VARIATIONAL THEOREMS FOR SUPERIMPOSED MOTIONS IN ELASTICITY, WITH APPLICATION TO BEAMS

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SUMMARY

This study presents variational theorems for a theory of small motions superimposed on large static deformations and governing equations for prestressed beams on the basis of 3-D theory of elastodynamics. First, the principle of virtual work is modified through Friedrichs's transformation so as to describe the initial stress problem of elastodynamics. Next, the modified principle together with a chosen displacement field is used to derive a set of 1-D macroscopic governing equations of prestressed beams. The resulting equations describe all the types of superimposed motions in elastic beams, and they include all the effects of transverse shear and normal strains, and the rotatory inertia. The instability of the governing equations is discussed briefly.

INTRODUCTION

Small motions superimposed upon large static deformations have been tackled by a variety of investigators. And differential as well as variational formulations have been derived for both the so-called initial stress and initial strain problems (see, e.g., refs. 1-3, and references cited there). A classical variational formulation for the initial stress problem is deduced from a general principle of physics and has certain advantages over a differential formulation (see, e.g., ref. 3, where the principle of virtual work is taken as fundamental). This yields only the stress equations of motion and the natural boundary conditions. The remaining equations of the initial stress problem should be introduced as constraints. The constraints, however, can be removed through Friedrichs's transformation. This has been illustrated by de Veubeke (ref. 4) for classical elastodynamics.

All the past efforts reveal how the static and dynamic behavior of structures may significantly change by the presence of initial stress or initial strain. Among those, we mention here references 5-8 and references 9-12 on initially stressed shells and plates, respectively. On initially stressed beams, the works of Brunelle (ref. 13) and Sun (ref. 14) are cited. Brunelle derived the governing equations for a prestressed, transversely isotropic beam via the direct integration of 3-D field equations. Sun studied the equations for a Timoshenko beam having an initial, in-plane compressive stress by the use of both Trefftz's and Biot's formulations.

The purpose of this investigation is twofold. The first aim is to modify the principle of virtual work, and then to obtain a generalized variational theorem which describes an arbitrary state of initial stress. The procedure used in achieving this is analogous to the one used in reference 4. The second aim is to construct the governing equations of anisotropic beams under initial stress by the use of the generalized variational theorem together with an incremental displacement field chosen a priori. The displacement field allows to include all the effects of transverse shear and normal strains, and the rotatory inertia for the prestressed thick beam in which they are significant. The resulting equations describe all the types of superimposed extensional, flexural, and torsional motions of thick anisotropic, elastic beam of uniform cross section. The dynamic instability of the prestressed beam is also discussed.

SYMBOLS

In a Euclidaen 3-space, Cartesian tensors are used, and Einstein's summation convention is implied for all repeated Latin (1,2,3) and Greek (1,2) indices, unless indices are put within parantheses.

L, A; C	length and cross-sectional area of beam; Jordan curve which bounds A
V, S	entire volume of beam and its total boundary surface
S', S"	complementary subsurfaces of S, where stresses and displacements are, respectively, prescribed
x_i; x_α, x_3	a system of right-handed Cartesian convected coordinates; lateral coordinates and beam axis
u_i, $u_i^{m,n}$	components of displacement vector, displacement functions of order (m,n)
ρ	mass density
n_i, ν_i	components of unit outward vector normal to S and C
ε_{ij}, s_{ij}	components of strain and symmetric stress tensors
Θ	prescribed steady temperature field

C_{ijkl}, α_{ij}	isothermal elastic stiffnesses and strain-temperature constants
I_{mn}	moment of inertia of order (m,n)
$a_i = \ddot{u}_i$, t_i	components of acceleration and traction vectors
$T_{ij}^{m,n}$	stress resultants of order (m,n)
$F_i^{m,n}$, $A_i^{m,n}$	body force and acceleration resultants of order (m,n)
$Q_i^{m,n}$, $P_i^{m,n}$	effective load and external force of order (m,n)
$(\dot{\ })$, $(\ )_{,i}$	partial differentiation with respect to time, t, and x_i
$(^0)$, $(^*)$	field quantities belong to the reference state and prescribed quantities
C_{mn}	functions with derivatives of order up to and including m and n with respect to space coordinates x_i and time,t

