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VARIATIONAL PROBLEMS WITH TWO PHASES AND THEIR FREE BOUNDARIES

BY

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ABSTRACT. The problem of minimizing $\int[|\nabla v|^2 + q^2(x)\lambda^2(v)]\,dx$ in an appropriate class of functions v is considered. Here $q(x) \neq 0$ and $\lambda^2(v) = \lambda_1^2$ if $v < 0$, $= \lambda_2^2$ if $v > 0$. Any minimizer u is harmonic in $\{u \neq 0\}$ and $|\nabla u|^2$ has a jump

$$q^2(x)(\lambda_1^2 - \lambda_2^2)$$

across the free boundary $\{u \neq 0\}$. Regularity and various properties are established for the minimizer u and for the free boundary.

Introduction. In this paper we consider the problem of minimizing

$$J(v) = \int_\Omega \left[|\nabla v|^2 + q^2(x)\lambda^2(v)\right] dx, \qquad v \in K,$$

where $q^2(x) \neq 0$,

$$\lambda^2(v) = \begin{cases} \lambda_1^2 & \text{if } v < 0, \\ \lambda_2^2 & \text{if } v > 0, \end{cases}$$

and $\lambda^2(v)$ is lower semicontinuous at $v = 0$; it is assumed that $\lambda_i^2 > 0$ and $\Lambda = \lambda_1^2 - \lambda_2^2 \neq 0$. The class K consists of all functions v in $L^1_{\mathrm{loc}}(\Omega)$, with $\nabla v \in L^2(\Omega)$ such that $v = u^0$ on a given open subset S of $\partial\Omega$, and Ω is a domain in R^n.

The analogous problem for functions in $K^+ = \{v \in K,\ v \geq 0 \text{ a.e.}\}$ was studied in **[1]**; in that paper it was proved that any (local) minimizer of $J(v)$ in K^+ is Lipschitz continuous and, if $n = 2$, the free boundary $\partial\{u > 0\}$ is analytic if $q(x)$ is analytic.

The present variational problem is motivated by applications to the flow of two liquids in models of jets and cavities; these applications will be studied in other forthcoming papers **[5, 6]**. The present work is aimed at extending results of **[1]**. In particular, we shall establish nondegeneracy theorems, the Lipschitz continuity of the solution, and some properties of the free boundary; for $n = 2$ the free boundary is proved to be continuously differentiable.

A new and rather powerful tool introduced in this paper is the monotonicity formula (Lemma 5.1) asserting that, for a minimizer u, if $u(x_0) = 0$ then

$$r^{-4}\int_{B_r(x_0)} \rho^{2-n}|\nabla u^+|^2\,dx \cdot \int_{B_r(x_0)} \rho^{2-n}|\nabla u^-|^2\,dx \nearrow \quad \text{if } r \nearrow.$$

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This is used in establishing Lipschitz continuity and in identifying blow-up limits. The differentiability of the free boundary for $n = 2$ also involves a new set of ideas, exploiting among other things, the monotonicity formula.

1. Existence. Let Ω be a domain in R^n with boundary $\partial\Omega$ which is locally a Lipschitz graph. Let S be a nonempty open subset of $\partial\Omega$ and let u^0 be a given function in $L^1_{\text{loc}}(\Omega)$ with $\nabla u^0 \in L^2(\Omega)$. Let $q(x)$ be a strictly positive uniformly Lipschitz continuous function in compact subsets of $\bar{\Omega}$, and let $\lambda(u)$ be the function

$$\lambda(u) = \begin{cases} \lambda_1 & \text{if } u < 0, \\ \lambda_2 & \text{if } u > 0, \end{cases} \tag{1.1}$$

where $\lambda_1, \lambda_2 \geqslant 0$, and define $\lambda(0)$ such that

$$0 \leqslant \lambda(0) \leqslant \min\{\lambda_1, \lambda_2\}. \tag{1.2}$$

We assume that

$$\Lambda = \lambda_1^2 - \lambda_2^2 \neq 0. \tag{1.3}$$

Finally, set $Q(u, x) = q(x)\lambda(u)$.

We introduce the convex set

$$K = \left\{ v \in L^1_{\text{loc}}(\Omega),\ \nabla v \in L^2(\Omega),\ v = u^0 \text{ on } S \right\}$$

and the functional

$$J(v) = \int_\Omega \left(|\nabla v|^2 + Q^2(v, x) \right) dx, \qquad v \in K.$$

Problem (J). Find $u \in K$ such that $J(u) = \min_{v \in K} J(v)$.

THEOREM 1.1. *If $J(u^0) < \infty$ then there exists a solution of Problem* (J).

PROOF. Take a minimizing sequence u_k. Then the ∇u_k are uniformly bounded in $L^2(\Omega)$. Since $u_k - u^0 = 0$ on S, S open and nonempty, we can estimate $u_k - u^0$ in $L^2(\Omega \cap B_R)$ for any ball $B_R = \{|x| < R\}$ and deduce that, for a subsequence,

$$\begin{aligned} \nabla u_k \to \nabla u \quad & \text{weakly in } L^2_{\text{loc}}(\Omega), \\ u_k \to u \quad & \text{a.e. in } \Omega, \\ Q^2(u_k, x) \to \gamma \quad & \text{weakly star in } L^\infty_{\text{loc}}(\Omega), \end{aligned}$$

where $\gamma = Q^2(u, x)$ if $u \neq 0$, and $\gamma \geqslant Q^2(u, x)$ if $u = 0$ (by (1.2)). Hence,

$$\begin{aligned} \int_{\Omega \cap B_R} \left(|\nabla u|^2 + Q^2(u, x) \right) &\leqslant \liminf_{k \to \infty} \int_{\Omega \cap B_R} |\nabla u_k|^2 + \lim_{k \to \infty} \int_{\Omega \cap B_R} Q^2(u_k, x) \\ &\leqslant \liminf_{k \to \infty} J(u_k). \end{aligned}$$

Letting $R \to \infty$ we see that u is an absolute minimum for J.

2. Continuity, subharmonicity and the free boundary condition. We denote a solution of Problem (J) by u.

THEOREM 2.1. *For any compact subset D of Ω there exists a constant C such that*

$$|u(x)-u(y)|\leqslant C|x-y|\log(1/|x-y|)$$

if $x, y\in D$, $|x-y|<\frac{1}{2}$.

PROOF. Let B_r be any ball of radius r in D and denote by v_r the solution of

$$\nabla v_r = 0 \quad \text{in } B_r, \qquad v_r = u \quad \text{on } \partial B_r. \tag{2.1}$$

Then, by the minimality of u,

$$\int_{B_r}\left(|\nabla u|^2 + Q^2(u,x)\right)\leqslant\int_{B_r}\left(|\nabla v_r|^2+Q^2(v_r,x)\right).$$

It follows that $\int_{B_r}(|\nabla u|^2-|\nabla v_r|^2)\leqslant Cr^n$. But the left-hand side is equal to

$$\int_{B_r}\nabla(u-v_r)\cdot\nabla(u+v_r)=\int_{B_r}|\nabla(u-v_r)|^2+2\int_{B_r}\nabla(u-v_r)\cdot\nabla v_r$$

and the last integral vanishes, by (2.1). Consequently, $\int_{B_r}|\nabla(u-v_r)|^2\leqslant Cr^n$.

Proceeding as in [**11**, Theorem, 5.3.6], one can establish that

$$\int_{B_r}|\nabla(u-v_r)|^2\leqslant C(R)r^n(\log R/r+1) \quad \text{if } 0<r<R,$$

so that

$$\int_{B_r}|\nabla u|^2\leqslant C(R)r^n\left(\log\frac{R}{r}+1\right),$$

from which the assertion follows as in [**11**, Theorem 3.5.2].

THEOREM 2.2. *The function u is harmonic in $\{u\neq 0\}$.*

PROOF. For any $\zeta\in C_0^1(\Omega\setminus\{u=0\})$, $u\pm\varepsilon\zeta$ is in K for any $\varepsilon>0$. Hence,

$$0=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\left(J(u+\varepsilon\zeta)-J(\zeta)\right)=\int_\Omega\nabla\zeta\cdot\nabla u.$$

THEOREM 2.3. *If $\lambda(0)=\lambda_1$ and $\Lambda<0$ ($\lambda(0)=\lambda_2$ and $\Lambda>0$) then u is subharmonic (superharmonic) in Ω.*

PROOF. Defining v by (2.1), $B_r\subset\Omega$, we have $J(u)\leqslant J(\min(u,v))$, which gives, if $\lambda(0)=\lambda_1$,

$$I\equiv\int_{B_r}\left[|\nabla u|^2-|\nabla\min(u,v)|^2\right]\leqslant\int_{B_r}\left[Q^2(\min(u,v),x)-Q^2(u,x)\right]$$

$$=\int_{B_r\cap\{u>v\}}\left[Q^2(v,x)-Q^2(u,x)\right]=\int_{B_r\cap\{u\geqslant 0>v\}}\Lambda q^2(x).$$

But

$$I = \int_{B_r} \nabla \max(u - v, 0) \cdot \nabla (u + v)$$

$$= \int_{B_r} \nabla \max(u - v, 0) \cdot \nabla (u - v) + 2\int_{B_r} \nabla \max(u - v, 0) \cdot \nabla v$$

$$= \int_{B_r} | \nabla \max(u - v, 0) |^2.$$

Hence, $\Lambda < 0$ implies $u \leqslant v$, i.e., u is subharmonic. Similarly, if $\lambda(0) = \lambda_2$ and $\Lambda > 0$, then u is superharmonic.

DEFINITION 2.1. The set $\Gamma = \partial\{u > 0\} \cup \partial\{u < 0\}$ is called the *free boundary*.

The next theorem shows that u satisfies, in a generalized sense, the equation

$$| \nabla u^- |^2 - | \nabla u^+ |^2 = \Lambda q^2(x) \quad \text{on } \Gamma,$$

provided the set $\{u = 0\}$ has zero measure.

THEOREM 2.4. *Suppose* $\operatorname{meas}\{u = 0\} = 0$. *Then, for any* $\eta \in C_0^1(\Omega, R^n)$,

(2.2)

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u < -\varepsilon\}} \left(| \nabla u |^2 - \lambda_1^2 q^2(x)\right) \eta \cdot \nu + \lim_{\delta \downarrow 0} \int_{\partial\{u > \delta\}} \left(| \nabla u |^2 - \lambda_2^2 q^2(x)\right) \eta \cdot \nu = 0$$

where ν *is the outward normal.*

PROOF. Let $\tau_\varepsilon(x) = x + \varepsilon\eta(x)$, $\varepsilon \neq 0$, and define $u_\varepsilon \in K$ by $u_\varepsilon(\tau_\varepsilon x) = u(x)$. Then

$$0 \leqslant J(u_\varepsilon) - J(u)$$

$$= \int_\Omega \left\{ \left[| \nabla u (D\tau_\varepsilon)^{-1} |^2 + Q^2(u, \tau_\varepsilon(x)) \right] \det(D\tau_\varepsilon) - \left(| \nabla u |^2 + Q^2(u, x) \right) \right\}$$

$$= \varepsilon \int_\Omega \left[| \nabla u |^2 + Q^2(u, x) \right] \nabla \cdot \eta$$

$$+ \varepsilon \int_\Omega \left[-2 \nabla u D\eta \nabla u + \nabla_x Q^2(u, x) \cdot \eta \right] + O(\varepsilon).$$

The linear term in ε must vanish, giving (since $\Delta u = 0$ in $\{u \neq 0\}$)

$$0 = \lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \int_{\Omega \setminus \{-\varepsilon < u < \delta\}} \nabla \cdot \left[\left(| \nabla u |^2 + Q^2(u, x) \right) \eta - 2\eta \cdot \nabla u \nabla u \right]$$

$$= \lim_{\varepsilon \downarrow 0} \int_{\partial\{u < -\varepsilon\}} \left[\left(| \nabla u |^2 + Q^2(u, x) \right) \eta - 2\eta \cdot \nabla u \nabla u \right] \cdot \nu$$

$$+ \lim_{\delta \downarrow 0} \int_{\partial\{u > \delta\}} \left[\left(| \nabla u |^2 + Q^2(u, x) \right) \eta - 2\eta \cdot \nabla u \nabla u \right] \cdot \nu$$

$$= \lim_{\varepsilon \downarrow 0} \int_{\partial\{u < -\varepsilon\}} \left[\lambda_1^2 q^2(x) - | \nabla u |^2 \right] \eta \cdot \nu$$

$$+ \lim_{\delta \downarrow 0} \int_{\partial\{u > \delta\}} \left[\lambda_2^2 q^2(x) - | \nabla u |^2 \right] \eta \cdot \nu.$$

REMARK 2.1. If $\text{meas}\{u=0\}>0$ and if $\{u=0\}$ is a limit of increasing open sets D_ρ ($\rho\downarrow 0$), then on the left-hand side of (2.2) there appears the additional term

$$\lim_{\rho\downarrow 0}\int_{\partial D_\rho}\left(|\nabla u|^2-\lambda^2(0)q^2(x)\right)\eta\cdot\nu$$

3. Nondegeneracy. For any function v and a ball $B_r=B_r(x^0)$ with center x^0 and radius r, we set

$$\fint_{\partial B_r} v=\frac{1}{|\partial B_r|}\int_{\partial B_r} v,\qquad |\partial B_r|=\text{surface area of } \partial B_r.$$

Let

$$0\leqslant q_1\leqslant q(x)\leqslant q_2<\infty,\qquad |\Lambda|\geqslant l_0>0. \tag{3.1}$$

THEOREM 3.1. *Suppose $\Lambda<0$. For any $0<\kappa<1$ there is a positive constant c depending only on κ and $q_1^2 l_0$ such that if $B_r\subset\Omega$ and $\frac{1}{r}\fint_{\partial B_r}u^+<c$, then $u^+=0$ in $B_{\kappa r}$.*

PROOF. Set $\gamma=\frac{1}{r}\fint_{\partial B_r}u^+$. The idea of this proof is to replace u in B_r by a function v satisfying

$$\begin{aligned} v&=0 \quad \text{on } \partial B_r,\\ v&=u \quad \text{in } B_r\cap\{u\leqslant 0\},\\ v&=0 \quad \text{in } B_{\kappa r}\cap\{u>0\},\\ \Delta v&=0 \quad \text{in } (B_r\setminus B_{\kappa r})\cap\{u>0\}\end{aligned}$$

and show that $J(v)<J(u)$ if γ is sufficiently small.

