# Variational Theory for the Total Scalar Curvature Functional for Riemannian Metrics and Related Topics 

Richard M. Schoen<br>Mathematics Department<br>Stanford University<br>Stanford, CA 94305

The contents of this paper correspond roughly to that of the author's lecture series given at Montecatini in July 1987. This paper is divided into five sections. In the first we present the Einstein-Hilbert variational problem on the space of Riemannian metrics on a compact closed manifold $M$. We compute the first and second variation and observe the distinction which arises between conformal directions and their orthogonal complements. We discuss variational characterizations of constant curvature metrics, and give a proof of Obata's uniqueness theorem. Much of the material in this section can be found in papers of Berger-Ebin [3], Fischer-Marsden [8], N. Koiso [14], and also in the recent book by A. Besse [4] where the reader will find additional references.

In $\S 2$ we give a general discussion of the Yamabe problem and its resolution. We also give a detailed analysis of the solutions of the Yamabe equation for the product conformal structure on $S^{1}(T) \times S^{n-1}(1)$, a circle of radius $T$ crossed with a sphere of radius one. These exhibit interesting variational features such as symmetry breaking and the existence of solutions with high Morse index. Since the time of the summer school in Montecatini, the beautiful survey paper of J. Lee and T. Parker [15] has appeared. This gives a detailed discussion of the Yamabe problem along with a new argument unifying the work of T. Aubin [1] with that of the author.
$\S 3$ contains an a priori estimate on arbitrary (nonminimizing) solutions of the Yamabe problem in terms of a bound on the energy. The estimate applies uniformly to solutions of the subcritical equation, and implies that solutions of the subcritical equation converge in $C^{2}$ norm to solutions of the Yamabe equation. These estimates hold on manifolds which are not conformally diffeomorphic to the standard sphere. We present here the result for locally conformally flat metrics. This estimate has not appeared in print prior to this paper although
we discovered it some time ago.
In $\S 4$ we discuss asymptotically flat manifolds and total energy for $n$-dimensional manifolds. We discuss the positive energy theorems which are needed for the Yamabe problem. We give a detailed $n$-dimensional proof of the author's work with S.T. Yau [25], [26] which proves the positive energy theorem through the use of volume minimizing hypersurfaces. The proof we give works for $n \leq 7$ in which dimensions we have complete regularity of volume minimizing hypersurfaces. Along with the locally conformally flat case which is treated in [29], this covers all cases which are used in the resolution of the Yamabe problem. We note that E. Witten's [34] proof implies this theorem under the (topological) assumption that the manifold is spin. The $n$-dimensional proof is given in $[2,15]$.

Finally in the last section we discuss weak solutions of the Yamabe equation on $S^{n}$ with prescribed singular set. We motivate this through the example of $\S 2$ which gives the solutions with two singular points. We also relate weak solutions to the geometry of locally conformally flat manifolds describing some of the results of [29]. Lastly we give a brief account of the author's existence theorem [24] for weak solutions with prescribed singular set.

## 1 The variational problem

Let $M$ ber a smooth $n$-dimensional compact manifold without bounday: For any smooth Riemanian metric $g$ on $M$ we let $w_{g}$ denote the volume form of $g$; thus if $x^{1}, \ldots, x^{n}$ are local coordinates on $M$ we have

$$
g=\sum_{i, j=1}^{n} g_{i j}(x) d x^{i} d x^{j}, \quad w_{g}=\sqrt{\operatorname{det}\left(g_{i j}\right)} d x^{1} \wedge \ldots \wedge d x^{n}
$$

Let $\mathcal{M}$ denote the space of all smooth Riemamian metrics on $M$, and let $\mathcal{M}_{1}$ denote the subset of $\mathcal{M}$ consisting of those metrics of total whmes one: that is,

$$
\operatorname{Vol}(g)=\int_{M} d w_{g}=1
$$

Let Ricm $(g), \operatorname{Ric}(g), R(g)$ denote the Riemann curvature tensor, the Riccitensor, and the scalar curvature respectively. In local coordinates we have

$$
\begin{aligned}
\operatorname{Riem}(g) & =\sum_{i, j, k, t} R_{i j k f}\left(d x^{i} \wedge d x^{j}\right) \sigma\left(d x^{k} \wedge d x^{\ell}\right) \\
\operatorname{Ric}(g) & =\sum_{i, j} R_{i j} d x^{i} d x^{j} . \quad R_{i j}=\sum_{k, f} g^{k t} R_{i k j \ell} \\
R(g) & =\sum_{i, j} g^{i j} R_{i j} .
\end{aligned}
$$

The (elliptic) Einstin equations then express the condition that the trace-free part of the Ricci tensor vanishes, that is

$$
\begin{equation*}
\operatorname{Ric}(g)=\frac{1}{n} R(g) g \tag{1.1}
\end{equation*}
$$

The contracted second Bianchi identity implies

$$
\sum_{j, k}^{n} g^{j k}\left(R_{i j}-\frac{1}{2} R(g) g_{i j}\right)_{k}=0 . \quad i=1, \ldots, n
$$

where the semi-colon denotes the covariant derivative of a tensor with respect to the LeviCivita connection of $g$. Thus for $n \geq 3$ we see that (1.1) implies

$$
\begin{equation*}
R(g) \equiv \mathcal{R}(g) \tag{1.2}
\end{equation*}
$$

where $\mathcal{R}(g)=\operatorname{Vol}(g)^{-1} \int_{M} R(g) d \omega_{g}$. It was shown by Hilbert that equation (1.1) arises as the Euler-Lagrange equations for the functional $\mathcal{R}(g)$ on the space $\mathcal{M}_{1}$. This may seem surprising since (1.1) is a second order equation for $g$ while the integrand $R(g)$ of $\mathcal{R}(g)$ also involves second derivatives of $g$. To see that this is correct, let $g \in \mathcal{M}_{1}$ and let $h$ be any smooth symmetric tensor of type $(0,2)$ on $\mathcal{M}$. We then set $g(t)=g+t h$ for $t \in(-\varepsilon, \varepsilon)$, and this gives us a family of Riemannian metrics. The normalized family $\bar{g}(t)=V(t)^{-2 / n} g(t)$,
$V(t)=\operatorname{Vol}(g+t h)$ is then a path in $\mathcal{M}_{1}$. We have the formulae

$$
\begin{aligned}
R_{i j} & =\sum_{k}\left\{\Gamma_{i, j, k}^{k}-\Gamma_{k i, j}^{k}+\sum_{\ell}\left(\Gamma_{k \ell}^{k} \Gamma_{j i}^{\ell}-\Gamma_{j \ell}^{k} \Gamma_{k i}^{\ell}\right)\right\} \\
\Gamma_{i j}^{k} & =\frac{1}{2} \sum_{\ell} g^{k \ell}\left(g_{i \ell, j}+g_{j \ell, i}-g_{i j, \ell}\right)
\end{aligned}
$$

where the comma denotes the partial derivative in a local coordinate system. Using an "upper dot" to denote the derivative with respect to $t$, we have

$$
\begin{align*}
& \dot{R}_{i j}=\sum_{k}\left(\dot{\Gamma}_{i ; ; k}^{k}-\dot{\Gamma}_{k i j j}^{k}\right)  \tag{1.3}\\
& \dot{\Gamma}_{i j}^{k}=\frac{1}{2} \sum g^{k \ell}\left(h_{i \ell ; j}+h_{j \ell ; i}-h_{i j ; \ell}\right)
\end{align*}
$$

Therefore we find that $\dot{R}$ has the expression

$$
\dot{R}=-\sum_{i, j} h^{i 3} R_{i j}+\text { divergence terms }
$$

where $h^{i j}=\sum_{k, \ell} g^{i k} g^{j \ell} h_{k \ell}$. Upon integration we find

$$
\begin{aligned}
\frac{d}{d t} \int_{M} R(g(t)) d \omega_{g(t)} & =-\int_{M}(h, \operatorname{Ric}(g(t))\rangle_{g(t)} d \omega_{g(t)}+\frac{1}{2} \int_{M} R(g(t)) \operatorname{Tr}_{g(t)}(h) d \omega_{g(t)} \\
& =-\int_{M}\left(h, \operatorname{Ric}(g(t))-\frac{1}{2} R(g(t)) g(t)\right\rangle_{g(t)} d \omega_{g(t)}
\end{aligned}
$$

where we have used Stoke's theorem together with the formulas

$$
\begin{aligned}
\dot{w}_{g(t)} & =\frac{1}{2} \operatorname{Tr}_{g(t)}(h) w_{g(t)}, \\
\operatorname{Tr}_{g(t)}(h) & =\sum_{i, j} g(t)^{i j} h_{i j} .
\end{aligned}
$$

Now we have $\mathcal{R}(\bar{g}(t))=V(t)^{(2-n) / n} \int_{M} R(g(t)) d \omega_{g(t)}$, and hence we find

$$
\begin{aligned}
\frac{d}{d t} \mathcal{R}(\bar{g}(t)) & =-V(t)^{(2-n) / n} \int_{M}\langle h, F(g(t))\rangle_{g(t)} d \omega_{g(t)} \quad \text { where } \\
F(g) & =\operatorname{Ric}(g)-\frac{1}{2} R(g) g+\frac{n-2}{2 n} \mathcal{R}(g) g
\end{aligned}
$$

To derive this expression we have used, in addition to our computation above, the formula $\dot{V}(t)=\frac{1}{2} \int_{M}\langle h, g(t)\rangle_{g(t)} d \omega_{g(t)}$. Therefore, if $g$ is a critical point for $\mathcal{R}(\cdot)$ on $\mathcal{M}_{1}$ we find, setting $t=0$, that $F(g) \equiv 0$. In particular, it follows that the trace-free part of $\operatorname{Ric}(g)$ vanishes and hence (1.1) holds.