FUNDAMENTAL EQUATIONS

Consider a simply connected elastic body V+S, with its boundary S, in a 3-D Euclidean space Ξ. The elastic body is referred to a x_i-fixed system of Cartesian convected coordinates in this space. When this body is prestressed, we distinguish two states of the body: its reference (or initial) and spatial (or final) state. The reference state is considered to be self-equilibrating following static loading in the natural (or undisturbed) state of the body at time, $t=t_0$. We may summarize, for ease of quick reference, the fundamental equations (see, e.g., ref. 2) in the form

$$s^0_{ij,i} + \rho^0 f^0_j = 0 \quad \text{in} \quad V \tag{1}$$

$$n_i s^0_{ij} - t^{0*}_j = 0 \quad \text{on} \quad S' \ , \quad u^0_i - u^{0*}_i = 0 \quad \text{on} \quad S'' \tag{2}$$

$$s^0_{ij} = C_{ijkl}\, \varepsilon^0_{kl} \qquad \varepsilon^0_{ij} = 1/2(u^0_{i,j} + u^0_{j,i}) \quad \text{in} \quad V \tag{3}$$

for this state. Here, ρ^0 is the known mass density of the body material, s^0_{ij} the symmetric stress tensor, f^0_i the body force vector per unit mass in V, u^0_i the displacement vector, n_i the unit outward vector normal to S, u^{0*}_i and t^{0*}_i the prescribed displacement and traction vectors on the complementary subsurfaces S" and S' of S, ε^0_{ij} the linear strain tensor, and C_{ijkl} ($C_{ijkl} = C_{jikl} = C_{klij}$) the isothermal elastic stiffnesses.

Now, suppose that an infinitesimal (or small) motion is superimposed upon the reference state. For this motion, we have the following fundamental equations:

$$(s_{ij} + s^0_{ir}u_{j,r})_{,i} + \rho^0(f_j - a_j) = 0 \quad \text{in} \quad V \tag{4}$$

$$n_i(s_{ij} + s^0_{ir}u_{j,r}) - t^*_j = 0 \quad \text{on} \quad S' \tag{5}$$

$$u_i - u^*_i = 0 \quad \text{on} \quad S'' \tag{6}$$

$$\varepsilon_{ij} = 1/2(u_{i,j} + u_{j,i}) \quad \text{in} \quad V \tag{7}$$

$$s_{ij} = C_{ijkl}(\varepsilon_{kl} - \Theta\alpha_{kl}) \quad \text{in} \quad V \tag{8}$$

$$u_i - v^*_i = 0 \quad \& \quad \dot{u}_i - w^*_i = 0 \quad \text{in} \quad V(t_0) \tag{9}$$

in the spatial state. In these equations, s_{ij}, u_i, t_i and so on indicate small incremental quantities superimposed upon those of the reference state (i.e., s^0_{ij}, u^0_i, t^0_i). And $a_i = \ddot{u}_i$ is the acceleration vector, v^*_i and w_i^* are the prescribed displacement vectors. Θ is an incremental prescribed steady temperature field and $\alpha_{ij} = \alpha_{ji}$ the strain-temperature coefficients at constant stress. Also, $V(t_0)$ is used to designate V at $t=t_0$.

Equations (1)-(9) describe completely the initial stress problem of interest.