For almost any $\varepsilon>0$ the surface $\{u=\varepsilon\}$ is smooth. Choose any such small ε and consider the function v_ε satisfying

$$\begin{aligned} v_\varepsilon&=u \quad \text{on } \partial B_r,\\ v_\varepsilon&=u \quad \text{in } B_r\cap\{u<\varepsilon\},\\ v_\varepsilon&=\varepsilon \quad \text{in } B_{\kappa r}\cap\{u>\varepsilon\},\\ \Delta v_\varepsilon&=0 \quad \text{in } D_\varepsilon^+\equiv(B_r\setminus B_{\kappa r})\cap\{u>\varepsilon\}.\end{aligned}$$

The function v_ε can be obtained by minimizing the Dirichlet integral over B_r subject to the above constraints. Also v_ε is continuous at $\{u=\varepsilon\}\cap(B_r\setminus\overline{B}_{\kappa r})$ and $\min(u,0)\leqslant v_\varepsilon\leqslant u$. Since ∇v_ε is bounded in $L^2(B_r)$, the limit $v=\lim_{\varepsilon\to 0}v_\varepsilon$ exists and $\min(u,0)\leqslant v\leqslant u$; hence v is continuous in B_r and has the desired properties.

We obtain

$$\begin{aligned}\int_{B_r}\left(|\nabla u|^2-|\nabla v|^2\right)&\leqslant\int_{B_r}q^2\left(\lambda^2(v)-\lambda^2(u)\right)\\ &\leqslant\int_{B_{\kappa r}\cap\{u>0\}}\Lambda q^2.\end{aligned}$$

Hence, setting $D^+ = (B_r \setminus B_{\kappa r}) \cap \{u > 0\}$,

$$\begin{aligned}\int_{B_{\kappa r}\cap\{u>0\}} (|\nabla u|^2 - \Lambda q^2) &\leq \int_{D^+} (|\nabla v|^2 - |\nabla u|^2) \\ &= \int_{D^+} \nabla(v-u)\cdot\nabla(u-v+2v) \\ &\leq 2\int_{D^+} \nabla v \cdot \nabla(v-u) \\ &\leq \liminf_{\varepsilon\to 0} 2\int_{D^+} \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u) \\ &= \liminf_{\varepsilon\to 0} 2\int_{\partial B_{\kappa r}\cap\{u>\varepsilon\}} (u-\varepsilon)|\nabla v_\varepsilon| \equiv M\end{aligned}$$

where in the last formula we have used the integration by parts

$$\int_{D_\varepsilon^+} \nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u) = \int_{\partial B_{\kappa r}} (u-\varepsilon)\left|\frac{\partial v_\varepsilon}{\partial \nu}\right|; \tag{3.2}$$

notice that $\partial v_\varepsilon/\partial\nu \leq 0$ on $\partial B_{\kappa r}$. Since $\partial B_{\kappa r}$ and $\partial\{u > \varepsilon\}$ form a corner at their intersection, one has to justify (3.2) by approximation. We shall do this later.

To estimate M we introduce the function w:

$$\begin{aligned}\Delta w = 0 &\quad \text{in } B_r \setminus B_{\kappa r}, \\ w = u &\quad \text{on } \partial B_r \cap \{u > \varepsilon\}, \\ w = \varepsilon &\quad \text{elsewhere on } \partial(B_r \setminus B_{\kappa r}).\end{aligned}$$

Clearly $w \geq v_\varepsilon$ and thus $|\nabla w| \geq |\nabla v_\varepsilon|$ on $\partial B_{\kappa r} \cap \{u > \varepsilon\}$. Since

$$|\nabla w| \leq \frac{C}{r} \fint_{\partial B_r} (u-\varepsilon)^+ \leq C\gamma \quad \text{on } \partial B_{\kappa r},$$

we get

$$|\nabla v_\varepsilon| \leq C\gamma \quad \text{on } \partial B_{\kappa r} \cap \{u > \varepsilon\}. \tag{3.3}$$

Hence

$$\begin{aligned}M &\leq C\gamma \int_{\partial B_{\kappa r}} u^+ \leq C\gamma\left(\int_{B_{\kappa r}} |\nabla u^+| + \frac{1}{r}\int_{B_{\kappa r}} u^+\right) \\ &\leq \frac{C\gamma}{|\Lambda|^{1/2} q_1} \int_{B_{\kappa r}} (|\nabla u^+|^2 + |\Lambda| q_1^2 I_{\{u^+>0\}}) \\ &\quad + \frac{C\gamma}{|\Lambda| q_1^2 r}\left(\sup_{B_{\kappa r}} u^+\right) \int_{B_{\kappa r}} |\Lambda| q_1^2 I_{\{u^+>0\}}.\end{aligned}$$

Since u is harmonic in $\{u > 0\}$, u^+ is subharmonic in Ω; therefore $\sup_{B_{\kappa r}} u^+ \leqslant C\gamma r$. Hence

$$\int_{B_{\kappa r}\cap\{u>0\}}\left(|\nabla u|^2 - \Lambda q_1^2\right) \leqslant \frac{C\gamma}{|\Lambda|^{1/2}q_1}\left(1 + \frac{\gamma}{|\Lambda|^{1/2}q_1}\right) \cdot \int_{B_{\kappa r}\cap\{u>0\}}\left(|\nabla u|^2 - \Lambda q_1^2\right).$$

Hence if $\gamma/(|\Lambda|^{1/2}q_1)$ is small enough then $u \leqslant 0$ in $B_{\kappa r}$.

It remains to justify (3.2). Approximate D_ε^+ by domains D_m by changing D_ε^+ near $\partial B_{\kappa r} \cap \partial\{u > \varepsilon\}$ so as to form a smooth boundary there. Denote the corresponding v_ε by $v_{\varepsilon m}$ ($v_{\varepsilon m} = \varepsilon$ on the modified boundary ∂D_m near $\partial B_{\kappa r} \cap \partial\{u > \varepsilon\}$). Then,

$$\begin{aligned} &Dv_{\varepsilon m} \to Dv_\varepsilon \quad \text{on } \partial B_{\kappa r}, \text{ away from } \partial\{u > \varepsilon\}, \\ &|Dv_{\varepsilon m}| \leqslant C \quad \text{on } \partial D_m, \text{ away from } \partial B_r \end{aligned} \tag{3.4}$$

(by (3.3)). Since (3.2) holds for $v_\varepsilon = v_{\varepsilon m}$, taking $m \to \infty$ and using (3.4), the assertion (3.2) for v_ε follows.

Theorem 3.1 may be considered as a *nondegeneracy* theorem. It implies

COROLLARY 3.2. *Suppose* $\Lambda < 0$. *If* $B_r \subset \Omega$ *with center in the free boundary* $\partial\{u > 0\}$ *then* $\frac{1}{r}\fint_{\partial B_r} u^+ \geqslant c$ $(c > 0)$; *c depends only on* $q_1^2 l_0$.

The analog of Theorem 3.1 and its corollary to the case $\Lambda > 0$ are obvious.

REMARK 3.1. If $\lambda^2(0) < \min\{\lambda_1^2, \lambda_2^2\}$ then the proof of Theorem 3.1 applies to both u^+ and u^-. Consequently, if $B_r \subset \Omega$ with center in the free boundary $\partial\{u > 0\}$ $(\partial\{u < 0\})$, then

$$\frac{1}{r}\fint_{\partial B_r} u^+ \geqslant c \qquad \left(\frac{1}{r}\fint_{\partial B_r} u^- \geqslant c\right)$$

where c is a positive constant depending only on $q_1^2\{\min(\lambda_1^2, \lambda_2^2) - \lambda^2(0)\}$.

4. Upper estimates on the averages. Let

$$\max\{\lambda_1^2, \lambda_2^2\} \leqslant l_1. \tag{4.1}$$

THEOREM 4.1. *Assume that* $\lambda(0) = \min(\lambda_1, \lambda_2)$. *There exists a positive constant C depending only on* q_2 *(in* (3.1)) *and* l_1 *such that, if* $B_r \subset \Omega$ *with center in* $\{u = 0\}$, *then*

$$\frac{1}{r}\left|\fint_{\partial B_r} u\right| \leqslant C. \tag{4.2}$$

We shall prove the theorem in case $\Lambda < 0$; the proof in case $\Lambda > 0$ is similar. Since $\Lambda < 0$, Δu is a (positive) measure (by Theorem 2.3). In order to prove the theorem we first estimate the measure Δu.

LEMMA 4.2. *If* $\Lambda < 0$ *and* $B_r \subset \Omega$ *then*

$$\Delta u(B_{r/2}) \leqslant Cr^{n-1}. \tag{4.3}$$

PROOF. Defining v as in (2.1) we have

$$\int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla v|^2 \leqslant \int_{B_r} \big(Q^2(v,x) - Q^2(u,x)\big) \leqslant Cr^n.$$

The left-hand side is equal to

$$\int_{B_r} \nabla(u-v)\cdot\nabla(u+v) = \int_{B_r} \nabla(u-v)\cdot\nabla u = \int_{B_r} (v-u)\Delta u$$

where Δu is a measure supported on $\{u=0\}$ (the continuity of $v-u$ is used in making sense out of the last integral); the last integral is equal to $\int_{B_r} v\Delta u$. Since $v \geqslant u$, also $v\Delta u \geqslant u\Delta u = 0$, and thus

$$\int_{B_{r/2}\cap\{u=0\}} v\Delta u \leqslant Cr^n. \tag{4.4}$$

We shall use the representation

$$u(x^0) = \int_{\partial B_r} P_{x^0}(y)u(y) - \int_{B_r} G_{x^0}(y)\Delta u(y) \tag{4.5}$$

where P_{x^0} and G_{x^0} are Poisson's kernel and Green's function (in B_r), respectively. This formula can be justified by approximating u by mollifiers $J_\varepsilon u$, applying the formula to $J_\varepsilon u$ at x^0 and taking $\varepsilon \to 0$. If $x^0 \in \{u=0\}$ then we obtain, from (4.5),

$$\int_{B_r} G_{x^0}(y)\Delta u(y) = \int_{\partial B_r} P_{x^0}(y)u(y) \tag{4.6}$$

and the right-hand side is precisely $v(x^0)$. Thus we can rewrite (4.4) in the form

$$\int_{B_{r/2}} \left(\int_{B_r} G_x(y)\Delta u(y) \right) \Delta u(x) \leqslant Cr^n.$$

Noting that $G_x(y) \geqslant cr^{2-n}$ if $x, y \in B_{r/2}$ $(c>0)$ we obtain $cr^{2-n}(\Delta u(B_{r/2}))^2 \leqslant Cr^n$, and the assertion (4.3) follows.

PROOF OF THEOREM 4.1. As before we take $\Lambda < 0$. We may assume that the center of B_r is in the origin. By (4.6),

$$\int_{\partial B_r} P_0 u = \int_{B_r} G_0(y)\Delta u(y). \tag{4.7}$$

Suppose first that Δu is smooth. Then

$$I \equiv \int_{B_r} G_0(y)\Delta u(y) = \int_0^r G(s)h(s)\,ds$$

with suitable functions G and h; $h(r) = r^{n-1}\int_{\partial B_1} \Delta u(r\xi)\,dH^{n-1}(\xi)$. By Lemma 4.2,

$$\int_0^s h(\tau)\,d\tau \leqslant Cs^{n-1}. \tag{4.8}$$

Hence,

$$I = \int_0^r G(s)\frac{d}{ds}\left(\int_0^s h(\tau)\,d\tau\right)ds = \left[G(s)\int_0^s h(\tau)\,d\tau\right]_0^r - \int_0^r G'(s)\int_0^s h(\tau)\,d\tau\,ds.$$

The expression in brackets vanishes at $s = r$ (since $G_0 = 0$ on ∂B_r) and at $s = 0$ (by (4.8) and $G(s) \leqslant Cs^{2-n}$). Hence,

$$\int_{B_r} G_0(y)\Delta u(y) \leqslant \int_0^r \frac{C}{s^{n-1}} Cs^{n-1}\,ds \leqslant Cr. \tag{4.9}$$

By using mollifiers $u_\varepsilon = u * \psi_\varepsilon$ we can establish the same estimate for the measure Δu. Here we use the estimate

$$\begin{aligned}\int_{B_r(x_0)} \Delta u_\varepsilon(x) &= \int_{B_r(x_0)}\int_{\{|y|<\varepsilon\}} \Delta u(x-y)\psi_\varepsilon(y)\,dy\\ &= \int_{\{|y|<\varepsilon\}}\int_{B_r(x_0)} \Delta u(x-y)\psi_\varepsilon(y)\,dy\\ &= \int_{\{|y|<\varepsilon\}} \Delta u(B_r(x_0-y))\psi_\varepsilon(y)\,dy \leqslant Cr^{n-1}.\end{aligned}$$

From (4.7) and (4.9) we see that $\frac{1}{r}\fint_{\partial B_r} u \leqslant C$. Since $u(0) = 0$ and u is subharmonic, the last integral is actually positive and therefore (4.2) follows.

