Variational solution of fractional advection dispersion equations on bounded domains in $\mathbb{R}^d$

Vincent J. Ervin *

John Paul Roop †

Abstract

In this paper, we discuss the steady state Fractional Advection Dispersion Equation (FADE) on bounded domains in $\mathbb{R}^d$. Fractional differential and integral operators are defined and analyzed. Appropriate fractional derivative spaces are defined, and shown to be equivalent to the fractional dimensional Sobolev spaces. A theoretical framework for the variational solution of the steady state FADE is presented. Existence and uniqueness results are proven, and error estimates obtained for the Finite Element approximation.

Key words: Finite element method, fractional differential operator, fractional diffusion equation, fractional advection dispersion equation.

AMS Mathematics subject classification: 65N30, 35J99

1 Introduction

In this paper, we investigate the variational solution to the steady state fractional advection dispersion equation (FADE) in $\mathbb{R}^d$, defined by

$$-\int_{\|\boldsymbol{v}\|=1} D_{\boldsymbol{v}}\, a\, D_{\boldsymbol{v}}^{-\beta} D_{\boldsymbol{v}} u \, M(d\boldsymbol{v}) + \mathbf{b} \cdot \nabla u + cu = f, \tag{1.1}$$

where $0 \leq \beta < 1$, $\mathbf{b}(x, y)$ is the velocity of the fluid, $c(x, y)u$ represents a reaction-absorption term, f is a source term, a is the diffusivity coefficient, $M(d\boldsymbol{v})$ is a probability density

*Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975. email: vjervin@clemson.edu. Partially supported by the National Science Foundation under award number DMS-0410792.

†Corresponding author. Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123. email: jroop@vt.edu. Partially supported by the Air Force Office of Scientific Research under award number F49620-1-00-0299.

function (p.d.f.) on the unit sphere in $\mathbb{R}^d$, $D_{\boldsymbol{v}}$ is the directional derivative in the direction of the unit vector $\boldsymbol{v}$, and $D_{\boldsymbol{v}}^{-\beta}$ is the β order fractional integral, given by

$$D_{\boldsymbol{v}}^{-\beta}u(\boldsymbol{x}) = \int_0^\infty \frac{w^{\beta-1}}{\Gamma(\beta)} u(\boldsymbol{x} - w\boldsymbol{v})\, dw. \tag{1.2}$$

Our interest in (1.1) arises from the application of FADEs as a model for physical phenomena exhibiting *anomalous* diffusion, i.e. diffusion not accurately modeled by the usual advection dispersion equation. Anomalous diffusion has been used in modeling turbulent flow [3, 17], and chaotic dynamics of classical conservative systems [18]. An application of particular interest is that of contaminant transport in groundwater flow. In [1] the authors state that solutes moving through aquifers do not generally follow a Fickian second-order governing equation, because of large deviations from the stochastic process of Brownian motion. This give rise to *superdiffusive* motion. In [10] the authors derive a general FADE, which is equivalent to (1.1), by modeling the diffusion for which the probability density function governing the underlying jump process follows the form of an arbitrary multivariate stable law [16].

To date most solution techniques for equations involving fractional differential operators have exploited the properties of the Fourier and Laplace transforms of the operators to determine a classical solution. Finite difference have also been applied to construct numerical approximation [12]. Finite difference quotients for multidimensional fractional differential operators have also been derived [11]. Aside from [5, 6] we are not aware of any other papers in the literature which investigate the Galerkin approximation and associated error analysis for the FADE.

There are two properties of fractional differential operators which make the analysis of the variational solution to the FADE more complicated than that for the usual advection dispersion equation. These are
(i) fractional differential operators are <u>not</u> local operators, and
(ii) the adjoint of a fractional differential operator is <u>not</u> the negative of itself.
Because of (i), (ii), and the fact that the FADE in (1.1) contains a probability measure over the unit sphere in $\mathbb{R}^d$, the correct function space setting for the variational solution is not obvious. In our analysis we use the spaces $J^\mu_{L,\theta}$, $J^\mu_{S,\theta}$ which are direct generalizations of the left and symmetric fractional derivative spaces introduced in [5]. Additionally, we define a fractional derivative space $J^\mu_{L,M}$, whose definition involves the p.d.f. M, and show the equivalence of these spaces to the fractional Sobolev spaces H^μ_0.

For clarity of exposition we present the analysis for the fractional operators in $\mathbb{R}^2$. The generalization from $\mathbb{R}^2$ to higher dimensions is obvious. This paper is organized as follows. In Section 2, we discuss directional integral and directional differential operators in two dimensions. In Section 3, the fractional integral and fractional differential operators are defined in terms of the directional integral and directional differential operators and the p.d.f. M. Section 4 contains a derivation/motivation for using (1.1) as a model for *superdiffusion.* Section 5 contains definitions of fractional derivative spaces which form the functional setting for the analysis of FADEs. In Section 6, we analyze the steady-state two-dimensional FADE, establishing the existence and uniqueness of a variational solution in the fractional Sobolev space $H_0^\alpha(\Omega)$. Section 7 contains the analysis for the Finite Element approximation, with convergence results. For a discussion of the computational implementation of this method see [14].

2 Directional Integrals and Directional Derivatives

In this section we introduce directional integral and directional derivative operators, and establish properties of these operators. To this end, we associate with each unit vector $\boldsymbol{v} = [v_1, v_2]^t \in \mathbb{R}^2$ a unique angle $\theta \in [0, 2\pi)$ such that $\boldsymbol{v} = [\cos\theta, \sin\theta]^t$. Also, let $C_0^\infty(G)$ denote the set of all functions $u \in C^\infty(G)$ that vanish outside a compact subset K of G. The following analysis directly generalizes to higher dimensions.

Definition 2.1 ***[Directional Integral]*** *Let $\alpha > 0, \theta \in [0, 2\pi)$ be given. The α^{th} order fractional integral in the direction of θ is given by*

$$D_\theta^{-\alpha} u(x, y) := \int_0^\infty \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} u(x - \xi\cos\theta, y - \xi\sin\theta) d\xi.$$

Remark: We note that for special directions the directional integral operator is equivalent to the left and right Riemann-Liouville integral operators (see (A.1),(A.2)), i.e.

$$\begin{aligned}
D_0^{-\alpha} u(x, y) &= {}_{-\infty}D_x^{-\alpha} u(x, y), \\
D_{\pi/2}^{-\alpha} u(x, y) &= {}_{-\infty}D_y^{-\alpha} u(x, y), \\
D_\pi^{-\alpha} u(x, y) &= {}_{x}D_\infty^{-\alpha} u(x, y), \\
D_{3\pi/2}^{-\alpha} u(x, y) &= {}_{y}D_\infty^{-\alpha} u(x, y).
\end{aligned}$$

Theorem 2.1 *The directional integral satisfies the semi-group property*

$$D_\theta^{-\alpha} D_\theta^{-\beta} u(x, y) = D_\theta^{-\alpha-\beta} u(x, y), \quad \forall\ \alpha, \beta > 0,\ \theta \in [0, 2\pi),\ u \in L^p(\mathbb{R}^2), p \geq 1.$$

Proof: Using the definition of the directional integral,

$$D_\theta^{-\alpha} D_\theta^{-\beta} u(x, y) = \int_0^\infty \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty \frac{\nu^{\beta-1}}{\Gamma(\beta)} u(x - (\xi+\nu)\cos\theta, y - (\xi+\nu)\sin\theta) d\nu d\xi.$$

Setting $\eta = \xi + \nu$ in the inner integral, we have

$$\begin{aligned}
D_\theta^{-\alpha} D_\theta^{-\beta} u(x, y) &= \int_0^\infty \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} \int_\xi^\infty \frac{(\eta-\xi)^{\beta-1}}{\Gamma(\beta)} u(x - \eta\cos\theta, y - \eta\sin\theta) d\eta d\xi \\
&= \int_0^\infty u(x - \eta\cos\theta, y - \eta\sin\theta) \int_0^\eta \frac{\xi^{\alpha-1}(\eta-\xi)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} d\xi d\eta \\
&= \int_0^\infty u(x - \eta\cos\theta, y - \eta\sin\theta) k(\eta)\, d\eta, \qquad (2.1)
\end{aligned}$$

where

$$k(\eta) := \int_0^\eta \frac{\xi^{\alpha-1}(\eta-\xi)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\, d\xi. \tag{2.2}$$

Substituting $s = \xi/\eta$ in (2.2), we see that

$$\begin{aligned} k(\eta) &= \eta^{\alpha+\beta-1}\int_0^1 \frac{s^{\alpha-1}(1-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}ds \\ &= \frac{\eta^{\alpha+\beta-1}B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\ &= \frac{\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}, \end{aligned} \tag{2.3}$$

where $B(\alpha,\beta)$ denotes the beta function satisfying

$$B(\alpha,\beta) := \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The stated result then follows from (2.1), (2.3) and the definition of the directional integral. ∎

Next, we establish the adjoint property of the directional integral operators for colinear directions, i.e. $D_\theta^{-\alpha}$ and $D_{\theta+\pi}^{-\alpha}$.

Theorem 2.2 *For all $u, v \in L^2(\mathbb{R}^2), \alpha > 0, \theta \in [0, 2\pi)$,*

$$\left(D_\theta^{-\alpha}u(x,y), v(x,y)\right) = \left(u(x,y), D_{\theta+\pi}^{-\alpha}v(x,y)\right),$$

where $(\cdot,\cdot)$ denotes the usual inner product on $L^2(\mathbb{R}^2)$.