VARIATIONAL THEOREMS

To begin with, we express a principle of virtual work as the assertation

$$\int_V (s^0_{ij} + s_{ij})\delta\gamma_{ij}dV = \int_V \rho^0(f^0_i + f_i)\delta u_i dV - \int_V \rho^0 a_i \delta u_i dV$$
$$+ \int_S (t^0_i + t^*_i)\delta u_i dS \tag{10}$$

in the spatial state. Here, γ_{ij} denotes the Lagrangian strain tensor, and it is given by

$$\gamma_{ij} = \varepsilon_{ij} + 1/2(u_{i,r}u_{j,r}) \tag{11}$$

In equation (10), through the use of equation (11), we first carry out the indicated variations, apply Green - Gauss integral transformations and combine the resulting surface and volume integrals. Next, we recall the usual arguments on incremental field quantities (see, e.g., ref. 2), take into account equations (1) and (2), and finally arrive at the variational equation of the form:

$$\delta J = \delta J_{\alpha\alpha} = 0 \qquad (12\ a)$$

with

$$\delta J_{11} = \int_V (s_{ij} + s^0_{ir} u_{j,r})_{,i} \delta u_i dV + \int_V \rho^0 (f_i - a_i) \delta u_i dV$$

$$\delta J_{22} = \int_S [(s_{ij} + s^0_{ir} u_{j,r}) n_i - t^*_j] \delta u_j dS \qquad (12\ b)$$

The variations of displacements are arbitrary and independent in this equation. Hence, equation (12) leads evidently to the stress equations of motion (4) in V and the natural boundary conditions (5) on S, as the appropriate Euler equations.

Variational Theorem: Let V+S denote a regular, finite region of space (see, e.g., ref. 15) in Ξ, with its boundary S, and define the functional J whose the first variation is given by equation (12). Then, of all the admissible displacement states $u_i \varepsilon C_{12}$, if and only if, the one which satisfies the stress equations of motion (4) and the natural boundary conditions (5) as the appropriate Euler equations, renders $\delta J = 0$.

This is a one-field variational theorem in which equations (6) -(9) of the initial stress problem remain to be satisfied as constraints.

To include the rest of equations of the initial stress problem in the variational formulation, we introduce dislocation potentials and use Friedrichs's transformation, and we closely follow de Veubeke (ref. 4). Thus, we obtain the following theorem.

Generalized Variational Theorem: Let V+S denote a regular, finite region of space in Ξ, with its boundary S ($S' \cap S''=0$ and $S' \cup S''=S$), and define the functional I whose first variation is given by

$$\delta I = \delta I_{ii} + \delta J_{11} \qquad (13\ a)$$

with

$$\delta I_{11} = \int_{S'} [(s_{ij} + s^0_{ir} u_{j,r}) n_i - t^*_j] \delta u_j dS$$

$$+ \int_{S''} (u_i - u^*_i) \delta t_i dS \qquad (13\ b)$$

$$\delta I_{22} = \int_V [s_{ij} - C_{ijkl}(\varepsilon_{kl} - \Theta\alpha_{kl})] \delta \varepsilon_{ij} dV \qquad (13\ c)$$

$$\delta I_{33} = \int_V [\varepsilon_{ij} - 1/2(u_{i,j} + u_{j,i})] \delta s_{ij} dV \qquad (13\ d)$$

Then, of all the admissible states of $u_i \varepsilon C_{12}$, $\varepsilon_{ij} \varepsilon C_{00}$, $t_i \varepsilon C_{00}$, and

$s_{ij} \epsilon C_{1o}$, if and only if, those which satisfy the stress equations of motion (4) in V, the natural boundary conditions (5) and (6) for displacements and tractions on S" and S', the strain-displacement relations (7) in V, and the constitutive equations (8 in V, as the appropriate Euler equations, render $\delta I=0$.

In the generalized variational equation (13), the incremental field quantities (s_{ij}, u_i, t_i, and ϵ_{ij}) are varied independently. And this is a four-field variational theorem. The admissible states are not required to meet any of the fundamental equations o the initial stress problem but the initial conditions (9) only.

BEAMS UNDER INITIAL STRESS

Geometry and Kinematics

A straight elastic beam is embedded in the space Ξ. The beam is of uniform cross section, A, and it occupies a regular, finite region of space V with its boundary S in Ξ. The total surface S consists of two right and left faces, A_r and A_1, and a cylindrical lateral surface S_1. The beam is referred to the x_i-system of Cartesian convected coordinates located at the centroid of A_1. The x_3-axis is chosen to be the beam axis, and the x_α-axes indicate the principal axes of A which is bounded by a Jordan curve C. The beam is under an initial stress field in the reference state.