Now suppose $g$ is a solution of (1.1) so that in particular $F(g) \equiv 0$. We compute the second variation of $\mathcal{R}(\cdot)$ at $g$. We have

$$
\begin{equation*}
\left.\frac{d^{2}}{d t^{2}} \mathcal{R}(\bar{g}(t))\right|_{t=0}=-\int_{M}\langle h, \mathcal{L} h\rangle_{g} d \omega_{g} \tag{1.4}
\end{equation*}
$$

where $\mathcal{L} h=\dot{F}(g(0))$. Thus $\mathcal{L}$ is a linear operator on symmetric $(0,2)$ tensors given by

$$
\begin{equation*}
\mathcal{L} h=\operatorname{Ric}(g)-\frac{1}{2} \dot{R} g-\frac{1}{n} R h \tag{1.5}
\end{equation*}
$$

which Ric $(g), \dot{R}$ may be computed from (1.3) and we have used (1.2) and the fact that $\dot{\mathcal{R}}(g)=0$. We write the space of symmetric $(0,2)$ tensors as a sum of three subspaces $S_{0}, S_{1}, S_{2}$ where $S_{0}$ denotes those $h$ which may be written $h=L_{\mathrm{X}} g$ (Lie derivative) for some vector field X on $M$, that is,

$$
h_{i j}=\mathbf{X}_{i, j}+\mathbf{X}_{j ; i}
$$

(The fact that this decomposition of smooth (0,2) tensors is valid is shown in [8].) The subspace $S_{1}$ denotes the pure trace tensors, that is, the $h$ of the form $h=\eta g$ where $\eta$ is a smooth function on $M$. Finally $S_{2}$ denotes those $h$ which are orthogonal to both $S_{0}$ and $S_{1}$, that is, those $h$ satisfying $\operatorname{Tr}_{g}(h)=0$ and $\sum_{j, k} g^{j k} h_{i j ; k}=0$. Tensors $h \in S_{2}$ are referred to as transverse traceless tensors. Note that the subspace $S_{0}$ consists of those infinitesimal deformations of $g$ which arise from diffeomorphisms of $M$. It follows that if $\mathbf{X}$ is a vector field on $M$ and $\Phi_{i}: M \rightarrow M$ is the one-parameter group of diffeomorphisms generated by $\mathbf{X}$, then we have for each $t \in \mathbf{R}, F\left(\Phi_{t}^{*}\right)=0$. Differentiating and setting $t=0$ we have $\mathcal{L} h=0$ where $h=L_{\mathbf{x}} g$. Thus $\mathcal{L} \equiv 0$ on $S_{0}$. We now compute $\mathcal{L} h$ for $h \in S_{1}$. Suppose $h=\eta g$ where $\eta$ is a smooth function. We then have from (1.3), (1.5)

$$
\begin{equation*}
\mathcal{L} h=\frac{n-2}{2}\left(\left(\Delta \eta+\frac{1}{n} R \eta\right) g-\operatorname{Hess}(\eta)\right) \tag{1.6}
\end{equation*}
$$

where Hess $(\eta)=\sum_{i, j} \eta_{i, j} d x^{i} d x^{j}$ is the Hessian of $\eta$. Now we have Hess $(\eta) \in S_{0}$, so we see that $S_{0}+S_{1}$ is invariant under $\mathcal{L}$.

Next we show that $\mathcal{L}$ is a self-adjoint operator. This may be seen from the variational definition of $\mathcal{L}$ by considering two symmetric $(0,2)$ tensors $h, k$ and the two parameter variation $g(t, s)=g+t h+s k$. Let $\bar{g}=V(t, s)^{-2 / n} g(t, s)$ be the normalized variation. We then have from above

$$
\frac{\partial \mathcal{R}(\bar{g}(t, s))}{\partial t}=-V(t, s)^{(2-n) / n} \int_{M}\langle h, F(g(t, s))\rangle_{g(t, s)} d \omega_{g(t, s)} .
$$

Differentiating in $s$ and setting $t=s=0$ we have

$$
\left.\frac{\partial^{2} \mathcal{R}(\bar{g}(t, s))}{\partial s \partial t}\right|_{t=s=0}=-\int_{M}\langle h, \mathcal{L} k\rangle_{g} d \omega_{g} .
$$

Reversing the order of differentiation for the smooth function $\mathcal{R}(\bar{g}(t, s))$ of two variables we get

$$
\int_{M}\langle h, \mathcal{L} k\rangle_{g} d \omega_{g}=\int_{M}\langle k, \mathcal{L} h\rangle_{g} d \omega_{g}
$$

for all $h, k$. Thus $\mathcal{L}$ is self-adjoint.

Now since $S_{0}+S_{1}$ is $\mathcal{L}$-invariant and $\mathcal{L}$ is self-adjoint it follows that $S_{2}=\left(S_{0}+S_{1}\right)^{\perp} \cap C^{\infty}$ is also $\mathcal{L}$-invariant. We compute $\mathcal{L} h$ for $h \in S_{2}$ using (1.3), (1.5)

$$
(\mathcal{L} h)_{i j}=-\frac{1}{2}(\Delta h)_{i j}+\frac{1}{2} \sum_{k, \ell} g^{k \ell}\left(h_{i k ; j \ell}+h_{j k ; i \ell}\right)-\frac{1}{n} R h_{i j}
$$

where $\Delta h$ is the trace Laplacian given by

$$
(\Delta h)_{i j}=\sum_{k, \ell} g^{k \ell} h_{i j ; k \ell}
$$

Using the transverse (divergence free) condition on $h$ we may interchange covariant derivatives and write the second term above as a zero order term in $h$

$$
\begin{equation*}
\mathcal{L} h=-\frac{1}{2} \Delta h+K(h) \tag{1.7}
\end{equation*}
$$

where $K(h)$ is the linear term

$$
(K(h))_{i j}=-\sum_{k, \ell} R_{i k j \ell} h^{k \ell}+\frac{1}{2} \sum_{k}\left(R_{i k} h_{j}^{k}+R_{j k} h_{i}^{k}\right)-\frac{1}{n} R h_{i j} .
$$

An important qualitative feature of the variational problem is apparent from (1.6) and (1.7), namely that a critical metric $g$ tends to minimize $\mathcal{R}$ among those metrics conformally equivalent to $g$ and to maximize $\mathcal{R}$ among metrics transversely related to $g$. In fact, for $h=\eta g \in S_{1}$, we denote by $\mathcal{L}_{1}$ the second variation operator on the conformal class of $g$. Thus $\mathcal{L}_{1}$ is the operator $\mathcal{L}$ followed by projection into $S_{1}$. Precisely $\mathcal{L}_{1}$ is the scalar operator

$$
\begin{equation*}
\mathcal{L}_{1} \eta=\frac{(n-1)(n-2)}{2 n}\left(\Delta \eta+\frac{1}{n-1} R \eta\right) . \tag{1.8}
\end{equation*}
$$

Thus if we consider the restricted variational problem for the functional $\mathcal{R}(\cdot)$ on the conformal class $[g]$ of $g$ we have for $h=\eta g$

$$
\left.\frac{d^{2}}{d t^{2}} \mathcal{R}(\bar{g}(t))\right|_{t=0}=-n \int_{M} \eta \mathcal{L}_{1} \eta d \omega_{g}
$$

Now the operator $-\mathcal{L}_{1}$ has eigenvalues tending to $+\infty$, and hence the metric $g$ locally minimizes $\mathcal{R}$ in $\mathcal{M}_{1} \cap[g]$ modulo a finite dimensional space of variations (finite Morse index). On the other hand, from (1.7) we see that the operator $-\mathcal{L}$ on $S_{2}$ has eigenvalues tending to $-\infty$ so that $\mathcal{R}(\cdot)$ is locally maximized among variations from $S_{2}$ modulo a finite dimensional subspace.

This dichotomy for the linearized operator $\mathcal{L}$ suggests the following global procedure for finding critical points of $\mathcal{R}(\cdot)$ on $\mathcal{M}_{1}$. For any $g_{0} \in \mathcal{M}_{1}$, let [ $\left.g_{0}\right]$ denote the conformal class of $g_{0}$, that is,

$$
\left[g_{0}\right]=\left\{g \in \mathcal{M}: g=e^{2 v} g_{0} \text { for some } v \in C^{\infty}(M)\right\}
$$

Let $\left[g_{0}\right]_{1}=\mathcal{M}_{1} \cap\left[g_{0}\right]$, and define $I\left(g_{0}\right)$ by

$$
I\left(g_{0}\right)=\inf \left\{\mathcal{R}(g): g \in\left[g_{0}\right]_{1}\right\}
$$

If $g \in\left[g_{0}\right]_{1}$ realizes the infimum, then we see from above that the Euler-Lagrange equation satisfied by $g$ is $\operatorname{Tr}_{g}(F(g)) \equiv 0$, that is. equation (1.2) bolds. If we write $g=u^{4 /(u-2)} g_{0}$ where $u$ is a positive smooth function then we have the formula

$$
R(g)=-C(m)^{-1} u^{-(n+2) /(n-2)} L_{0} u
$$

where $c(n)=\frac{n-2}{4(n-1)}$ and $L_{0}$ is the "conformal Laplacian" for the metric $g_{0}$

$$
L_{0} u=د_{y_{0}} u-c(n) R\left(g_{0}\right) u .
$$

Thus our functional $\mathcal{R}(\cdot)$ becomes $\mathcal{R}(g)=r(n)^{-1} E(u)$ where

$$
E(u)=\int_{M}\left[\mid \nabla_{g_{0}} u \|^{2}+c(n) R\left(g_{0}\right) u^{2}\right] d \omega_{g_{0}} .
$$

The volume constraint on $g$ then becomes $\int_{M} u^{2 n /(n-2)} d \omega_{y_{0}}=1$. The equation (1.2) may then be written

$$
\begin{equation*}
L_{0} u+c(n) \mathcal{R}(g) u^{(n+2) /(n-2)}=0 . \tag{1.9}
\end{equation*}
$$

Since

$$
E(u) \geq \lambda_{0}\left(L_{0}\right) \int_{M} u^{2} d u_{y_{0}} \geq \min \left\{0, \lambda_{0}\left(L_{0}\right)\right\}
$$

where $\lambda_{0}\left(L_{0}\right)$ denotes the lowest eigenvalue of $L_{0}$, we see that $I\left(g_{0}\right)>-\infty$ for any $g_{0}$. We then define $\sigma(M)$ to be the supremum of $I\left(g_{0}\right)$ over all $g_{0} \in \mathcal{M}_{1}$,

$$
\sigma(M)=\sup \left\{\left(g_{0}\right): g_{0} \in \mathcal{M}_{1}\right\}
$$

If we consider constant curvature metrics $g_{0}$ on $S^{n}$ normalized to have volume one, then we have $\mathcal{R}\left(g_{0}\right)=n(n-1) \operatorname{Vol}\left(S^{n}(1)\right)^{2 / n}$ where $S^{n}(1)$ denotes the sphere of radius 1 . The following lemma tells us that the standard metric on $S^{n}$ in fact realizes $\sigma\left(S^{n}\right)$ and provides an upper bound for $\sigma(M)$ for any $n$ dimensional manifold $M$.