5. Lipschitz continuity.

LEMMA 5.1. *Let u be any function in $C^0(B_R) \cap H^{1,2}(B_R)$, where B_r is a ball with radius r and center x^0, $u(x^0) = 0$, and u is harmonic in $B_R \setminus \{u = 0\}$. Set*

$$\phi(r) = \frac{1}{r^2}\int_{B_r} \rho^{2-n}|\nabla u^+|^2\,dx \cdot \frac{1}{r^2}\int_{B_r} \rho^{2-n}|\nabla u^-|^2\,dx$$

where $\rho = |x - x^0|$. Then $\phi(r) < \infty$ and $\phi(r)$ is increasing in r, $r \in (0, R)$.

We shall refer to this result as the *monotonicity lemma*.

PROOF. Set $S_r = \partial B_r$. We first assume that

$$\min_{S_r} u < 0 < \max_{S_r} u \quad \text{for all } r \in (0, R). \tag{5.1}$$

Notice that the distribution Δu^+ is a measure. Denote by v_m mollifiers of u^+. Then $\Delta v_m^2 = 2|\nabla v_m|^2 + 2v_m\Delta v_m \geqslant 2|\nabla v_m|^2$, so that

$$2\int_{B_r\setminus B_\varepsilon} |\nabla v_m|^2\rho^{2-n} \leqslant \int_{B_r\setminus B_\varepsilon} \Delta(v_m^2)\rho^{2-n} = r^{2-n}\int_{S_r}\frac{\partial}{\partial r}v_m^2 + (n-2)r^{1-n}\int_{S_r} v_m^2 - I_\varepsilon$$

where

$$I_\varepsilon = \varepsilon^{2-n}\int_{S_\varepsilon}\frac{\partial}{\partial r}v_m^2 + (n-2)\varepsilon^{1-n}\int_{S_\varepsilon} v_m^2.$$

Since $|Dv_m|$ is bounded, $I_\varepsilon \to (n-2)|S_1|v_m^2(0)$ as $\varepsilon \to 0$. Hence,

$$2\int_{B_r\setminus B_\varepsilon} |\nabla v_m|^2\rho^{2-n} \leqslant r^{2-n}\int_{S_r}\frac{\partial}{\partial r}v_m^2 + (n-2)r^{1-n}\int_{S_r} v_m^2.$$

Integrating with respect to r, $r_0 < r < r_0 + \delta$, and dividing by δ, and then letting $m \to \infty$, we obtain

$$\frac{2}{\delta}\int_{r_0}^{r_0+\delta} dr \int_{B_r \setminus B_\varepsilon} |\nabla u^+|^2 \rho^{2-n} \leq \frac{1}{\delta}\int_{r_0}^{r_0+\delta} r^{2-n}\, dr \int_{S_r} 2u^+ u_r^+ + \frac{n-2}{\delta}\int_{r_0}^{r_0+\delta} r^{1-n}\, dr \int_{S_r} (u^+)^2.$$

Taking $\delta \to 0$ we obtain for a.a. r_0

$$2\int_{B_{r_0}\setminus B_\varepsilon} |\nabla u^+|^2 \rho^{2-n} \leq r_0^{2-n}\int_{S_{r_0}} 2u^+ u_r^+ + (n-2) r_0^{1-n}\int_{S_{r_0}} (u^+)^2.$$

Hence, for a.a. r,

$$2\int_{B_r} |\nabla u^+|^2 \rho^{2-n} \leq r^{2-n}\int_{S_r} 2u^+ u_r^+ + (n-2) r^{1-n}\int_{S_r} (u^+)^2. \tag{5.2}$$

Since a similar inequality holds for u^-, it follows that $\psi(r)$ is finite.

Since $r \to \int_{S_r} |\nabla u^+|^2 \rho^{2-n}$ is in $L^1(0, R)$, we have

$$\frac{d}{dr}\int_{B_r} \rho^{2-n} |\nabla u^+|^2 = \int_{S_r} r^{2-n} |\nabla u^+|^2 \quad \text{a.e.}$$

It follows that a.e.

$$\phi'(r) = -\frac{4}{r^5}\int_{B_r} \rho^{2-n}|\nabla u^+|^2 \cdot \int_{B_r} \rho^{2-n} |\nabla u^-|^2 + \frac{1}{r^4}\int_{S_r} r^{2-n}|\nabla u^+|^2 \cdot \int_{B_r} \rho^{2-n}|\nabla u^-|^2 + \frac{1}{r^4}\int_{B_r}\rho^{2-n}|\nabla u^+|^2 \cdot \int_{S_r} r^{2-n}|\nabla u^-|^2. \tag{5.3}$$

We shall prove that $\phi'(r) \geq 0$ a.e. in $(0, R)$. By scaling, we may assume that $r = 1$.

Denote by $\nabla_\theta v$ the gradient of a function v on S_1. Denote by Γ_1 the support of u^+ on S_1, and by Γ_2 the support of u^- on S_1. By assumption,

$$\text{meas}(\Gamma_i) \neq 0 \quad \text{for } i = 1, 2, \ldots. \tag{5.4}$$

We introduce the constants

$$\frac{1}{\alpha_i} = \inf_{v \in H_0^{1,2}(\Gamma_i)} \frac{\int_{\Gamma_i} |\nabla_\theta v|^2}{\int_{\Gamma_i} v^2}.$$

For any $0 < \beta_1 < 1$ we can write

$$\int_{S_1}\left((u_r^+)^2 + \beta_1^2 |\nabla_\theta u^+|^2\right) \geq 2\left\{\int_{S_1} (u_r^+)^2 \cdot \int_{S_1} \beta_1^2 |\nabla_\theta u^+|^2\right\}^{1/2} \geq 2\frac{\beta_1}{\sqrt{\alpha_1}}\left\{\int_{S_1}(u_r^+)^2 \cdot \int_{S_1}(u^+)^2\right\}^{1/2} \geq \frac{2\beta_1}{\sqrt{\alpha_1}}\int_{S_1} |u^+ u_r^+|$$

and

$$\int_{S_1} (1-\beta_1^2) |\nabla_\theta u^+|^2 \geq \frac{1-\beta_1^2}{\alpha_1} \int_{S_1} (u^+)^2$$

Choosing

$$\frac{1-\beta_i^2}{\alpha_i} = (n-2)\frac{\beta_i}{\sqrt{\alpha_i}} \tag{5.5}$$

we find that

$$\int_{S_1} |\nabla u^+|^2 \geq \frac{\beta_1}{\sqrt{\alpha_1}} \left\{ \int_{S_1} 2|u^+ u_r^+| + (n-2)\int_{S_1} (u^+)^2 \right\}. \tag{5.6}$$

The relations (5.2) and (5.6) hold also for u^-. Comparing with (5.3) we see that $\phi'(r) \geq 0$ provided

$$\frac{\beta_1}{\sqrt{\alpha_1}} + \frac{\beta_2}{\sqrt{\alpha_2}} \geq 2. \tag{5.7}$$

We easily compute that the β_i satisfy (5.5) if

$$\frac{\beta_i}{\sqrt{\alpha_i}} = \frac{1}{2}\left\{ \left[(n-2)^2 + \frac{4}{\alpha_i} \right]^{1/2} - (n-2) \right\}.$$

If γ_i is defined by

$$\gamma_i(\gamma_i + n - 2) = 1/\alpha_i, \qquad \gamma_i > 0, \tag{5.8}$$

then we obtain

$$\frac{\beta_i}{\sqrt{\alpha_i}} = \gamma_i. \tag{5.9}$$

The set function γ_1 as a function of Γ_1 was studied by Sperner [**12**] and by Friedland and Hayman [**8**]. In [**12**] it is proved that $\gamma_1(E) \geq \gamma_1(E^*)$ where E, $E^* \subset S_1$ provided E^* is a spherical cap having the same $(n-1)$-dimensional Hausdorff measure as E. In [**8**] it is proved that $\gamma_1(E) \geq \psi(s)$ where $s = \text{meas}(E)/\text{meas}(S_1)$, and $\psi(s)$ is convex and decreasing:

$$\psi(s) = \begin{cases} \frac{1}{2}\log\frac{1}{4s} + \frac{3}{2} & \text{if } s < \frac{1}{4}, \\ 2(1-s) & \text{if } \frac{1}{4} < s < 1. \end{cases}$$

Setting $s_1 = \text{meas}(\Gamma_i)/\text{meas}(S_1)$, we then have

$$\gamma_1 + \gamma_2 \geq \psi(s_1) + \psi(s_2) \geq 2\psi[(s_1+s_2)/2] \geq 2\psi(1/2) = 2;$$

in view of (5.9), this completes the proof of (5.7), provided (5.1) is satisfied.

If (5.1) is not satisfied, let R_0 be the smallest value of r for which at least one of the inequalities in (5.1) is invalid. Suppose for definiteness that $\min_{S_{R_0}} u \geq 0$. Then

u^- is harmonic in $D = B_{R_0} \cap \{u < 0\}$, vanishing on ∂D; hence $u^- = 0$ in D, which gives $\phi(r) = 0$ if $0 < r \leq R_0$. Since $\phi'(r) \geq 0$ for a.a. $R_0 < r < R$ (by the previous proof), the proof of the lemma is complete.

We shall now use Theorem 4.1 in order to establish Lipschitz continuity for any minimizer u.

LEMMA 5.2. *Assume that* $\lambda(0) = \min(\lambda_1, \lambda_2)$. *Then for any domain* $D \Subset \Omega$ *there exists a positive constant* C *such that if* $B_r \subset D$ *with center in* $\{u = 0\}$ *then*

$$\frac{1}{r} \fint_{\partial B_r} |u| \leq C. \tag{5.10}$$

PROOF. By Green's formula $(0 < \alpha < 1)$

$$\begin{aligned} \fint_{\partial B_r} r^\alpha u^- &= \int_{B_r} G_0 \Delta(\rho^\alpha u^-) = -\int_{B_r} \nabla G_0 \cdot \nabla(\rho^\alpha u^-) \\ &= -\int_{B_r} \rho^\alpha \nabla G_0 \cdot \nabla u^- + c_n \int_{B_r} \rho^{1-n} \alpha \rho^{\alpha-1} u^- \equiv J_1 + J_2, \end{aligned} \tag{5.11}$$

and $G_0(\rho) = c\rho^{2-n}$, $c > 0$ (we take for definiteness $n \geq 3$).Clearly,

$$\begin{aligned} |J_1| &\leq C\left(\int_{B_r} \rho^{2-n} |\nabla u^-|^2\right)^{1/2} \left(\int_{B_r} \rho^{2\alpha-n}\right)^{1/2} \\ &\leq Cr^\alpha \left(\int_{B_r} \rho^{2-n} |\nabla u^-|^2\right)^{1/2}. \end{aligned}$$

Introducing the function $\phi_\varepsilon(r) = (r^\varepsilon/r)\fint_{\partial B_r} u^-$ $(0 < \varepsilon < \alpha)$ we also have

$$J_2 \leq c_n \alpha \int_0^r \rho^{\alpha-\varepsilon} \phi_\varepsilon(\rho) \leq \frac{c_n \alpha}{1 + \alpha - \varepsilon} r^{1+\alpha-\varepsilon} \sup_{\rho \leq r} \phi_\varepsilon(\rho);$$

notice that $\phi_\varepsilon(\rho)$ is bounded since u^- is Hölder continuous with any exponent < 1. Dividing both sides of (5.11) by $r^{1+\alpha-\varepsilon}$ we then have

$$\phi_\varepsilon(r) \leq c_n \alpha \sup_{\rho \leq r} \phi_\varepsilon(\rho) + \frac{cr^\varepsilon}{r}\left(\int_{B_r} \rho^{2-n} |\nabla u^-|^2\right)^{1/2}. \tag{5.12}$$

Similarly, if $\psi_\varepsilon(r) = (r^\varepsilon/r)\fint_{\partial B_r} u^+$ then

$$\psi_\varepsilon(r) \leq c_n \alpha \sup_{\rho \leq r} \psi_\varepsilon(\rho) + \frac{Cr^\varepsilon}{r}\left(\int_{B_r} \rho^{2-n} |\nabla u^+|^2\right)^{1/2}.$$

By Theorem 4.1 $\phi_\varepsilon(r) = \psi_\varepsilon(r) + O(r^\varepsilon)$. Hence,

$$\phi_\varepsilon(r) \leq c_n \alpha \sup_{\rho \leq r} \phi_\varepsilon(\rho) + \frac{Cr^\varepsilon}{r}\left(\int_{B_r} \rho^{2-n} |\nabla u^+|^2\right)^{1/2} + O(r^\varepsilon). \tag{5.13}$$

Taking the product of the left-hand sides of (5.12) and (5.13), we obtain

$$(\phi_\varepsilon(r))^2 \leq C\alpha^2 \left(\sup_{\rho \leq r} \phi_\varepsilon(\rho)\right)^2 + Cr^{2\varepsilon} + C\frac{r^{2\varepsilon}}{r^2}\left(\int_{B_r} \rho^{2-n} |\nabla u^+|^2 \cdot \int_{B_r} \rho^{2-n} |\nabla u^-|^2\right)^{1/2}.$$

Using Lemma 5.1 and choosing α small enough, we obtain $(\phi_\varepsilon(r))^2 \leqslant Cr^{2\varepsilon}$, C independent of ε. Hence, $\frac{1}{r}⨍_{\partial B_r} u^- \leqslant C$ and therefore, also, $\frac{1}{r}⨍_{\partial B_r} u^+ \leqslant C$.