The incremental displacements of the prestressed elastic beam are taken of the form:

$$u_i(x_j,t) = \sum_{m,n=0}^{M=1} [x_1^m x_2^n \; u_i^{(m,n)}] \tag{14}$$

Here, the $u_i^{(m,n)}$ are functions of x_3 and time, t, only. These terms readily accommodate low-frequency extensional, flexural and torsional superimposed motions. However, it should be kept in mind that, in the case of torsion, equation (14) can represent only the displacements of beams of elliptic and circular cross-sections, and for all other sections, more terms should be retaine in the expansion. The displacement field (14) is like the one Mindlin (ref. 16) used in his recent derivation of the governing equations for a non-initially stressed elastic bar.

Stress and Load Resultants

We define the stress resultants of order (m,n):

$$T_{ij}^{(m,n)} = \int_A x_1^m x_2^n \; s_{ij} dA \tag{15 a}$$

This represents the weighted, averaged values of stress tensor over a cross section of the prestressed beam in the reference state.

In addition, we introduce the body force, acceleration and load resultants, and the moment of inertia of order (m,n):

$$F_i^{(m,n)} = \int_A x_1^m x_2^n \, f_i dA, \quad T_i^{*(m,n)} = \int_A x_1^m x_2^n \, t_i^* dA$$

$$I_{mn} = \int_A x_1^m x_2^n \, dA, \quad A_i^{(m,n)} = \sum_{p,q=0}^{M=1} I_{m+p,n+q} \ddot{u}_i^{(p,q)}$$

$$[P_i^{(m,n)}, \; P_i^{0(m,n)}] = \oint_C x_1^m x_2^n \nu_\alpha \, [s_{\alpha i}, \; s_{\alpha i}^0] ds$$

$$R_i^{0(m,n)} = \sum_{p,q=0}^{M=1} [\; (pP_1^{0(m+p-1,n+q)} + qP_2^{0(m+p,n+q-1)}) u_i^{(p,q)} + P_3^{0(m+p,n+q)} u_{i,3}^{(p,q)} \;]$$

$$V_i^{0(m,n)} = \sum_{p,q=0}^{M=1} \{ \; [mpT_{11}^{0(m+p-2,n+q)} + (np+mq) T_{12}^{0(m+p-1,n+q-1)} + qnT_{22}^{0(m+p,n+q-1)} + pT_{31,3}^{0(m+p-1,n+q)} + qT_{32,3}^{0(m+p,n+q-1)}] u_i^{(p,q)} + T_{33}^{0(m+p,n+q)} u_{i,33}^{(p,q)} + [(p+m) T_{13}^{0(m+p-1,n+q)} + (q+n) T_{23}^{0(m+p,n+q-1)} + T_{33,3}^{0(m+p,n+q-1)}] u_{i,3}^{(p,q)} \; \}$$

$$\;{}_{3i}^{0(m,n)} = \sum_{p,q=0}^{M=1} [\; (pT_{31}^{0(m+p-1,n+q)} + qT_{32}^{0(m+p,n+q-1)}) u_i^{(p,q)} + T_{33}^{0(m+p,n+q)} u_{i,3}^{(p,q)} \;] \qquad (15\text{ b})$$

Prestressed Beam Equations – Instability

Now, we shall derive the prestressed beam equations by the se of the generalized variational theorem (13) together with the ncremental displacement field (14). First, upon substituting he expansion (14) into equation (13 a), we find the variational quation (16). In this equation, the variations $\delta u_i^{m,n}$ are rbitrary and independent for any choice of m(=0,1) and n(=0,1), nd hence it evidently leads to the macroscopic equations of motion (17) as follows:

$$\int_0^L \sum_{m,n=0}^{M=1} U_i^{(m,n)} \delta u_i^{(m,n)} \, dx_3 = 0, \quad m,n=0,1 \tag{16}$$

$$U_i^{(m,n)} = T_{3i,3}^{(m,n)} - mT_{1i}^{(m-1,n)} - nT_{2i}^{(m,n-1)} + P_i^{(m,n)} + Q_i^{0(m,n}$$
$$+ \rho^0(F_i^{(m,n)} - A_i^{(m,n)}) = 0, \quad m,n=0,1 \tag{17}$$