Lemma 1.1. We have $\sigma\left(S^{n}\right)=n(n-1) V o l\left(S^{n}(1)^{2 / n}\right.$, and for any $n$-manifold $M$ we have $\sigma(M) \leq \sigma\left(S^{n}\right)$.

Proof: Let $g_{0} \in \mathcal{M}_{1}$ be a metric on $M$. We may show that $l\left(g_{0}\right) \leq n(n-1)$ Vol $\left(S^{2 n}\right)^{2 / n}$ by constructing a metric $g \in[g]$, which is a concentrated spherical metric near a point of $M$. We omit the details and refer the reader to [1, 15,23].

Now let $g_{0}$ be a constant curvature, unit volume metric on $S^{n}$. The fact that $\mathcal{R}\left(g_{0}\right)=I\left(g_{0}\right)$ follows from a symmetrization argument ([21,31]) or from the existence theory together with a uniqueness theorem of M . Obata (see later discussion) as in [15]. Combining these two facts we see that $\sigma\left(S^{n}\right)=\mathcal{R}\left(g_{0}\right) \geq \sigma(M)$ for any $n$-manifold $M$. This completes the proof of Lemma 1.1.

If we have a metric $g \in \mathcal{M}_{1}$ which realizes $\sigma(M)$, that is, $\mathcal{R}(g)=I(g)=\sigma(M)$, we should hope that $g$ is Einstein. This is gencrally true if $\sigma(M) \leq 0$ but is not clear for $\sigma(M)>0$.

To see this for $\sigma(M) \leq 0$, we use the fact that for $I\left(g_{0}\right) \leq 0$ there is a unique solution of (1.9). The existence follows from [32] and uniqueness from the maximum principle. Thus if $h$ is any trace-free $(0,2)$ tensor, we consider the deformed metric $g^{(t)}=g+t h$. There is then a unique function $v^{(t)}>0$ such that $\left(v^{(t)}\right)^{4 /(n-2)} g^{(t)}$ has constant scalar curvature equal to $I\left(g^{(t)}\right)$ since $I\left(g^{(t)}\right) \leq \sigma(M) \leq 0$. The family $v^{(t)}$ is smooth as a function of $t$ (see [14]), so we have $\frac{d}{d t} I\left(g^{(t)}\right)=0$ at $t=0$, and this tells us that the trace-free Ricci tensor of $g$ vanishes and $g$ is Einstein.

We now discuss properties of $\sigma(M)$ and various uniqueness theorems.
Lemma 1.2. Let $M$ be a smooth, closed $n$-dimensional manifold. The invariant $\sigma(M)$ is positive if and only if $M$ admits a metric of positive scalar curvature.

Proof: If $\sigma(M)>0$, then by definition there is a metric $g_{0} \in \mathcal{M}_{1}$ with $I\left(g_{0}\right)>0$. This implies that $\lambda_{0}\left(L_{0}\right)>0$, and hence the lowest eigenfunction $u_{0}$, which may be taken to be positive, satisfies $L_{0} u_{0}<0$. Thus the metric $u_{0}^{4 /(n-2)} g_{0}$ has positive scalar curvature.

Conversely, if $g_{0} \in \mathcal{M}_{1}$ has positive scalar curvature, then $I\left(g_{0}\right)>0$ (see [1]) and hence $\sigma(M)>0$. This completes the proof of Lemma 1.2.

Since many topological obstructions are known for manifolds to admit metrics of positive scalar curvature (see [13,28]), Lemma 1.2 indicates that the invariant $\sigma(M)$ is quite nontrivial. We prove the following uniqueness theorem for constant curvature metrics.

## Proposition 1.3.

1. Let $M=S^{n}$. Any metric $g \in \mathcal{M}_{1}$ which satisfies $\mathcal{R}(g)=I(g)=\sigma(g)$ has constant positive sectional curvature.
2. Suppose that $M$ admits a flat metric. Any metric $g \in \mathcal{M}_{1}$ satisfing $\mathcal{R}(g)=I(g)=$ $\sigma(M)$ is a flat metric. In particular, $\sigma(M)=0$ and any flat metric $g \in \mathcal{M}_{1}$ satisfies $\mathcal{R}(g)=I(g)=\sigma(M)$.

Proof: The proof of the first statement is a consequence of the work $([1,23])$ on the Yamabe problem which shows that $I\left(g_{0}\right)<\sigma\left(S^{n}\right)$ for any $g_{0} \in \mathcal{M}_{1}$ unless $g_{0}$ has constant curvature $\left(M=S^{n}\right)$. Statement 2 follows from $[13,28]$ where it is shown that a flat manifold does not admit a metric of positive scalar curvature (i.e. $\sigma(M) \leq 0$ ), and any scalar flat metric on $M$ is flat. This completes the proof of Proposition 1.3.

There are two obvious uniqueness questions left unresolved for metrics of constant curvature. The first is whether the constant positive curvature metrics $g$ on non-simply connected manifolds achieve the same characterization as the standard metrics on $S^{n}$, i.e. $\mathcal{R}(g)=I(g)=$ $\sigma(M)$. The second question is whether a hyperbolic metric $g$ on $M$ can be characterized similarly. We conjecture that the answer is yes to these questions.

As a final topic in this section we discuss the uniqueness theorem of Obata [18] and its relevance to our variational problem. Let $g_{0} \in \mathcal{M}$, and let $g=u^{4 /(n-2)} g_{0}$ where $u$ is a smooth positive function. Let $T_{0}, T$ denote the trace-free part of the Ricci tensors of $g_{0}, g$ respectively. We then have the formula

$$
\begin{equation*}
T=T_{0}+(n-2) u^{2 /(n-2)}\left[\| \operatorname{less}\left(u^{-2 /(n-2)}\right)-\frac{1}{n} \Delta\left(u^{-2 /(n-2)}\right) g\right] \tag{1.10}
\end{equation*}
$$

which follows from direct computation (see [4]). In (1.10) the Hessian and Laplacian are with respect to $g_{0}$. Assume $g_{0}$ has constant scalar curvature. We then have by the contracted Bianchi identity $\sum_{j, k} g_{0}^{j k}\left(T_{0}\right)_{i j ; k}=0$ for $i=1, \ldots, n$. It follows then from Stoke's theorem

$$
\int_{M}\left\langle T_{0}, \operatorname{Hess}\left(u^{-2 /(n-2)}\right)\right\rangle_{g_{0}} d \omega_{g_{0}}=0
$$

Therefore, we multiply (1.10) by $u^{-2 /(n-2)}$ and integrate its inner product with $T_{0}$ to get

$$
\begin{equation*}
\int_{M} u^{-2 /(n-2)}\left\langle T, T_{0}\right\rangle_{g_{0}} d \omega_{g_{0}}=\int_{M} u^{-2 /(n-2)}\left\|T_{0}\right\|_{g_{0}}^{2} d \omega_{g_{0}} . \tag{1.11}
\end{equation*}
$$

Combining (1.11) with the Schwarz inequality we see that for any constant scalar curvature metric $g_{0}$ and for any $g=u^{4 /(n-2)} g_{0}$ we have

$$
\begin{equation*}
\int_{M} u^{-2 /(n-2)}\left\|T_{0}\right\|_{g_{0}}^{2} d \omega_{g_{0}} \leq \int_{M} u^{-2 /(n-2)}\|T\|_{g_{0}}^{2} d \omega_{g_{0}} \tag{1.12}
\end{equation*}
$$

In particular, if $g$ were Einstein then go would necessarily also be Einstein.
Proposition 1.4. For an Einstein metric $g$ (mit volume) on $M$ we necessarily have $\mathcal{R}(g)=$ $I(g)$. Moreover, any constant scalar curvature metric $g_{0} \in[g]_{1}$ is Einstein. We then have $g_{0}=g$ unless $(M, g)$ is isometric to a round $S^{n}$ in which case $g_{0}$ is a constant curvature metric on $S^{n}$ which is pointwise conformal to $g$.

The main step in the proof of this result is (1.12) which shows that $g_{0}$ is Einstein if it has constant scalar curvature. The analysis of conformally related Einstein metrics on a closed manifold is fairly straightforward (again based on (1.10)) and we omit the details referring the reader to Obata [18] for the complete proof.

A consequence of Proposition 1.4 is that any critical point $g \in \mathcal{M}_{1}$ of $\mathcal{R}(\cdot)$ automatically minimizes in its conformal class and hence has conformal Morse index zero.

We also observe that for $n=3$ inequality (1.12) gives a strong a priori estimate on solutions of the Yamabe equation (1.9). To agree with our earlier notation we let $g_{0} \in \mathcal{M}_{1}$ be a fixed metric and let $g=u^{4 /(n-2)} g_{0}$ have constant scalar curvature. Inequality (1.12) then says for $n=3$ (note that $u$ of (1.12) becomes $u^{-1}$ )

$$
\int_{M} u^{2}\|T\|_{g}^{2} d \omega_{g} \leq \int_{M} u^{2}\left\|T_{0}\right\|_{g}^{2} d \omega_{g}
$$

Since $n=3$ we also have

$$
\int_{M}\left\|T_{0}\right\|_{g_{0}}^{2} d \omega_{g}=\int_{M} u^{2}\left\|T_{0}\right\|_{g}^{2} d \omega_{g}
$$

Therefore we have

$$
\begin{align*}
\left(\int_{M}\left(\|T\|_{g}\right)^{3 / 2} d \omega_{g}\right)^{2 / 3} & \leq\left(\int_{M} u^{-6} d \omega_{g}\right)^{1 / 6}\left(\int_{M} u^{2}\|T\|_{g}^{2} d \omega_{g}\right)^{1 / 2} \\
& \leq\left(\int_{M}\left\|T_{0}\right\|_{g_{0}}^{2} d \omega_{g_{0}}\right)^{1 / 2} \tag{1.13}
\end{align*}
$$

since $u^{-6} d \omega_{g}=d \omega_{g_{0}}$, and we assume $\operatorname{Vol}\left(g_{0}\right)=1$. It follows that the quantity $\int_{M}\left(\|T\|_{g}\right)^{3 / 2} d \omega_{g}$ is a priori bounded (depending only on the background $g_{0}$, hence the conformal class) for any metric $g \in\left[g_{0}\right]$ of constant scalar curvature. Note that $\int_{M}\left(\|T\|_{g}\right)^{3 / 2} d \omega_{g}$ is a dimensionless quantity for $n=3$.