Proof: From the definition of the fractional integral, we have that

$$\left(D_\theta^{-\alpha}u(x,y), v(x,y)\right) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty \frac{\xi^{\alpha-1}}{\Gamma(\alpha)}u(x-\xi\cos\theta, y-\xi\sin\theta)v(x,y)d\xi\, dx\, dy.$$

Setting $\tilde{x} = x - \xi\cos\theta, \tilde{y} = y - \xi\sin\theta$, we have

$$\begin{aligned} \left(D_\theta^{-\alpha}u(x,y), v(x,y)\right) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty \frac{\xi^{\alpha-1}}{\Gamma(\alpha)}u(\tilde{x},\tilde{y})v(\tilde{x}+\xi\cos\theta, \tilde{y}+\xi\sin\theta)d\xi\, d\tilde{x}\, d\tilde{y} \\ &= \left(u(x,y), D_{\theta+\pi}^{-\alpha}v(x,y)\right). \end{aligned}$$

∎

Corollary 2.1 *For $\alpha > 0$, $u, v \in L^2(\Omega)$,*

$$\left(D_\theta^{-\alpha}u,\, v\right)_{L^2(\Omega)} = \left(u,\, D_{\theta+\pi}^{-\alpha}v\right)_{L^2(\Omega)}.$$

Proof: Let $\tilde{u}, \tilde{v}$ denote the extensions of u, v by zero outside of Ω. Then, Theorem 2.2 implies

$$\left(D_\theta^{-\alpha} u,\, v\right)_{L^2(\Omega)} = \left(D_\theta^{-\alpha} \tilde{u},\, \tilde{v}\right)_{L^2(\mathbb{R}^2)} = \left(\tilde{u},\, D_{\theta+\pi}^{-\alpha} \tilde{v}\right)_{L^2(\mathbb{R}^2)} = \left(u,\, D_{\theta+\pi}^{-\alpha} v\right)_{L^2(\Omega)} .$$

∎

Theorem 2.3 *The fractional directional integral operator $D_\theta^{-\alpha}$ satisfies the following Fourier transform property*

$$\mathcal{F}(D_\theta^{-\alpha} u(x,y)) = (i\omega_1 \cos\theta + i\omega_2 \sin\theta)^{-\alpha}\, \hat{u}(\omega_1, \omega_2),$$

where

$$\mathcal{F}\left(u(x,y)\right) \;=\; \int_{\mathbb{R}^2} e^{-i(\omega_1 x + \omega_2 y)} u(x,y)\, dx\, dy \;:=\; \hat{u}(\omega_1, \omega_2).$$

Proof: Introduce the linear mapping

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} . \tag{2.4}$$

Then,

$$\begin{aligned} D_\theta^{-\alpha} u(x,y) &= \int_0^\infty \frac{\omega^{\alpha-1}}{\Gamma(\alpha)} u(\tilde{x}\cos\theta - \tilde{y}\sin\theta - \omega\cos\theta, \tilde{x}\sin\theta + \tilde{y}\cos\theta - \omega\sin\theta)\, d\omega \\ &= \int_0^\infty \frac{\omega^{\alpha-1}}{\Gamma(\alpha)} u((\tilde{x}-\omega)\cos\theta - \tilde{y}\sin\theta, (\tilde{x}-\omega)\sin\theta + \tilde{y}\cos\theta)\, d\omega\ . \qquad (2.5) \end{aligned}$$

Let $\eta := \tilde{x} - \omega \quad \Leftrightarrow \quad \omega = \tilde{x} - \eta$. Rewriting (2.5), we have

$$D_\theta^{-\alpha} u(x,y) \;=\; \int_{-\infty}^{\tilde{x}} \frac{(\tilde{x}-\eta)^{\alpha-1}}{\Gamma(\alpha)} u(\eta\cos\theta - \tilde{y}\sin\theta, \eta\sin\theta + \tilde{y}\cos\theta)\, d\eta\ .$$

Let $v(\eta) := u(\eta\cos\theta - \tilde{y}\sin\theta, \eta\sin\theta + \tilde{y}\cos\theta)$. Then

$$\begin{aligned} D_\theta^{-\alpha} u(x,y) &= \int_{-\infty}^{\tilde{x}} \frac{(\tilde{x}-\eta)^{\alpha-1}}{\Gamma(\alpha)} v(\eta)\, d\eta \\ &= {}_{-\infty}D_{\tilde{x}}^{-\alpha} v(\tilde{x})\, . \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{F}(D_\theta^{-\alpha} u(x,y)) &= \int_{\mathbb{R}^2} e^{-i(\omega_1 x + \omega_2 y)} D_\theta^{-\alpha} u(x,y)\, dx\, dy \\ &= \int_{\mathbb{R}^2} e^{-i(\omega_1(\tilde{x}\cos\theta - \tilde{y}\sin\theta) + \omega_2(\tilde{x}\sin\theta + \tilde{y}\cos\theta))} D_\theta^{-\alpha} u(x,y)\, d\tilde{x}\, d\tilde{y} \\ &= \int_{\mathbb{R}^2} e^{-i((\omega_1\cos\theta + \omega_2\sin\theta)\tilde{x} + (-\omega_1\sin\theta + \omega_2\cos\theta)\tilde{y})} D_\theta^{-\alpha} u(x,y)\, d\tilde{x}\, d\tilde{y}\ . \end{aligned}$$

Let

$$\tilde{\omega}_1 := \omega_1 \cos\theta + \omega_2 \sin\theta \quad \text{and} \quad \tilde{\omega}_2 := -\omega_1 \sin\theta + \omega_2 \cos\theta\,.$$

Thus,

$$\begin{aligned}
\mathcal{F}(D_\theta^{-\alpha} u(x,y)) &= \int_{\mathbb{R}^2} e^{-i(\tilde{\omega}_1\tilde{x}+\tilde{\omega}_2\tilde{y})}\, D_\theta^{-\alpha} u(x,y)\, d\tilde{x}\, d\tilde{y} \\
&= \int_{\mathbb{R}} e^{-i\tilde{\omega}_2\tilde{y}} \int_{\mathbb{R}} e^{-i\tilde{\omega}_1\tilde{x}}\ {}_{-\infty}D_{\tilde{x}}^{-\alpha} v(\tilde{x})\, d\tilde{x}\, d\tilde{y} \\
&= \int_{\mathbb{R}} e^{-i\tilde{\omega}_2\tilde{y}}\ (i\tilde{\omega}_1)^{-\alpha} \mathcal{F}(v(\tilde{x}))\, d\tilde{y} \text{ (using (A.7))} \\
&= (i\tilde{\omega}_1)^{-\alpha} \int_{\mathbb{R}} e^{-i\tilde{\omega}_2\tilde{y}} \int_{\mathbb{R}} e^{-i\tilde{\omega}_1\tilde{x}} v(\tilde{x})\, d\tilde{x}\, d\tilde{y} \\
&= (i\tilde{\omega}_1)^{-\alpha} \int_{\mathbb{R}^2} e^{-i\tilde{\omega}_2\tilde{y}} e^{-i\tilde{\omega}_1\tilde{x}} u(x,y)\, d\tilde{x}\, d\tilde{y}.
\end{aligned}$$

Finally, using $\tilde{\omega}_1\tilde{x} + \tilde{\omega}_2\tilde{y} \ = \ \omega_1 x + \omega_2 y$, we obtain.

$$\begin{aligned}
\mathcal{F}(D_\theta^{-\alpha} u(x,y)) &= (i\tilde{\omega}_1)^{-\alpha} \int_{\mathbb{R}^2} e^{-i(\omega_1 x+\omega_2 y)} u(x,y)\, dx\, dy \\
&= (i\tilde{\omega}_1)^{-\alpha} \hat{u}(\omega_1,\omega_2) \\
&= (i\omega_1\cos\theta + i\omega_2\sin\theta)^{-\alpha} \hat{u}(\omega_1,\omega_2).
\end{aligned}$$

■

Next we introduce directional derivatives of arbitrary order.

Definition 2.2 *Let $n \in \mathbb{N}, \theta \in [0, 2\pi)$ be given. The n^{th} order derivative in the direction of θ is given by*

$$D_\theta^n u(x,y) := \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)^n u(x,y) = ([\cos\theta, \sin\theta]^t \cdot \nabla)^n u(x,y). \tag{2.6}$$

Definition 2.3 ***[Directional Derivative]*** *Let $\alpha > 0, \theta \in [0, 2\pi)$ be given. Let n be the smallest integer greater than α, $n-1 \le \alpha < n$, and define $\sigma = n - \alpha$. Then the α^{th} order directional derivative in the direction of θ is defined by*

$$D_\theta^\alpha u(x,y) := D_\theta^n D_\theta^{-\sigma} u(x,y).$$

The Fundamental Theorem of Calculus generalizes to the directional integral and derivative.

Lemma 2.1 *For $u \in L^p(\mathbb{R}^2)$, $p \ge 1$, we have*

$$D_\theta D_\theta^{-1} u(x,y) = u(x,y).$$

Proof: As before, define $\tilde{x}, \tilde{y}$ by the mapping (2.4). Then

$$D_\theta^{-1} u(x,y) = {}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y}),$$

where, $v(\tilde{x},\tilde{y}) = u(\tilde{x}\cos\theta - \tilde{y}\sin\theta, \tilde{x}\sin\theta + \tilde{y}\cos\theta)$. Then, using by the chain rule, we have

$$\begin{aligned} \frac{\partial}{\partial x}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right) &= \frac{\partial \tilde{x}}{\partial x}\frac{\partial}{\partial \tilde{x}}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right) + \frac{\partial \tilde{y}}{\partial x}\frac{\partial}{\partial \tilde{y}}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right) \\ &= \cos\theta\, v(\tilde{x},\tilde{y}) - \sin\theta \frac{\partial}{\partial \tilde{y}}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right). \end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial y}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right) &= \frac{\partial \tilde{x}}{\partial y}\frac{\partial}{\partial \tilde{x}}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right) + \frac{\partial \tilde{y}}{\partial y}\frac{\partial}{\partial \tilde{y}}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right) \\ &= \sin\theta\; (v(\tilde{x},\tilde{y})) + \cos\theta \frac{\partial}{\partial \tilde{y}}\left({}_{-\infty}D_{\tilde{x}}^{-1} v(\tilde{x},\tilde{y})\right). \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) with Definition 2.2 we obtain the stated result. ■

Theorem 2.4 *For $u \in L^p(\mathbb{R}^2)$, $p \geq 1$, we have*

$$D_\theta^\alpha D_\theta^{-\alpha} u(x,y) = u(x,y).$$

Proof: Taking $n \in \mathbb{N}$ such that $n-1 \leq \alpha < n$, with $\sigma = n - \alpha$, and using the semi-group property, we have

$$D_\theta^\alpha D_\theta^{-\alpha} u(x,y) = D_\theta^n D_\theta^{-\sigma} D_\theta^{-\alpha} u(x,y) = D_\theta^n D_\theta^{-n} u(x,y) = u(x,y),$$

by repeated application of Lemma 2.1. ■

The Fourier transform property also generalizes to the fractional directional derivative. We show this property holds over $C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}^2$.

Theorem 2.5 *For $u \in C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}^2$, we have*

$$\mathcal{F}(D_\theta^\alpha u(x,y)) = (i\omega_1\cos\theta + i\omega_2\sin\theta)^\alpha \hat{u}(\omega_1,\omega_2). \tag{2.9}$$

Proof: Firstly, note that

$$\begin{aligned} \mathcal{F}(D_\theta u(x,y)) &= \int_{\mathbb{R}^2} e^{-i(\omega_1 x + \omega_2 y)}\left(\cos\theta\frac{\partial u(x,y)}{\partial x} + \sin\theta\frac{\partial u(x,y)}{\partial y}\right) dA \\ &= \int_{\mathbb{R}} e^{-i\omega_2 y}\cos\theta \int_{\mathbb{R}} e^{-i\omega_1 x}\frac{\partial u(x,y)}{\partial x}\, dx\, dy \\ &\quad + \int_{\mathbb{R}} e^{-i\omega_1 x}\sin\theta \int_{\mathbb{R}} e^{-i\omega_2 y}\frac{\partial u(x,y)}{\partial y}\, dy\, dx. \end{aligned}$$

Integrating by parts, we have

$$\mathcal{F}(D_\theta u(x,y)) = (i\omega_1 \cos\theta + i\omega_2 \sin\theta)\hat{u}(\omega_1,\omega_2).$$

Then,
$$\begin{aligned}\mathcal{F}(D_\theta^\alpha u(x,y)) &= \mathcal{F}(D_\theta^n D_\theta^{-\sigma} u(x,y)) \\ &= (i\omega_1\cos\theta + i\omega_2\sin\theta)^n \mathcal{F}(D_\theta^{-\sigma}u(x,y)) \\ &= (i\omega_1\cos\theta + i\omega_2\sin\theta)^{n-\sigma}\hat{u}(\omega_1,\omega_2). \qquad \blacksquare\end{aligned}$$

3 Fractional Integrals and Fractional Derivatives

In this section we introduce the fractional integral and fractional derivative operators..