THEOREM 5.3. *If* $\lambda(0) = \min(\lambda_1, \lambda_2)$, *then* u *is Lipschitz continuous in any compact subset of* Ω.

PROOF. Let K be a compact subset of Ω and introduce $d = \operatorname{dist}(K, \partial\Omega)$. For any $x \in \Omega$, $x \notin \{u = 0\}$, denote by $\rho = \rho(x)$ the distance from x to $\{u = 0\}$ and let x^0 be such that $\rho = |x - x^0|$, $u(x^0) = 0$. If $\rho > d/6$ then u is harmonic in $B_{d/6}(x)$, and thus $|Du(x)| \leqslant C/d$. Suppose next that $\rho(x) < d/6$.

Representing u by Poisson's formula, $u(x) = \int_{\partial B_\rho(x)} P_x(y) u(y)$, we conclude that

$$(5.14) \qquad |Du(x)| \leqslant \frac{C}{\rho} ⨍_{\partial B_\rho(x)} |u(y)| .$$

The function $|u|$ is subharmonic. Representing $|u(y)|$ $(y \in \partial B_\rho(x))$ by Green's function in $B_\sigma(x^0)$ we get

$$|u(y)| \leqslant ⨍_{\partial B_\sigma(x^0)} \tilde{P}_y(z) |u(z)|, \qquad 3\rho < \sigma < 5\rho;$$

thus, $|u(y)| \leqslant C⨍_{\partial B_\sigma(x^0)} |u(z)| \leqslant C\rho$ by Lemma 5.2. Substituting this into (5.14) we conclude that $|Du(x)| \leqslant C$ if $x \in K$, $u(x) \neq 0$. Since $u \in H^{1,2}_{\mathrm{loc}}(\Omega)$, $Du = 0$ a.e. on $\{u = 0\}$, and thus $Du \in L^\infty(K)$.

Another Proof of Theorem 5.3. We shall give another proof, also based on Theorem 4.1 and Lemma 5.1.

Suppose $0 \in \Omega$, $u(0) = M > 0$, and let x^0 be the nearest point to 0 on $\{u = 0\}$. We assume first that $|x^0| = 1$ and $B_2 \subset \Omega$. By Harnack's inequality $u > c_0 M$ in $B_{3/4}$ $(c_0 > 0)$ and therefore $⨍_{\partial B_1(x^0)} u^+ > cM$ $(c > 0)$. From Theorem 4.1 it follows that

$$(5.15) \qquad \int_{\partial B_1(x^0)} u^- > cM$$

with another $c > 0$, provided M is large enough.

Let $y \in \partial B_{1/2}$ be a point on $\overrightarrow{0x^0}$. Then $u > c_0 M > 0$ in $B_{1/4}(y)$. We shall use polar coordinates (r, ω) about y. Denote by Γ the set of ω's such that if $(r, \omega) \in \partial B_1(x^0)$ then $u(r, \omega) < 0$.

We integrate $u_r^-(r, \omega)$ over $(r, \omega) \in B_1(x^0)$, $\omega \in \Gamma$. Using (5.15) and the fact that $u > 0$ in $B_{1/4}(y)$ we obtain

$$(5.16) \qquad cM \leqslant \int_{\partial B_1(y)} u^- = \int_\Gamma d\omega \int u_r^- \leqslant |\Gamma|^{1/2} \left\{ \int_{B_1(x^0)} |\nabla u^-|^2 \right\}^{1/2}.$$

Next we integrate $u_r^+(r, \omega)$ and $(r, \omega) \in \{B_1(x^0) \setminus B_{1/4}(y)\}$, $\omega \in \Gamma$, and notice that $u^+(r, \omega) \geqslant c_0 M$ in $B_{1/4}(y)$. We obtain

$$(5.17) \qquad cM|\Gamma| \leqslant \int_\Gamma d\omega \int u_r^+ \leqslant |\Gamma|^{1/2} \left\{ \int_{B_1(x^0)} |\nabla u^+|^2 \right\}^{1/2}.$$

Taking the product of both sides of the inequalities in (5.16) and (5.17), we get

$$cM^4 \leqslant \int_{B_1(x^0)} |\nabla u^+|^2 \cdot \int_{B_1(x^0)} |\nabla u^-|^2.$$

Using Lemma 5.1 we then obtain $M \leqslant C$. We have thus proved that

(5.18) $$u(z) \leqslant C\rho(z) \qquad (\rho(z) = \operatorname{dist}(z, \{u = 0\}))$$

if $z = 0$, $\rho(z) = 1$, $u(z) > 0$, $B_{2\rho(z)}(z) \subset \Omega$. The proof for general z follows by considering $\tilde{u}(x) \equiv u(z + \rho(z)x)/\rho(z)$. From (5.18) we deduce that $|\nabla u(z)| \leqslant C$; the same estimate holds if $u(z) < 0$. The proof that $u \in C^{0,1}_{\mathrm{loc}}$ now readily follows.

6. Blow-up limits. The rest of this paper is devoted to the study of the free boundary. For definiteness we shall always assume that

(6.1) $$\Lambda < 0, \quad \lambda(0) = \lambda_1;$$

all the results obviously extend to the case $\Lambda > 0$, $\lambda(0) = \lambda_2$. When (6.1) holds the free boundary coincides with

(6.2) $$\Gamma^+ = \partial\{u > 0\}.$$

Indeed, for the remaining free boundary

(6.3) $$\Gamma^- = \partial\{u < 0\} \setminus \partial\{u > 0\},$$

we obviously have

(6.4) $$u \leqslant 0 \quad \text{in a neighborhood } N \text{ of } \Gamma^-.$$

But since $\lambda(0) = \lambda_1$, the minimizer u must be harmonic in N. Consequently, Γ^- is empty.

DEFINITION 6.1. A function u is called a minimizer (of J) in R^n if for any $B_r \subset R^n$ and for any $v \in H^{1,2}(B_r)$, $v = u$ on ∂B_r,

$$J_{B_r}(u) \leqslant J_{B_r}(v)$$

where $J_{B_r}(v)$ is the functional $J(v)$ with Ω replaced by B_r.

Suppose u is a minimizer, $u(x_k) = 0$, $x_k \to x_0 \in \Omega$, $\rho_k \downarrow 0$, and set

(6.5) $$u_k(x) = \frac{1}{\rho_k} u(x_i + \rho_k x).$$

We call $\{u_k\}$ a *blow-up sequence* with respect to $B_{\rho_k}(x_k)$. Since $|\nabla u_k(x)| \leqslant C$ in any bounded set and $u_k(0) = 0$, we have, for a subsequence,

(6.6) $$u_k(x) \to u_0(x) \quad \text{unformly in bounded sets,}$$
$$\nabla u_k \to \nabla u_0 \quad \text{weakly in } L^\infty_{\mathrm{loc}}(R^n);$$

u_0 is called a *blow-up limit*.

LEMMA 6.1. *There holds*

(6.7) $$\partial\{u_k > 0\} \to \partial\{u_0 > 0\} \quad \textit{locally in the Hausdorff metric},$$

(6.8) $$\nabla u_k \to \nabla u_0 \quad \textit{a.e. in } R^n.$$

PROOF. Suppose a ball $\overline{B}_r$ does not intersect $\partial\{u_0 > 0\}$. Then either $u_0 > 0$ in $\overline{B}_r$ or $u_0 \leqslant 0$ in $\overline{B}_r$. In the first case $u_k > 0$ in $\overline{B}_r$ if k is large enough. In the second case $\frac{1}{r}⨍_{\partial B_r} u_k^+ < \varepsilon$ for any $\varepsilon > 0$ if k is large enough, so that, by nondegeneracy, $u_k \leqslant 0$ in $B_{r/2}$.

In both cases we conclude that $B_{r/2}$ does not intersect $\partial\{u_k > 0\}$ if k is large enough.

Conversely, if B_r does not intersect $\partial\{u_k > 0\}$ for any large k then either $u_k > 0$ in B_r or $u_k \leqslant 0$ in B_r. In the first case u_k is harmonic in B_r and then so is u_0; thus either $u_0 > 0$ in B_r or $u_0 \equiv 0$ in B_r, so that B_r does not intersect $\partial\{u_0 > 0\}$. In the second case we have $u_0 \leqslant 0$ in B_r so that again B_r does not intersect $\partial\{u_0 > 0\}$.

To prove (6.8) notice that, in every compact subset of $\{u_0 \neq 0\}$, (6.8) is certainly valid. Next consider a density point x^0 of the set $\{u_0(x) = 0\}$. By the Lipschitz continuity of u_0, we then deduce that $|u_0| = o(r)$ in B_r, and therefore, $\frac{1}{r}⨍_{\partial B_r} u_0^+ = o(1)$ as $r \to 0$.

Since $u_k \to u_0$ uniformly in B_1, we get $\frac{1}{r}⨍_{\partial B_r} u_k^+ < \varepsilon$ for any small $\varepsilon > 0$, provided k is large enough; hence by nondegeneracy, $u_k \leqslant 0$ in B_r. But then (since $\lambda(0) = \lambda_1$) u_k is harmonic in B_r and then so is u_0. Consequently, $\nabla u_k \to \nabla u_0$ uniformly in $B_{r/2}$. We have thus proved that almost all the set $\{u_0 = 0\}$ can be covered by balls B_{r_i} with suitable centers such that $\nabla u_k \to \nabla u_0$ in each B_{r_i}. It follows that $\nabla u_k \to \nabla u_0$ a.e. in the set $\{u_0 = 0\}$. This completes the proof of (6.8).

LEMMA 6.2. *u_0 is a minimizer in R^n with respect to the function $Q_0(u, \lambda) = q(x_0)\lambda(u)$.*

Indeed, the proof is similar to the proof of Lemma 5.4 in [**1**]; that proof can be slightly simplified by using (6.8).

THEOREM 6.3. *Suppose $D \Subset \Omega$, $B_r \subset D$ with center x^0 in $\partial\{u > 0\}$. Then*

$$\frac{1}{r}⨍_{\partial B_r(x^0)} u \geqslant c, \qquad c > 0. \tag{6.9}$$

Strictly speaking, this result does not include Corollary 3.2 since the constant c in (6.9) depends also on D and on the Lipschitz coefficient of u.

PROOF. Suppose the assertion is not true. Then there exist points $x_m^0 \in D$ and $r_m \downarrow 0$ such that

$$\frac{1}{r_m}⨍_{\partial B_{r_m}(x_m^0)} u \to 0, \qquad x_m^0 \in \partial\{u > 0\}.$$

Setting $u_m(x) = u(x_m^0 + r_m x)/r_m$ we may suppose that $x_m^0 \to 0$, $u_m \to u_0$ uniformly in bounded sets. Then u_0 is subharmonic (since u_m is subharmonic) and $⨍_{\partial B_1} u_0 = 0 = u_0(0)$. By the maximum principle it then follows that u_0 is harmonic in B_1.

Now u_0 is a local minimizer and $0 \in \partial\{u_0 > 0\}$, by (6.7). It follows that the free boundary $\partial\{u_0 > 0\}$ is nonempty; this set must be piecewise analytic since u_0 is harmonic. But then Theorem 2.4 shows that $|\nabla u_0|^2$ has jump $\Lambda q^1(0)$ across smooth parts of the free boundary. Since, however, u_0 is harmonic, $|\nabla u_0|^2$ cannot have a jump, i.e., $\Lambda q^2(0) = 0$, a contradiction.

Consider a blow-up family

$$u_\varepsilon(x) = \frac{1}{\varepsilon}u(x^0 + \varepsilon x), \qquad x^0 \in \partial\{u > 0\}, \varepsilon > 0,$$

and let

$$I_\varepsilon(r) \equiv \frac{1}{r^4}\int_{B_r}\rho^{2-n}|\nabla u_\varepsilon^+|^2 \cdot \int_{B_r}\rho^{2-n}|\nabla u_\varepsilon^-|^2$$

$$= \frac{1}{(\varepsilon r)^4}\int_{B_{\varepsilon r}(x^0)}\rho^{2-n}|\nabla u^+|^2 \cdot \int_{B_{\varepsilon r}(x^0)}\rho^{2-n}|\nabla u^-|^2 \equiv \tilde{I}_{\varepsilon r}.$$

By Lemma 5.1, $\tilde{I}_\rho$ is an increasing function of ρ. Consequently there exists a nonnegative constant γ such that

$$(6.10) \qquad I_\varepsilon(r) \downarrow \gamma \quad \text{if } \varepsilon \downarrow 0.$$

Now take a sequence $\varepsilon = \varepsilon_k \downarrow 0$ such that

$$(6.11) \qquad u_{\varepsilon_k}(x) \to u_0(x) \quad \text{uniformly in bounded subsets of } R^n.$$

LEMMA 6.4. *If* (6.11) *holds then, as* $\varepsilon_k \downarrow 0$,

$$(6.12) \qquad I_{\varepsilon_k}(r) \to \frac{1}{r^4}\int_{B_r}\rho^{2-n}|\nabla u_0^+|^2 \cdot \int_{B_r}\rho^{2-n}|\nabla u_0^-|^2.$$

PROOF. By the Lipschitz continuity of u, $|\nabla u_\varepsilon^+| \leqslant C$, and by Lemma 6.1, $\nabla u_\varepsilon^\pm \to \nabla u_0^\pm$ a.e. Hence, (6.12) follows by the Lebesgue bounded convergence theorem.