Here, $Q_i^{0(m,n)}$ is the effective initial load given by

$$Q_i^{0(m,n)} = N_i^{0(m,n)} + R_i^{0(m,n)} \tag{18}$$

Similarly, we evaluate the variational equation (13 b) and obtain the natural displacement and traction boundary conditions in the form

$$u_i^{(m,n)} - u_i^{*(m,n)} = 0, \quad m,n=0,1 \quad \text{on} \quad S_1$$
$$T_i^{*(m,n)} + n_3(T_{3i}^{(m,n)} + N_{3i}^{0(m,n)}) = 0, \quad m,n=0,1 \text{ on } A_r \text{ and } A \tag{19}$$

Here, $S'=A_r \cup A_1$ and $S''=S_1$, and $n_3=+1$ for A_r and $n_3=-1$ for A_1.

Upon using of equations (13 c) and (13 d) together with (14), we have the strain distribution:

$$\varepsilon_{ij} = \sum_{m,n=0,1}^{M=1} x_1^m x_2^n \, \varepsilon_{ij}^{(m,n)} (x_3,t) \tag{20 a}$$

with

$$\varepsilon_{ij}^{(m,n)} = 1/2 \, [u_{i,j}^{(m,n)} + u_{j,i}^{(m,n)}$$
$$+ (m+1)(\delta_{1i}u_j^{(m+1,n)} + \delta_{1j}u_i^{(m+1,n)})$$
$$+ (n+1)(\delta_{2i}u_j^{(m,n+1)} + \delta_{2j}u_i^{(m,n+1)}) \,] \tag{20 b}$$

and the macroscopic constitutive equations:

$$T_{ij}^{(m,n)} = C_{ijkl} \sum_{p,q=1}^{M=1} I_{m+p,n+q}(\varepsilon_{kl}^{(p,q)} - \alpha_{kl}\theta^{(p,q)}) \tag{21}$$

where we take the temperature increment of the form:

$$\Theta(x_i) = \sum_{m,n=0}^{M=1} x_1^m x_2^n \, \Theta^{(m,n)}(x_3) \qquad (22)$$

Lastly, the initial conditions, based on equations (9) and (14),

$$u_i^{(m,n)} - v_i^{*(m,n)} = 0 \quad \dot{u}_i^{(m,n)} - w_i^{*(m,n)} = 0 \quad \text{in } L(t_o) \qquad (23)$$

complete the beam equations (cf., ref. 17, where non-initially stressed beams are treated) under an arbitrary state of initial stress field.

The beam equations of equilibrium may be derived similarly on the basis of equations (1)-(3); they are not written out here in order to conserve space.

To examine the stability of the prestressed beam equations, we first consider the beam with a set of initial forces χ. Next, we replace χ by a prescribed set χ^*. And, as usual, we arrive at a system of linear homogeneous differential equations which describes the instability problem under consideration. The sets are defined by

$$\chi = (T_{ij}^{0(m,n)} \text{ in } L,\ F_i^{0(m,n)} \text{ in } L,\ T_i^{0(m,n)} \text{ on } A)$$

$$\chi^* = \lambda(T_{ij}^{0*(m,n)} \text{ in } L,\ F_i^{0*(m,n)} \text{ in } L,\ T_i^{0*(m,n)} \text{ on } A)$$

here λ is a monotonically increasing factor, and whenever it eaches certain values the equilibrating reference configuration ecomes unstable. The behavior of the eigenvalues of this factor s to be investigated in each particular case of interest. Some xamples of instability will be reported elsewhere.

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