## 2 The Yamabe problem

In this section we discuss solvability of (1.2), or equivalently (1.9). From the previous section we know that (1.2), (1.9) is the Euler-Lagrange equation for the functional $\mathcal{R}(\cdot)$ on $\left[g_{0}\right]_{1}$. An approach to producing solutions of this equation would be to construct a minimizer; that is, a metric $g \in\left[g_{0}\right]_{1}$ such that $\mathcal{R}(g)=I\left(g_{0}\right)$. This approach has been successful as we will outline here.

Historically this problem was studied by H. Yamabe [35] in the early sixties, and was claimed to have been solved in [35]. During the sixties there was substantial development in partial differential equations, and nonlinear problems were being understood more deeply. In particular it was realized [20] that, in many situations, equations such as (1.9) do not have positive solutions. In light of these developments, N . Trudinger re-examined Yamabe's paper and discovered that it contained a serious error. In [32] Trudinger developed analytic machinery relevant to (1.9) and showed that a solution (in fact a minimizer) exists if $I\left(g_{0}\right) \leq 0$ (or if $I\left(g_{0}\right)$ is not too positive). He also proved regularity of $W^{1,2}$ weak solutions of (1.9). This left open the general case with $I\left(g_{0}\right)>0$. The fact that this case is subtle is apparent from the example of $\left(S^{n}, g_{0}\right)$ where $g_{0} \in \mathcal{M}_{1}$ has constant sectional curvature. In this case, $g_{0}$ is itself a solution of (1.2) but is by no means the only solution in $\left[g_{0}\right]_{1}$. In fact, given any conformal transformation $F: S^{n} \rightarrow S^{n}$ we have $F^{*}\left(g_{0}\right) \in\left[g_{0}\right]_{1}$ is another solution of (1.2). Thus if we take a divergent sequence of conformal transformations $F_{i}$ ( such as dilations $F_{i}(x)=i \cdot x$ in stereographic coordinates) we get a divergent sequence of minima for the Yamabe problem on ( $S^{n}, g_{0}$ ). In particular, one cannot obtain uniform estimates on solutions such as would be required to prove existence by usual analytic methods. It follows that any method which produces solutions "with bounds" must distinguish $\left(S^{n}, g_{0}\right)$ from the conformal class one considers. In 1976, T. Aubin [1] proved a general existence result in the positive case. He showed that if $n \geq 6$ and $g_{0}$ is not locally conformally flat then (1.2) has a solution (in fact, a minimizer) $g \in\left[g_{0}\right]_{1}$. A metric $g_{0}$ is said to be locally conformally flat if in a neighborhood of any point of $M$, there exists local coordinate $x^{1}, \ldots, x^{n}$ such that $g_{0}$ is given by

$$
g_{0}=\lambda^{2}(x) \sum_{i=1}^{n}\left(d x^{i}\right)^{2}
$$

for a locally defined positive function $\lambda(x)$. Alternatively, a metric $g_{0}$ is locally conformally flat if any point $p_{0} \in M$ has a neighborhood $\vartheta$ such that $\left(\vartheta, g_{0}\right)$ is conformally equivalent to a subdomain of the standard sphere. In particular, the assumption that $g_{0}$ be not locally conformally flat should be viewed as requiring ( $M, g_{0}$ ) to be far from the standard sphere (which we've seen is a bad case). By a purely local computation Aubin showed that a manifold $\left(M, g_{0}\right)$ with $n \geq 6$ and $g_{0}$ not l.c.f. satisfies $I\left(g_{0}\right)<\sigma\left(S^{n}\right)$ and thus one can derive the necessary estimates to construct a minimizer. We refer the reader to [15] for details and merely describe developments here in a general way. Because Aubin's argument is purely local, there
was no chance that it could work for a locally conformally flat metric, and all attempts to weaken the dimensional restriction ( $n \geq 6$ ) have failed. In 1984 (see [23]) we developed a new global attack on the problem and succeeded in solving (1.9) (again producing a minimizer) for $n=3,4,5$ and for locally conformally flat metrics. We present here the general idea and refer the reader to [15] for details. (In the next section we present an a priori estimate for solutions of (1.2) which are not necessarily minimizers.) The critical metrics one must consider in the Yamabe problem are those which are concentrated near a point $p_{0}$ of $M$, and are very small away from the point. If $g$ denotes such a metric, then we may choose a point $p \neq p_{0}$, and rescale $g$ by multiplication by a large constant so that $g$ agrees with our background metric $g_{0}$ at $p$. If we imagine a sequence of metrics $\left\{g_{i}\right\} \in \mathcal{M}_{1}$ which concentrate near $p_{0}$ and tend to zero at $p$, then by rescaling we get a sequence $\left\{\bar{g}_{i}\right\}$ which are uniformly controlled near $p$. If the scalar curvatures of the $g_{i}$ were bounded, then the scalar curvatures of $\bar{g}_{i}$ tend to zero, and we expect the $\bar{g}_{i}$ to converge to a metric $\bar{g}$ of zero scalar curvature with $\bar{g}$ being a complete metric on $M-\left\{p_{0}\right\}$. (We rigorously carry out this type of rescaling in the next section.) If we write $\bar{g}=G^{4 /(n-2)} g_{0}$ as a function times our background $g_{0}$, then $G$ satisfies $L_{0} G=0$ on $M-\left\{p_{0}\right\}$, and $G>0$. Thus $G$ must be a (multiple of) the fundamental solution of $L_{0}$ with pole at $p_{0}$. Near $p_{0}$, the function $G$ has the behavior $G(x)=|x|^{2-n}+\alpha(x)$ where $\alpha$ has a milder singularity at $x=0$ than $|x|^{2-n}$. Thus near $p_{0}$, the metric $\bar{g}$ approximates $|x|^{-4} \sum_{i} d x_{i}^{2}$ which is simply the metric $\sum d y_{i}^{2}$ on $\mathbf{R}^{n}$ written in the inverted coordinates $y=|x|^{-2} x$. Thus ( $M-\left\{p_{0}\right\}, \bar{g}$ ) is scalar flat and asymptotically llat. In such a situation (in certain cases) there is a number which can be attached to $\bar{g}$ which is referred to as total energy. The reason for this name is that for $n=3$, asymptotically flat manifolds arise as initial data for asymptotically flat spacetimes which model finite isolated gravitating systems in general relativity. The scalar curvature assumption corresponds to (a special case of) the physical assumption that the local energy density of the matter fields be nonnegative. The total energy of a system measures the deviation of $\bar{g}$ from the Euclidean metric a.t infinity, and "positive energy" theorems assert that the total energy is strictly positive unless $\left(M-\left\{p_{0}\right\}, \bar{g}\right)$ is isometric to $\mathbf{R}^{n}$. In [23], it is shown that if $g_{0}$ is locally conformally flat or if $n=3,4,5$ the energy term can be used to show that $I\left(g_{0}\right)<\sigma\left(S^{n}\right)$ unless $\left(M, g_{0}\right)$ is conformally equivalent to the standard $S^{n}$. This implies existence of a minimizer for the Yamabe problem with appropriate estimate. In $\S 4$ we discuss the positive energy theorems which are needed for the Yamabe problem.

For a compact, closed manifold $M$, let $g_{0} \in \mathcal{M}_{1}$, and let $\mathcal{F}$ be given by

$$
\mathcal{F}=\left\{u: u^{4 /(n-2)} g_{0} \in \mathcal{M}_{1}, \mathcal{R}\left(u^{4 /(n-2)} g_{0}\right)=I\left(g_{0}\right)\right\} .
$$

Thus $\mathcal{F}$ is the set of solutions of (1.9) which arise as minimizers for the Yamabe problem. The following compactness theorem is a standard consequence (see [23]) of the inequality $I\left(g_{0}\right)<\sigma\left(S^{n}\right)$.

Proposition 2.1. Suppose ( $M, g_{0}$ ) is not conformally equivalent to the standard sphere. The
set $\mathcal{F}$ is a nonempty compact subset of $C^{2}(M)$, the set of twice continuously differentiable functions on $M$ with the usual $C^{2}$ norm.

As we have observed, the above result is false for the standard sphere because the conformal group of $S^{n}$ is noncompact. We give a geometric corollary which says that any manifold except the standard sphere has a compact conformal group. This result is a theorem of J. LelongFerrand [16].

Corollary 2.2. If ( $M, g_{0}$ ) is not conformally equivalent to the standard sphere, then the group of conformal automorphisms of ( $M, g_{0}$ ) is compact.

Proof: Let $\mathcal{D}$ be the group of conformal diffeomorphisms of ( $M, g_{0}$ ). It suffices to show that $\mathcal{D}$ is compact in the $C^{0}$ topology. The main point is that $\mathcal{D}$ acts on the set $\mathcal{F}$ by pullback; that is, given $F \in \mathcal{D}, u \in \mathcal{F}$ we have $F^{*}\left(u^{4 /(n-2)} g_{0}\right)=\left(u_{F}\right)^{4 /(n-2)} g_{0}, u_{F}=\left|F^{\prime}\right|^{(n-2) / 2} u_{0} F \in \mathcal{F}$. Here we write $F^{*} g_{0}=\left|F^{\prime}\right|^{2} g_{0}$ so that $\left|F^{\prime}\right|$ is a function which measures the stretch factor of $F$ measured with respect to $g_{0}$. Thus the compactness of $\mathcal{F}$ implies that $u_{F} \leq c$ for all $F \in \mathcal{D}$, and hence $\left|F^{\prime}\right|$ is uniformly bounded for all $F \in \mathcal{D}$. Therefore, by the Arzela-Ascoli theorem, $\mathcal{D}$ is a compact subset of $C^{0}(M, M)$. This completes the proof of Corollary 2.2.