COROLLARY 6.5. *For any blow-up limit* u_0 *of* u_ε *there holds*

$$(6.13) \qquad \frac{1}{r^4}\int_{B_r}\rho^{2-n}|\nabla u_0^+|^2 \cdot \int_{B_r}\rho^{2-n}|\nabla u_0^-|^2 = \gamma$$

for all $r > 0$.

LEMMA 6.6. (i) *If* $\gamma = 0$ *then* $u_0 \geqslant 0$ *in* R^n; (ii) *if* $\gamma > 0$ *and* $n = 2$ *then* $u_0(x) = \mu_2(x \cdot e)^+ - \mu_1(x \cdot e)^-$ *in* R^n *where* e *is a constant unit vector*, μ_i *are positive constants, and* $\mu_1^2 - \mu_2^2 = \Lambda q^2(x^0)$.

The function u_0 in (ii) is called a 2-*plane solution*; if $\mu_1 = 0$ or $\mu_2 = 0$ then we call it a 1-*plane solution*.

PROOF. If $\gamma = 0$ then either $u_0^+ = 0$ or $u_0^- = 0$ in R^n. Since u_0 is subharmonic and $u_0(0) = 0$, we conclude that $u_0 \geqslant 0$. To prove (ii) we check the proof of Lemma 5.1 and find that equality can hold in (6.13) only if equality holds in the various Cauchy-Schwarz inequalities and $s_1 = s_2 = 1/2$. Thus, with S_1 replaced by S_r,

$$|u_r^+| = Cu^+, \qquad u^+u_r^+ \geqslant 0, \ C \text{ constant},$$

$$\int (u_r^+)^2 = \beta_1^2 \int |\nabla_\theta u^+|^2, \quad \int |\nabla_\theta u^+|^2 = \frac{1}{\alpha_1 r^2}\int (u^+)^2.$$

It follows that $u_r^+ = cu^+/r$ ($c = c_n > 0$); a similar relation holds for u^-. Thus $u = r^b g(\theta)$ if $u \neq 0$. Since u is bounded, $b \geqslant 0$. By nondegeneracy and Lipschitz continuity, $b = 1$. Thus $u = rg(\theta)$ if $u \neq 0$ and then,

$$Ag + (n-1)g = 0 \quad \text{where } g(\theta) \neq 0, \tag{6.14}$$

where A is the Laplacian restricted to ∂B_1. If $n = 2$ then $Ag = g''$ and the assertions easily follow using Lemma 6.2 and Theorem 2.4.

REMARK 6.1. We do not know whether the isoperimetric inequality $\gamma_1(E) \geqslant \gamma_1(E^*)$ used in the proof of Lemma 5.1 is a strict inequality whenever E is not a spherical cap. If this is indeed the case then Lemma 6.6(ii) is valid for any $n \geqslant 2$. Indeed, from the proof of Lemma 5.1 we then conclude that, for any r, $u_0 = l_2(x \cdot e)^+ - l_1(x \cdot e)^-$ on ∂B_r where $e = e(r)$, $l_i = l_i(r) > 0$. Setting

$$\begin{aligned} f(r) &= l_1 e \quad \text{if } u_0 > 0, \\ &= l_2 e \quad \text{if } u_0 < 0, \end{aligned}$$

we have $\Delta(x \cdot f(r)) = 0$ on $\{u_0 \neq 0\}$, which gives

$$\sum x_i\left(f_i''(r) + \frac{n+1}{r} f_i'(r)\right) = 0$$

where $f = (f_1, \ldots, f_n)$. It follows that $f'' + (n+1)f'/r = 0$, or $f(r) = Cr^{-n-1} + c$ where C, c are constant vectors in any component of $\{u_0 \neq 0\}$. Since u_0 is bounded, $C = 0$ and the assertions in (ii) easily follow.

7. Properties of the free boundary.

THEOREM 7.1. *There exists a positive constant $c \in (0,1)$ such that for any ball $B_r \subset \Omega$ with center in $\partial\{u > 0\}$*

$$c \leqslant \frac{\mathfrak{L}^n(B_r \cap \{u > 0\})}{\mathfrak{L}^n(B_r)} \leqslant 1 - c. \tag{7.1}$$

PROOF. By nondegeneracy there exists a point $y \in \partial B_{r/2}$ with $u(y) \geqslant cr$. Since u is Lipschitz, $u(x) > 0$ in $B_{\kappa r}(y)$, for some small enough κ. This establishes the left-hand side of (7.1). To obtain the second inequality, let

$$\begin{aligned} \Delta v &= 0 \quad \text{in } B_r, \\ v &= u \quad \text{on } \partial B_r. \end{aligned}$$

Then $v \geqslant u$ in B_r and (cf. the proof of Theorem 2.1)

$$\int_{B_r \cap \{u \leqslant 0 < v\}} |\Lambda| \geqslant \int_{B_r} |\nabla(u-v)|^2 \geqslant \frac{c}{r^2}\int_{B_r} |u-v|^2 \geqslant \frac{c}{r^2} \fint_{B_{\kappa r}} |u-v|^2, \tag{7.2}$$

for any $0 < \kappa < 1$.

If $y \in B_{\kappa r}$ then (we take the center of B_r to be at the origin)

$$|v(y) - v(0)| \leqslant |y|\,|\nabla v| \leqslant \kappa r \frac{C}{r} \fint_{\partial B_r} |u| \leqslant C\kappa r,$$

and $v(0) = \fint_{\partial B_r} v = \fint_{\partial B_r} u$, $|u(y)| \leqslant C\kappa r$. It follows that $|v(y) - u(y)| \geqslant \fint_{\partial B_r} u - C\kappa r$. Recalling Theorem 6.3 we obtain

$$|v(y) - u(y)| \geqslant cr - C\kappa r \geqslant cr/2$$

if κ is small enough. Using this estimate in (7.2) we find that

$$\mathcal{L}^n(B_r \cap \{u \leqslant 0\}) \geqslant \frac{C}{r^2}\int_{B_{\kappa r}} c^2 r^2 \geqslant c r^n \qquad (c > 0).$$

Since $u^{\pm}$ are continuous and subharmonic, the measures $d\lambda^+ = \Delta u^+$ and $d\lambda^- = \Delta u^-$ are Radon measures supported on $\Omega \cap \partial\{u > 0\}$ and $\Omega \cap \partial\{u < 0\}$, respectively.

THEOREM 7.2. *For any $D \Subset \Omega$ there exist positive constants c, C such that, for any $B_r \subset D$ with center in $\partial\{u > 0\}$,*

$$cr^{n-1} \leqslant \int_{B_r} d\lambda^+ \leqslant Cr^{n-1}, \tag{7.3}$$

$$\int_{B_r} d\lambda^- \leqslant Cr^{n-1}. \tag{7.4}$$

PROOF. Let $x \in \partial\{u > 0\}$. For almost all r with $B_r(x) \subset \Omega$,

$$\int_{B_r(x)} d\lambda^+ = \int_{\partial B_r(x)} \nabla u^+ \cdot \nu \, dH^{n-1} \leqslant Cr^{n-1}$$

since u^+ is Lipschitz continuous. This proves the second inequality in (7.3). The proof of (7.4) is similar. The proof of the first inequality in (7.3) is similar to the proof given in [**1**, Theorem 4.3], with u replaced by u^+.

THEOREM 7.3 (REPRESENTATIVE THEOREM). (i) *If $D \Subset \Omega$ then*

$$H^{n-1}(D \cap \partial\{u > 0\}) < \infty.$$

(ii) *There exist Borel functions $q_u^{\pm}$ such that*

$$\Delta u^{\pm} = q_u^{\pm} H^{n-1} L \partial\{u > 0\}, \tag{7.5}$$

that is, for every $\zeta \in C_0^\infty(\Omega)$,

$$-\int_\Omega \nabla u^{\pm} \cdot \nabla\zeta = \int_{\Omega \cap \partial\{u>0\}} \zeta q_u^{\pm} \, dH^{n-1}. \tag{7.6}$$

(iii) *For any $D \Subset \Omega$ there exist positive constants c, C depending on D, Ω, the constant c in Corollary 3.2 and any bound on $|\nabla u|_{L^\infty(D)}$, such that for any ball $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$,*

$$c \leqslant q_u^+ \leqslant C, \tag{7.7}$$

$$cr^{n-1} \leqslant H^{n-1}(B_r(x) \cap \partial\{u > 0\}) \leqslant Cr^{n-1}, \tag{7.8}$$

$$0 \leqslant q_u^- \leqslant C. \tag{7.9}$$

PROOF. For any compact set $E \subset D \cap \partial\{u > 0\}$ and small r choose a covering of E with balls $B_r(y_i)$ such that $\sum I_{B_{2r}(y_i)} \leqslant C$. Choosing $x_i \in B_r(y_i) \cap E$ we have, by Theorem 7.2,

$$\sum_i r^{n-1} \leqslant C \sum_i \lambda^+(B_r(x_i)) \leqslant C\lambda^+(B_{4r}(E))$$

which gives

$$(7.10)\qquad H^{n-1}(E) \leqslant C\lambda^+(E).$$

Thus (i) holds and $H^{n-1}L(D \cap \partial\{u > 0\})$ is absolutely continuous with respect to λ^+.

Next, the support of λ^+ is contained in $\partial\{u > 0\}$ and, by Theorem 7.2,

$$(7.11)\qquad \lambda^+(B_r) \leqslant Cr^{n-1} \quad \text{for any ball } B_r \subset D;$$

from this it follows that $\lambda^+(E) \leqslant CH^{n-1}(E)$. We have thus shown that the Radon measure λ^+ is absolutely continuous with respect to the Radon measure

$$H^{n-1}L\partial\{u > 0\}$$

and vice versa. Setting $q_u^+ = d\lambda^+/dH^{n-1}\mathfrak{L}(\partial\{u > 0\})$ we see that (7.5) holds (for Δu^+), and (7.10) and (7.11) establish (7.7) and (7.8).

Using the assertion (i) we can now proceed with proving (ii) and (iii) for λ^- by the same proof as for λ^+.

Since $\partial\{u > 0\}$ has finite H^{n-1} measure, the set $A = \Omega \cap \{u > 0\}$ has finite perimeter locally in Ω, that is, $\mu_u \equiv -\nabla I_A$ is a Borel measure and the total variation $|\mu_u|$ is a radon measure. We denote by $\partial_{\text{red}}\{u > 0\}$ the reduced boundary of $\partial\{u > 0\}$.

THEOREM 7.4 (IDENTIFICATION THEOREM). *Let* $x_0 \in \partial_{\text{red}}\{u > 0\}$ *with*

$$(7.12)\qquad \theta^{*n-1}\big(H^{n-1}L\partial\{u > 0\}, x_0\big) \leqslant 1,$$

$$(7.13)\qquad \int_{B_r(x_0)\cap\partial\{u>0\}} |q_u^{\pm} - q_u^{\pm}(x_0)| = o(1), \qquad r \to 0.$$

(i) *If* $\gamma > 0$ (*in Corollary* 6.5) *and* $n = 2$, *then*

$$u(x_0 + x) = \mu_2(x \cdot e(x_0))^+ - \mu_1(x \cdot e(x_0))^- + o(|x|) \quad \text{as } |x| \to 0,$$

where $\mu_i > 0$, $\mu_1^2 - \mu_2^2 = \Lambda q^2(x_0)$, *and*

$$(\mu_2 - \mu_1)e(x_0) = (q_u^+(x_0) - q_u^-(x_0))\nu_u(x_0).$$

(ii) *If* $\gamma = 0$, *then*

$$u(x_0 + x) = q_u^+(x_0)\max\{-x \cdot \nu_u(x_0), 0\} + o(|x|)$$

as $|x| \to 0$, *and* $(q_u^+(x_0))^2 = (\lambda_2^2 - \lambda^2(0))q^2(x_0)$.

Here $\nu_u(x_0)$ *is the outward normal to* $\partial\{u > 0\}$ *at* x_0.

PROOF. Take a blow-up sequence $u_\varepsilon(x) = u(x_0 + \varepsilon x)/\varepsilon$ with $u_\varepsilon \to u_0$ uniformly in compact subsets. Then $\Delta u_\varepsilon \to \Delta u_0$ as distributions, and thus also as measures. From (7.5) we deduce that

$$\Delta u_\varepsilon^{\pm} = q_u^{\pm}(x_0 + \varepsilon x)H^{n-1}L\partial\{u^\varepsilon > 0\}.$$

If $\gamma > 0$ and $n = 2$ then, by Lemma 6.6,

$$u_0 = \mu_2(x \cdot e)^+ - \mu_1(x \cdot e)^- \qquad (e \text{ constant vector})$$

and therefore,

$$\Delta u_0^+ - \Delta u_0^- = (\mu_1 - \mu_2) e dH^{n-1} L \Pi_e$$

where Π_e is the hyperplane orthogonal to e. We thus conclude that

$$\left[q_u^+(x_0 + \varepsilon x) - q_u^-(x_0 + \varepsilon x)\right] dH^{n-1} L \partial\{u^\varepsilon > 0\} \to (\mu_1 - \mu_2) e dH^{n-1} L \Pi_e.$$

Since $x_0 \in \partial_{\text{red}}\{u > 0\}$ we have [**10**, Theorem 3.7]

$$dH^{n-1} L \partial\{u^\varepsilon > 0\} \to dH^{n-1} L \Pi_0$$

where $\Pi_0 = \{x; \nu_u(x_0) \cdot x = 0\}$. Recalling (7.12) we deduce that

$$(q_u^+(x_0) - q_u^-(x_0))\, dH^{n-1} L \Pi_0 = (\mu_2 - \mu_1) e dH^{n-1} L \Pi_e$$

so that

$$(\mu_2 - \mu_1) e = (q_u^+(x_0) - q_u^-(x_0)) \nu_u(x_0);$$

also $\mu_1^2 - \mu_2^2 = \Lambda q^2(x_0)$. Thus the μ_i and e are uniquely determined, independently of the blow-up sequence, and assertion (i) follows.

