z_k), \quad k = 0, 1, \ldots.$$

Summing up the last two inequalities and combing this with the triangle one, we get

$$z - z_k + d(x, x_k)\xi \le 0, \text{ i.e., } x \in T(x_k, z_k) \text{ for all } k = 0, 1, \ldots.$$

This means then x from (3.11) belongs to the set intersection in (3.9). Thus $x = \bar{x}$ by (3.9), which justifies (3.5).

To complete the proof of the theorem, it remains to estimate the distance $d(\bar{x}, x_0)$ when (x_0, z_0) is an approximate $\varepsilon\xi$-minimizer to F. Arguing by contradiction, suppose that (3.6) does not hold, i.e., $d(\bar{x}, x_0) > \lambda$. Since $\bar{x} \in T(x_0, z_0)$, we have

$$\bar{z} - z_0 + \varepsilon\xi \le \bar{z} - z_0 + \frac{\varepsilon}{\lambda} d(\bar{x}, x_0)\xi \le 0,$$

which contradicts the approximate minimum assumption on (x_0, z_0). Thus (3.6) holds, and the proof of the theorem is finished. △

Note that, by the order definition (2.7), conclusion (3.4) of Theorem 3.2 immediately implies that $\bar{z} \le z_0$. When $F = f\colon X \to Z$ is *single/vector-valued*, we have the following corollary (and simplification) of Theorem 3.2, which agrees with the classical Ekeland variational principle for scalar functions.

Corollary 3.3 (Ekeland variational principle for vector-valued mappings). *Let (X, d), Z, and Θ be as in Theorem 3.2, and let $f\colon X \to Z$ be a single-valued mapping, which is level-closed and Θ-bounded from below. Take any $\varepsilon > 0$, $\lambda > 0$, $\xi \in -\Theta$ with $\|\xi\| = 1$, and $x_0 \in X$ that is assumed to be an approximate $\varepsilon\xi$-minimizer to f, i.e.,*

$$f(x) + \varepsilon\xi \not\le f(x_0) \text{ whenever } x \in X \setminus \{x_0\}.$$

Then there is an approximate $\varepsilon\xi$-minimizer $\bar{x}$ such that $d(\bar{x}, x_0) \le \lambda$, $f(\bar{x}) \le f(x_0)$, and

$$f(x) - f(\bar{x}) + \frac{\varepsilon}{\lambda} d(\bar{x}, x)\xi \not\le 0 \text{ for all } x \in X \setminus \{\bar{x}\}.$$

Proof. It follows directly from Theorem 3.2 by observing that $\operatorname{Min} f(x) \neq \emptyset$ whenever $x \in X$ for single-valued mappings. In this case the part in the proof of Theorem 3.2 related to considering the sets $R(x, z)$ in (3.9) is not needed. △

Remark 3.4 (comparison with other extensions of the Ekeland principle). Note that the proof of Theorem 3.2 (and its important Corollary 3.3), based on the iteration technique (3.8) involving the mapping $T(x, z)$ in (3.7), does not use any scalarization and/or vector metric as in [5, 8, 12, 13, 15] and does not impose any assumptions on nonempty interior, upper semicontinuity/demicontinuity, compactness, boundedness from below (instead of Θ-boundedness from below), etc. as in many previous results. The principal new feature of Theorem 3.2 is condition (3.5), which can be equivalently written as

$$z \notin \bar{z} + \frac{\varepsilon}{\lambda} d(\bar{x}, x)\xi + \Theta \text{ for all } (x, z) \in \operatorname{gph} F \text{ with } (x, z) \neq (\bar{x}, \bar{z}).$$

In comparison we observe that the corresponding condition of [13] can be written in our setting as

$$z \notin \bar{z} + \frac{\varepsilon}{\lambda} d(\bar{x}, x)\xi + \Theta \text{ for all } (x, z) \in \operatorname{gph} F \text{ with } x \neq \bar{x}, \tag{3.12}$$

while the one in [9] is equivalent to

$$F(\bar{x}) \not\subset F(x) + \frac{\varepsilon}{\lambda} d(\bar{x}, x)\xi + \Theta \quad \text{for all} \quad x \in X \quad \text{with} \quad x \neq \bar{x}. \tag{3.13}$$

We can easily see that (3.5)$\Longrightarrow$ (3.13) $\Longrightarrow$(3.12). Furthermore, (3.5) is *strictly better* than both (3.12) and (3.13). Indeed, considering $F\colon I\!R \rightrightarrows I\!R$ given by

$$F(x) := \begin{cases} [-1, 1] & \text{for } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

with $\Theta = I\!R_-$, we see that $\bar{x} = 0$ satisfies (3.13) and $(\bar{x}, \bar{z}) = (0, 0)$ satisfies (3.12) while not (3.5). We finally emphasize that the new condition (3.5) plays a crucial role in the proof of the following Theorem 3.5: it allows us to organize an *extremal system of sets* (see the proof), which does not seem to be possible by using conditions (3.12) and (3.13).

Next we establish a new set-valued extension of the *subdifferential variational principle* from [17, 19] by using the Fréchet subdifferential of set-valued/vector-valued mappings introduced in Definition 5.4. Previous versions of this result, which either follow from our theorem or different from it in both assumptions and conclusions, can be found in [9, 10, 11].

Theorem 3.5 (subdifferential variational principle for set-valued mappings). *Let $F\colon X \rightrightarrows Z$ be a set-valued mapping between Asplund spaces that is epiclosed, Θ-bounded from below and satisfies (3.3), where the ordering cone Θ of Z satisfies the assumptions of Theorem 3.2. Take any $\varepsilon > 0$, $\lambda > 0$, $\xi \in -\Theta$ with $\|\xi\| = 1$ and consider a strict approximate $\varepsilon\xi$-minimizer $(x_0, z_0) \in \operatorname{gph} F$ to the mapping F. Then there is $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ such that $\|\bar{x} - x_0\| \le \lambda$ and the subdifferential condition*

$$\widehat{\partial} F(\bar{x}, \bar{z}) \cap \frac{\varepsilon}{\lambda} I\!B^* \neq \emptyset \tag{3.14}$$

is satisfied. If furthermore $\xi \in -\operatorname{int}\Theta$, then the pair $(\bar{x}, \bar{z})$ above can be selected as an approximate $\varepsilon\xi$-minimizer to F.