Definition 3.1 *Let $u : \mathbb{R}^2 \to \mathbb{R}$, $\alpha < 0$ ($\alpha > 0$) be given. Then the α order fractional integral (derivative) with respect to the measure M is defined as*

$$D_M^\alpha u(x,y) := \int_0^{2\pi} D_\theta^\alpha u(x,y)\, M(d\theta), \tag{3.1}$$

where $M(d\theta)$ is a probability measure on $[0, 2\pi)$.

Note: From (2.9) and (3.1), the α order fractional derivative operator with respect to M satisfies

$$\mathcal{F}(D_M^\alpha u(x,y)) = \left[\int_0^{2\pi} (i\omega_1\cos\theta + i\omega_2\sin\theta)^\alpha M(d\theta)\right]\hat{u}(\omega_1,\omega_2). \tag{3.2}$$

Remark: Definition 3.1 is equivalent to the definition of the fractional order derivative in [10],

$$\mathcal{F}(D_M^\alpha u(\mathbf{x})) = \left[\int_{\|\mathbf{v}\|=1} (i\boldsymbol{\omega}\cdot\mathbf{v})^\alpha M(d\mathbf{v})\right]\hat{u}(\boldsymbol{\omega}).$$

Definition 3.2 *Let $u : \mathbb{R}^d \to \mathbb{R}, \alpha > 0$ be given. Then the α order Riesz fractional integral is defined as [15]*

$$\mathcal{D}^{-\alpha}u(\mathbf{x}) = \frac{1}{\gamma_d(\alpha)}\int_{\mathbb{R}^d} |\mathbf{x}-\mathbf{y}|^{\alpha-d}u(\mathbf{y})d\mathbf{y}, \tag{3.3}$$

where

$$\gamma_d(\alpha) = \begin{cases} 2^\alpha \pi^{d/2}\,\Gamma(\frac{\alpha}{2})/\Gamma(\frac{d-\alpha}{2}), & \text{if } \alpha \neq d+2k, k \in \mathbb{N} \\ 1, & \text{if } \alpha = d+2k, k\in\mathbb{N} \end{cases}.$$

This definition stems from the facts that, [7, 15], $\mathcal{F}(-\Delta u(\mathbf{x})) = |\boldsymbol{\omega}|^2$ and

$$\mathcal{F}^{-1}(|\boldsymbol{\omega}|^{-\alpha}) = \frac{1}{\gamma_d(\alpha)}\begin{cases} |\mathbf{x}|^{\alpha-d}, & \text{if } \alpha\neq d+2k, \alpha\neq -2k, k\in\mathbb{N} \\ |\mathbf{x}|^{\alpha-d}\ln\frac{1}{|\mathbf{x}|}, & \text{if } \alpha = d+2k, k\in\mathbb{N}\end{cases}. \tag{3.4}$$

Correspondingly, a Riesz fractional order derivative may be defined as a power of the Laplace operator composed with a Riesz fractional integral operator.

Definition 3.3 *Let $u : \mathbb{R}^d \to \mathbb{R}, \alpha > 0$ be given, n the smallest integer greater than $\alpha/2$, ($n - 1 < \alpha/2 \leq n$,) and $\sigma = 2n - \alpha$. Then the α order Riesz fractional derivative is defined as*

$$\mathcal{D}^\alpha u := (-\Delta)^n \mathcal{D}^{-\sigma} u = \frac{(-\Delta)^n}{\gamma_d(\sigma)} \int_{\mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^{\sigma - d} u(\mathbf{y}) d\mathbf{y}. \tag{3.5}$$

Lemma 3.1 *For $u : \mathbb{R}^2 \to \mathbb{R}$ and M the uniform probability measure, the fractional integral (3.1) is a constant multiple of the Riesz fractional integral (3.5).*

Proof: If $M(d\theta)$ is constant, then $M(d\theta) = d\theta/(2\pi)$. From the definition of the directional integral, we have

$$D_M^\alpha u(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{r^{\alpha-1}}{\Gamma(\alpha)} u(x + r\cos\theta, y + r\sin\theta) dr d\theta.$$

Changing to Cartesian coordinates, this is just

$$D_M^\alpha u(\mathbf{x}) = \frac{1}{2\pi\Gamma(\alpha)} \int_{\mathbb{R}^2} |\mathbf{x} - \mathbf{y}|^{\alpha-2} u(\mathbf{y}) d\mathbf{y},$$

which is a constant multiple of the Riesz fractional integral. ∎

The following theorem shows that the Riesz fractional integral of a function of two variables which is constant with respect to one of the variables can be rewritten as a directional integral.

Theorem 3.1 *Let $u : \mathbb{R}^2 \to \mathbb{R}$, and $\tilde{u} : \mathbb{R} \to \mathbb{R}$ satisfy $u(x, y) = \tilde{u}(x)$. Then*

$$\mathcal{D}^{-\alpha} u(x, y) = \mathcal{D}^{-\alpha} \tilde{u}(x).$$

Proof: If u is constant in y, then changing to polar coordinates yields

$$\begin{aligned}
\mathcal{D}^{-\alpha} u(x, y) &= \frac{1}{\gamma_2(\alpha)} \int_0^{2\pi} \int_0^\infty r^{\alpha-1} u(x - r\cos\theta, y - r\sin\theta) dr d\theta \\
&= \frac{1}{\gamma_2(\alpha)} \int_0^{2\pi} \int_0^\infty r^{\alpha-1} \tilde{u}(x - r\cos\theta) dr d\theta \\
&= \frac{1}{\gamma_2(\alpha)} \int_{E_1} \int_x^\infty \frac{(\eta - x)^{\alpha-1}}{(\cos\theta)^\alpha} \tilde{u}(\eta) d\eta d\theta \\
&\quad + \frac{1}{\gamma_2(\alpha)} \int_{E_2} \int_{-\infty}^x \frac{(x - \eta)^{\alpha-1}}{(-\cos\theta)^\alpha} \tilde{u}(\eta) d\eta d\theta,
\end{aligned}$$

where $E_1 := (0, \pi/2) \cup (3\pi/2, 2\pi)$, $E_2 := (\pi/2, 3\pi/2)$. By symmetry, this can be rewritten as

$$\mathcal{D}^{-\alpha} u(x, y) = c_1 \int_{-\infty}^\infty |x - \eta|^{\alpha-1} \tilde{u}(\eta) d\eta,$$

where

$$c_1 := \frac{2}{\gamma_2(\alpha)} \int_0^{\pi/2} (\cos\theta)^{-\alpha} d\theta.$$

What remains is to show that $c_1 = 1/\gamma_1(\alpha)$.

Using the substitution $x = \sin^2\theta$, we have

$$\begin{aligned} c_1 &= \frac{1}{\gamma_2(\alpha)} \int_0^1 (1-x)^{-\alpha/2-1/2} x^{-1/2} dx \\ &= \frac{1}{\gamma_2(\alpha)} B(1/2-\alpha, 1/2). \end{aligned}$$

Using the definitions of the function $\gamma_n(\alpha)$, and the beta function $B(\cdot,\cdot)$, we have

$$c_1 = \left(\frac{\Gamma(1-\alpha/2)}{2^\alpha \pi \Gamma(\alpha/2)}\right)\left(\frac{\Gamma(1/2-\alpha/2)\Gamma(1/2)}{\Gamma(1-\alpha/2)}\right).$$

As $\Gamma(1/2) = \sqrt{\pi}$, the stated result follows. ■

4 Derivation of the FADE

In this section, we motivate the definition of the fractional advection dispersion equation via a continuous time random walk (CTRW) model. For the CTRW we consider the jump probability density function (p.d.f.) to be an arbitrary bivariate stable law. We then rewrite the FADE in terms of the composition of the traditional equation for conservation of mass, as well as a *fractional Fick's law.*

For a particle undergoing a CTRW, let $P(x, y, t)$ denote the p.d.f. describing the probability of the particle being at position (x, y) at time t. We denote by $f(\Delta x, \Delta y, \Delta t)$ the transitional probability density of the particle being displaced $(\Delta x, \Delta y)$ units over the time interval Δt. We assume that $f(\cdot,\cdot,\cdot)$ is both spatially and temporally independent. The Chapman-Kolmogorov equation gives

$$P(x, y, t+\Delta t) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-\xi, y-\psi, \Delta t) P(\xi, \psi, t)\, d\xi\, d\psi,$$

from which we obtain

$$\frac{P(x,y,t+\Delta t) - P(x,y,t)}{\Delta t} = \frac{1}{\Delta t}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f(x-\xi, y-\psi, \Delta t) P(\xi,\psi,t)\, d\xi\, d\psi - P(x,y,t)\right). \tag{4.1}$$

Taking the Fourier transform of both sides of (4.1) yields

$$\frac{\hat{P}(\omega_1,\omega_2,t+\Delta t) - \hat{P}(\omega_1,\omega_2,\Delta t)}{\Delta t} = \frac{1}{\Delta t}\left(\hat{f}(\omega_1,\omega_2,\Delta t)\hat{P}(\omega_1,\omega_2,t) - \hat{P}(\omega_1,\omega_2,t)\right). \tag{4.2}$$

If we view $f(\Delta x, \Delta y, \Delta t)$ as an arbitrary bivariate stable distribution with index $\alpha \neq 1$, [10], we have

$$\begin{aligned}
\hat{f}(\omega_1, \omega_2, \Delta t) &= \exp\left[-i\Delta t\,\mathbf{b}\cdot\boldsymbol{\omega} + a\,\Delta t\int_0^{2\pi}(i\omega_1\cos\theta + i\omega_2\sin\theta)^\alpha M(d\theta)\right] \\
&= 1 - i\Delta t\,\mathbf{b}\cdot\boldsymbol{\omega} + a\,\Delta t\int_0^{2\pi}(i\omega_1\cos\theta + i\omega_2\sin\theta)^\alpha M(d\theta) + o(\Delta t). \quad (4.3)
\end{aligned}$$

Substituting (4.3) into (4.2), using (3.2), and taking the limit as $\Delta t \to 0$, we obtain the two dimensional FADE

$$\frac{\partial P}{\partial t} = -\mathbf{b}\cdot\nabla P + a\,D_M^\alpha P. \quad (4.4)$$

We remark that the FADE in two dimensions can take a variety of forms depending upon the structure of the probability measure M. For example, if M is uniform on $[0, 2\pi]$, we obtain a FADE with a Riesz fractional derivative in the dispersive term. However, if M is discrete p.d.f. over the set $\{0, \pi/2, \pi, 3\pi/2\}$ with probabilities $\{p_1, p_2, p_3, p_4\}$ respectively, we obtain

$$D_M^\alpha P = (p_1\ {}_{-\infty}D_x^\alpha + p_2\ {}_{-\infty}D_y^\alpha + p_3\ {}_xD_\infty^\alpha + p_4\ {}_yD_\infty^\alpha)P,$$

which corresponds to the FADE presented in [9], and represents a continuous time random walk in which the jumps are restricted to the x (horizontal) and y (vertical) directions [10].