There are very few (conformal) manifolds on which one can analyze all solutions of (1.2). Besides the standard sphere, where Obata's theorem tells us that all solutions are minimizing and have constant sectional curvature, the product metrics on $S^{1} \times S^{n-1}$ seem to be the only manifolds where all solutions can be analyzed. In particular, on $S^{1} \times S^{n-1}$ we see many solutions of (1.2) which are not minimizing, and we see situations where the most symmetric solutions are not the minima. For convenience of notation, we dispense with the volume constraint and normalize solutions of (1.2) so that their scalar curvature is equal to $n(n-1)$, the scalar curvature of the unit $n$-sphere. Equation (1.9) then becomes

$$
\begin{equation*}
L_{0} u+\frac{n(n-2)}{4} u^{(n+2) /(n-2)}=0 \tag{2.1}
\end{equation*}
$$

We analyze $S^{1} \times S^{n-1}$ by looking for solutions on the universal covering space $\mathbf{R} \times S^{n-1}$, and we choose $S^{n-1}$ to have unit radius. If we consider the $n-$ sphere to be $\mathbf{R}^{n} \cup\{\infty\}$ where the coordinates $x \in \mathbf{R}^{n}$ arise from stereographic projection, then the manifold $\mathbf{R} \times S^{n-1}$ is conformally equivalent to $S^{n}-\{0, \infty\}=\mathbf{R}^{n}-\{0\}$. The conformal diffeomorphism is given explicitly by sending the point $x \in \mathbf{R}^{n}-\{0\}$ to the point $(\log |x|, x /|x|) \in \mathbf{R} \times S^{n-1}$. Thus the analysis of solutions of (1.9) on $\mathbf{R} \times S^{n-1}$ is completely equivalent to the analysis of solutions of (1.9) on $\mathbf{R}^{n}-\{0\}$. An important method was introduced into the subject, by Gidas, Ni , and Nirenberg [9] which enables one to show that, under suitable conditions, arbitrary solutions of (1.9) have a maximal amount of symmetry. For solutions on $S^{n}-\{0, \infty\}$ it has been shown by Caffarelli, Gidas, Spruck [5] that any solution of (1.9) which is singular at either 0 or $\infty$ is necessarily singular at both 0 and $\infty$, and such a solution is a radial function, that is, a function of $|x|$. We are interested in complete metrics on $\mathbf{R} \times S^{n-1}$ and hence we want
solutions singular at both 0 and $\infty$. We will write (2.1) with respect to the product metric $g_{0}=d t^{2}+d \xi^{2}$ on $\mathbf{R} \times S^{n-1}$ where ( $t, \xi$ ) denote coordinates on $\mathbf{R} \times S^{n-1}$, and $d \xi^{2}$ is used to denote the metric on the unit, $S^{n-1}$. We then have $R\left(g_{0}\right)=(n-1)(n-2)$, and for a function $u(t)$ (which any global solution will be from the above discussion), equation (2.1) becomes

$$
\begin{equation*}
\frac{d^{2} u}{d t^{2}}-\frac{(n-2)^{2}}{4} u+\frac{n(n-2)}{4} u^{(n+2) /(n-2)}=0 . \tag{2.2}
\end{equation*}
$$

We are interested in positive solutions of (2.2) defined on all of $\mathbf{R}$. There are two obvious nonzero solutions of (2.2). The first is the constant solution

$$
\begin{equation*}
u(t) \equiv u_{0}=\left(\frac{n-2}{n}\right)^{(n-2) / 4} \tag{2.3}
\end{equation*}
$$

Geometrically, $u_{0}^{4 /(n-2)} g_{0}$ is that multiple of $g_{0}$ having scalar curvature $n(n-1)$. The second explicit solution is a solution of constant sectional curvature. The spherical metric $g_{1}$ on $\mathbf{R}^{n}$ takes the form $g_{1}=4\left(1+|x|^{2}\right)^{-2} \sum_{i}\left(d x^{i}\right)^{2}$. Writing this metric as a function times $g_{0}$ we get

$$
g_{1}=4\left(|x|+|x|^{-1}\right)^{-2} g_{0}=(\cosh t)^{-2} g_{0}
$$

Therefore the function $u_{1}(t)$ given by

$$
\begin{equation*}
u_{1}(t)=(\cosh t)^{-(n-2) / 2} \tag{2.4}
\end{equation*}
$$

is a solution of (2.2). Of course the metric $g_{1}$ is not a complete metric on $\mathbf{R} \times S^{n-1}$. We convert (2.2) to a first order system by setting $v=\frac{d u}{d t}$, and defining the vector field $\mathbf{X}(u, v)$ in the $u v$-plane by

$$
\mathbf{X}(u, v)=\left(v, \frac{(n-2)^{2}}{4} u-\frac{n(n-2)}{4} u^{(n+2) /(n-2)}\right)
$$

Equation (2.2) then becomes the autonomous system

$$
\frac{d}{d t}(u, v)=\mathbf{X}(u, v)
$$

The vector field X has critical points at $(0,0)$ and $\left(u_{0}, 0\right)$. The linearized equation at $(0,0)$ is

$$
\frac{d u}{d t}=v, \quad \frac{d v}{d t}=\frac{(n-2)^{2}}{4} u
$$

which has a saddle point at the origin. At $\left(x_{0}, 0\right)$ the linearized system becomes

$$
\frac{d u}{d t}=v, \quad \frac{d v}{d t}=(2-n) u
$$

which has a proper node at the origin. The orbit corresponding to the solution $u_{1}(t)$ contains the point $(1,0)$, is symmetric under reflection in the $u$-axis, and approaches $(0,0)$ as $t$ approaches both $+\infty$ and $-\infty$. Therefore, this orbit (together with $(0,0)$ ) bounds a region $\Omega$, and the point $\left(u_{0}, 0\right)$ lies in $\Omega$. Thus the region $\Omega$ is invariant under the flow, and
it is elementary that any orbit on which " remains positive for all time must lie in $\bar{\Omega}$. We may parametrize the orbits in $\Omega$ by letting $\gamma_{\alpha}(1)$ denote the orbit with $\gamma_{\alpha}(0)=(\alpha, 0)$ where $\alpha \in\left[u_{0}, 1\right]$. Thus $\gamma_{u_{0}}(t) \equiv\left(u_{0}, 0\right)$, and $\gamma_{1}(t)=\left(u_{1}(t), \frac{d u_{1}}{d t}(t)\right)$. For $\alpha \in\left(u_{0}, 1\right)$, there is a first positive time, which we denote $\frac{1}{2} T(\alpha)$, at which $\gamma_{0}$ intersects the $u$-axis. We also see that if we denote the coordinates of $\gamma_{\alpha}(t)$ by $\left(u_{\alpha}(t), v_{\alpha}(t)\right)$, then we have $\gamma_{\alpha}(-t)=\left(u_{\alpha}(t),-v_{\alpha}(t)\right)$. Therefore it follows that $\gamma_{\alpha}(t)$ is periodic with period $T(\alpha)$. It should be true that $T(\alpha)$ is an increasing function of $\alpha$, but we have not checked this. It is elementary that $\lim _{\alpha 11} T(\alpha)=\infty$, and $\lim _{\alpha u_{0}} T(\alpha)=(n-2)^{-1 / 2} 2 \pi$. The quantity $(n-2)^{-1 / 2} 2 \pi$ is the fundamental period of the linearized operator at $u_{0}$, which is $\frac{d^{2}}{d t^{2}}+(n-2)$.

We now summarize the consequences of the above discussion for solutions of (1.2) on $S^{1} \times S^{n-1}$. We normalize the radius of $S^{n-1}$ to be one, and let the length of $S^{1}$ be a parameter $T$, so our manifold is $S^{1}(T) \times S^{n-1}$. We take our background metric $g_{0}$ to be the product metric. We assume in this discussion that $T(\alpha)$ is increasing for $\alpha \in\left[u_{0}, 1\right]$, otherwise one can make the obvious modifications. There is a number $T_{0}=(n-2)^{-1 / 2} 2 \pi$ such that for $T \leq T_{0}$ the manifold $S^{1}(T) \times S^{n-1}$ has a unique solution for (2.1) hence for the Yamabe problem. This solution is a constant times $g_{0}$. For $T \in\left(T_{0}, 2 T_{0}\right]$ equation (2.1) has two inequivalent solutions, the constant solution and also the solution with fundamental period $T$. Actually, since the solution with fundamental period $T$ is not invariant under rotation about $S^{1}$ we actually have an $S^{1}$ parameter family of solutions. For $T \in\left(2 T_{0}, 3 T_{0}\right]$ we have 3 inequivalent solutions, the constant solution, two periods of the solution with fundamental period $T / 2$, and the solution with fundamental period $T$. Again the last two lie in $S^{1}$ parameter families of solutions. Generally, we see that for $T^{\prime} \in\left((k-1) T_{0}, k T_{0}\right]$ we have $k$ inequivalent solutions given by the constant solution, together with $i$ periods of a solution with fundamental period $T / i$ for $i=1, \ldots, k-1$. Each of these $(k-1)$ solutions lies in an $S^{1}$ parameter family of equivalent solutions. All of the solutions for $T>T_{0}$ are variationally unstable except the solutions with fundamental period $T$, and hence these solutions are minimizing for the Yamabe problem (after one normalizes the volume). The instability of the constant solution is elementary, and for a solution consisting of $i$ periods of a solution with fundamental period $T / i(i \geq 2)$ we can use the following argument: Let $u(t)$ be such a solution. Then we have $u(t+T / i)=u(t)$, and hence $v(t)=\frac{d_{u}}{d t}$ has the property that $\left\{t \in S^{1}(T): v(t)>0\right\}$ consists of at least $i$ disjoint intervals. On the other hand $v$ satisfies the lincarized equation

$$
L v=\frac{d^{2} v}{d t^{2}}+\left(\frac{n(n+2)}{4} u^{4 /(n-2)}-\frac{(n-2)^{2}}{4}\right) v=0 .
$$

It follows from Sturm-Liouville theory that there are at least $i(\geq 2)$ eigenvalues of $-L$ which are less than zero. This implies instability for the constrained variational problem.