Proof. Since (x_0, z_0) is a *strict* approximate $\varepsilon\xi$-minimizer to F, there is a positive number $\widetilde{\varepsilon} < \varepsilon$ such that (x_0, z_0) is an approximate $\widetilde{\varepsilon}\xi$-minimizer to F. Define the number

$$\widetilde{\lambda} := \frac{\varepsilon + \widetilde{\varepsilon}}{2} \lambda \quad \text{with} \quad 0 < \widetilde{\lambda} < \lambda \tag{3.15}$$

and apply to the mapping F and its approximate $\widetilde{\varepsilon}\xi$-minimizer (x_0, z_0) the generalized Ekeland variational principle from Theorem 3.2 with the parameters $(\widetilde{\varepsilon}, \widetilde{\lambda})$. Then we find by (3.4)–(3.6) a point $(\bar{u}, \bar{v}) \in \operatorname{gph} F$ satisfying the conditions

$$\bar{v} - z_0 + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}} \|x_0 - \bar{u}\| \xi \in \Theta, \quad \|x_0 - \bar{u}\| \le \widetilde{\lambda}, \tag{3.16}$$

$$z - \bar{v} + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}} \|x - \bar{u}\| \xi \notin \Theta \quad \text{for all} \quad (x, z) \in \operatorname{gph} F \quad \text{with} \quad (x, z) \neq (\bar{u}, \bar{v}). \tag{3.17}$$

Define further a set-valued mapping $G\colon X \rightrightarrows Z$ by

$$G(x) := \bar{v} - \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}} \|x - \bar{u}\| \xi + \Theta \tag{3.18}$$

and consider the following two *closed* subsets of the product space $X \times Z$ (which is well known to be *Asplund*; see, e.g., [21]):

$$\Omega_1 := \operatorname{epi} F \quad \text{and} \quad \Omega_2 := \operatorname{gph} G. \tag{3.19}$$

Let us check that $(\bar{u}, \bar{v})$ is an *extremal point* of the set system $\{\Omega_1, \Omega_2\}$ from (3.19) in the sense of [17, Definition 2.1]. Indeed, we obviously have $(\bar{u}, \bar{v}) \in \Omega_1 \cap \Omega_2$, and thus the extremality of this system follows from the fact that

$$\Omega_1 \cap \big(\Omega_2 + (0, k^{-1}\zeta)\big) = \emptyset \quad \text{for all} \quad k \in I\!N, \tag{3.20}$$

where $\zeta \neq 0$ is an arbitrary fixed element of the cone Θ. Suppose that (3.20) does not hold for some fixed $k \in I\!N$. By the constructions of Ω_1 and Ω_2 in (3.19) and the fact that

$$\operatorname{epi} F = \big\{(x, w) \in X \times Z \big|\ \exists z \in Z,\ \exists \vartheta \in \Theta \ \text{ with } \ w = z - \vartheta,\ (x, z) \in \operatorname{gph} F\big\}$$

our assumption means that there are $(x - \vartheta, z) \neq (\bar{u}, \bar{v})$ and $\vartheta \in \Theta$ such that

$$(x, z - \vartheta) \in \operatorname{epi} F \quad \text{and} \quad \big(x, z - \vartheta - k^{-1}\zeta\big) \in \operatorname{gph} G.$$

By the structure of G in (3.18) and the convexity of the cone Θ we have

$$z - \vartheta - \frac{\zeta}{k} \in \bar{v} - \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}\|x - \bar{u}\|\xi + \Theta \ \text{ and hence } \ z - \bar{v} + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}\|x - \bar{u}\|\xi \in \vartheta + \frac{\zeta}{k} + \Theta \subset \Theta$$

for the point $(x, z) \in \operatorname{gph} F$ under consideration, which implies by (3.17) that $(x, z) = (\bar{u}, \bar{v})$. Since $\bar{v} - \vartheta = z - \vartheta \in F(x) = F(\bar{u})$, we get from (3.17) that $\bar{v} - \vartheta - \bar{v} = -\vartheta \leq 0$, and so $\vartheta = 0$. This clearly contradicts the above relationship $(x - \vartheta, z) \neq (\bar{u}, \bar{v})$ and justifies therefore the extremality of the system (3.19) at $(\bar{u}, \bar{v})$.

Thus we can apply the (approximate) *extremal principle* from [17, Theorem 2.20] to the extremal system $\{\Omega_1, \Omega_2, (\bar{u}, \bar{v})\}$ of the closed sets (3.19) at $(\bar{u}, \bar{v})$ in the Asplund space $X \times Z$ with the norm $\|(x, z)\| := \|x\| + \|z\|$ for $(x, z) \in X \times Z$. Observe the corresponding *dual norm* on $X^* \times Z^*$ is

$$\|(x^*, z^*)\| = \max\big\{\|x^*\|, \|z^*\|\big\} \quad \text{for} \quad (x^*, z^*) \in X^* \times Z^*.$$

Employing the extremal principle, for any $\nu > 0$ we find $(x_i, z_i, x_i^*, z_i^*) \in X \times Z \times X^* \times Z^*$ with $i = 1, 2$ satisfying the relationships

$$\begin{cases} (x_i, z_i) \in \Omega_1 \times \Omega_2, \quad \|x_i - \bar{u}\| + \|z_i - \bar{v}\| \leq \nu, \quad i = 1, 2, \\ (x_i^*, z_i^*) \in \widehat{N}\big((x_i, z_i); \Omega_i\big), \quad i = 1, 2, \\ \dfrac{1}{2} - \nu \leq \max\big\{\|x_i^*\|, \|z_i^*\|\big\} \leq \dfrac{1}{2} + \nu, \quad i = 1, 2, \\ \max\big\{\|x_1^* + x_2^*\|, \|z_1^* + z_2^*\|\big\} \leq \nu. \end{cases} \tag{3.21}$$

Observe that $(x_2^*, z_2^*) \neq 0$ whenever $\nu > 0$ is sufficiently small in (3.21). It also follows from the second line in (3.21), the graphical structure of the set Ω_2 in (3.19), and the coderivative construction (2.4) as $\varepsilon = 0$ that

$$x_2^* \in \widehat{D}^* G(x_2, z_2)(-z_2^*). \tag{3.22}$$

To proceed further with (3.21) and (3.22), let us check that the set-valued mapping G is *Lipschitz continuous* on X with the (global) Lipschitz constant $\ell := \widetilde{\varepsilon}/\widetilde{\lambda}$, i.e.,

$$G(x) \subset G(y) + \ell\|x - y\| I\!B \quad \text{whenever} \quad x, y \in X. \tag{3.23}$$

To justify (3.23), take any $z \in G(x)$ and find by (3.18) and the definition of ℓ such $\zeta \in \Theta$ that $z = \bar{v} - \ell\|x - \bar{u}\|\xi + \zeta$. Then we have the following relationships:

$$\begin{aligned} z &= \bar{v} - \ell\|x - \bar{u}\|\xi + \zeta \\ &= \bar{v} - \ell\|y - \bar{u}\|\xi + \ell\|x - y\|\xi + \ell\big(\|x - \bar{u}\| - \|y - \bar{u}\| - \|x - y\|\big) + \zeta \\ &\subset \bar{v} - \ell\|y - \bar{u}\|\xi + \ell\|x - y\|\xi + \Theta + \zeta \\ &\subset \bar{v} - \ell\|y - \bar{u}\|\xi + \Theta + \ell\|x - y\|\xi = G(y) + \ell\|x - y\|\xi, \end{aligned}$$

where the first inclusion holds due to

$$\|x - \bar{u}\| - \|y - \bar{u}\| - \|x - y\| \le 0 \quad \text{and} \quad \xi \in -\Theta$$

and the second one holds due to the convexity of Θ. Since $\|\xi\| = 1$, we arrive at (3.23).