Note that for a random walk in one spatial dimension, M must be discrete with $P(\theta = 0) = p$ (jump to the left), $P(\theta = \pi) = q$ (jump to the right), with $p + q = 1$.

For a general FADE in two dimensions, consider the conservation of mass equation

$$\frac{\partial P}{\partial t} + \nabla\cdot\mathbf{F} + cu = f, \quad (4.5)$$

where $\mathbf{F}$ denotes mass flux, cu a reaction-absorption term, and f a source term. Observe that the dispersive term in (4.4) can be rewritten as

$$\begin{aligned}
D_M^\alpha P &= \int_0^{2\pi} D_\theta^\alpha P\ M(d\theta) \\
&= \int_0^{2\pi}(\nabla\cdot[\cos\theta, \sin\theta]^t D_\theta^{\alpha-1}P\ M(d\theta) \\
&= \nabla\cdot\left(\int_0^{2\pi}[\cos\theta D_\theta^{\alpha-1}P, \sin\theta D_\theta^{\alpha-1}P]^t M(d\theta)\right).
\end{aligned}$$

Thus, we can view the fractional advection dispersion equation (4.4) as (4.5) with the relationship

$$\mathbf{F} = \mathbf{F}_d$$

where $\mathbf{F}_d$ denotes the *fractional Fick's law*

$$\mathbf{F}_d = -a\int_0^{2\pi}[\cos\theta D_\theta^{\alpha-1}P, \sin\theta D_\theta^{\alpha-1}P]^t M(d\theta) + \mathbf{b}P, \quad (4.6)$$

$\mathbf{b}$ is the velocity of the fluid, and a denotes the coefficient of diffusivity.

Thus, we take as the general form for the time dependent fractional advection dispersion equation in two dimensions

$$\frac{\partial P}{\partial t} - \nabla \cdot \left(a \int_0^{2\pi} [\cos\theta D_\theta^{\alpha-1} P, \sin\theta D_\theta^{\alpha-1} P]^t M(d\theta) \right) + \mathbf{b} \cdot \nabla P + cu = f. \tag{4.7}$$

Note that if $\alpha = 2$, then the traditional advection dispersion equation is obtained, i.e.

$$\frac{\partial P}{\partial t} - \nabla \cdot A \nabla P + \mathbf{b} \cdot \nabla P + cu = f,$$

where

$$A = a \int_0^{2\pi} \Sigma M(d\theta),$$

with

$$\Sigma = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

represents the covariance of the distribution.

5 Fractional Derivative Spaces

In this section we function spaces needed for the analysis of the variational solution to FADEs. We show that these spaces equivalence to the fractional order Sobolev spaces. The following lemma is helpful in establishing equivalence of the spaces.

Lemma 5.1 *Let $\alpha > 0$ be given. Then for each $\theta \in [0, 2\pi)$,*

$$\left(D_\theta^\alpha u, D_{\theta+\pi}^\alpha u \right) = cos(\pi\alpha) \| D_\theta^\alpha u \|_{L^2(\mathbb{R}^2)}^2. \tag{5.1}$$

Proof: We will make use of the property that for $\xi \in \mathbb{R}$

$$\overline{(i\xi)^\alpha} = \begin{cases} \exp(-i\pi\alpha)\ \overline{(-i\xi)^\alpha} & \text{if } \xi \geq 0 \\ \exp(i\pi\alpha)\ \overline{(-i\xi)^\alpha} & \text{if } \xi < 0 \end{cases}. \tag{5.2}$$

($\overline{}$ denotes complex conjugate.)

We have,

$$\begin{aligned} \left(D_\theta^\alpha u, D_{\theta+\pi}^\alpha u \right) &= \int_{\mathbb{R}^2} (i(\omega_1\cos\theta + \omega_2\sin\theta))^\alpha\, \hat{u}(\boldsymbol{\omega})\ \overline{(-i(\omega_1\cos\theta + \omega_2\sin\theta))^\alpha\, \hat{u}(\boldsymbol{\omega})} dA \\ &= \int_{E_1} (i(\omega_1\cos\theta + \omega_2\sin\theta))^\alpha\, \hat{u}(\boldsymbol{\omega})\ \overline{(-i(\omega_1\cos\theta + \omega_2\sin\theta))^\alpha\, \hat{u}(\boldsymbol{\omega})}\, dA \\ &\quad + \int_{E_2} (i(\omega_1\cos\theta + \omega_2\sin\theta))^\alpha\, \hat{u}(\boldsymbol{\omega})\ \overline{(-i(\omega_1\cos\theta + \omega_2\sin\theta))^\alpha\, \hat{u}(\boldsymbol{\omega})}\, dA, \end{aligned}$$

where

$$E_1 := \{(\omega_1, \omega_2) \in \mathbb{R}^2 |\, \omega_1 \cos\theta + \omega_2 \sin\theta \geq 0\},$$
$$E_2 := \{(\omega_1, \omega_2) \in \mathbb{R}^2 |\, \omega_1 \cos\theta + \omega_2 \sin\theta < 0\}.$$

Using (5.2), this becomes

$$\begin{aligned}\left(D_\theta^\alpha u, D_{\theta+\pi}^\alpha u\right) &= \exp(-i\pi\alpha) \int_{E_2} |(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})|^2 \, dA \\ &\quad + \exp(i\pi\alpha) \int_{E_1} |(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})|^2 \, dA \\ &= \cos(\pi\alpha) \int_{\mathbb{R}^2} |(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})|^2 \, dA \\ &\quad + i \sin(\pi\sigma) \left(\int_{E_1} |(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})|^2 \, dA \right. \\ &\quad \left. - \int_{E_2} |(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})|^2 \, dA \right). \end{aligned} \tag{5.3}$$

For $f(\boldsymbol{x}) \in \mathbb{R}$, we have that $\overline{\mathcal{F}(f)(-\boldsymbol{\omega})} = \mathcal{F}(f)(\boldsymbol{\omega})$. Thus,

$$\begin{aligned}&(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega}) \overline{(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})} \\ &= \overline{(-i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(-\boldsymbol{\omega})} (-i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(-\boldsymbol{\omega}) \\ &= |(-i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(-\boldsymbol{\omega})|.\end{aligned}$$

Integrating over E_1, we have

$$\begin{aligned}\int_{E_1} |(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})|^2 \, dA &= \int_{E_1} |(-i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(-\boldsymbol{\omega})|^2 \, dA \\ &= \int_{E_2} |(i(\omega_1 \cos\theta + \omega_2 \sin\theta))^\alpha \hat{u}(\boldsymbol{\omega})|^2 \, dA.\end{aligned} \tag{5.4}$$

Therefore, combining (5.3) and (5.4), we have the stated result. ■

Next, we introduce two spaces which depend on the θ-directional derivative and then show that these spaces are equivalent.

Definition 5.1 *Let $\alpha > 0$, $\theta \in [0, 2\pi)$ be given. Define the semi-norm*

$$|u|_{J^\alpha_{L,\theta}(\mathbb{R}^2)} := \|D^\alpha_\theta u\|_{L^2(\mathbb{R}^2)},$$

and norm

$$\|u\|_{J^\alpha_{L,\theta}(\mathbb{R}^2)} := (\|u\|^2_{L^2(\mathbb{R}^2)} + |u|^2_{J^\alpha_{L,\theta}(\mathbb{R}^2)})^{1/2}, \tag{5.5}$$

and let $J^\alpha_{L,\theta}(\mathbb{R}^2)$ denote closure of $C^\infty(\mathbb{R}^2)$ with respect to $\|\cdot\|_{J^\alpha_{L,\theta}(\Omega)}$.

Definition 5.2 *Let $\alpha > 0$, $\alpha \neq n - 1/2, n \in \mathbb{N}, \theta \in [0, 2\pi)$ be given. Define the semi-norm*

$$|u|_{J^\alpha_{S,\theta}(\mathbb{R}^2)} := |\left(D^\alpha_\theta u, D^\alpha_{\theta+\pi} u\right)_{L^2(\mathbb{R}^2)}|^{1/2},$$

and norm

$$\|u\|_{J^\alpha_{S,\theta}(\mathbb{R}^2)} := (\|u\|^2_{L^2(\mathbb{R}^2)} + |u|^2_{J^\alpha_{S,\theta}(\Omega)})^{1/2}, \tag{5.6}$$

and let $J^\alpha_{S,\theta}(\mathbb{R}^2)$ denote the closure of $C^\infty(\mathbb{R}^2)$ with respect to $\|\cdot\|_{J^\alpha_{S,\theta}(\mathbb{R}^2)}$.

Lemma 5.2 *Let $\alpha > 0$, $\alpha \neq n - 1/2, n \in \mathbb{N}, \theta \in [0, 2\pi)$ be given. Then the spaces $J^\alpha_{L,\theta}(\mathbb{R}^2)$, $J^\alpha_{L,\theta+\pi}(\mathbb{R}^2)$, and $J^\alpha_{S,\theta}(\mathbb{R}^2)$ are equal, with equivalent semi-norms and norms.*

Proof: This result follows directly from the relation (5.1). ∎

Let Ω denote a bounded open, convex set in $\mathbb{R}^2$. We now show that the same norm equivalence holds for functions defined with support in Ω.

Definition 5.3 *Define the spaces $J^\alpha_{L,\theta}(\Omega)$, $J^\alpha_{S,\theta}(\Omega)$ to be the closures of $C^\infty_0(\Omega)$ under (5.5) and (5.6), respectively.*

Lemma 5.3 *Let $\alpha \neq n - 1/2, n \in \mathbb{N}$, $u \in C^\infty_0(\Omega)$. Then there exists a constant C such that*

$$\|D^\alpha_\theta u\|_{L^2(\mathbb{R}^2)} \leq C \|D^\alpha_\theta u\|_{L^2(\Omega)}. \tag{5.7}$$

Proof: Under the change of variables

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}, \tag{5.8}$$

D^α_θ maps to D^α_0. As $u \in C^\infty_0(\Omega)$, $D^\alpha_0 u = \mathbf{D}^\alpha u$, the one-dimensional left Riemann-Liouville fractional derivative (see (A.3)). By Fubini's theorem

$$\|D^\alpha_\theta u\|^2_{L^2(\Omega)} = \int_\Omega |D^\alpha_\theta u|^2 \, dxdy = \int_{\tilde{\Omega}} |D^\alpha_0 u|^2 \, d\tilde{x}d\tilde{y} = \int_{\tilde{y}_{\min}}^{\tilde{y}_{\max}} \int_{f_1(\tilde{y})}^{f_2(\tilde{y})} |D^\alpha_0 u|^2 \, d\tilde{x}d\tilde{y}, \tag{5.9}$$

and

$$\|D^\alpha_\theta u\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}} \int_{\mathbb{R}} |D^\alpha_\theta u|^2 \, dxdy = \int_{\mathbb{R}} \int_{\mathbb{R}} |D^\alpha_0 u|^2 \, d\tilde{x}d\tilde{y}.$$