Since the solution with fundamental period $T$ approaches $u_{1}$ as $T \rightarrow \infty$, we also see that

$$
\lim _{T \rightarrow \infty} I\left(S^{1}(T) \times S^{n-1}\right)=\sigma\left(S^{n}\right)
$$

and in particular we have $\sigma\left(S^{1} \times S^{n-1}\right)=\sigma\left(S^{n}\right)$ since we have exhibited a maximizing sequence of conformal classes of metrics on $S^{1} \times S^{n-1}$. We see that $\sigma\left(S^{1} \times S^{n-1}\right)$ is not achieved by a smooth metric on $S^{1} \times S^{n-1}$.

## 3 A priori estimates on nonminimal solutions

In this section we will derive estimates on metrics in a given conformal class which satisfy a generalization of equation (1.9). It will be essential for these estimates that ( $M, g_{0}$ ) be conformally inequivalent to the standard sphere, as they are false on $S^{n}$. While analogues of these estimates hold in general, we restrict ourselves here to metrics $g_{0}$ which are locally conformally flat. This case contains the main ideas without as many technical complications as one encounters generally. We begin with a geometric Pohozacv type identity which holds in exact form for a locally conformally flat metric $g$. Throughout this section we will assume that ( $M, g_{0}$ ) is a locally conformally flat manifold and $g \in\left[g_{0}\right]$. Assume that $x^{1}, \ldots, x^{n}$ are local coordinates on $M$ in which $g$ takes the form $\lambda^{4 /(n-2)}(x) \sum_{i}\left(d x^{i}\right)^{2}$. Let. $r^{2}=\sum_{i}\left(x^{i}\right)^{2}$ be the square of the Euclidean length of $x$, and let $D_{\sigma}$ denote the open Euclidean ball centered at $x=0$ of radius $\sigma$. The following identity holds

$$
\begin{equation*}
\int_{D_{\sigma}} r \frac{\partial R(g)}{\partial r} d \omega_{g}=\frac{2 n}{n-2} \int_{\partial D_{\sigma}} T\left(r \frac{\partial}{\partial r}, \lambda^{-2 /(n-2)} \frac{\partial}{\partial r}\right) d \Sigma_{g} \tag{3.1}
\end{equation*}
$$

where $d \Sigma_{g}$ is surface measure on $\partial D_{\sigma}$ determined by $g$ and $T(\cdot, \cdot)$ is the trace free Riccitensor of $g$ considered as a symmetric bilinear form on tangent vectors. The identity (3.1) reduces to the standard Pohozaev [20] identity for the function $\lambda(x)$. In this form it is derived in [24, Proposition 1.4] where the conformal Killing vector field is $\mathbf{X}=r \frac{\partial}{\partial r}$, the generator of dilations centered at. 0 (locally defined). Suppose $g_{0}=\lambda_{0}^{4 /(n-2)}(x) \sum_{i}\left(d x^{i}\right)^{2}$ and $g=u^{4 /(n-2)} g_{0}$ so that $\lambda=u \lambda_{0}$. We may rewrite (3.1)

$$
\begin{equation*}
\int_{D_{\sigma}} r \frac{\partial R(g)}{\partial r}\left(\lambda_{0} u\right)^{2 n /(n-2)} d x=\frac{2 n}{n-2} \int_{\partial D_{\sigma}} \sigma^{n}\left(\lambda_{0} u\right)^{2} T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) d \xi \tag{3.2}
\end{equation*}
$$

where $d \xi$ denotes the volume measure on the unit ( $n-1$ )-sphere. Equation (1.10) gives us an expression for $T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$

$$
\begin{equation*}
T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)=(n-2)\left(\lambda_{0} u\right)^{2 /(n-2)}\left[\frac{\partial^{2}}{\partial r^{2}}\left(\left(\lambda_{0} u\right)^{-2 /(n-2)}\right)-\frac{1}{n} \Delta\left(\left(\lambda_{0} u\right)^{-2 /(n-2)}\right)\right] \tag{3.3}
\end{equation*}
$$

where $\Delta$ denotes the Euclidean Laplace operator $\sum_{i} \frac{\partial^{2}}{\left(\partial x^{1}\right)^{2}}$.
A common method of attack on the existence of solutions of (1.9), which was in fact used by Yamabe, is to regularize the problem by lowering the exponent of the nonlinear term. Thus one introduces the equation

$$
\begin{equation*}
L u+K u^{p}=0, \quad u>0 \tag{3.4}
\end{equation*}
$$

where $K$ is a positive constant and $p \in(1,(n+2) /(n-2)]$. For $p<(n+2) /(n-2)$ it is standard to construct a nonzero solution which minimizes the associated constrained variational problem. More generally, the associated variational problem satisfies the PalaisSmale condition, and hence the methods of nonlinear functional analysis and the calculus of
variations may be applied. We will derive uniform estimates on solutions of (3.4) which have bounded energy. In particular, these estimates imply that solutions of (3.4) converge in $C^{2}$ norm as $p \uparrow(n+2) /(n-2)$ to solutions of (1.9). We define for $\Lambda>0$ a set of solutions $\mathcal{S}_{\Lambda}$

$$
\mathcal{S}_{\Lambda}=\left\{u: u \text { satisfies (3.4) for some } p \in\left(1, \frac{n+2}{n-2}\right], E(u) \leq \Lambda, K \leq \Lambda\right\}
$$

We will show that, if ( $M, g_{0}$ ) is not conformally equivalent to $S^{n}$, then $S_{\Lambda}$, is a compact subset of $C^{2}(M)$. We first state, without giving a detailed proof, a general weak compactness theorem for metrics $g \in\left[g_{0}\right]$ whose scalar curvatures are controlled. This type of result is at present well known to experts in several areas. An analogous theorem is proven by Sacks-Uhlenbeck [22] for harmonic maps in two variables, by Uhlenbeck [33] for Yang-Mills connections in four variables, and by several authors [7], [12], [17] in various contexts.

Proposition 3.1. Let $\left\{u_{i}\right\}$ be a sequence of positive $C^{2}$ functions on $M$ such that

$$
\left\{\operatorname{Vol}\left(u_{i}^{4 /(n-2)} g_{0}\right)\right\}, \quad\left\{R\left(u_{i}^{4 /(n-2)} g_{0}\right)\right\}
$$

are both uniformly bounded sequences. There is a subsequence $\left\{u_{i^{\prime}}\right\}$ which converges weakly in $W^{1,2}(M)$ to a limit function $u$. The function $u$ is $C^{1}$ on $M$, and there is a finite set of points $\left\{p_{1}, \ldots, p_{k}\right\}$ such that $\left\{u_{i^{\prime}}\right\}$ converges in $C^{1}$ norm to $u$ on compact subsets of $M-\left\{p_{1}, \ldots, p_{k}\right\}$.

Since our arguments will be geometric in nature, it will be convenient to estimate

$$
R\left(u^{4 /(n-2)} g_{0}\right)
$$

for $u \in \mathcal{S}_{\Lambda}$. This can be done based on "subcritical" estimates.
Proposition 3.2. Suppose $u \in \mathcal{S}_{\Lambda}$. There is a constant $C$ depending only on $g_{0}, \Lambda$ such that $\max \left|R\left(u^{4 /(n-2)} g_{0}\right)\right| \leq C$. Similarly all derivatives of $R\left(u^{4 /(n-2)} g_{0}\right)$ with respect to $g_{0}$ can be bounded in terms of $g_{0}, A$.

Proof: Let $\delta=(n+2) /(n-2)-p$ where $u$ satisfies (3.4) with exponent $p$. If $\delta=0$, then $R\left(u^{4 /(n-2)} g_{0}\right)=c(n)^{-1} K$ and our result is trivial. Thus we assume $\delta>0$, and we derive estimates on $u$ keeping track of the $\delta$-dependence. We first derive an upper bound on $u$ by a scaling argument. Let $\bar{u}=\max \{u(p): p \in M\}$ and let $\bar{p} \in M$ be a point with $u(\bar{p})=\bar{u}$. Let $x^{1}, \ldots, x^{n}$ be coordinates centered at $\bar{p}$. Observe that for $a>0$ the function $u_{a}(x)$ defined (locally) by $u_{a}(x)=a^{2 /(p-1)} u(a x)$ satisfies the equation $L_{a} u_{a}+K u_{a}^{p}=0$ where $L_{a}$ is the operator

$$
L_{a} v(x)=\frac{1}{\sqrt{\operatorname{det} g_{0}(a x)}} \sum_{i, j} \frac{\partial}{\partial x^{i}}\left(\sqrt{\operatorname{det} g_{0}(a x)} g_{0}^{i j}(a x) \frac{\partial v}{\partial x^{j}}\right)-c(n) a^{2} R\left(g_{0}\right)(a x) v(x)
$$

We choose $a$ such that $u_{a}(0)=1$, that is, we set $a=(\bar{u})^{-(p-1) / 2}$. We assume $\bar{u}$ is large so that $u_{a}$ is defined on the unit ball in $\mathbf{R}^{n}$. Since $x=0$ is the maximum point of $u_{a}$ in $B_{1}$, we have
$u_{a} \leq 1$ and standard elliptic estimates imply

$$
u_{a}(0) \leq c\left(\int_{B_{1}} u_{a}^{2 n /(n-2)} d x\right)^{(n-2) /(2 n)}
$$

Now from the definition of $u_{a}$ we have after a change of variable,

$$
\int_{B_{1}(0)} u_{a}^{2 n /(n-2)} d x=a^{\frac{n}{p-1}\left(\frac{n+2}{n-2}-r\right)} \int_{B_{a}(0)} u^{2 n /(n-2)} d x
$$

This then implies

$$
1=u_{0}(0) \leq c \cdot(\bar{u})^{-\frac{n-2}{4}\left(\frac{n+2}{n-2}-p\right)}
$$

and hence we have

$$
\begin{equation*}
\max _{M} u \leq c_{1}^{1 / s}, \quad \delta=\frac{n+2}{n-2}-p \tag{3.5}
\end{equation*}
$$

for a constant $c_{1}$.
We may derive a lower bound on $u$ of a similar type by observing that $L u \leq 0$, and so standard estimates (see [10]) give us

$$
\min _{M} u \geq c \int_{M} u d \omega_{g_{0}}
$$

From (3.5) we have

$$
\int_{M} u^{2 n /(n-2)} d \omega_{9_{0}} \leq c_{1}^{\frac{n+2}{n-2} \cdot \frac{1}{\delta}} \int_{M} u d \omega_{g_{0}}
$$