Employing now the *coderivative estimate* for Lipschitzian mappings from [17, Theorem 1.43], we get from (3.22) that

$$\|x_2^*\| \le \ell\|z_2^*\| \quad \text{and hence} \quad \|z_2^*\| \ne 0, \quad \frac{\|x_2^*\|}{\|z_2^*\|} \le \ell \tag{3.24}$$

by the third line in (3.21) for $i = 2$. Furthermore, it gives

$$\|z_2^*\| \ge \min\Big\{\ell\Big(\frac{1}{2} - \nu\Big), \Big(\frac{1}{2} - \nu\Big)\Big\}. \tag{3.25}$$

This inequality together with the last line of (3.21) ensure that $z_1^* \ne 0$ whenever ν is sufficiently small. Then by the structure of Ω_1 and the second line of (3.21) for $i = 1$ we have $(x_1^*, z_1^*) \in \widehat{N}((x_1, z_1)); \operatorname{epi} F)$, which implies—by the construction of the Fréchet normal cone in (2.1) and the structure of $\operatorname{epi} F$—that there is $\widetilde{z}_1 \in F(x_1)$ and $\vartheta \in \Theta$ with

$$\widetilde{z}_1 = z_1 + \vartheta, \quad (x_1^*, z_1^*) \in \widehat{N}\big((x_1, \widetilde{z}_1); \operatorname{epi} F\big), \quad \text{and} \quad -z_1^* \in \widehat{N}(0; \Theta).$$

Taking (2.4) and (2.8) into account, we thus have

$$\frac{x_1^*}{\|z_1^*\|} \in \widehat{D}^*\mathcal{E}_F(x_1, \widetilde{z}_1)\Big(\frac{-z_1^*}{\|z_1^*\|}\Big) \quad \text{with} \quad (x_1, \widetilde{z}_1) \in \operatorname{gph} F. \tag{3.26}$$

It follows from (3.25) that $\nu/\|z_2^*\| \to 0$ as $\nu \downarrow 0$ and that, by the above estimates,

$$\frac{\|x_1^*\|}{\|z_1^*\|} < \frac{\|x_2^*\| + \nu}{\|z_2^*\| - \nu} = \Big(\frac{\|x_2^*\|}{\|z_2^*\|} + \frac{\nu}{\|z_2^*\|}\Big)\Big/\Big(1 - \frac{\nu}{\|z_2^*\|}\Big) < \frac{\varepsilon}{\lambda}$$

for all $\nu > 0$ sufficiently small. Observe also that

$$\|x_1 - x_0\| \le \|\bar{u} - x_0\| + \|x_1 - \bar{u}\| \le \widetilde{\lambda} + \nu < \lambda$$

for all small $\nu > 0$ by the second inequality in (3.16), the first line in (3.21) for $i = 1$, and the choice of $\widetilde{\lambda}$ in (3.15). Denoting $(\bar{x}, \bar{z}) := (x_1, \widetilde{z}_1)$ and taking into account the subdifferential construction (2.9), we get from (3.26) and the subsequent estimates that the desired relationship (3.14) is satisfied with $\|\bar{x} - x_0\| < \lambda$.

To complete the proof of the theorem, it remains to justify that $(\bar{x}, \bar{z}) = (x_1, \widetilde{z}_1)$ is a $\varepsilon\xi$-minimizer to F provided that $\xi \in -\operatorname{int}\Theta$. In this case $-(\varepsilon - \widetilde{\varepsilon})\xi \in \operatorname{int}\Theta$, and for all $\nu > 0$ sufficiently small we obviously have

$$\nu I\!\!B \subset (\varepsilon - \widetilde{\varepsilon})\xi + \Theta. \tag{3.27}$$

It follows from the first line of (3.21) that $\|z_1 - \bar{v}\| \le \nu$. By (3.27) we find $\zeta \in \Theta$ such that $z_1 - \bar{v} = (\varepsilon - \widetilde{\varepsilon})\xi + \zeta$. If $(x_1, \widetilde{z}_1)$ is not an approximate $\varepsilon\xi$-minimizer to F, then there is $(x, z) \in \mathrm{gph}\, F$ satisfying

$$z + \varepsilon\xi \in \widetilde{z}_1 + \Theta = z_1 + \theta + \Theta = \bar{v} + (\varepsilon - \widetilde{\varepsilon})\xi + \zeta + \theta + \Theta.$$

Since $\bar{v} \in z_0 + \Theta$ by (3.16), we get in this case that

$$z + \widetilde{\varepsilon}\xi \in z_0 + \Theta,$$

which contradicts the strict approximate $\varepsilon\xi$-minimality of the initial pair (x_0, z_0) to F and thus ends the proof of the theorem. $\triangle$

4 Existence of optimal solutions to multiobjective problems

In this section we study the *existence of optimal solutions* to the constrained multiobjective (set-valued and vector-valued) optimization problem:

$$\text{minimize } F(x) \text{ subject to } x \in \Omega, \tag{4.1}$$

where $F\colon X \rightrightarrows Z$ is a mapping from a *complete* metric space (X, d) to a *partially ordered* linear topological space Z with the ordering cone $\Theta \subset Z$ assumed to be closed, convex, and pointed. Our goal in this section is to establish efficient conditions for the existence of *weak minimizers* to (4.1), and thus we impose the *interiority* requirement on the ordering cone: $\mathrm{int}\,\Theta \neq \emptyset$. The afore-mentioned assumptions are *standing* throughout this section.

In what follows we present three results for the existence of weak minimizers to (4.1). The first two results unified in one theorem employ our *basic construction* in the proof of Theorem 3.2—an extension of the Ekeland variational principle to set-valued mappings. We start with the *compactness* requirement on the *constraint set* Ω and then replace it by a certain *coercivity condition* imposed on the *cost mapping*. The third existence result is based on the application of the *subdifferential variational principle* from Theorem 3.5 combined with an appropriate subdifferential extension of the *Palais-Smale condition* and generalized differential calculus rules developed in [17].

Theorem 4.1 (existence of weak minimizers under either compactness of constraint sets or coercivity of cost mappings). *Consider the constrained multiobjective optimization problem* (4.1) *under the standing assumptions made in this section. Then this problem admits a weak minimizer in each of the following cases:*

(i) *Let the constraint set Ω be compact, and let the cost mapping F satisfy the* LIMITING MONOTONICITY CONDITION *as $k \to \infty$:*

$$\big[x_k \to \bar{x},\ \ z_k \in F(x_k)\ \ \textit{with}\ \ z_{k+1} \le z_k\big] \Longrightarrow \big[\exists\, \bar{z} \in \mathrm{Min}\, F(\bar{x})\ \ \textit{with}\ \ \bar{z} \le z_k\big] \tag{4.2}$$

for all $k \in I\!N$; the latter is implied by condition (3.3) *of Theorem* 3.2 *provided that F is level-closed.*

(ii) *Let the cost mapping F satisfy* (4.2) *and the* COERCIVITY CONDITION*: there is a compact set $\Xi \subset X$ such that*

$$\big[x \in X \setminus \Xi,\ \ z \in F(x)\big] \Longrightarrow \big[\exists\,(y, v) \in \mathrm{gph}\, F\ \ \textit{with}\ \ y \in \Xi\ \ \textit{and}\ \ v \le z\big]. \tag{4.3}$$