Consider next the case $\gamma = 0$. By Lemma 6.6 we then have $u_0 \geqslant 0$ for any blow-up limit of the u_ε. We can then proceed as in Theorem 4.8 of [**1**]. Thus, taking $\nu_u(x_0) = e_n$, the proof that $u_0 > 0$ if $x_n < 0$, $u_0 = 0$ if $x_n > 0$ is the same as in [**1**]. Next, setting

$$\mu_w = -\nabla I_{(\Omega \cap \{w > 0\})} \tag{7.14}$$

for any function w for which $\partial\{w > 0\}$ has finite H^{n-1} measure, we have, for every compact subset $E \subset B_r'$ (B_r' is the ball in R^{n-1}),

$$\begin{aligned} H^{n-1}(E) &= \mu_{u_\varepsilon^+}(Ex(-1,1)) \cdot e_n \leqslant H^{n-1}(\partial\{u_\varepsilon^+ > 0\} \cap (Ex(-1,1))) \\ &= H^{n-1}(\partial\{u_\varepsilon > 0\} \cap (Ex(-1,1))) \end{aligned}$$

and we can again proceed as in [**1**], thereby establishing that $u_0(x) = u_0^+(x) = -q_u(x_0) x_n$ if $x_n < 0$, and the proof of (ii) thereby follows; the last assertion in (ii) follows from Lemma 6.2 and Remark 2.1.

REMARK 7.1. From Theorem 7.1 it follows (by [**7**, 4.5.6(3)]) that

$$H^{n-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0.$$

From [**7**, 4.5.6(2), 2.9.8 and 2.9.9] applied to $H^{n-1} L \partial\{u > 0\}$ and the Vitali relation $\{(x, B_r(x))\colon x \in \partial\{u > 0\} \text{ and } B_r(x) \subset \Omega\}$ it follows that for H^{n-1} a.a. $x_0 \in \partial_{\text{red}}\{u > 0\}$ the assumptions (7.12) and (7.13) are satisfied. Thus Theorem 7.4 shows that for H^{n-1} a.a. $x \in \partial\{u > 0\}$ the free boundary in a neighborhood of x_0 is approximately a hyperplane.

REMARK 7.2. In special models arising in jet flows [**5, 6**] it has been shown that the free boundary is a continuous graph. In the next section we prove, more generally, that the free boundary is C^1 if $n = 2$.

8. Differentiability of the free boundary ($n = 2$). In this section we prove that, in case $n = 2$, the free boundary is continuously differentiable. The first lemma is valid for any $n \geqslant 2$. In proving it we shall use the fact that

(8.1) the sets $\{u > 0\}$ and $\{u < 0\}$ are connected to the boundary of Ω.

To show this, suppose K is a component of $\{u > 0\}$ which is not connected to the boundary. Then, by replacing u in K by 0 we obtain a new function $\tilde{u}$ with smaller functional $J(\tilde{u})$, which is a contradiction.

LEMMA 8.1. *If u and $\tilde{u}$ are both minimizers of J in a bounded domain D, and if $\tilde{u} > u$ on ∂D, then $\tilde{u} > u$ in $\{u \neq 0\}$.*

PROOF. Set $v_1 = \min\{u, \tilde{u}\}$ and $v_2 = \max\{u, \tilde{u}\}$. Then $v_1 = u$ on ∂D and therefore, $J(v_1) \geq J(u)$. Similarly, $v_2 = \tilde{u}$ on ∂D and therefore, $J(v_2) \geq J(\tilde{u})$. However, $J(v_1) + J(v_2) = J(u) + J(\tilde{u})$ as seen by writing explicitly the terms in each J. It follows that $J(v_1) = J(u)$.

Suppose $u(x^0) = \tilde{u}(x^0) \neq 0$ and $u - \tilde{u}$ changes sign in any neighborhood of x^0. Then v_1 is not harmonic in any neighborhood of x^0. We introduce the function w defined by

$$\Delta w = 0 \quad \text{in } B_r(x^0),$$
$$w = v_1 \quad \text{on } \partial B_r(x^0)$$

for some small $r > 0$, and $w = v_1$ in $D \setminus B_r(x^0)$. By the Dirichlet principle we find that $J(w) < J(v_1) = J(u)$, contradicting the minimality of u. Thus we conclude that either $\tilde{u} \geq u$ or $u \geq \tilde{u}$ in some neighborhood of x^0. Starting with x^0 near ∂D and recalling (8.1), we deduce that $\tilde{u} \geq u$ on the set $\{u \neq 0\}$; furthermore, by the strong maximum principle, $\tilde{u} > u$ in this set.

From now on we make the assumptions

$$n = 2, \quad q(x) \equiv 1. \tag{8.2}$$

For definiteness we shall also assume that $\Lambda < 0$. We denote points in R^2 by X or (x, y). Set $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

LEMMA 8.2. *For any $\varepsilon_0 > 0$, $\eta > 0$ there is a $\delta = \delta(\varepsilon_0, \eta) > 0$ such that for any minimizer u in the rectangle $I = \{-3 < x < 3, -1 < y < 1\}$ satisfying*

(i) *the free boundary contains* $(0, 0)$ *and lies in the strip* $\{|y| < \delta\}$,

(ii) $u(A) < -\eta$ *where* $A = (0, -\frac{1}{2})$,

the free boundary in $I_0 = \{-1 < x < 1, -1 < y < 1\}$ is a graph in any direction $\varepsilon e_2 \pm e_1$, $\varepsilon \geq \varepsilon_0$.

PROOF. Take a circle K_μ^1: $(x + 2)^2 + (y - \mu)^2 < \delta^{-3/2}$ with center $(-2, \mu)$ and radius $\delta^{-3/4}$ and increase μ from $-\infty$ until, at $\mu = \mu_1$, $\partial K_{\mu_1}^1$ touches the free boundary of u for the first time. Since $\partial K_{\mu_1}^1 \cap \{x = -2\}$ lies in $\{y < \delta\}$,

$$\partial K_{\mu_1}^1 \cap \left\{-3 < x < -\frac{5}{2}\right\} \quad \text{and} \quad \partial K_{\mu_1}^1 \cap \left\{-\frac{3}{2} < x < 3\right\}$$

both lie below $y = \delta - C\delta^{3/4}$ and thus, also below $y = -\delta$ if δ is small enough. Consequently, $\partial K_{\mu_1}^1 \cap \partial\{u > 0\}$ lies in $\{-\frac{5}{2} < x < -\frac{3}{2}\}$ and contains a point $E_1 = (x_1, y_1)$ with $-\frac{5}{2} < x_1 < -\frac{3}{2}$, $-\delta < y_1 < \delta$.

Similarly, we construct a circle $K_{\mu_2}^2$ whose closure intersects the free boundary only at points of $\partial K_{\mu_2}^2$ lying in $\{\frac{3}{2} < x < \frac{5}{2}\}$, and a point $E_2 = (x_2, y_2)$ on $\partial K_{\mu_2}^2 \cap \partial\{u > 0\}$, with $\frac{3}{2} < x_2 < \frac{5}{2}$, $-\delta < y_2 < \delta$; further, $K_{\mu_1}^1 \cap \{|y| \leq \delta\}$ and $K_{\mu_2}^2 \cap \{|y| \leq \delta\}$ are disjoint. We denote by σ the curve consisting of (i) three line segments on

$y = -\delta$, from $(-3, -\delta)$ to the left endpoint of $\{y = -\delta\} \cap \partial K_{\mu_1}^1$, from the right endpoint of $\{y = -\delta\} \cap K_{\mu_1}^1$ to the left endpoint of $\{y = -\delta\} \cap \partial K_{\mu_2}^2$ and from the right endpoint of $(y = -\delta) \cap \partial K_{\mu_2}^2$ to $(3, -\delta)$, and (ii) the arcs of $\partial K_{\mu_i}^i$ lying in $\{|y| < \delta\}$.

Denote by Σ_- the part of I lying below σ. Notice that $u < 0$ in Σ_-.

From assumption (ii) and Harnack's inequality we get

$$u(X) \leqslant -c\eta \operatorname{dist}(X, \sigma) \quad \text{if } X \in \Sigma_- \ (c > 0). \tag{8.3}$$

We next claim

(8.4) there exists a C^1 curve σ_i: $y = f_i(x)$ in I such that $E_i \in \sigma_i$ and $u > 0$ above σ_i (in I), for $i = 1, 2$; furthermore, $f_i'(x) - f_i'(x_i) \to 0$ as $x - x_i \to 0$, uniformly with respect to u.

Notice that σ and E_i depend on u and so does f_i. To prove (8.4) suppose first that there exist sequences $E_1 = E_1(m) = (x_1(m), y_1(m))$, $u = u_m$ and $Z_m = (\tilde{x}_m, \tilde{y}_m)$ with $u_m(Z_m) \leqslant 0$, such that $|Z_m - E_1(m)| \to 0$ and the angle between $\overrightarrow{E_1(m)Z_m}$ and the tangent to σ at $E_1(m)$ does not converge to zero as $m \to \infty$. Set $r_m = |Z_m - E_1(m)|$ and consider a blow-up sequence of u_m with respect to $B_{r_m}(E_1(m))$. Let w be a blow-up limit. We can rotate the coordinates in such a way that

$$w(x, y) \leqslant 0 \quad \text{if } y \leqslant 0, \tag{8.5}$$

and then $w(x_0, y_0) \leqslant 0$ for some point (x_0, y_0) with $y_0 > 0$. Consequently,

$$w \text{ is not a 2-plane solution.} \tag{8.6}$$

In view of (8.3) and the assumption $\Lambda < 0$, w does have two phases.

By Corollary 6.5 and Lemma 6.6,

$$w(X) = \alpha y + o(|X|) \quad \text{if } y < 0, |X| \to 0 \tag{8.7}$$

where α is determined by $\alpha^2(\alpha^2 + |\Lambda|) = \gamma$, $\gamma = \lim_{r \to 0} \psi(r)$, where

$$\psi(r) = \frac{1}{r^4} \int_{B_r} |\nabla w^+|^2 \cdot \int_{B_r} |\nabla w^-|^2.$$

Similarly, working with blow-up sequences $\frac{1}{m} w(mX)$ $(m \to \infty)$ we find that

$$w(X) = \beta y + o(|X|) \quad \text{if } y < 0, |X| \to \infty \tag{8.8}$$

where $\beta^2(\beta^2 + |\Lambda|) = \gamma_0$, $\gamma_0 = \lim_{r \to \infty} \psi(r)$. Since, by (8.6), w is not a 2-plane solution, Lemma 6.6 shows that $\gamma_0 > \gamma$ and, consequently,

$$\beta > \alpha. \tag{8.9}$$

Let $\Omega_R = \{w < 0\} \cap B_R$. If we formally apply Green's formula to w and $G = y/(x^2 + y^2) - y/R^2$ in $\Omega_R \setminus B_\varepsilon$, we obtain

$$\int_{\partial \Omega_R \setminus B_\varepsilon} (G w_\nu - w G_\nu) + \int_{\partial B_\varepsilon \cap \{w < 0\}} (G w_\nu - w G_\nu) = 0 \tag{8.10}$$

where ν is the inner normal. In order to justify (8.10) and make sense of the integrals over the free boundary we apply (7.6) with $u^- = w^-$ and $\zeta = \eta G$ where $\eta = \eta(r)$ is given by

$$\eta(r) = \begin{cases} 1 & \text{if } r \leqslant R, \\ 1 - (r-R)^2/\delta^2 & \text{if } R < r < R + \delta, \\ 0 & \text{if } r > R + \delta, \end{cases}$$

and then let $\delta \to 0$. We then obtain (8.10) with $\int G_\nu w = 0$ on the free boundary and $\int G w_\nu = -\int q_w^- G dH^1$ on the free boundary. By (8.5), $G \geqslant 0$ on the free boundary and therefore the last integral is nonnegative. We thus conclude from (8.10) that

$$(8.11) \qquad \int_{\partial B_r \cap \{w<0\}} w G_\nu \leqslant \int_{\partial B_\varepsilon \cap \{w<0\}} (G w_\nu - w G_\nu).$$

Using (8.8) we compute that

$$\int_{\partial B_R \cap \{w<0\}} w G_\nu = \int 2 \frac{\sin\theta}{R^2} (\beta y + o(R)) = 2\beta \int \sin^2\theta \, d\theta + \eta(R)$$

where $\eta(R) \to 0$ if $R \to \infty$. Similarly,

$$-\int_{\partial B_\varepsilon \cap \{w<0\}} w G_\nu = \alpha \int \sin^2\theta \, d\theta + \eta_0(\varepsilon)$$

where $\eta_0(\varepsilon) \to 0$ if $\varepsilon \to 0$. Finally, for a sequence $\varepsilon_i \to 0$ we have

$$\int_{\partial B_{\varepsilon_i} \cap \{w<0\}} G w_\nu = \alpha \int \sin^2\theta \, d\theta + \eta_1(\varepsilon_i)$$

with $\eta_1(\varepsilon_i) \to 0$ if $\varepsilon_i \to 0$. Indeed, this follows from

$$\frac{1}{\bar\varepsilon} \int_0^{\bar\varepsilon} d\varepsilon \int G w_\nu \, ds = \frac{1}{\bar\varepsilon} \int \sin\theta [w]_{r=\bar\varepsilon} \, d\theta + \frac{o(\bar\varepsilon)}{\bar\varepsilon}$$
$$= \alpha \int \sin^2\theta \, d\theta + \frac{o(\bar\varepsilon)}{\bar\varepsilon}.$$

Using the preceding estimates in (8.11) we get

$$2\alpha \int_{\pi + o(1)}^{2\pi + o(1)} \sin^2\theta \geqslant 2\beta \int_\pi^{2\pi} \sin^2\theta + o(1)$$

where $o(1) \to 0$ if $\varepsilon \to 0$, $R \to \infty$; this contradicts (8.9).