The Sobolev inequality implies

$$
\left(\int_{M} u^{2 n /(n-2)} d \omega_{g_{0}}\right)^{(n-2) / n} \leq c E(u)=c K \int_{M} u^{p+1} d \omega_{g_{0}}
$$

Since $p+1 \leq 2 n /(n-2)$ and $K \leq A$ we have

$$
1 \leq c \lambda\left(\int_{M} u^{2 n /(n-2)} d \omega_{9_{0}}\right)^{\frac{n-2}{2 n}(p-1)}
$$

Combining the above inequalities we get

$$
\begin{equation*}
\min _{M} u \geq c_{2}^{-1 / \delta} \tag{3.6}
\end{equation*}
$$

for a constant $c_{2}$. Rescaling as above with $a=\bar{u}^{-(p-1) / 2}$ and with center any given point of $M$ we get from elliptic theory $\left|\nabla u_{u}(0)\right| \leq c u_{a}(0) \leq c$ which implies in light of (3.5)

$$
\begin{equation*}
\max _{M}\left|\nabla_{g_{0}} u\right| \leq c_{3}^{1 / \delta} \tag{3.7}
\end{equation*}
$$

Higher derivatives can be similarly estimated. To complete the proof we observe that

$$
R\left(u^{4 /(n-2)} g_{0}\right)=c(n)^{-1} K u^{-\delta}
$$

from (3.4). Therefore (3.5), (3.6), (3.7) imply that $R\left(u^{4 /(n-2)} g_{0}\right)$ and its first derivative with respect to $g_{0}$ are bounded. Higher derivatives of $R\left(u^{4 /(n-2)} g_{0}\right)$ are similarly bounded, and we have completed the proof of Proposition 3.2.

We now prove the main result of this section.
Theorem 3.3. Suppose ( $M, g_{0}$ ) is not conformally equivalent to the standard $n$-sphere and $g_{0}$ is locally conformally flat. For any $\Lambda>0$, the set $\mathcal{S}_{\Lambda}$ is a bounded subset of $C^{3}(M)$.

Proof: We prove the theorem by contradiction. Suppose $\left\{u_{i}\right\}$ is a sequence in $\mathcal{S}_{\mathrm{A}}$ with $\lim \left\|u_{i}\right\|_{C(M)}=\infty$. From Proposition 3.1 we may require the sequence $u_{i}$ to converge weakly in $W^{1,2}(M)$ to a limit $u$, and uniformly on compact subsets of $M-\left\{P_{1}, \ldots, P_{k}\right\}$ for some collection of points $P_{1}, \ldots, P_{k} \in M$. The function $u$ is smooth on $M$, and the sequence $u_{i}$ converges in $C^{3}$ norm to $u$ on compact subsets of $M-\left\{P_{1}, \ldots, P_{k}\right\}$ by elliptic estimates. If we can show that the sequence $u_{i}$ converges uniformly on all of $M$, then we conclude that max $u_{i}$ are bounded, and standard elliptic theory implics $\left\|u_{i}\right\|_{C^{3}(M)}$ are bomded contrary to assumption.

We divide the proof into two steps. We first show that $u$ is nonzero. This is where we use the global hypothesis that $\left(M, g_{0}\right)$ is not conformally $S^{n}$. Assume $u \equiv 0$, and choose a point $Q \in M$ different from $P_{1}, \ldots, P_{k}$. Let $\varepsilon_{i}=u_{i}(Q)$, so by assumption $\lim \varepsilon_{i}=0$. Define $v_{i}$ by $v_{i}=\varepsilon_{i}^{-1} u_{i}$. and observe that the $v_{i}$ satisfy the equation

$$
\begin{equation*}
L v_{i}+\varepsilon_{i}^{p_{i}-1} K_{i} v_{i}^{p_{i}}=0 . \tag{3.8}
\end{equation*}
$$

Since $\left\{u_{i}\right\}$ is uniformly bounded on compact subsets of $M-\left\{P_{1}, \ldots, P_{k}\right\}$, we have from (3.4) a Harnack inequality for $u_{i}$ on compact subsets of $M-\left\{P_{1}, \ldots, P_{k}\right\}$. Thus the $v_{i}$ satisfy a Harnack inequality, and $v_{i}(Q)=1$. Therefore the $v_{i}$ are locally uniformly bounded on $M-\left\{P_{1}, \ldots, P_{k}\right\}$. From (3.8) we then get bounds on all derivatives of $v_{i}$ away from $\left\{P_{1}, \ldots, P_{k}\right\}$. Therefore a subsequence, again denoted $v_{i}$, converges in $C^{3}$ norm on compact subsets of $M-\left\{P_{1}, \ldots, P_{k}\right\}$ to a smooth positive solution $G$ of $L G=0$ on $M-\left\{P_{1}, \ldots, P_{k}\right\}$. Since we are assuming $R\left(g_{0}\right)>0, G$ must be singular at one or more of the points $P_{1}, \ldots, P_{k}$. Suppose $G$ is singular at $P_{1}, \ldots, P_{\ell}$. It then follows that $G$ is a positive linear combination of (positive) fundamental solutions $G_{\alpha}$ with poles at $P_{\alpha \alpha}$ for $\alpha=1, \ldots, \ell$. That is, there exist positive constants $a_{1}, \ldots, a_{\ell}$ such that $G=\sum_{\alpha=1}^{\ell} a_{\alpha \alpha} G_{\alpha}$. Let $x^{2}, \ldots, x^{n}$ be conformally flat coordinates centered at $P_{1}$. Let $\sigma>0$ be a number which will be chosen small, and apply (3.2) with $u=u_{i}$ on $D_{\sigma}$. For a solution $u$ of (3.4), we have $R\left(u^{1 /(n-2)} g_{0}\right)=c(n)^{-1} K u^{-\delta}$ where $\delta=(n+2) /(n-2)-p$, and thus the left hand side of (3.2) can be written

$$
\begin{aligned}
& c(n)^{-1} K \int_{D_{\sigma}} r \frac{\partial}{\partial r}\left(u^{-\delta}\right)\left(\lambda_{0} u\right)^{2 n /(n-2)} \\
& \quad=-c(n)^{-1} K \delta(p+1)^{-1} \int r \frac{\partial}{\partial r}\left(u^{p+1}\right) \lambda_{0}^{2 n /(n-2)} d x .
\end{aligned}
$$

Since $r \frac{\partial}{\partial r}=\sum_{i} x^{i} \frac{\partial}{\partial z^{i}}$, we may integrate by parts to obtain

$$
\begin{gathered}
\int_{D_{\sigma}} r \frac{\partial}{\partial r}\left(u^{p+1}\right) \lambda_{0}^{2 n /(n-2)} d x=-\int_{D_{\sigma}} u^{p+1}\left(n+\frac{2 n}{n-2} r \frac{\partial \log \lambda_{0}}{\partial r}\right) \lambda_{0}^{2 n /(n-2)} d x \\
+\sigma^{n} \int_{\partial D_{\sigma}} u^{p+1} \lambda_{0}^{2 n /(n-2)} d \xi
\end{gathered}
$$

For $\sigma$ small $n+\frac{2 n}{n-2} r \cdot \frac{\partial \log \lambda \theta}{\partial r}>0$, and hence (3.2) implies the inequality

$$
\frac{2 n \sigma^{n}}{n-2} \int_{\partial D_{\sigma}}\left(\lambda_{0} u\right)^{2} T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) d \xi \geq-c(n)^{-1} k \delta(p+1)^{-1} \sigma^{n} \int_{\partial D_{\sigma}} u^{p+1} \lambda_{0}^{2 n /(n-2)} d \xi
$$

for any solution $u$ of (3.4). Applying this with $u=u_{i}$ and multiplying by $\varepsilon_{i}^{2}$ we get in the limit

$$
\begin{equation*}
\sigma^{n} \int_{\partial D_{\sigma}}\left(\lambda_{0} C i\right)^{2} T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) d \xi \geq 0 \tag{3.9}
\end{equation*}
$$

where

$$
T\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)
$$

 scalar curvature, $\lambda_{0} G$ is a positive Euclidean harmonic function on $D_{\sigma}-\{0\}$ which is singular at $x=0$. It follows that $\left(\lambda_{0} G\right)(x)$ has the expansion

$$
\left(\lambda_{0} G\right)(x)=a_{1}|x|^{2-n}+A+\alpha(x)
$$

where $\alpha(x)$ is a harmonic function with $\alpha(0)=0$. Using this expression in (3.9) we get $-A+O(\sigma) \geq 0$ by elementary calculation using (3.3). Since $\sigma$ is arbitrarily small we get $A \leq 0$. On the other hand we have $G \geq a_{1} G_{1}$, and

$$
\lambda_{0} G_{1}(x)=|x|^{2-n}+E\left(P_{1}\right)+O(|x|)
$$

where $E(\cdot)$ is the energy function discussed in $\xi 4$. Thus $A \geq a_{1} E\left(P_{1}\right)$ which is strictly positive since ( $M, g_{0}$ ) is not conformally equivalent to $S^{n}$. We discuss this positive energy statement, in the next section. This contradiction shows that $u>0$ on $M$.