Note that as

$$\mathrm{supp}(u(\tilde{x}, \tilde{y})) \subseteq \{(\tilde{x}, \tilde{y} \,|\, f_1(\tilde{y}) \leq x \leq f_2(\tilde{y}),\, \tilde{y}_{\min} \leq \tilde{y} \leq \tilde{y}_{\max}\},$$

then,

$$\mathrm{supp}(D^\alpha_0 u(\tilde{x}, \tilde{y})) \subseteq \{(\tilde{x}, \tilde{y} \,|\, -\infty < x < \infty,\, \tilde{y}_{\min} \leq \tilde{y} \leq \tilde{y}_{\max}\}.$$

Hence, by Property A.10,

$$\begin{aligned} \|D^\alpha_\theta u\|^2_{L^2(\mathbb{R}^2)} &= \int_{\tilde{y}_{\min}}^{\tilde{y}_{\max}} \int_{\mathbb{R}} |D^\alpha_0 u|^2 \, d\tilde{x}d\tilde{y} \\ &\leq \int_{\tilde{y}_{\min}}^{\tilde{y}_{\max}} C(\tilde{y}) \int_{f_1(\tilde{y})}^{f_2(\tilde{y})} |D^\alpha_0 u|^2 \, d\tilde{x}d\tilde{y}. \end{aligned}$$

As Ω is bounded, and $C(\tilde{y})$ is finite for all $\tilde{y} \in [\tilde{y}_{\min}, \tilde{y}_{\max}]$, there exists a constant C such that $C(\tilde{y}) \leq C$ for all $\tilde{y}$. Hence, by (5.9),

$$\|D_\theta^\alpha u\|^2_{L^2(\mathbb{R}^2)} \leq C \int_{\tilde{y}_{\min}}^{\tilde{y}_{\max}} \int_{f_1(\tilde{y})}^{f_2(\tilde{y})} |D_0^\alpha u|^2 \, d\tilde{x} d\tilde{y} = C\|D_\theta^\alpha u\|^2_{L^2(\Omega)}.$$

■

Lemma 5.4 *Let $\alpha > 0$, $\alpha \neq n - 1/2, n \in \mathbb{N}, \theta \in [0, 2\pi)$ be given. The spaces $J_{L,\theta}^\alpha(\Omega)$, $J_{L,\theta+\pi}^\alpha(\Omega)$, and $J_{S,\theta}^\alpha(\Omega)$ are equal, with equivalent semi-norms and norms.*

Proof: We show the stated result for $J_{L,\theta}^\alpha(\Omega)$ and $J_{S,\theta}^\alpha(\Omega)$. The proof for $J_{L,\theta+\pi}^\alpha(\Omega)$ and $J_{S,\theta}^\alpha(\Omega)$ follows analogously.

Let $\tilde{u}$ be the extension of u by zero outside of Ω. Note that

$$\text{supp}(D_\theta^\alpha u \, D_{\theta+\pi}^\alpha u) \subseteq \Omega,$$

thus

$$|u|_{J_{S,\theta}^\alpha(\Omega)} = |\tilde{u}|_{J_{S,\theta}^\alpha(\mathbb{R}^2)}.$$

We have from Lemma 5.2 that $|\cdot|_{J_{L,\theta}^\alpha(\mathbb{R}^2)}$ and $|\cdot|_{J_{S,\theta}^\alpha(\mathbb{R}^2)}$ are equivalent. Thus, it suffices to show that $|\cdot|_{J_{L,\theta}^\alpha(\mathbb{R}^2)}$ and $|\cdot|_{J_{L,\theta}^\alpha(\Omega)}$ are equivalent.

We immediately have that for $u \in C_0^\infty(\Omega)$,

$$|u|_{J_{L,\theta}^\alpha(\Omega)} \leq |\tilde{u}|_{J_{L,\theta}^\alpha(\mathbb{R}^2)}.$$

Also, for $\alpha \neq n - 1/2, n \in \mathbb{N}$, Lemma 5.3 implies

$$|\tilde{u}|_{J_{L,\theta}^\alpha(\mathbb{R}^2)} \leq C|u|_{J_{L,\theta}^\alpha(\Omega)}.$$

■

Next we define the fractional dimension Sobolev spaces and relate $J_{L,\theta}^\alpha(\Omega)$ ($J_{L,\theta+\pi}^\alpha(\Omega)$, $J_{S,\theta}^\alpha(\Omega)$) to $H_0^\alpha(\Omega)$ [8].

Definition 5.4 *Let $\Omega \subset \mathbb{R}^2$ and $\mu > 0$. Define the semi-norm*

$$|u|_{H^\mu(\Omega)} := \| \, |\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})}, \tag{5.10}$$

and norm

$$\|u\|_{H^\mu(\Omega)} := (\|u\|^2_{L^2(\Omega)} + |u|^2_{H^\mu(\Omega)})^{1/2},$$

and let $H_0^\mu(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{H^\mu(\Omega)}$.

Lemma 5.5 *Let $\alpha > 0$, $\alpha \neq n - 1/2, n \in \mathbb{N}, \theta \in [0, 2\pi)$ be given. The spaces $J_{L,\theta}^\alpha(\Omega)$, and $H_0^\alpha(\Omega)$ are equal, with equivalent semi-norms and norms.*

Proof: For $\Omega = \mathbb{R}^2$ the statement follows directly from (5.10), Plancherel's theorem, and (2.9). For $\Omega \subset \mathbb{R}^2$, with $\tilde{u}$ the extension of u by zero outside of Ω, we have

$$\begin{aligned} |u|_{J^{\alpha}_{L,\theta}(\Omega)} &\equiv |u|_{J^{\alpha}_{S,\theta}(\Omega)} = |\tilde{u}|_{J^{\alpha}_{S,\theta}(\mathbb{R}^2)} \equiv |\tilde{u}|_{J^{\alpha}_{L,\theta}(\mathbb{R}^2)} \\ &\equiv |\tilde{u}|_{H^{\alpha}(\mathbb{R}^2)} = |u|_{H^{\alpha}(\Omega)} . \end{aligned}$$

■

We now present two properties for the directional derivative operator, D^{α}_{θ}, (inverse and adjoint) which we use in our subsequent analysis.

Lemma 5.6 *For $\alpha > 0$, $u \in J^{\alpha}_{L,\theta}(\Omega)$,*

$$D^{-\alpha}_{\theta} D^{\alpha}_{\theta} u = u,$$

and for $0 < s < \alpha$,

$$D^{s}_{\theta} D^{\alpha - s}_{\theta} u = D^{\alpha}_{\theta} u.$$

Proof: The stated results follow directly from Property A.6 and Property A.7 ■

Lemma 5.7 *For $\alpha > 0$, $u, v \in J^{\alpha}_{L,\theta}(\Omega)$,*

$$(D^{\alpha}_{\theta} u,\, v)_{L^2(\Omega)} = \left(u,\, D^{\alpha}_{\theta+\pi} v\right)_{L^2(\Omega)} .$$

Proof: Applying Corollary 2.1 and Lemma 5.6, we have

$$\begin{aligned} (D^{\alpha}_{\theta} u,\, v)_{L^2(\Omega)} &= \left(D^{\alpha}_{\theta} u,\, D^{-\alpha}_{\theta+\pi} D^{\alpha}_{\theta+\pi} v\right)_{L^2(\Omega)} \\ &= \left(D^{-\alpha}_{\theta} D^{\alpha}_{\theta} u,\, D^{\alpha}_{\theta+\pi} v\right)_{L^2(\Omega)} \\ &= \left(u,\, D^{\alpha}_{\theta+\pi} v\right)_{L^2(\Omega)} . \end{aligned}$$

■

For the next step in our analysis, we define a semi-norm by integrating $|\cdot|^2_{J^{\alpha}_{L,\theta}(\Omega)}$ with respect to the probability measure $M(d\theta)$. We then show that the space defined using this semi-norm is equivalent to $H^{\alpha}_0(\Omega)$.

Definition 5.5 *For $\alpha > 0$, define the semi-norm*

$$|u|_{J^{\alpha}_{M}(\mathbb{R}^2)} := \left(\int_0^{2\pi} |u|^2_{J^{\alpha}_{L,\theta}(\mathbb{R}^2)} M(d\theta)\right)^{1/2},$$

and norm

$$\|u\|_{J^{\alpha}_{M}(\mathbb{R}^2)} := (\|u\|^2_{L^2(\mathbb{R}^2)} + |u|^2_{J^{\alpha}_{M}(\mathbb{R}^2)})^{1/2},$$

and let $J^{\alpha}_{M}(\mathbb{R}^2)$ denote closure of $C^{\infty}(\mathbb{R}^2)$ with respect to $\|\cdot\|_{J^{\alpha}_{M}(\mathbb{R}^2)}$.

Lemma 5.8 *For an arbitrary measure M, there exists a constant C such that*

$$|u|_{J^{\alpha}_{M}(\mathbb{R}^2)} \leq C|u|_{H^{\alpha}(\mathbb{R}^2)}.$$

Proof: Using (2.9) and Plancherel's theorem, for θ fixed we have

$$\begin{aligned} |u|_{J^{\alpha}_{L,\theta}(\mathbb{R}^2)} &= \|D^{\alpha}_{\theta} u\|_{L^2(\mathbb{R}^2)} \\ &= \|(i\omega_1 \cos\theta + i\omega_2 \sin\theta)^{\alpha} \hat{u}\|_{L^2(\mathbb{R}^2)} \\ &\leq \||\boldsymbol{\omega}|^{\alpha} \hat{u}\|_{L^2(\mathbb{R}^2)} = |u|_{H^{\alpha}(\mathbb{R}^2)}. \end{aligned}$$

Integration with respect to $M(d\theta)$ implies $|u|_{J^{\alpha}_{M}(\mathbb{R}^2)} \leq C|u|_{H^{\alpha}(\mathbb{R}^2)}$. ∎

Definition 5.6 *Define the space $J^{\alpha}_{M}(\Omega)$ as the closure of $C^{\infty}_{0}(\Omega)$ under the $J^{\alpha}_{M}(\Omega)$ norm.*

Corollary 5.1 *For an arbitrary measure M, there exists a constant C such that*

$$|u|_{J^{\alpha}_{M}(\Omega)} \leq C|u|_{H^{\alpha}(\Omega)}.$$

Proof: Let $\tilde{u}$ be the extension of u by zero outside of Ω. Then we immediately obtain

$$\begin{aligned} |u|_{J^{\alpha}_{M}(\Omega)} &\leq |\tilde{u}|_{J^{\alpha}_{M}(\mathbb{R}^2)} \\ &\leq C|\tilde{u}|_{H^{\alpha}(\mathbb{R}^2)} \\ &= C|u|_{H^{\alpha}(\Omega)}. \end{aligned}$$

∎

In order to show existence and uniqueness for (1.1) over the fractional Hilbert space $H^{\alpha}_{0}(\Omega)$, we must show $\|u\|_{J^{\alpha}_{M}(\Omega)} \geq C\|u\|_{H^{\alpha}(\Omega)}$. This, however, is dependent upon the form of the measure M. We introduce the condition on M that

$$\int_{0}^{2\pi} |\sin\theta|^{2\alpha}\, M(d(\theta - \psi)) \geq C_1, \tag{5.11}$$

independent of the value of $\psi \in [-\pi/2, \pi/2]$.