Proof. Since Ω is a closed subset of the complete metric space (X,d), the space (Ω,d) is complete metric as well. Consider the *unconstrained* mapping $F_\Omega\colon X \rightrightarrows Z$ defined by

$$F_\Omega(x) := F(x) + \Delta(x;\Omega) \ \text{ with } \ \Delta(x;\Omega) := \begin{cases} 0 \in Z & \text{if } x \in \Omega, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.4}$$

Modify sequentially the mapping $T(x,z)$ from (3.7) in the proof of Theorem 3.2 by

$$T_n(x,z) := \big\{y \in X \big|\ \exists\, v \in F_\Omega(y) \ \text{ with } \ v - z + n^{-1}d(x,y)\xi \le 0\big\}, \quad n \in I\!N. \tag{4.5}$$

Fixing $n \in I\!N$ and following the proof of Theorem 3.2 with T_n defined in (4.5), we find a sequence $\{(x_k,z_k)\}$ satisfying

$$(x_{k+1},z_{k+1}) \in \operatorname{gph} F, \quad x_k \in \Omega, \quad z_{k+1} - z_k + n^{-1}d(x_{k+1},x_k)\xi \le 0 \tag{4.6}$$

for all $k = 0,1,\ldots$. Furthermore, we get $\bar{x} \in \Omega$ (depending on $n \in I\!N$) by

$$\bigcap_{k=0}^{\infty} T_n(x_k,z_k) = \{\bar{x}\} \ \text{ for any fixed } \ n \in I\!N. \tag{4.7}$$

Since (4.6) obviously implies that $z_{k+1} \le z_k$, we find by assumption (4.2) such $\bar{z} \in \operatorname{Min} F(\bar{x})$ that $\bar{z} \le z_k$ for all $k = 0,1,\ldots$. It is not hard to observe, arguing by contradiction and employing (4.6) and (4.7) together with the triangle inequality for the metric $d(\cdot,\cdot)$, that

$$T_n(\bar{x},\bar{z}) = \{\bar{x}\} \ \text{ for all } \ n \in I\!N.$$

Since the pair $(\bar{x},\bar{z})$ constructed above depends on $n \in I\!N$, we denote it by (x_n,z_n) and hence have a sequence $\{(x_n,z_n)\}$ satisfying

$$x_n \in \Omega, \quad z_n \in F(x_n), \quad z_{n+1} \le z_n, \quad T_n(x_n,z_n) = \{x_n\} \tag{4.8}$$

for all $n \in I\!N$. By the *compactness* of Ω, we suppose without loss of generality that $x_n \to \widetilde{x}$ as $n \to \infty$ for some $\widetilde{x} \in \Omega$. Then conditions (4.2) and (4.8) ensure the existence of $\widetilde{z}$ satisfying the relationships

$$\widetilde{z} \in \operatorname{Min} F(\widetilde{x}) \ \text{ and } \ \widetilde{z} - z_n \in \Theta \ \text{ for all } \ n \in I\!N. \tag{4.9}$$

We claim that the pair $(\widetilde{x},\widetilde{z})$ is a *weak minimizer* to the multiobjective problem (4.1). Indeed, taking an arbitrary $(x,z) \in \operatorname{gph} F$ with $x \in \Omega$ and $(x,z) \ne (\widetilde{x},\widetilde{z})$ and employing (4.8) and (4.9), we have by elementary transformations that

$$z - \widetilde{z} + n^{-1}d(x_{n+1},x)\xi \in z_{n+1} - \widetilde{z} + Z \setminus \Theta$$

for all $n \in I\!N$, which easily implies that

$$z - \widetilde{z} + n^{-1}d(x_{n+1},x)\xi \in -\Theta + Z \setminus \Theta \ \text{ and hence } \ z - \widetilde{z} + n^{-1}d(x_{n+1},x)\xi \in Z \setminus \Theta$$

due to the convexity of the cone Θ. Now passing to the limit in the last inclusion as $n \to \infty$, we get $z - \widetilde{z} \in Z \setminus (\operatorname{int}\Theta)$, which justifies the weak minimality of $(\widetilde{x},\widetilde{z})$ to (4.1).

To complete the proof of assertion (i), it remains to justify that the limiting monotonicity condition (4.2) is implied by condition (3.3) of Theorem 3.2, where the minimum set

$\operatorname{Min} F(\bar{x})$ is assumed to be closed. Indeed, having the sequence $\{(x_k, z_k)\}$ from the left-hand side of (4.2), define the sets

$$Q(x_k) := \min F(\bar{x}) \cap (z_k + \Theta), \quad k \in I\!N,$$

which are obviously nonempty, closed, and nonincreasing $Q(x_{k+1}) \subset Q(x_k)$ by the monotonicity $z_{k+1} \in z_k + \Theta$ as $k \in I\!N$ in (4.2). Hence

$$\bigcap_{k=0}^{\infty} Q(x_k) \neq \emptyset,$$

and any $\bar{z}$ from the above intersection satisfies $\bar{z} \le z_k$ for all $k \in I\!N$.

Let us next proceed with the proof of assertion (ii). Having the compact set Ξ from the coercivity condition (4.3), we consider the auxiliary problem:

$$\text{minimize } F(x) \text{ subject to } x \in \Xi. \tag{4.10}$$

By assertion (i) of the theorem applied to (4.10) there is $\bar{x} \in \Xi$ and $\bar{z} \in F(\bar{x})$ such that $(\bar{x}, \bar{z})$ is a weak minimizer to problem (4.10). We claim that $(\bar{x}, \bar{z})$ is a weak minimizer to F over Ω as well. Arguing by contradiction, suppose it does not hold and then find $x \notin \Xi$ and $z \in F(x)$ with $z \in \bar{z} + \operatorname{int}\Theta$. By the coercivity condition (4.3), there are $y \in \Xi$ and $v \in F(y)$ such that $v \le z$, i.e., $v \in z + \Theta$. The last two inclusions give

$$v \in z + \Theta \subset \bar{z} + \operatorname{int}\Theta + \Theta \subset \bar{z} + \operatorname{int}\Theta,$$

which means that $(\bar{x}, \bar{z})$ is not a weak minimizer to F over Ξ. This contradiction completes the proof of (ii) and of the whole theorem. $\triangle$

Note that for scalar cost functions the coercivity condition of Theorem 4.1(ii) agrees with those from [2, 3]; see also the references therein. Observe also that the limiting monotonicity condition (4.2) in Theorem 4.1 is *strictly better* than condition (3.3) in Theorem 3.2. We illustrate this by the mapping $F\colon I\!R^2 \rightrightarrows I\!R^2$ defined as

$$F(x) = F(x_1, x_2) := \begin{cases} (|x_1|, |x_2|) & \text{if } (x_1, x_2) \neq 0, \\ I\!B \setminus \{(-1,0), (0,-1)\} & \text{otherwise.} \end{cases}$$

It is easy to check that the limiting monotonicity condition (4.2) and relationships in (3.3) are satisfied, while the minimum set $\operatorname{Min} F(0)$ is not closed.