We have thus proved that there cannot exist sequences $E_1(m)$, Z_m, u_m as above. It follows that, for each u, $\{u < 0\} \cap \{x > x_1\}$ lies below a polygonal curve π_0 with sides $\overline{Z^m Z^{m+1}}$ having slope ϕ_m which decreases to the slope ϕ_∞ of σ at E_1, uniformly with respect to u, as $|Z^m - E_1| \to 0$. We modify π_0 near its vertices so as to obtain a C^1 curve $y = f_1(x)$ lying above π_0 with slope converging to ϕ_∞ as $x \downarrow x_1$. Similarly, we can construct $y = f_1(x)$ for $x < x_2$, and this completes the construction of σ_1 as asserted in (8.4). σ_2 is constructed similarly.

REMARK 8.1. The assertion (8.4) remains valid also if condition (ii) is dropped. Indeed, if in the previous proof w is a 2-phase solution, then the proof is the same. If, on the other hand, w is a 1-phase solution (and then $w \geqslant 0$ since $\Lambda < 0$) then we get a contradiction to Lemma 8.4 below; Lemma 8.4 is proved independently of Lemmas 8.1–8.3. This remark will be used in proving Lemma 8.11 (which is an extension of Lemma 8.2 to the case where condition (ii) is dropped).

Now consider in the strip $I^\delta = I \cap \{|y| < \sqrt{\delta}\} \cap \{x_1 < x < x_2\}$ the quotient difference $\Delta_{h,l}u$ of u in the direction l of $\varepsilon e_2 \pm e_1$, with increment h, where $0 < h < 2\delta$, i.e.,

$$\Delta_{h,l}u(X) = (u(X + hl) - u(X))/h.$$

We claim that

$$\Delta_{h,l}u \geqslant c > 0 \tag{8.12}$$

in I^δ. We first prove (8.12) on $y = \sqrt{\delta}$. If the assertion is not true then for sequences u_m, $X_m = (x_m, \delta_m^{1/2})$ with $\delta_m \to 0$ there holds $\Delta_{h_m,l_m}u_m(X_m) \to 0$, with $0 < h_m < 2\delta_m$ and $l_m \to l$, l in direction $\varepsilon e_2 \pm e_1$, $\varepsilon \geqslant \varepsilon_0$. Take a blow-up about free boundary points of u_m on $\{x = x_m\}$ with radii $\leqslant 2\delta_m^{1/2}$. Since the free boundary of u_m lies in $\{|y| < \delta_m\}$, the blow-up limit w is a 2-plane solution (we use here (8.3) and the assumption $\Lambda < 0$) and its free boundary is the x-axis. Since

$$\Delta_{h_m,l_m}u_m(X_m) \to \partial w(X_0)/\partial l$$

where $X_0 = (0, 1)$ and $\varepsilon \neq 0$, we get a contradiction.

Similarly, we can establish (8.12) on $y = -\sqrt{\delta}$. Consider now the quotient difference on the vertical line V_1 of ∂I^δ passing through E_1. If (8.12) does not hold on V_1, say

$$\Delta_{h_m,l_m}u_m(X_m) \to 0 \qquad (X_m \in V_1),$$

then we make a blow-up about E_1 with radii $r_m = \max\{h_m, |X_m - E_1|\}$. Recalling that near E_1 the free boundary lies between σ and σ_1 and using (8.4), we again deduce that the blow-up limit w is a 2-plane solution, with the x-axis as the free boundary; Further,

$$\frac{\partial w}{\partial l}(X_0) = 0 \qquad \text{if } h_m = o(|X_m - E_1|),$$
$$\Delta_{h_0,l_*}(X_0) = 0 \quad \text{if } |h_m| \geqslant c_0|X_m - E_1| \, (c_0 > 0)$$

for some h_0, X_0 and some l in direction $\varepsilon e_2 \pm e_1$, $\varepsilon \geqslant \varepsilon_0$. But this is impossible since w is a function of y only.

Having proved (8.12) on ∂I^δ we now translate u in the direction l by considering

$$u_\tau(X) = u(X + \tau l), \qquad \tau > 0.$$

In view of (8.12), $u_\tau > u$ on ∂I^δ if $0 < \tau < 2\delta$. Since $q(x) \equiv 1$, u_τ is a minimizer for the same functional J as u. Appealing to Lemma 8.1 we conclude that $u_\tau > u$ in $\{u \neq 0\}$, from which the assertion follows.

LEMMA 8.3. *Any global minimizer u with two phases must be a 2-plane solution.*

PROOF. For a sequence $m \to \infty$ we have $u_m(X) \equiv u(mX)/m \to v(X)$ where $v(X)$ is a 2-plane solution. Indeed,

$$\psi(r) \equiv \frac{1}{r^4}\int_{B_r} |\nabla u^-|^2 \int_{B_r} |\nabla u^+|^2 \uparrow \gamma \quad \text{as } r \uparrow \infty,$$

and since $\psi(r_0) > 0$ for some $r_0 > 0$ (since u has two phases) it follows that $\gamma > 0$. On the other hand, v satisfies (6.13) and, by Lemma 6.6(ii),

$$v(x, y) = \begin{cases} \mu_1 y & \text{if } y > 0, \\ \mu_2 y & \text{if } y < 0, \end{cases}$$

where $\mu_1 > 0$, $\mu_2 > 0$, $\mu_1^2 \mu_2^2 = \gamma$.

Given $\varepsilon_0 > 0$ and $\eta = \mu_2/2$, if m is large enough then the u_m restricted to I satisfy the conditions of Lemma 8.2 (recall that the lemma is valid uniformly with respect to the class of all minimizers u). Hence, the free boundary $\partial\{u_m > 0\}$ (for $m \geqslant m(\varepsilon_0)$) in I_0 is a graph in the direction $(\pm 1, \varepsilon)$ for any $\varepsilon \geqslant \varepsilon_0$. It follows that $\partial\{u > 0\} \cap \{|x| < m, |y| < m\}$ is a graph in any direction $(\pm 1, \varepsilon)$ where $\varepsilon \geqslant \varepsilon_0$. Since ε_0 can be chosen arbitrarily small (and $m \geqslant m(\varepsilon_0)$), $\partial\{u > 0\}$ must coincide with the x-axis. By uniqueness for the Cauchy-Kowalewski theorem u is thus linear in y for $y > 0$ and for $y < 0$.

LEMMA 8.4. *Any global minimizer u with one phase must be a* 1*-plane solution.*

Naturally, to exclude a trivial case we assume that $u \geqslant 0$ in R^2 with (say) $0 \in \partial\{u > 0\}$ and with $\lambda > 0$, where $J(u) = \int(|\nabla u|^2 + \lambda^2 I_{\{u>0\}})$.

PROOF. The function $|\nabla u|$ is subharmonic and $|\nabla u| = \lambda$ on $\partial\{u > 0\}$. Proceeding as in [**2**] (see also [**9**, p. 327]) we deduce that $|\nabla u|$ takes its maximum on the free boundary and, consequently, the free boundary is convex to $\{u > 0\}$. If $\partial\{u > 0\}$ is not a straight line then the blow-up limit of a subsequence of $u_m(X) = u(mX)/m$ converges to a minimizer v whose free boundary includes two rays forming an angle $\neq \pi$ at the origin; this contradicts the Cauchy-Kowalewski theorem since $u = 0$, $\partial u/\partial \nu = 0$ on each of these rays.

LEMMA 8.5. *For any $\gamma > 0$ and $C_0 > 0$ there is a $\delta = \delta(\gamma, C_0)$ such that if u is a minimizer in B_1 with $|\nabla u| \leqslant C_0$ then for any ball $B_\delta(X^0) \subset B_{1/2}$ with center in the free boundary, the γ-flatness condition holds, i.e., the free boundary of u in $B_\delta(X^0)$ lies in a strip with center X^0 and width 2γ.*

PROOF. If the assertion is not true then there is a sequence $B_{\delta_m}(X_m) \subset B_{1/2}$ with $\delta_m \to 0$ such that the flatness condition does not hold for some u_m; $X_m \in \partial\{u_m > 0\}$. A blow-up sequence with respect to $B_{\delta_m}(X_m)$ is convergent to a minimizer v in R^2 and the free boundary of v in $B_1(0)$ does not lie in a (2γ)-strip with 0 in the centerline of the strip. If v has two phases, this contradicts Lemma 8.3, whereas if v has one phase, Lemma 8.4 is contradicted.

LEMMA 8.6. *If u satisfies the γ-flatness condition in $B_1 = B_1(0)$ in direction* (0, 1) *and if*

$$u(A) > Mu(P) \quad \text{where } A = \left(0, \tfrac{1}{2}\right), P \in \{u > 0\} \cap B_{1/2}, \tag{8.13}$$

then, for some absolute constant C, $\operatorname{dist}(P, \partial\{u > 0\}) < 2\gamma + C/M$.

PROOF. By the flatness assumption $u > 0$ in $B_1 \cap \{y > \varepsilon/2\}$ for any $\varepsilon > 2\gamma$. Suppose $\operatorname{dist}(P, \partial\{u > 0\}) > \varepsilon$; then also $\operatorname{dist}(P, \{y < \varepsilon/2\}) > \varepsilon/2$. Applying Harnack's inequality in $B_1 \cap \{y > \varepsilon/2\}$ we get $u(P) > c\varepsilon u(A)$. Hence, by (8.13), $1/M > c\varepsilon$, i.e., $\varepsilon < 1/cM$.

LEMMA 8.7. *For* γ *sufficiently small let* $\delta = \delta(\gamma, C_0)$ *be as in Lemma* 8.5, *and let* $B_\delta(X_0)$ *be any ball in* $B_{1/2}$ *with* X_0 *for which the* γ*-flatness holds in the direction* (0, 1), *say, and* $u(A) > 0$ *where* $A = X_0 + (0, \delta/2)$. *Then*

$$u(A) \geqslant \gamma \sup_{B_{\delta/2}(X_0)} u. \tag{8.14}$$

PROOF. Take, for simplicity, $X_0 = 0$ and normalize by taking $\delta = 1$. Set $A_0 = A$. If the assertion (8.14) is not true then there exists a point $P_0 \in B_{1/2} \cap \{u > 0\}$ such that

$$u(P_0) > \frac{1}{\gamma} u(A_0). \tag{8.15}$$

By Lemma 8.6

$$\operatorname{dist}(P_0, \{y > \gamma\}) < C_0\gamma. \tag{8.16}$$

Let E be a point on the free boundary with

$$|E - P_0| < (C_0 + 2)\gamma. \tag{8.17}$$

By the γ-flatness about E, the direction of flatness ν_E at E differs from the direction (0, 1) by at most $C\gamma$.

We fix η small, to be determined later (independently of γ) and take $\gamma \ll \eta$. By Harnack's inequality in $B_1 \cap \{y > \eta/2\}$ we have

$$\frac{1}{N} u(A_0) \leqslant u(X) \leqslant N u(A_0) \quad \text{if } X \in B_{1/2} \cap \{y > \eta\} \tag{8.18}$$

where $N = N(\eta)$. Denoting by G the Green function for $-\Delta$ in $\tilde{B} \equiv B_{1/4}(E) \cap \{y > -2\gamma\}$, we can represent the subharmonic function u^+ at P_0 in the form

$$u(P_0) = -\int_{\partial \tilde{B}} \frac{\partial G}{\partial \nu} u^+ = -\int_S - \int_T$$

where $S = \partial B_{1/4}(E) \cap \{y > \eta\}$ and $T = \partial B_{1/4}(E) \cap \{y < \eta\} \cap \{(X - E) \cdot \nu_E > \gamma\}$ (notice that $u^+ = 0$ on $\partial \tilde{B} \cap \{(X - E) \cdot \nu_E \leqslant \gamma\}$ and, in particular, on $\partial \tilde{B} \cap \{y = -2\gamma\}$). Setting $\sigma(\eta) = \operatorname{meas}(T)$, we have $\sigma(\eta) \to 0$ if $\eta \to 0$.