Remark 5.1: The condition (5.11) holds if $M(d\theta)$ is non-zero over a connected set of positive measure in $[0, 2\pi]$. As $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$, we have

$$\begin{aligned} \int_{0}^{2\pi} |\sin\theta|^{2\alpha}\, M(d(\theta - \psi)) \geq \frac{2}{\pi} \int_{0}^{\pi/2} \theta^{2\alpha} \quad & [M(d(\theta - \psi)) + M(d(\pi - \theta + \psi)) \\ & + M(d(\pi + \theta - \psi)) + M(d(2\pi - \theta + \psi))]\,, \end{aligned}$$

which is positive if $M(d\theta)$ is non-zero over a connected set of positive measure.

Remark 5.2: The condition (5.11) holds if $M(d\theta)$ is atomic with at least two atoms, θ_i, θ_j, such that $\theta_i \neq \theta_j + \pi$. In this case, (5.11) reduces to

$$\int_{0}^{2\pi} |\sin\theta|^{2\alpha}\, M(d(\theta - \psi)) = \sum_{i=1}^{n} P(\theta = \theta_i) |\sin(\theta_i + \psi)|^{2\alpha},$$

which is positive for all such ψ if and only if $\theta_i \neq \theta_j + \pi$ for some i and j.

Theorem 5.1 *Let M satisfy (5.11). Then the spaces $H^\alpha(\mathbb{R}^2)$ and $J_M^\alpha(\mathbb{R}^2)$ are equal, with equivalent semi-norms and norms.*

Proof: We already have from Lemma 5.8 that

$$|u|_{J_M^\alpha(\mathbb{R}^2)} \le C|u|_{H^\alpha(\mathbb{R}^2)}.$$

In order to show the reverse inequality, consider $u \in C_0^\infty(\mathbb{R}^2)$. Then, we have

$$\begin{aligned}
|u|^2_{J_M^\alpha(\mathbb{R}^2)} &= \int_0^{2\pi} |u|^2_{J_{L,\theta}^\alpha(\mathbb{R}^2)}\, M(d\theta) \\
&= \int_0^{2\pi} \|D_\theta^\alpha u\|^2\, M(d\theta) \\
&= \int_0^{2\pi} \|\mathcal{F}\left(D_\theta^\alpha u\right)\|^2\, M(d\theta) \\
&= \int_0^{2\pi}\int_{\mathbb{R}^2} |(i\omega_1\cos\theta \,+\, i\omega_2\sin\theta)^\alpha\, \hat{u}(\omega_1,\omega_2)|^2\, d\boldsymbol{\omega}\, M(d\theta) \\
&= \int_0^{2\pi}\int_{\mathbb{R}^2} |\boldsymbol{\omega}|^{2\alpha}\, |\sin(\theta \,+\, \arctan(\omega_1/\omega_2))|^{2\alpha}\, |\hat{u}(\omega_1,\omega_2)|^2\, d\boldsymbol{\omega}\, M(d\theta) \\
&= \int_{\mathbb{R}^2} \left(\int_0^{2\pi} |\sin(\theta \,+\, \arctan(\omega_1/\omega_2))|^{2\alpha}\, M(d\theta)\right) |\boldsymbol{\omega}|^{2\alpha}\, |\hat{u}(\omega_1,\omega_2)|^2\, d\boldsymbol{\omega}\ .
\end{aligned}$$

As for all $(\omega_1,\omega_2) \in \mathbb{R}^2$, $\psi = \arctan(\omega_1/\omega_2) \in [-\pi/2, \pi/2]$, (5.11) implies

$$|u|^2_{J_M^\alpha(\mathbb{R}^2)} \ge C_1|u|^2_{H^\alpha(\mathbb{R}^2)}.$$

Finally, as $J_M^\alpha(\mathbb{R}^2)$ and $H^\alpha(\mathbb{R}^2)$ are the closures of $C_0^\infty(\mathbb{R}^2)$, the equivalence of semi-norms follows. The equivalence of norms then follows from the definitions of the $J_M^\alpha(\mathbb{R}^2)$ and $H^\alpha(\mathbb{R}^2)$ norms. ∎

Corollary 5.2 *Let M satisfy (5.11) and $\alpha \ne n - 1/2, n \in \mathbb{N}$. Then the spaces $J_M^\alpha(\Omega)$ and $H_0^\alpha(\Omega)$ are equivalent, with equivalent semi-norms and norms.*

Proof: Let $u \in C_0^\infty(\Omega)$, and $\tilde{u}$ denote its extension by zero to all of $\mathbb{R}^2$. Using Lemma 5.4 and Theorem 5.1, we have

$$\begin{aligned}
|u|^2_{J_M^\alpha(\Omega)} &= \int_0^{2\pi} |u|^2_{J_{L,\theta}^\alpha(\Omega)}\, M(d\theta) \\
&\ge C\int_0^{2\pi} |u|^2_{J_{S,\theta}^\alpha(\Omega)}\, M(d\theta).
\end{aligned}$$

As

$$\left(D_\theta^\alpha u, D_{\theta+\pi}^\alpha u\right)_{L^2(\Omega)} = \left(D_\theta^\alpha \tilde{u}, D_{\theta+\pi}^\alpha \tilde{u}\right)_{L^2(\mathbb{R}^2)},$$

we have

$$
\begin{aligned}
|u|^2_{J^\alpha_M(\Omega)} &\geq C \int_0^{2\pi} |\tilde{u}|^2_{J^\alpha_{S,\theta}(\mathbb{R}^2)} \, M(d\theta) \\
&\geq C_2 \int_0^{2\pi} |\tilde{u}|^2_{J^\alpha_{L,\theta}(\mathbb{R}^2)} \, M(d\theta), \text{ from Lemma 5.2,} \\
&= C_2 |\tilde{u}|^2_{J^\alpha_M(\mathbb{R}^2)} \\
&\geq C_3 |\tilde{u}|^2_{H^\alpha(\mathbb{R}^2)}, \text{ from Lemma 5.8,} \\
&= C_3 |u|^2_{H^\alpha(\Omega)} \; .
\end{aligned}
$$

Finally, as $J^\alpha_M(\Omega)$ and $H^\alpha_0(\Omega)$ are closures of $C^\infty_0(\Omega)$ under equivalent norms, we have

$$|u|_{J^\alpha_M(\Omega)} \geq C|u|_{H^\alpha(\Omega)}.$$

The reverse inequality is given in Corollary 5.1, which implies that the semi-norms are equivalent. By the definition of the $J^\alpha_M(\Omega)$ and $H^\alpha_0(\Omega)$ norms, norm equivalence follows. ∎

We now may state a fractional Poincaré-Friedrichs inequality for this set of spaces.

Theorem 5.2 *Let $u \in J^\alpha_{L,\theta}(\Omega)$, then*

$$\|u\|_{L^2(\Omega)} \leq C|u|_{J^\alpha_{L,\theta}(\Omega)}, \tag{5.12}$$

and for $0 < s < \alpha$,

$$|u|_{J^s_{L,\theta}(\Omega)} \leq C|u|_{J^\alpha_{L,\theta}(\Omega)}. \tag{5.13}$$

Proof: Using the change of variables

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} . \tag{5.14}$$

$D^{-\alpha}_\theta$ maps to $D^{-\alpha}_0$, a one-dimensional fractional differential operator. In view of (5.9), the conclusions (5.12) and (5.13) then follow from Property A.8 and Property A.9, respectively. ∎

Corollary 5.3 *[Fractional Poincaré-Friedrichs] For $u \in J^\alpha_M(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C|u|_{J^\alpha_M(\Omega)}, \tag{5.15}$$

and for $0 < s < \alpha$,

$$|u|_{J^s_M(\Omega)} \leq C|u|_{J^\alpha_M(\Omega)}. \tag{5.16}$$

Proof: The conclusions (5.15) and (5.16) follow from (5.12), (5.13), and the definition of the $J^\alpha_M(\Omega)$ semi-norm. ∎

6 Variational Formulation

Let Ω denote an bounded, open connected region in $\mathbb{R}^2$. In this section we show that there exists a unique variational solution of (1.1) in the space $H_0^\alpha(\Omega)$.

Problem 6.1 *[Steady State Fractional Advection Dispersion Equation]*
Given $f \in H^{-\alpha}(\Omega)$, find $u : \bar{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned} Lu &= f, \quad in \ \ \Omega, && (6.1)\\ u &= 0, \quad on \ \ \partial\Omega. && (6.2) \end{aligned}$$

where

$$Lu := -\int_0^{2\pi} \left(D_\theta \, a \, D_\theta^\beta u \right) \, M(d\theta) + \mathbf{b}(x,y) \cdot \nabla u + c(x,y)u,$$

$0 < \beta \le 1$, $\alpha := \frac{\beta+1}{2}$, $a > 0$, $\mathbf{b}(x,y) := [b_1(x,y), b_2(x,y)]^t \in (C^1(\bar{\Omega}))^2$, $c(x,y) \in C(\bar{\Omega})$ with $c - \frac{1}{2}\nabla \cdot \mathbf{b} \ge 0$ and $M(d\theta)$ satisfies (5.11).

In order to derive a variational form for (6.1)-(6.2), we assume that u is a sufficiently smooth solution of (6.1)-(6.2) and multiply by an arbitrary $v \in C_0^\infty(\Omega)$ to obtain

$$\begin{aligned} \int_\Omega f \, v \, dx &= \int_\Omega -\left(\int_0^{2\pi} D_\theta a D_\theta^\beta u \, M(d\theta) \right) v + \mathbf{b} \cdot \nabla u \, v + c \, u \, v \ dx \\ &= \int_\Omega -\int_0^{2\pi} a D_\theta^\alpha u \, D_{\theta+\pi}^\alpha v \, M(d\theta) + D_0 u \ b_1 \, v \ + \ D_{\pi/2} u \ b_2 \, v \ + c \, u \, v \ dx \\ &= \int_\Omega -\left(\int_0^{2\pi} a D_\theta^\alpha u \, D_{\theta+\pi}^\alpha v \, M(d\theta) \right) + D_0^\alpha u \, D_\pi^{1-\alpha}(b_1 \, v) \ + \ D_{\pi/2}^\alpha u \, D_{3\pi/2}^{1-\alpha}(b_2 \, v) \ + c \, u \, v \ dx \, . \end{aligned}$$

Thus, we define the associated bilinear form $B : H_0^\alpha(\Omega) \times H_0^\alpha(\Omega) \to \mathbb{R}$ as

$$B(u,v) := -\int_0^{2\pi} a \left(D_\theta^\alpha u, \, D_{\theta+\pi}^\alpha v \right) \, M(d\theta) + \left(D_0^\alpha u, \, D_\pi^{1-\alpha}(b_1 \, v) \right) + \left(D_{\pi/2}^\alpha u, \, D_{3\pi/2}^{1-\alpha}(b_2 \, v) \right) + (c \, u, \, v) \, .$$

For $f \in H^{-\alpha}(\Omega)$ we define the associated linear functional $F : H_0^\alpha(\Omega) \to \mathbb{R}$ as

$$F(v) := \langle f, v \rangle \ ,$$

where $< \cdot, \cdot >$ denotes the duality pairing of $H^{-\mu}(\Omega)$ and $H_0^\mu(\Omega)$, $\mu \ge 0$.