Our next result establishes the *existence of weak minimizers* to the constrained multiobjective problem (4.1) under a new *subdifferential extension* of the classical *Palais-Smale condition* to set-valued (and vector-valued) mappings. To formulate this condition, we use the *normal subdifferential* (2.11) for set-valued mappings with values in partially ordered spaces introduced in Section 2. Note that new Palais-Smale condition and its application to the proof of the existence theorem rely on the *subdifferential variational principle for set-valued mappings* established in Theorem 3.5 and on the *basic intersection rule* for the limiting normal cone (2.3) derived in [17, Chapter 3].

Recall that the *classical Palais-Smale* condition for differentiable real-valued function $\varphi\colon X \to I\!R$ asserts that if a sequence $\{x_k\} \subset X$ is such that $\{\varphi(x_k)\}$ is bounded and $\|\nabla\varphi(x_k)\| \to 0$ as $k \to \infty$ for the corresponding derivative sequence, then $\{x_k\}$ contains a convergent subsequence. Our *subdifferential extension* for *set-valued mappings* is as follows.

Definition 4.2 (subdifferential Palais-Smale condition for set-valued mappings). *A set-valued mapping $F\colon X \rightrightarrows Z$ from a Banach space X to a partially ordered Banach space Z with the ordering cone $\Theta \subset Z$ satisfies the* SUBDIFFERENTIAL PALAIS-SMALE CONDITION *if any sequence $\{x_k\} \subset X$ such that*

$$\text{there are } z_k \in F(x_k) \text{ and } x_k^* \in \partial_N F(x_k, z_k) \text{ with } \|x_k^*\| \to 0 \text{ as } k \to \infty \tag{4.11}$$

contains a convergent subsequence, where $\{z_k\}$ is selected to be Θ-bounded from below.

The subdifferential Palais-Smale condition introduced clearly reduces to the classical one for smooth functions $F = \varphi$. The next theorem employs the subdifferential Palais-Smale condition to establish the existence of weak minimizers via advanced techniques of variational analysis and generalized differentiation. For simplicity and without loss of generality we consider the (formally) *unconstrained* case of $\Omega = X$ in (4.1). As in the proof of Theorem 4.1, the general constrained case of (4.1) can be obviously reduced to the unconstrained one via the *restriction* F_Ω of F to Ω defined in (4.4).

Theorem 4.3 (existence of weak minimizers under the subdifferential Palais-Smale condition). *Let all the assumptions of Theorem 3.5 be satisfied together with the subdifferential Palais-Smale condition (4.11). Then F admits a weak minimizer.*

Proof. As in the proof of Theorem 4.1, define the mapping $T_n\colon X \times Z \rightrightarrows X$ by (4.5) with $F_\Omega = F$ and $d(x,y) = \|x-y\|$; then construct a sequence $\{(x_n, z_n)\}$ satisfying relationships (4.8), where the condition $x_n \in \Omega$ can be omitted. Following the proof of assertion (i) in Theorem 4.1, we establish the *existence of weak minimizers* to F *provided* that the above sequence $\{x_n\}$ contains a *convergent subsequence*. Let us justify the latter by using the *subdifferential Palais-Smale condition* of Definition 4.2, the *subdifferential variational principle* from Theorem 3.5, and the *basic intersection rule* from [17, Theorem 3.4].

To proceed, consider for each $n \in I\!N$ the set-valued mapping $F_n\colon X \rightrightarrows Z$ given by

$$F_n(x) := F(x) + g_n(x) \text{ with } g_n(x) := n^{-1}\|x - x_n\|\xi \tag{4.12}$$

and conclude from (4.8) and from the structure of T_n in (4.5) that (x_n, z_n) is a *strict approximate $n^{-2}\xi$-minimizer* to F_n. Fix $n \in I\!N$ and apply Theorem 3.5 to F_n and its strict approximate $\varepsilon\xi$-minimizer (x_n, z_n) with $\varepsilon = n^{-2}$ and $\lambda = n^{-1}$. Taking into account the structure of F_n in (4.12) and the subdifferential construction (2.9), we find $(\bar{x}_n, \bar{z}_n, \bar{v}_n, \bar{x}_n^*, \bar{z}_n^*) \in X \times Z \times Z \times X^* \times Z^*$ satisfying the relationships

$$\bar{z}_n \in F(\bar{x}_n), \quad \bar{v}_n = g_n(\bar{x}_n), \quad (\bar{x}_n, \bar{z}_n + \bar{v}_n) \in \operatorname{gph} F_n, \quad \|\bar{x}_n - x_n\| \le n^{-1}, \tag{4.13}$$

$$(\bar{x}_n^*, -\bar{z}_n^*) \in \widehat{N}\big((\bar{x}_n, \bar{z}_n + \bar{v}_n); \operatorname{epi} F_n\big), \quad \bar{z}_n^* \in N(0;\Theta), \quad \|\bar{z}_n^*\| = 1, \quad \|\bar{x}_n^*\| \le n^{-1}. \tag{4.14}$$

Define now the following two subsets of the product space $X \times Z \times Z$, which is *Asplund*:

$$\Omega_1 := \big\{(x,z,v) \in X \times Z \times Z \big|\ (x,z) \in \operatorname{epi} F\big\}, \tag{4.15}$$

$$\Omega_2 := \big\{(x,z,v) \in X \times Z \times Z \big|\ (x,v) \in \operatorname{epi} g_n\big\}. \tag{4.16}$$

It is easy to see that $(\bar{x}_n,\bar{z}_n,\bar{v}_n)\in\Omega_1\cap\Omega_2$ and both sets Ω_i, $i=1,2$, are locally closed around this point by the epiclosedness of F and the Lipschitz continuity of g_n. Observe also that the implication

$$(x,z,v)\in\Omega_1\cap\Omega_2\Longrightarrow z\in F(x)-\Theta,\ v\in g_n(x)-\Theta,$$

which ensures therefore that $(x,z+v)\in\operatorname{epi}F_n$. Thus we have from (4.14) that

$$\limsup_{\substack{(x,z,v)\to(\bar{x}_n,\bar{z}_n,\bar{v}_n)\\(x,z,v)\in\Omega_1\cap\Omega_2}}\frac{\big\langle(\bar{x}_n^*,-\bar{z}_n^*,-\bar{z}_n^*),(x,z,v)-(\bar{x}_n,\bar{z}_n,\bar{v}_n)\big\rangle}{\|(x,z,v)-(\bar{x}_n,\bar{z}_n,\bar{v}_n)\|}$$

$$\le\limsup_{\substack{(x,z)\to(\bar{x}_n,\bar{z}_n+\bar{v}_n)\\(x,z)\in\operatorname{epi}F_n}}\frac{\big\langle(\bar{x}_n^*,\bar{z}_n^*),(x,z)-(\bar{x}_n,\bar{z}_n+\bar{v}_n)\big\rangle}{\|(x,z)-(\bar{x}_n,\bar{z}_n+\bar{v}_n)\|}\le 0,$$

which implies the inclusions

$$(\bar{x}_n^*,-\bar{z}_n^*,-\bar{z}_n^*)\in\widehat{N}\big((\bar{x}_n,\bar{z}_n,\bar{v}_n);\Omega_1\cap\Omega_2\big)\subset N\big((\bar{x}_n,\bar{z}_n,\bar{v}_n);\Omega_1\cap\Omega_2\big).\tag{4.17}$$