By (8.17), $-\partial G(P_0, X)/\partial \nu \leqslant C\gamma$ if $X \in S \cup T$. Consequently, $-\int_T \leqslant C\gamma\delta(\eta)\sup_T u^+$ and (using (8.18))

$$-\int_S \leqslant N u(A_0)(1 - \sigma(\eta)) C\gamma.$$

Recalling (8.15) we conclude that

$$\frac{1}{\gamma} u(A_0) \leqslant C u(P_0) \leqslant N C\gamma u(A_0) + C\gamma\sigma(\eta) \sup_T u^+ .$$

Choosing η such that $2C\sigma(\eta) < 1$ we find that, provided $NC\gamma/2\gamma$, there holds $u(A_0)/\gamma^2 \leqslant \sup_T u^+$. Thus, there is a point $P_1 \in T$ such that

$$u(P_1) > \frac{1}{\gamma^2} u(A_0). \tag{8.19}$$

Let A_1 be the point in $B_{1/2}(E)$ such that $\overrightarrow{A_1E}$ is in the direction $-\nu_E$, with $|A_1 - E| = 1/8$. Then, by Harnack's inequality,

$$u(A_1) \leqslant Nu(A_0) < \frac{1}{\gamma} u(A_0) \tag{8.20}$$

with the same N as before (if η is small enough). The previous setting for A_0, P_0 occurs also for A_1, P_1 since, by (8.19) and (8.20),

$$u(P_1) > \frac{1}{\gamma^2 N} u(A_1) > \frac{1}{\gamma} u(A_1).$$

We can now repeat the previous proof with 0, A_0, P_0 replaced by E, A_1, P_1 and $B_{1/2}(0)$ replaced by $B_{1/4}(E)$. Thus there is a triple E_2, A_2, P_2 such that

$$u(P_2) > \frac{1}{\gamma} \frac{1}{\gamma^2 N} u(A_1)$$

and $u(P_2) > u(A_2)/\gamma > 0$, $E_2 \in \partial\{u > 0\}$, $\overrightarrow{E_2A_2}$ is in the direction ν_{E_2} of γ-flatness at E_2, $|A_2 - E_2| = \frac{1}{2} \cdot \frac{1}{4}$.

Continuing in this way, step by step, we construct a sequence (E_n, A_n, P_n) such that

$$u(P_n) > \frac{1}{\gamma^{n+1}} \frac{1}{N^{n-1}} u(A_{n-1}) \tag{8.21}$$

and $u(P_n) > u(A_n)/\gamma > 0$, $u(E_n) = 0$, $\overrightarrow{E_nA_n}$ is in the direction ν_{E_n} of γ-flatness about E_n, $|A_n - E_n| = \frac{1}{2}2^{-n}$. Recall that, by Harnack's inequality, $u(A_1) > u(A_0)/N$. Since the configuration of each pair A_n, A_{n-1}, with respect to the free boundary, is similar (after scaling) to that of A_1, A_0 (using the γ-flatness in each ball $B_{2^{-n}}(E_n)$ and the fact that the directions ν_{E_n}, $\nu_{E_{n-1}}$ differ by at most $C\gamma/2^n$), we also have, by Harnack's inequality, $u(A_n) > u(A_{n-1})/N$ (with N independent of n). Recalling (8.21) we obtain

$$u(P_n) \geqslant \frac{1}{N^{n-1}} \frac{1}{\gamma^{n+1}} \frac{1}{N^{n-1}} u(A_0) = \frac{1}{\gamma^2} \frac{1}{(\gamma N^2)^{n-1}} u(A).$$

Choosing $\gamma < N^2$ we conclude that $u(P_n) \to \infty$ if $n \to \infty$, which is impossible. This completes the proof of (8.14).

Lemma 8.7 extends to u^-, that is, if $A_* = X_0 - (0, \delta/2)$ then

$$u(A_*) < 0, \quad u^-(A_*) > \gamma \sup_{B_{\delta/2}(X_0)} u^-. \tag{8.22}$$

COROLLARY 8.8. *If γ is small enough, say $\gamma < \gamma_0$, then*

$$\int_{B_R(X_0)} |\nabla u^+|^2 \leqslant C(u^+(A))^2, \quad \int_{B_R(X_0)} |\nabla u^-|^2 \leqslant C\big(u^-(A_*)\big)^2 \tag{8.23}$$

where $C = C(\gamma_0)$ and $R = \delta(\gamma_0)/4$.

Indeed, introducing $G(X) = \log 2R/|X - X_0|$ in $B_{2R}(X_0)$, we have, by Green's formula,

$$-\int_{\partial B_{2R}(X_0)} (u^{\pm})^2 \frac{\partial G}{\partial \nu} = \iint_{B_{2R}(X_0)} \left[\Delta(u^{\pm})^2 G - (u^{\pm})^2 \Delta G\right]$$

$$= 2 \iint_{B_{2R}(X_0)} |\nabla u^{\pm}|^2 G \geqslant c \iint_{B_R(X_0)} |\nabla u^{\pm}|^2,$$

and the left-hand side is estimated by (8.14) and (8.22).

LEMMA 8.9. *If* $X_0 \in \partial\{u > 0\}$ *and*

$$\limsup_{X \to X_0} |\nabla u^-(X)| = \alpha, \quad \limsup_{X \to X_0} |\nabla u^+(X)| = \beta,$$

then

$$\alpha^2(\alpha^2 + |\Lambda|) \leqslant \frac{1}{R^4} \int_{B_R(X_0)} |\nabla u^-|^2 \cdot \int_{B_R(X_0)} |\nabla u^+|^2, \tag{8.24}$$

$$\beta^2(\beta^2 - |\Lambda|) \leqslant \frac{1}{R^4} \int_{B_R(X_0)} |\nabla u^-|^2 \cdot \int_{B_R(X_0)} |\nabla u^+|^2; \tag{8.25}$$

here u is any minimizer in $B_R(X_0)$.

PROOF. It suffices to prove (8.24). We take $X_0 = 0$ and $X_n \to 0$ with $|\nabla u^-(X_n)| \to \alpha$; we may suppose that $\alpha > 0$. Let Y_n be the nearest point to X_n on the free boundary. Consider a blow-up sequence with respect to $B_{r_n}(Y_n)$, $r_n = |X_n - Y_n|$. Since $\alpha > 0$ and $\Lambda < 0$, the blow-up limit has two phases and, by Lemma 8.3, it is a 2-plane solution with slopes α and $\bar{\alpha}$ satisfying $\alpha^2 - \bar{\alpha}^2 = \Lambda$. It easily follows that, as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon^2} \int_{B_\varepsilon} |\nabla u^-|^2 \to \alpha^2, \quad \frac{1}{\varepsilon^2} \int_{B_\varepsilon} |\nabla u^+|^2 \to \bar{\alpha}^2.$$

The assertion (8.24) now follows using the monotonicity lemma.

LEMMA 8.10. *Under the conditions of Corollary* 8.8 (*with* $\gamma < \gamma_0$),

$$|\nabla u^-(X)| \leqslant Cu^-(A_*)(u^+(A) + 1) \quad \text{in } B_{R/2}(X_0), \tag{8.26}$$

where C is a constant (*depending on* γ_0).

PROOF. The function $w = |\nabla u^-|$ is subharmonic. By Lemma 8.9 and Corollary 8.8,

$$\limsup_{X \to X_0} w(X) \leqslant C\left[u^+(A)u^-(A_*)\right]^{1/2}$$

where $A = A(X_0)$, $A_* = A_*(X_0)$. If γ is small enough then by Harnack's inequality

$$u^+(A(X_0)) \leqslant Cu^+(A), \quad u^-(A_*(X_0)) \leqslant Cu^-(A_*) \tag{8.27}$$

where A, A_* correspond to the free boundary point 0 and $X_0 \in B_R$. Hence,

$$\limsup_{X \in B_R,\ \mathrm{dist}(X, \partial\{u>0\}) \to 0} w(X) \leqslant C\left[u^+(A)u^-(A_*)\right]^{1/2}. \tag{8.28}$$

On the other hand, by Corollary 8.8 and (8.27),

$$\int_{B_R\cap\{u<0\}} w^2 \leqslant C\big(u^-(A_*)\big)^2.$$

Set $W=\max\{w, C[u^+(A)u^-(A_*)]^{1/2}\}$ in B_R. By (8.28), W is a continuous subharmonic function and, therefore, by elliptic estimates,

$$W^2(X)\leqslant C\int_{B_R} W^2 \leqslant Cu^+(A)u^-\big(A_*\big)+C\big(u^-\big(A_*\big)\big)^2,$$

and (8.26) follows.

LEMMA 8.11. *Lemma* 8.2 *remains true without the assumption* (ii).

PROOF. It suffices to establish that

$$\Delta_{h,l}u>0 \quad \text{on } \partial I^\delta \tag{8.29}$$

for all h, l, u. Suppose this is not true for a sequence u_m with $X=X_m$, $h=h_m$, $l=l_m$. If the intervals $\tilde{l}_m$: $(X_m, X_m+h_m l_m)$ lie in $\{u_m>0\}$ then we can proceed as before. Indeed, the blow-up limit w with respect to $B_{\delta_m^{1/2}}(X_m)$ (or $B_{r_m}(E_i)$, E_i depends on m) is either a 1-plane solution with $w\geqslant 0$ (since $\Lambda<0$) or a 2-plane solution and its free boundary is $\{y=0\}$ (here we use Remark 8.1); thus we get a contradiction as before.

If $\tilde{l}_m$ lies in $\{u_m<0\}$ and if a blow-up limit w turns out to be a 1-plane solution with $w=0$ if $\{y<0\}$, we do not get a contradiction. In order to derive a contradiction we shall work with $U_m=u_m/u_m^-(A_*)$ instead of u_m, where A_* is chosen as in Lemma 8.10 (A_* depends on m). Then $U_m(A_*)=-1$ and U_m^- is uniformly Lipschitz continuous (by Lemma (8.10)). Taking a blow-up limit W of U_m^- with respect to $B_{\delta_m^{1/2}}(X_m)$ (or $B_{r_m}(E_i)$) we find that the free boundary of W is $\{y=0\}$; hence, by Liouville's theorem (reflecting first W across $\{y=0\}$) $W\equiv cy$ if $y<0$ ($c>0$), and therefore, $\Delta_{h_m,l_m}U_m\geqslant c$ uniformly with respect to $\tilde{l}_m$ in $\{u_m<0\}$, that is,

$$\Delta_{h_m,l_m}u_m>cu_m\big(A_*\big)>0 \tag{8.30}$$

uniformly with respect to h_m, l_m, X_m.

It remains to establish uniform positivity (in the sense of (8.30)) in case $\tilde{l}_m$ lies partially in $\{u_m>0\}$ and partially in $\{u_m<0\}$. In this case we can write it as a disjoint union of intervals $\tilde{l}_m=l_m^1+l_m^2+l_m^3$ where $l_m^1\subset\{u_m>0\}$, $l_m^2\subset\{u_m<0\}$ and l_m^3 is an interval with endpoints on σ and σ_i. By Remark 8.1, $\operatorname{meas}(l_m^3)=o(h_m)$ and thus either $\operatorname{meas}(l_m^1)>ch_m$, or $\operatorname{meas}(l_m^2)>ch$, or both inequalities hold. By the previous arguments for $\tilde{l}_m$ in $\{u_m>0\}$ and for $\tilde{l}_m$ in $\{u_m<0\}$ we deduce that the incremental quotients $\Delta_{l_m^i}u$ with respect to l_m^i satisfy

$$\Delta_{l_m^1}u\geqslant c\operatorname{meas}\big(l_m^1\big)/\operatorname{meas}\big(\tilde{l}_m\big),$$

$$\Delta_{l_m^2}u\geqslant cu_m\big(A_*\big)\operatorname{meas}\big(l_m^2\big)/\operatorname{meas}\big(\tilde{l}_m\big).$$

Since also $\Delta_{l_m^3}u\geqslant 0$, the assertion (8.29) holds. We can now proceed as in Lemma 8.2 to complete the proof of Lemma 8.11.

THEOREM 8.12. *The free boundary $\partial\{u > 0\} \cap \Omega$ is continuously differentiable.*

PROOF. By Lemma 8.5, for any small $\gamma > 0$ there is a $\delta = \delta(\gamma) > 0$ ($\delta \downarrow 0$ if $\gamma \downarrow 0$) such that the γ-flatness condition holds in every ball B_δ with center in the free boundary. Take such a ball B_δ and suppose for simplicity that its center is at the origin and that the flatness direction is (0, 1). By Lemma 8.11 the free boundary in $B_{\delta/2}$ has the form $y = f(x)$ with $f(x)$ Lipschitz continuous.

Denote by $\gamma = \gamma(\delta)$ the inverse of the function $\delta = \delta(\gamma)$.

Take x_1, x_2 in $(-\delta/4, \delta/4)$ and set

$$r = |x_1 - x_2|, \quad X_i = (x_i, f(x_i)), \quad B_i = B_{2r}(X_i).$$

Each X_i must lie in the flatness strip of the disc B_j ($j \neq i$). Therefore, the angles between the directions of flatness at X_1 and X_2 are bounded by $C\gamma(r)$. It follows that $|f'(x_1) - f'(x_2)| \leq C\gamma(r)$ for any two points x_1, x_2 where $f(x)$ is differentiable. Thus $f'(x)$ has a continuous version.

The next result is concerned with the continuity of the normal derivative of u. Letting

$$\gamma = \lim_{r\uparrow\infty} \frac{1}{r^4} \int_{B_r(X_0)} |\nabla u^-|^2 \cdot \int_{B_r(X_0)} |\nabla u^+|^2$$

where X_0 is a free boundary point, we define $\beta = \beta(\gamma) > 0$ by $\beta^2(\beta^2 - |\Lambda|) = \gamma$ and denote by $\nu = \nu_{X_0}$ the normal to the free boundary at X_0 (pointing into $\{u > 0\}$).

THEOREM 8.13. *For any sector*

$$\Sigma_c = \{X; (X - X_0) \cdot \nu > c|X - X_0|\}, \qquad c > 0,$$

there holds $u_\nu(X) \to \beta$ if $X \in \Sigma_c$, $X \to X_0$.

PROOF. Let $X_m \in \Sigma_c$, $X_m \to X_0$ and take a blow-up sequence u_m with respect to $B_{|X_m - X_0|}(X_0)$. Then $u_m(X) \to v(X) = \beta y$ ($y > 0$) and $\partial u_m(X_m)/\partial\nu \to \partial v(Y_0)/\partial y$ since Y_0 lies in $\{y > 0\}$. Since $\partial v(Y_0)/\partial y = \beta$, the assertion follows.

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