Definition 6.1 *[Variational Solution] A function $u \in H_0^\alpha(\Omega)$ is a variational solution of (6.1)-(6.2) provided that*

$$B(u,v) = F(v), \quad \forall v \in H_0^\alpha(\Omega). \tag{6.3}$$

Using the results of Section 5, we show that there exists a unique variational solution to (6.1)-(6.2).

Lemma 6.1 *The bilinear form $B(\cdot,\cdot)$ is coercive over $H_0^\alpha(\Omega)$, i.e. there exists a constant C_0 such that*

$$B(u,u) \geq C_0 \|u\|^2_{H^\alpha(\Omega)}. \tag{6.4}$$

Proof: With $\alpha > 1/2$, and $u \in H_0^\alpha(\Omega)$ the limit of $\{\phi_n\}_{n=0}^\infty$, $\phi_n \in C_0^\infty(\Omega)$, it is straight forward to establish that

$$\left(D_0^\alpha u,\, D_\pi^{1-\alpha}(b_1\, v)\right) \;+\; \left(D_{\pi/2}^\alpha u\, D_{3\pi/2}^{1-\alpha}(b_2\, v)\right) \;=\; -\frac{1}{2}\left(\left(\nabla \cdot \mathbf{b}\right) u,\, u\right) \;.$$

Hence

$$\begin{aligned} B(u,u) \;&=\; -\int_0^{2\pi} a\left(D_\theta^\alpha u,\, D_{\theta+\pi}^\alpha v\right)\, M(d\theta) \;+\; \left((c - \frac{1}{2}\nabla \cdot \mathbf{b})u, u\right) \\ &\geq\; a \int_0^{2\pi} |u|^2_{J^\alpha_{S,\theta}(\Omega)} M(d\theta). \end{aligned}$$

By norm equivalence (Lemma 5.4), Corollary 5.2, and the fractional Poincaré-Friedrichs inequality (Corollary 5.3), we obtain

$$\begin{aligned} B(u,u) \;&\geq\; C_1 \int_0^{2\pi} |u|^2_{J^\alpha_{L,\theta}(\Omega)} M(d\theta) \\ &=\; C_1 |u|^2_{J^\alpha_M(\Omega)} \\ &\geq\; C_2 \|u\|^2_{J^\alpha_M(\Omega)} \\ &\geq\; C \|u\|^2_{H^\alpha(\Omega)}. \end{aligned}$$

■

Lemma 6.2 *The bilinear form $B(\cdot,\cdot)$ is continuous on $H_0^\alpha(\Omega) \times H_0^\alpha(\Omega)$, i.e. there exists a constant C_1 such that*

$$|B(u,v)| \leq C_1 \|u\|_{H^\alpha(\Omega)} \|v\|_{H^\alpha(\Omega)}. \tag{6.5}$$

Proof: First, we apply the triangle inequality and introduce terms I, II and III:

$$\begin{aligned} |B(u,v)| \;&\leq\; \left|-\int_0^{2\pi} a\left(D_\theta^\alpha u,\, D_{\theta+\pi}^\alpha v\right)\, M(d\theta)\right| \;+\; \left|\left(D_0^\alpha u,\, D_\pi^{1-\alpha}(b_1\, v)\right) \;+\; \left(D_{\pi/2}^\alpha u\, D_{3\pi/2}^{1-\alpha}(b_2\, v)\right)\right| \\ &\quad\; + \; |(c\, u,\, v)| \\ &:=\; \mathrm{I} + \mathrm{II} + \mathrm{III} \end{aligned}$$

In order to bound III, we use Cauchy-Schwarz to obtain

$$\begin{aligned} \mathrm{III} \;&\leq\; \|c\, u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq\; \|c\|_\infty \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq\; \|c\|_\infty \|u\|_{H^\alpha(\Omega)} \|v\|_{H^\alpha(\Omega)}. \end{aligned} \tag{6.6}$$

Next, using (2.9) we bound I by

$$\begin{aligned} \mathrm{I} &\leq a\left(\int_0^{2\pi} |u|^2_{J^\alpha_{L,\theta}(\Omega)} M(d\theta)\right)^{1/2} \left(\int_0^{2\pi} |v|^2_{J^\alpha_{L,\theta+\pi}(\Omega)} M(d\theta)\right)^{1/2} \\ &\leq a\|u\|_{J^\alpha_M(\Omega)} \left(\int_0^{2\pi} |v|^2_{J^\alpha_{L,\theta}(\Omega)} M(d\theta)\right)^{1/2} \leq a\|u\|_{J^\alpha_M(\Omega)} \|v\|_{J^\alpha_M(\Omega)} \\ &\leq C_1 \|u\|_{H^\alpha(\Omega)} \|v\|_{H^\alpha(\Omega)}. \end{aligned} \tag{6.7}$$

To bound II note that from Property A.11 in the appendix, there exists constants C_{b_1} and C_{b_2} such that $\|b_1 v\|_{H^{1-\alpha}(\Omega)} \leq C_{b_1}\|v\|_{H^{1-\alpha}(\Omega)}$ and $\|b_2 v\|_{H^{1-\alpha}(\Omega)} \leq C_{b_2}\|v\|_{H^{1-\alpha}(\Omega)}$. Hence, using Lemmas 5.4 and 5.5,

$$\begin{aligned} \mathrm{II} &\leq \|u\|_{J^\alpha_{L,0}(\Omega)} \|b_1 v\|_{J^{1-\alpha}_{L,\pi}(\Omega)} + \|u\|_{J^\alpha_{L,pi/2}(\Omega)} \|b_2 v\|_{J^{1-\alpha}_{L,3\pi/2}(\Omega)} \\ &\leq C_1 \|u\|_{H^\alpha(\Omega)} \left(\|b_1 v\|_{H^{1-\alpha}(\Omega)} + \|b_2 v\|_{H^{1-\alpha}(\Omega)}\right) \\ &\leq C_2 \|u\|_{H^\alpha(\Omega)} \|v\|_{H^{1-\alpha}(\Omega)} \leq C \|u\|_{H^\alpha(\Omega)} \|v\|_{H^\alpha(\Omega)}. \end{aligned} \tag{6.8}$$

Combining (6.6) - (6.8), we have (6.5). ■

Lemma 6.3 *The linear functional $F(\cdot)$ is continuous over $H^\alpha_0(\Omega)$.*

Proof: The result follows from the fact that

$$F(v) = \langle f, v\rangle \leq \|f\|_{H^{-\alpha}(\Omega)} \|v\|_{H^\alpha(\Omega)}, \quad \forall\, v \in H^\alpha_0(\Omega). \tag{6.9}$$

■

Theorem 6.1 *There exists a unique solution $u \in H^\alpha_0(\Omega)$ to (6.3) satisfying*

$$\|u\|_{H^\alpha(\Omega)} \leq C\|f\|_{H^{-\alpha}(\Omega)}. \tag{6.10}$$

Proof: By Lemmas 6.1, 6.2, 6.3, the operators B, F satisfy the hypotheses of the Lax-Milgram theorem, from which existence and uniqueness of a solution to (6.3) immediately follow. The estimate (6.10) is obtained from combining (6.4), (6.5), and (6.9). ■

7 Finite Element Convergence Estimates

Let $\{S_h\}$ denote a family of partitions of Ω, with grid parameter h. Associated with S_h, define the finite dimensional subspace X_h to be the basis of piecewise polynomials of order $m-1$, where $m \geq 1 \in \mathbb{N}$. Denote by $\mathcal{I}^h u$ the piecewise polynomial interpolant of u in S_h.

Let u_h be the solution to the finite dimensional variational problem

$$B(u_h, v_h) = F(v_h), \quad \forall\, v_h \in X_h. \tag{7.1}$$

We define the energy norm associated with (6.3) as

$$\|u\|_E := B(u,u)^{1/2}. \tag{7.2}$$

Note that from (6.4) and (6.5) we have norm equivalence of $\|\cdot\|_{H^\alpha(\Omega)}$ and $\|\cdot\|_E$.

Theorem 7.1 *Let u denote the solution to (6.3). There exists a unique solution to (7.1) which satisfies the estimate*

$$\|u-u_h\|_E \le C_I \inf_{v\in X_h} \|u-v\|_E \le C_I\|u-\mathcal{I}^h u\|_E. \tag{7.3}$$

Proof: Existence and uniqueness follow from the fact that X_h is a subspace of the space $H_0^\alpha(\Omega)$, and thus (7.1) satisfies the hypotheses of the Lax-Milgram lemma over the finite dimensional subspace X_h. The estimate (7.3) is a result of Ceá's lemma. ■

The finite dimensional subspace X_h and the interpolant $\mathcal{I}^h u$ are chosen specifically so that they satisfy an approximation property over subspaces of $H^m(\Omega)$. That is to say that $\mathcal{I}^h u$ satisfies the following theorem [2].

Theorem 7.2 *[Approximation Property] Let $u \in H^r(\Omega), 0 < r \le m$, and $0 \le s \le r$. Then there exists a constant C_A depending only on Ω such that*

$$\|u-\mathcal{I}^h u\|_{H^s(\Omega)} \le C_A h^{r-s}\|u\|_{H^r(\Omega)}.$$

■

We can combine the previous results into an estimate for $e := u - u_h$ in the energy norm.