Next we are going to express basic normals to the set intersection in (4.17) via basic normals to Ω_1 and Ω_2 and then—by taking into account the structures of these sets—to arrive at the desired conclusions in terms of the mapping F under consideration. To apply the basic intersection rule from [17, Theorem 3.4] to the intersection $\Omega_1\cap\Omega_1$, let us first check that the set system $\{\Omega_1,\Omega_2\}$ satisfies the *limiting qualification condition* at $(\bar{x}_n,\bar{z}_n,\bar{v}_n)$ required in the afore-mentioned theorem. The latter means that for any sequences

$$(x_{ik},z_{ik},v_{ik})\xrightarrow{\Omega_i}(\bar{x}_n,\bar{z}_n,\bar{v}_n)\ \text{ and }\ (x_{ik}^*,z_{ik}^*,v_{ik}^*)\xrightarrow{w^*}(x_i^*,z_i^*,v_i^*)\ \text{ as }\ k\to\infty$$

with $(x_{ik}^*,z_{ik}^*,v_{ik}^*)\in\widehat{N}((\bar{x}_n,\bar{z}_n,\bar{v}_n);\Omega_i)$, $k\in I\!N$, $i=1,2$, one has the implication

$$\big[\|(x_{1k}^*,z_{1k}^*,v_{1k}^*)+(x_{2k}^*,z_{2k}^*,v_{2k}^*)\|\to 0\ \text{ as }\ k\to\infty\big]\Longrightarrow(x_i^*,z_i^*,v_i^*)=0\tag{4.18}$$

for $i=1,2$. To proceed, we observe from the structures of Ω_i in (4.15) and (4.16) that $v_{1k}^*=z_{2k}^*=0$ for all $k\in I\!N$, and hence (4.18) reduces to

$$\big[\|x_{1k}^*+x_{2k}^*\|\to 0,\ \|z_{1k}^*\|\to 0,\ \|v_{2k}^*\|\to 0\big]\Longrightarrow x_1^*=x_2^*=z_1^*=v_2^*=0.\tag{4.19}$$

Since the conclusions $z_1^*=v_2^*=0$ are obvious, it remains to show that $x_1^*=x_2^*=0$. To this end, observe similarly to the proof of estimate (3.24) in Theorem 3.5—based on [17, Theorem 1.43]—that

$$(x_{2k}^*,v_{2k}^*)\in\widehat{N}\big((x_{2k},g_n(x_{2k}));\operatorname{epi}g_n\big)\Longrightarrow\|x_{2k}^*\|\le n^{-1}\|v_{2k}^*\|\ \text{ for all }\ k\in I\!N,$$

since the mapping $g_n\colon X\to Z$ from (4.12) is Lipschitz continuous with modulus $\ell=n^{-1}$. This gives $\|x_{2k}^*\|\to 0$ and hence $\|x_{1k}^*\|\to 0$ as $k\to\infty$ by (4.19), which justifies the fulfillment of the limiting qualification condition for $\{\Omega_1,\Omega_2\}$ at $(\bar{x}_n,\bar{z}_n,\bar{v}_n)$.

To apply the intersection rule from [17, Theorem 3.4], we need also to check that Ω_1 is *strongly PSNC* at $(\bar{x}_n,\bar{z}_n,\bar{v}_n)$ with respect to the last component Z in the product $X\times Z\times Z$ and that Ω_2 is PSNC at this point with respect to $X\times Z$. The former is obvious from the structure of (4.15), while the latter follows from (4.16) due to the Lipschitz continuity of g_n; see [17, Corollary 1.69(i)]. Thus we have

$$N\big((\bar{x}_n, \bar{z}_n, \bar{v}_n); \Omega_1 \cap \Omega_2\big) \subset N\big((\bar{x}_n, \bar{z}_n, \bar{v}_n); \Omega_1\big) + N\big((\bar{x}_n, \bar{z}_n, \bar{v}_n); \Omega_2\big). \tag{4.20}$$

It follows from (4.17), (4.20), and the structures of Ω_i that there are $u_n^*, p_n^* \in X^*$ satisfying

$$(u_n^*, -\bar{z}_n^*) \in N\big((\bar{x}_n, \bar{z}_n); \operatorname{epi} F\big), \quad (p_n^*, -\bar{z}_n^*) \in N\big((\bar{x}_n, g_n(\bar{x}_n)); \operatorname{epi} g_n\big) \tag{4.21}$$

and such that $\bar{x}_n^* = u_n^* + p_n^*$. By the condition on $\bar{z}_n^*$ in (4.14) and definition (2.11) of the normal subdifferential we get from (4.21) the relationships

$$u_n^* \in \partial_N F(\bar{x}_n, \bar{z}_n), \quad p_n^* \in \partial_N g_n(\bar{x}_n), \quad u_n^* + p_n^* = \bar{x}_n^*. \tag{4.22}$$

It is easy to observe from the form of g_n in (4.12) with $\|\xi\| = 1$ that $\|p_n^*\| \le n^{-1}$, and thus—by using the last estimate in (4.14)—one has

$$\|u_n^*\| = \|\bar{x}_n^* - p_n^*\| \le \|\bar{x}_n^*\| + \|p_n^*\| \le n^{-1} + n^{-1} = 2n^{-1}.$$

Summarizing the above derivation, we have a sequence of triples $\{(\bar{x}_n, \bar{z}_n, u_n^*)\} \subset X \times Z \times X^*$ satisfying the relationships

$$(\bar{x}_n, \bar{z}_n) \in \operatorname{gph} F, \quad u_n^* \in \partial_N F(\bar{x}_n, \bar{z}_n), \quad \text{and} \quad \|u_n^*\| \to 0 \ \text{ as } \ n \to \infty. \tag{4.23}$$

Furthermore, the sequence $\{\bar{z}_n\}$ in (4.23) is Θ-bounded from below due to this assumption on F induced by Theorem 3.5. Thus the sequence $\{\bar{x}_n\}$ contains a convergent subsequence as $n \to \infty$ by the *subdifferential Palais-Smale condition* from Definition 4.2. Employing the estimate $\|x_n - \bar{x}_n\| \le n^{-1}$ from (4.13), we conclude that the initial sequence $\{x_n\}$ selected in the beginning of the proof of this theorem also contains a convergent subsequence. This completes the proof of the theorem. △

5 Necessary optimality and suboptimality conditions for constrained multiobjective problems

In the concluding section of the paper we employ the *variational principles* established in Section 3 and the *tools of generalized differentiation* from Section 2 to deriving new *necessary optimality conditions* and *suboptimality conditions* for general constrained problems of multiobjective optimization. The necessary optimality conditions established below concern *minimizers* (not just weak minimizers) to multiobjective problems *without any interiority requirements* imposed on the ordering cone Θ of Z. The (strong) suboptimality conditions are derived in this section for arbitrary *approximate $\varepsilon\xi$-minimizers* to multiobjective problems defined by ordering cones with possible *empty interiors.*

For simplicity we mainly focus in what follows on the class of constrained multiobjective problems given in the form:

$$\text{minimize } \ f(x) \ \text{ subject to } \ x \in \Omega \tag{5.1}$$

with a *single-valued* cost mapping $f\colon X \to Z$ between *Asplund spaces* and with *geometric constraints* described by a closed subset Ω of X. The results obtained can be extended to more general problems of *set-valued optimization* with various constraints (of *operator,* *functional*, and *equilibrium* types) based on the extremal and variational principles and on the corresponding generalized differential and SNC calculus rules (the latter calculus is needed only in infinite dimensions)—similarly to the developments and applications in [17, 18] for other classes of optimization and equilibrium problems. To illustrate this approach, we present some necessary optimality and suboptimality conditions derived in this way for *multiobjective* problems with *functional constraints* given by *finitely many equalities and inequalities* via (generally nonsmooth) real-valued functions.