Corollary 7.1 *Let $u \in H_0^\alpha(\Omega)\cap H^r(\Omega)$ ($\alpha \le r \le m$) solve (6.3), and u_h solve (7.1). Then there exists a constant C such that the error $e = u - u_h$ satisfies*

$$\|e\|_{H^\alpha(\Omega)} \le Ch^{r-\alpha}\|u\|_{H^r(\Omega)}. \tag{7.4}$$

Proof: From Theorem 7.1, we have that the error satisfies

$$\|e\|_E \le C_I\|u-\mathcal{I}^h u\|_E.$$

Applying the approximation property and continuity yields

$$\|e\|_E \le \sqrt{C_1}C_I C_A h^{r-\alpha}\|u\|_{H^r(\Omega)}. \tag{7.5}$$

Finally, we obtain (7.4) via the norm equivalence of $\|\cdot\|_{H^\alpha(\Omega)}$ and $\|\cdot\|_E$. ■

We now apply the Aubin-Nitsche trick to obtain a convergence estimate in the L^2 norm. First, we must make an assumption concerning the regularity of the solution to the adjoint problem:

$$-\int_0^{2\pi} \left(D^{\beta}_{\theta+\pi}\, a\, D_{\theta+\pi} w\right) M(d\theta) + (-\nabla\cdot \mathbf{b}(x,y) + c(x,y))w \;=\; g, \quad in\ \Omega \tag{7.6}$$

$$w \;=\; 0, \quad on\ \partial\Omega. \tag{7.7}$$

Assumption ADRG: For w solving (7.6) with $g \in L^2(\Omega)$, we have

$$\|w\|_{H^{2\alpha}(\Omega)} \leq C_E \|g\|_{L^2(\Omega)}.$$

Theorem 7.3 *Let $u \in H^{\alpha}_0(\Omega) \cap H^r(\Omega)$ $(\alpha \leq r \leq m)$ solve (6.3), and u_h solve (7.1). Then, under Assumption ADRG, there exists a constant C such that the error $e = u - u_h$ satisfies*

$$\|e\|_{L^2(\Omega)} \leq C h^r \|u\|_{H^r(\Omega)}. \tag{7.8}$$

Proof: Introduce w as the solution of (7.6) with $g = e = u - u_h \in L^2(\Omega)$. Then w satisfies the variational form

$$B(v, w) = (e, v), \quad \forall\, v \in H^{\alpha}_0(\Omega), \tag{7.9}$$

and the regularity estimate

$$\|w\|_{H^{2\alpha}(\Omega)} \leq C_E \|e\|_{L^2(\Omega)}.$$

Substitute $v = e$ in (7.9), and applying Galerkin orthogonality, we have

$$\begin{aligned}
\|e\|^2_{L^2(\Omega)} &= B(e, w) \\
&= B(e, w - \mathcal{I}^h w) \\
&\leq C_1 \|e\|_{H^{\alpha}(\Omega)} \|w - \mathcal{I}^h w\|_{H^{\alpha}(\Omega)} \\
&\leq C_1 C_A h^{\alpha} \|e\|_{H^{\alpha}(\Omega)} \|w\|_{H^{2\alpha}(\Omega)} \\
&\leq C_1 C_A C_E h^{\alpha} \|e\|_{H^{\alpha}(\Omega)} \|e\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, dividing through by $\|e\|_{L^2(\Omega)}$ yields the estimate

$$\|e\|_{L^2(\Omega)} \leq C_1 C_A C_E h^{\alpha} \|e\|_E,$$

and applying (7.4) we obtain (7.8). ■

Remark: For a Finite Element approximation to (6.3), $u_h \in S_h$, $v \in S_h$,

$$\tilde{B}(u_h, v) \;:=\; \int_0^{2\pi} \int_{\Omega} a\, D^{\beta}_{\theta} u_h\, D_{\theta} v\, dx\, M(d\theta) \;+\; \int_{\Omega} \mathbf{b}\cdot\nabla u_h\, v\, dx \;+\; \int_{\Omega} c\, u_h\, v\, dx\ ,$$

is computationally more suitable than $B(u_h, v)$. See [14] for a discussion on Finite Element implementation issues for FADEs.

A Riemann-Liouville Fractional Integral Operators

In this section we present the Riemann-Liouville fractional integral and differential operators and several properties which they satisfy.

Definition A.1 *[Left Riemann-Liouville Fractional Integral] Let u be a function defined on (a,b), and $\sigma > 0$. Then the left Riemann-Liouville fractional integral of order σ is defined to be*

$$ {}_aD_x^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)} \int_a^x (x-s)^{\sigma-1}u(s)ds. \tag{A.1}$$

Definition A.2 *[Right Riemann-Liouville Fractional Integral] Let u be a function defined on (a,b), and $\sigma > 0$. Then the right Riemann-Liouville fractional integral of order σ is defined to be*

$$ {}_xD_b^{-\sigma}u(x) := \frac{1}{\Gamma(\sigma)} \int_x^b (s-x)^{\sigma-1}u(s)ds. \tag{A.2}$$

Definition A.3 *[Left Riemann-Liouville Fractional Derivative] Let u be a function defined on $\mathbb{R}$, $\mu > 0$, n be the smallest integer greater than μ ($n-1 \leq \mu < n$), and $\sigma = n - \mu$. Then the left fractional derivative of order μ is defined to be*

$$ \mathbf{D}^{\mu}u := {}_{-\infty}D_x^{\mu}u = D^n \, {}_{-\infty}D_x^{-\sigma}u(x) = \frac{1}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_{-\infty}^x (x-\xi)^{\sigma-1}u(\xi)d\xi. \tag{A.3}$$

Definition A.4 *[Right Riemann-Liouville Fractional Derivative] Let u be a function defined on $\mathbb{R}$, $\mu > 0$, n be the smallest integer greater than μ ($n-1 \leq \mu < n$), and $\sigma = n - \mu$. Then the right fractional derivative of order μ is defined to be*

$$ \mathbf{D}^{\mu*}u := {}_xD_{\infty}^{\mu}u = (-D)^n \, {}_xD_{\infty}^{-\sigma}u(x) = \frac{(-1)^n}{\Gamma(\sigma)} \frac{d^n}{dx^n} \int_x^{\infty} (\xi-x)^{\sigma-1}u^{(n)}(\xi)d\xi. \tag{A.4}$$

Note. If supp $(u) \subset (a,b)$, then $\mathbf{D}^{\mu}u = {}_aD_x^{\mu}u$ and $\mathbf{D}^{\mu*}u = {}_xD_b^{\mu}u$, where ${}_aD_x^{\mu}$ and ${}_xD_b^{\mu}u$ are the left and right Riemann-Liouville fractional derivatives of order μ [13].

With these definitions, we note several properties of the Riemann-Liouville fractional integral and differential operators [13, 15].

Property A.1 *[Semigroup Property] The left and right Riemann-Liouville fractional integral operators satisfy the semigroup properties: for $u \in L^p(a,b)$, $p \geq 1$,*

$$ \begin{aligned} {}_aD_x^{-\mu} \, {}_aD_x^{-\sigma}u(x) &= {}_aD_x^{-\mu-\sigma}u(x), \ \forall\, x \in (a,b), \ \forall\, \mu, \sigma > 0, \\ {}_xD_b^{-\mu} \, {}_xD_b^{-\sigma}u(x) &= {}_xD_b^{-\mu-\sigma}u(x), \ \forall\, x \in (a,b), \ \forall\, \mu, \sigma > 0. \end{aligned} \tag{A.5}$$

Property A.2 *[Adjoint Property] The left and right Riemann-Liouville fractional integral operators are adjoints in the L^2 sense, i.e. for all $\sigma > 0$,*

$$\left({}_aD_x^{-\sigma} u, v \right)_{L^2(a,b)} = \left(u, {}_xD_b^{-\sigma} v \right)_{L^2(a,b)}, \ \forall \, u, v \in L^2(a,b). \tag{A.6}$$

Property A.3 *[Fourier Transform Property] Let $\sigma > 0$, $u \in L^p(\mathbb{R})$, $p \geq 1$. The Fourier transform of the left and right Riemann-Liouville fractional integral satisfy the following,*

$$\begin{aligned} \mathcal{F}({}_{-\infty}D_x^{-\sigma} u(x)) &= (i\omega)^{-\sigma} \hat{u}(\omega), \\ \mathcal{F}({}_xD_\infty^{-\sigma} u(x)) &= (-i\omega)^{-\sigma} \hat{u}(\omega), \end{aligned} \tag{A.7}$$

where $\hat{u}(\omega)$ denotes the Fourier transform of u.

Property A.4 *The left (right) Riemann-Liouville fractional derivative of order μ acts as a left inverse of the left (right) Riemann-Liouville fractional integral of order μ, i.e.*

$$ {}_aD_x^{\mu} \, {}_aD_x^{-\mu} u(x) = u(x), \tag{A.8}$$

$$ {}_xD_b^{\mu} \, {}_xD_b^{-\mu} u(x) = u(x), \ \forall \, \mu > 0. \tag{A.9}$$

Property A.5 *[Fourier Transform Property] Let $\mu > 0$, $u \in C_0^\infty(a,b)$. The Fourier transform of the left and right Riemann-Liouville fractional integral satisfy the following,*

$$\begin{aligned} \mathcal{F}({}_{-\infty}D_x^{\mu} u(x)) &= (i\omega)^{\mu} \hat{u}(\omega), \\ \mathcal{F}({}_xD_\infty^{\mu} u(x)) &= (-i\omega)^{\mu} \hat{u}(\omega). \end{aligned} \tag{A.10}$$

In one space dimension the following norms and spaces are useful. For I denoting the interval $(a,b) \subset \mathbb{R}^1$

$$|u|_{J_L^\mu(I)} := \|\mathbf{D}^\mu u\|_{L^2(I)}, \qquad \|u\|_{J_L^\mu(I)} := \left(\|u\|_{L^2(I)}^2 + \|\mathbf{D}^\mu u\|_{L^2(I)}^2 \right)^{1/2}, \tag{A.11}$$

$$|u|_{J_R^\mu(I)} := \|\mathbf{D}^{\mu *} u\|_{L^2(I)}, \qquad \|u\|_{J_R^\mu(I)} := \left(\|u\|_{L^2(I)}^2 + \|\mathbf{D}^{\mu *} u\|_{L^2(I)}^2 \right)^{1/2}, \tag{A.12}$$

and $J_L^\mu(I)$, $J_R^\mu(I)$, denotes the closure of $C_0^\infty(I)$ with respect to (A.11) and (A.12), respectively.

Property A.6 *[5] For $u \in J_L^\mu(I)$, we have $\mathbf{D}^{-\mu}\mathbf{D}^\mu u = u$, and for $u \in J_R^\mu(I)$, we have $\mathbf{D}^{-\mu *}\mathbf{D}^{\mu *} u = u$.*

Property A.7 *[5] For $u \in J_L^\mu(I)$, $0 < s < \mu$,*

$$\mathbf{D}^{-s} \, \mathbf{D}^{s} \, \mathbf{D}^{\mu - s} u = \mathbf{D}^{\mu - s} u.$$

Property A.8 *[5] [Fractional Poincaré-Friedrichs] For* $u \in J_L^\mu(I)$, *we have*

$$\|u\|_{L^2(I)} \leq C|u|_{J_L^\mu(I)}, \tag{A.13}$$

and for $u \in J_R^\mu(I)$,

$$\|u\|_{L^2(I)} \leq C|u|_{J_R^\mu(I)} \,. \tag{A.14}$$

Property A.9 *[5] For* $u \in J_L^\mu(I)$, $0 < s < \mu$, *we have*

$$|u|_{J_L^s(I)} \leq C|u|_{J_L^\mu(I)},$$

and for $u \in J_R^\mu(I)$, $0 < s < \mu$,

$$|u|_{J_R^s(I)} \leq C|u|_{J_R^\mu(I)}.$$

We also state an additional result regarding the diminishing nature of ${}_aD_x^\mu$ outside of (a, b) for functions $u \in C_0^\infty(a, b)$.

Property A.10 *[5] Let* $\mu \neq n - 1/2$, $n \in \mathbb{N}$, $u \in C_0^\infty(I)$. *Then there exists a constant* C *depending only on* u *such that*

$$\| {}_aD_x^\mu u\|_{L^2(\mathbb{R})} \leq C\| {}_aD_x^\mu u\|_{L^2(a,b)}.$$

Property A.11 *[5] Let* $\mu \in [0, 1]$, $b \in C^1(\bar{I})$ *and* $v \in H^\mu(I)$. *Then there exists a constant* C *depending only on* b *and* μ *such that*

$$\|bv\|_{H^\mu(I)} \leq C\|v\|_{H^\mu(I)}.$$

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