Let us start with necessary optimality conditions for *local minimizers* to problem (5.1), where an optimal solution (minimizer) is understood in the sense of Definition 3.1(i) with the usual neighborhood localization. Recall that the corresponding subdifferential constructions and SNC properties used in the theorem are defined and discussed in Section 2.

Theorem 5.1 (necessary optimality conditions for multiobjective problems with geometric constraints). *Let $\bar{x}$ be a local minimizer to problem* (5.1) *with $\bar{z} := f(\bar{x})$, where the ordering cone $\Theta \subset Z$ satisfies the standing convexity, closedness and pointedness assumptions, where f is locally epiclosed around $(\bar{x}, \bar{z})$, and where Ω is locally closed around $\bar{x}$. Suppose also that Θ is SNC at the origin, that either Ω is SNC at $\bar{x}$ or f is partially SNEC at $(\bar{x}, \bar{z})$, and that the qualification condition*

$$\partial^{\infty} f(\bar{x}) \cap \big(-N(\bar{x}; \Omega)\big) = \{0\} \tag{5.2}$$

is satisfied. Then one has the inclusion

$$0 \in \partial_N f(\bar{x}) + N(\bar{x}; \Omega). \tag{5.3}$$

Proof. Consider the restriction f_Ω of the mapping f to the set Ω given by

$$f_\Omega(x) := f(x) + \Delta(x; \Omega), \tag{5.4}$$

where the indicator mapping $\Delta(\cdot; \Omega)$ of Ω is defined in (4.4). Taking any $\xi \in -\Theta$ with $\|\xi\| = 1$ and any $k \in I\!N$, observe that $(\bar{x}, \bar{z})$ is a (local) strict approximate $k^{-1}\xi$-minimizer to f_Ω in the sense of Definition 3.1(iv). It is easy to see that f_Ω satisfies (locally) all the assumptions required by Theorem 3.5 except that of $\xi \in -\text{int}\,\Theta$, which is *not needed* in what follows. Employing the latter theorem and relationships (3.21) of the extremal principle in its proof, we find sequences $\{(x_k, x_k^*)\} \in \Omega \times X^*$ such that

$$x_k^* \in \widehat{\partial} f_\Omega(x_k), \quad \|(x_k, z_k) - (\bar{x}, \bar{z})\| \le k^{-1}, \quad \text{and} \quad \|x_k^*\| \le k^{-1} \quad \text{for all } k \in I\!N. \tag{5.5}$$

Passing to the limit as $k \to \infty$ in (5.5) and taking into account the construction of the limiting subdifferential in (2.10), we get $0 \in \partial_L f_\Omega(\bar{x})$, which implies by Theorem 2.2 (since Θ is assumed to be SNC at the origin) and the structure of f_Ω in (5.3) that

$$0 \in \partial_N f_\Omega(\bar{x}) = \partial_N \big[f + \Delta(\cdot; \Omega)\big](\bar{x}). \tag{5.6}$$

It follows from (2.11) and (5.6)

$$(0, -z^*) \in N\big((\bar{x}, \bar{z}); \text{epi}\, f \cap (\Omega \times Z)\big) \quad \text{for some } z^* \in N(0; \Theta), \quad \|z^*\| = 1. \tag{5.7}$$

Employing now in (5.7) the *basic intersection rule* from [17, Theorem 3.4] whose requirements are satisfied due to the qualification condition (5.2) and the SNC assumptions of this theorem, we get from (5.7) that

$$(0, -z^*) \in N\big((\bar{x}, \bar{z}); \text{epi}\, f\big) + N(\bar{x}; \Omega) \times \{0\} \quad \text{with} \quad z^* \in N(0; \Theta), \quad \|z^*\| = 1,$$

which is obviously equivalent to (5.3). This completes the proof of the theorem. $\triangle$

It occurs that the qualification condition (5.2) and the partial SNEC condition of Theorem 5.1 are *automatically fulfilled* for a major class of *epi-Lipschitz-like* (ELL) cost mappings $f\colon X \to Z$ described in Section 2.

Corollary 5.2 (necessary optimality conditions for multiobjective problems with Lipschitzian costs). *Let $\bar{x}$ be a local minimizer to* (5.1), *where the ordering cone Θ satisfies the assumptions of Theorem* 5.1, *where the constraint set Ω is locally closed around $\bar{x}$, and where the cost mapping f is epiclosed and ELL around $(\bar{x}, \bar{z})$ with $\bar{z} = f(\bar{x})$. Then the necessary optimality condition* (5.3) *is satisfied.*

Proof. This follows from Theorem 5.1 due to Proposition 2.3 ensuring simultaneously the partial SNEC property and the qualification condition (5.2) for ELL mappings. $\triangle$

Next we present a specification of Theorem 5.1 for multiobjective problems (5.1) with *functional constraints* given in the conventional form of *mathematical programming*:

$$\Omega := \big\{x \in X \big|\ \varphi_i(x) \le 0, \quad i = 1, \ldots, m; \quad \varphi_i(x) = 0, \quad i = m+1, \ldots, m+r\big\}. \tag{5.8}$$

For simplicity we assume that all the functions $\varphi_i\colon X \to I\!R$ are *locally Lipschitzian* around the reference point; more general non-Lipschitzian settings can be also considered based on the calculus rules of [17]. The following consequence of Theorem 5.1 holds.

Corollary 5.3 (necessary optimality conditions in multiobjective mathematical programming). *Let $\bar{x}$ be a local minimizer to problem* (5.1) *with the constraint set Ω given by* (5.8), *where the ordering cone Θ satisfies the assumptions of Theorem* 5.1, *where the cost mapping f is epiclosed around $(\bar{x}, \bar{z})$ with $\bar{z} = f(\bar{x})$, and where all the functions φ_i are locally Lipschitzian around $\bar{x}$. Impose the two qualifications conditions*

$$\begin{array}{l}\Big[\ 0 \in \displaystyle\sum_{i=1}^{m} \lambda_i \partial\varphi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i\big(\partial\varphi_i(\bar{x}) \cup \partial(-\varphi_i)(\bar{x})\big), \\ \quad \lambda_i \ge 0 \ \textit{for}\ i = 1, \ldots, m+r, \quad \lambda_i\varphi_i(\bar{x}) = 0 \ \textit{for}\ i = 1, \ldots, m\Big] \\ \quad \Longrightarrow \lambda_i = 0 \ \textit{for all}\ i = 1, \ldots, m+r; \end{array} \tag{5.9}$$

$$\begin{array}{l}\Big[\ -\partial^\infty f(\bar{x}) \ni -x^* \in \displaystyle\sum_{i=1}^{m} \lambda_i \partial\varphi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i\big(\partial\varphi_i(\bar{x}) \cup \partial(-\varphi_i)(\bar{x})\big) \ \textit{with} \\ \quad \lambda_i \ge 0 \ \textit{as}\ i = 1, \ldots, m, \quad \lambda_i\varphi_i(\bar{x}) = 0 \ \textit{as}\ i = 1, \ldots, m\Big] \Longrightarrow x^* = 0 \end{array} \tag{5.10}$$

formulated via the basic subdifferential [17] *of Lipschitzian functions φ_i. Then there are $\lambda_i \ge 0$ for $i = 1, \ldots, m+r$ such that $\lambda_i\varphi_i(\bar{x}) = 0$ as $i = 1, \ldots, m$ and*

$$0 \in \partial_N f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial\varphi_i(\bar{x}) + \sum_{i=m+1}^{m+r} \lambda_i\big(\partial\varphi_i(\bar{x}) \cup \partial(-\varphi_i)(\bar{x})\big). \tag{5.11}$$

Proof. First observe that the basic normal cone $N(\cdot;\Omega)$ to the constraint set Ω given in (5.8) satisfies the inclusion

$$N(\bar{x};\Omega)\subset\Big\{\sum_{i=1}^{m}\lambda_i\partial\varphi_i(\bar{x})+\sum_{i=m+1}^{m+r}\lambda_i\big(\partial\varphi_i(\bar{x})\cup\partial(-\varphi_i)(\bar{x})\big)\Big|\; \lambda_i\ge 0\ \text{as}\ i=1,\ldots,m+r,\quad \lambda_i\varphi_i(\bar{x})=0\ \text{as}\ i=1,\ldots,m\Big\} \tag{5.12}$$

provided the fulfillment of the qualification condition (5.9); see, e.g., [17, Corollary 4.36]. Substituting (5.12) into (5.2) and (5.3), we get the qualification condition (5.10) and optimality condition (5.11), respectively. Finally, the qualification condition (5.9) ensures the SNC property of the constraint set (5.8) at $\bar{x}$; this follows from [17, Theorem 3.86]. Thus we meet all the requirements of Theorem 5.1 and complete the proof of the corollary. △

Note that the qualification condition (5.9) reduces to the classical *Mangasarian-Fromovitz constraint qualification* when the functions φ_i are strictly differentiable at $\bar{x}$ (in particular, when $\varphi_i\in C^1$ around $\bar{x}$); in this case $\partial\varphi(\bar{x})=\{\nabla\varphi(\bar{x})\}$. Note furthermore that, by Corollary 5.2, the qualification condition (5.10) is automatic if the cost mapping f is ELL around $\bar{x}$. For the latter class we also have the partial SNEC property of f at $(\bar{x},\bar{z})$, which is *not needed* in the framework of Corollary 5.3 under the generalized Mangasarian-Fromovitz constraint qualification (5.9).

Our final result concerns *suboptimality conditions* for problem (5.1) applied to its *approximate solutions*—the exact minimizers may *not even exist.*

Theorem 5.4 (suboptimality conditions in multiobjective optimization). *Let $\bar{x}$ be a local approximate $\varepsilon\xi$-minimizer to problem* (5.1) *in the sense of Definition* 3.1(ii) *with $\varepsilon>$ and $0\ne\xi\in-\Theta$, let $\lambda>0$, and let the ordering cone $\Theta\subset Z$ satisfy the requirements of Theorem* 5.1. *Suppose furthermore that for any approximate $\varepsilon\xi$-minimizer $x\in\Omega\cap(\bar{x}+\eta I\!B)$ with some $\eta>\lambda$ and with $z:=f(x)\le f(\bar{x})=:\bar{z}$ the following assumptions hold:*
—Ω is locally closed around x and f is epiclosed around (x,z);
—either Ω is SNC at x, or f is partially SNEC at (x,z);
—one has the qualification condition

$$\partial^\infty f(x)\cap\big(-N(x;\Omega)\big)=\{0\}. \tag{5.13}$$

Then there is a local approximate $\varepsilon\xi$-minimizer $\widehat{x}\in\Omega$ to problem (5.1) *with $\|\widehat{x}-\bar{x}\|\le\lambda$ and $f(\widehat{x})\le f(\bar{x})$ satisfying the suboptimality relationships*

$$\|\widehat{x}^*_f+\widehat{x}^*_\Omega\|\le\frac{\varepsilon}{\lambda}\ \text{ for some }\ \widehat{x}^*_f\in\partial_N f(\widehat{x})\ \text{ and }\ \widehat{x}^*_\Omega\in N(\widehat{x};\Omega). \tag{5.14}$$

Proof. Employing Corollary 3.3 with $x_0:=\bar{x}$ to the restricted mapping f_Ω in (5.4), we find $\widehat{x}\in\Omega\cap(\bar{x}+\lambda I\!B)$ with $f(\widehat{x})\le f(\bar{x})$, which is obviously a local approximate $\varepsilon\xi$-minimizer to f on Ω; furthermore, it provides an *exact local minimum* to the *perturbed* mapping

$$g(x):=f(x)+\frac{\varepsilon}{\lambda}\|x-\widehat{x}\|\xi\ \text{ over }\ x\in\Omega. \tag{5.15}$$

Applying now Theorem 5.1 to (5.15), we get the optimality condition

$$0\in\partial_N g(\widehat{x})+N(\widehat{x};\Omega) \tag{5.16}$$

under the assumptions of the latter theorem imposed on g. It follows from definition (2.12) of the singular subdifferential, the Lipschitz continuity of the perturbation in (5.15), and the mixed coderivative sum rule from [17, Theorem 3.10] that $\partial^\infty g(\bar{x}) = \partial^\infty f(\bar{x})$, and thus the qualification condition (5.2) for g is equivalent to (5.13) at $x = \widehat{x}$. Taking into account the SNC calculus result of [17, Theorem 3.88], we easily conclude from (5.15) that the SNEC requirement on g agrees with that on f at $\widehat{x}$. Finally, it follows from the normal subdifferential construction (2.11) and from the normal coderivative sum rule in [17, Theorem 3.10] that

$$\partial_N g(\widehat{x}) \subset \partial_N f(\widehat{x}) + \frac{\varepsilon}{\lambda} I\!B^*. \tag{5.17}$$

Substituting (5.17) into (5.16), we arrive at the suboptimality relationships in (5.14) and thus finish the proof of theorem. △

Similarly to Corollaries 5.2 and 5.3, we can establish the corresponding consequences of Theorem 5.4 that provide suboptimality conditions to multiobjective problems with Lipschitzian costs and with functional constraints.

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