The second step in our proof deals with the remaining case $u>0$. In this case our argument is local. The sequence $\left\{u_{i}\right\}$ must be unbounded near one of the points $\left\{P_{1}, \ldots, P_{k}\right\}$, for otherwise we have uniform convergence. Assume that $\lim \left\{\sup _{B_{\sigma}\left(P_{1}\right)} u_{i}\right\}=\infty$ for any $\sigma>0$. Since $u>0$, the metrics $g_{i}=u_{i}^{4 /(n-2)} g_{0}$ have uniformly bounded curvature away from the points $P_{\mathrm{f}}, \ldots, P_{k}$. Let $x^{1}, \ldots, x^{n}$ be conformally flat coordinates centered at $P_{1}$. Let $\lambda_{0}(x)>0$ be such that $g_{0}=\lambda_{0}^{4 /(n-2)} \sum\left(d x^{i}\right)^{2}$, and assume $\lambda_{0}$ is bounded above and below (locally). The functions $w_{i}=\lambda_{0} u_{i}$ then satisfy

$$
\Delta w_{i}+c(n) R_{i} w_{i}^{(n+2) /(n-2)}=0
$$

where $\Delta$ is the Euclidean Laplace operator and $R_{i}=R\left(g_{i}\right)$. In particular, $w_{i}$ is superharmonic and by assumption $w_{i}$ is bounded below on $\partial D_{\sigma}$ for $i$ large. Therefore $w_{i}$ has a lower bound on $D_{\sigma}$. If the Ricci curvature of $g_{i}$ were bounded in $D_{\sigma}$, then we can use the gradient estimate [6] on the solution $w_{i}^{-1}$ of the equation $L_{g_{i}}\left(w_{i}^{-1}\right)=0$. Note that this equation holds because $w_{i}^{-4 /(n-2)} g_{i}$ is the Euclidean metric. The gradient estimate can be applied because of Proposition 3.2 which gives us a bound on $R_{i}$ and $\left|\nabla_{g_{0}} R_{i}\right|$. We have

$$
\left|\nabla_{g_{i}} R_{i}\right|=u_{i}^{-2 /(n-2)}\left|\nabla_{g_{0}} R_{i}\right|
$$

which is bounded since $u_{i}$ has a lower bound. The gradient estimate then gives

$$
\left|\nabla_{g_{i}} w_{i}^{-1}\right| \leq c w_{i}^{-1}
$$

Writing this in terms of the Euclidean metric we have

$$
\left|\partial\left(w_{i}^{-2 /(n-2)}\right)\right| \leq c
$$

where $\partial$ denotes the Euclidean gradient. Note that the gradient bound depends on the geodesic distance to $\partial D_{\sigma}$. Since $w_{i}$ is bounded below we have $\sup _{D_{\sigma / 2}}\left|\partial\left(w_{i}^{-2 /(n-2)}\right)\right| \leq c \sigma^{-1}$. This implies an upper bound on $w_{i}$ near 0 in terms of an upper bound at a fixed small distance from 0 . Since $w_{i}$ are converging away from 0 , we get an upper bound independent of $i$. This contradiction shows that $\lim \left\{\sup _{D_{\sigma}}\left\|\operatorname{Ric}\left(g_{i}\right)\right\|_{g_{i}}\right\}=\infty$ for any $\sigma>0$. Therefore we can choose a sequence of points $Q_{i} \rightarrow P_{1}$ such that

$$
c_{i}=\sup _{D_{\sigma}}\left\|\operatorname{Ric}\left(g_{i}\right)\right\|_{g_{i}}=\left\|\operatorname{Ric}\left(g_{i}\right)\right\|_{g_{i}}\left(Q_{i}\right)
$$

with $c_{i} \rightarrow \infty$. We then let $\widetilde{g}_{i}=c_{i} g_{i}$, and observe that we have

$$
1=\sup _{D_{\sigma}}\left\|\operatorname{Ric}\left(\bar{g}_{i}\right)\right\|_{\bar{g}_{i}}=\left\|\operatorname{Ric}\left(\bar{g}_{i}\right)\right\|_{\bar{g}_{i}}\left(Q_{i}\right)
$$

Thus we have $\bar{g}_{i}=\bar{w}_{i}^{4 /(n-2)} \sum_{j}\left(d x^{j}\right)^{2}$ where $\bar{w}_{i}=c_{i}^{(n-2) / 4} w_{i}$. Let $x_{i}$ denote the point in $D_{\sigma}$ corresponding to $Q_{i}$, so that we have $\lim x_{i}=0$. Let

$$
\begin{aligned}
& v_{i}(y)=\lambda_{i}^{(n-2) / 2} w_{i}\left(\lambda_{i} y+x_{i}\right) \\
& \bar{v}_{i}(y)=c_{i}^{(n-2) / 4} v_{i}(y)
\end{aligned}
$$

where we choose $\lambda_{i}=\left(\bar{w}_{i}\left(x_{i}\right)\right)^{-2 /(n-2)}$ so that $\bar{v}_{i}(0)=1$. Notice that $\lambda_{i} \rightarrow 0$ so that $v_{i}$ is defined on increasingly large balls in $\mathbf{R}^{n}$. Moreover, $v_{i}^{4 /(n-2)} \Sigma\left(d y^{j}\right)^{2}$ is the pullback of $g_{i}$ under the map $y \mapsto \lambda_{i} y+x_{i}$, and hence the scalar curvature and volume of $v_{i}^{4 /(n-2)} \sum\left(d y^{j}\right)^{2}$ are bounded. Thus by Proposition 3.1, a subsequence of $\left\{v_{i}\right\}$ converges uniformly away from a finite set of points $y_{1}, \ldots, y_{r} \in \mathbf{R}^{n}$. In particular, $v_{i}$ satisfies a Harnack inequality away from $y_{1}, \ldots, y_{r}$. Since the Ricci curvature of $\bar{v}_{i}^{4 /(n-2)} \sum\left(d y^{j}\right)^{2}$ is bounded and the metric is complete,
the Harnack inequality of [6] holds on unit geodesic balls. In particular, $\bar{v}_{i}$ remains bounded in a uniform neighborhood of $y=0$, and 0 is distinct from $y_{1} \ldots, y_{r}$. Therefore a subsequence of $\left\{\bar{v}_{i}\right\}$, again denoted $\left\{\bar{v}_{i}\right\}$, converges to a limit $h$. From the construction $h$ is a positive harmonic function on $\mathbf{R}^{n}-\left\{y_{1}, \ldots, y_{r}\right\}$ with $h(0)=1$. Moreover, the metric $h^{4 /(n-2)} \sum\left(d y^{i}\right)^{2}$ has Ricci curvature of length one at $y=0$ and in particular is not flat. It follows that $h$ has at least two singularities in $S^{n}=\mathbf{R}^{n} \cup\{\infty\}$. Let $y_{1}, \ldots, y_{s}$ denote the singular points of $h$ in $S^{n}$. It follows that $h$ is a positive linear combination of fundamental solutions with poles at $y_{1}, \ldots, y_{s}$. Thus there are positive numbers $a_{1}, \ldots, a_{s}$ such that $h(y)=\sum_{\alpha=1}^{s} a_{\alpha} G_{\alpha}$ where $G_{\alpha}(y)=\left|y-y_{\alpha}\right|^{2-n}$ if $y_{\alpha} \in \mathbf{R}^{n}$ and $G_{\alpha}(y) \equiv 1$ if $y_{\alpha}=\infty$. Assume $y_{1} \in \mathbf{R}^{n}$ so that

$$
h(y)=a_{1}\left|y-y_{1}\right|^{2-n}+A+\alpha(y)
$$

where $\alpha\left(y_{1}\right)=0$ and $A>0$ because $s \geq 2$. Now the same argument as in the previous step, using (3.2), gives us a contradiction. This shows that our initial assumption of nonconvergence of $\left\{u_{i}\right\}$ is violated and we have completed the proof of Theorem 3.3.

There is an obvious question which is left unresolved by Theorem 3.3, and this is the question of whether one can remove the assumed bound on the energy $E(u)$ which is required in Theorem 3.3. It seems likely that the energy of solutions of (1.9) will be bounded by a constant depending only on $g_{0}$. Inequality (1.13) gives a very strong a priori integral estimate on solutions of (1.2) for $n=3$. It may be possible to use this in place of the energy bound in Theorem 3.3.

## 4 The relevant positive energy theorems

In this section we give a discussion of the the total energy of an asymptotically flat $n$-manifold and discuss the positive energy theorems which are relevant to the Yamabe problem. Let $\left(M^{n}, g\right)$ be a Riemannian manifold. $(M, g)$ is said to be asymptotically flat if there is a compact subset $K \subset M$ such that $M-K$ is diffeomorphic to $\mathbf{R}^{n}-\{|x| \leq 1\}$, and a diffeomorphism $\Phi: M-K \rightarrow \mathbf{R}^{n}-\{|x| \leq 1\}$ such that, in the coordinate chart defined by $\Phi$, we have $g=\sum_{i, j} g_{i j}(x) d x^{i} d x^{j}$ where $g_{i j}(x)=\delta_{i j}+O\left(|x|^{-p}\right)$ as $x \rightarrow \infty$ for some $p>0$. We also assume that

$$
|x|\left|g_{i j, k}(x)\right|+|x|^{2}\left|g_{i j, k \ell}(x)\right|=O\left(|x|^{-p}\right)
$$

where we use commas to denote partial derivatives as in §1. Assuming that $|R(g)|=O\left(|x|^{-q}\right)$, $q>n$, and $p>(n-2) / 2$ it is possible to define the total energy of $M$. To do this we recall the expression for $R(g)$ in the $x$-coordinates

$$
\begin{aligned}
R(g) & =\sum_{i, j, k} g^{i j}\left(\Gamma_{i j, k}^{k}-\Gamma_{i k, j}^{l k}+\sum_{\ell}\left(\Gamma_{k \ell}^{k} \Gamma_{i j}^{\ell}-\Gamma_{j \ell}^{k} \Gamma_{i k}^{\ell}\right)\right) \\
\Gamma_{i j}^{k} & =\frac{1}{2} \sum_{m} g^{k m}\left(g_{i m, j}+g_{j m, i}-g_{i j, m}\right) .
\end{aligned}
$$

Using the asymptotic assumptions we find

$$
R(g)=\sum_{i, j}\left(g_{i j, i j}-g_{i i . j 3}\right)+O\left(|x|^{-2 p-2}\right) .
$$

Since $2 p+2>n$ we therefore have the divergence term absolutely integrable near infinity. Thus the divergence theorem implies the existence of the following limit

$$
\lim _{\sigma \rightarrow \infty} \int_{\{|x|=\sigma\}} \sum_{i, j}\left(g_{i, i, i} \nu_{j}-g_{i i, j} \nu_{j}\right) d \xi(\sigma)
$$

where $\nu=\sigma^{-1} x$ is the Euclidean unit normal to $\{|x|=\sigma\}$ and $d \xi(\sigma)$ denotes the Euclidean area element on $\{|x|=\sigma\}$. Moreover, the family of spheres $S_{\sigma}=\{|x|=\sigma\}$ may be replaced by any sequence of boundaries which go uniformly to infinity, and the limit will exist and have the same value (see [2]). We define the total energy $E=E(M, g)$ by

$$
E=\left(4(n-1) w_{n-1}\right)^{-1} \lim _{\sigma \rightarrow \infty} \int_{S_{\sigma}} \sum_{i, j}\left(g_{i j, i} \nu_{j}-g_{i i, j} \nu_{j}\right) d \xi(\sigma)
$$

where $w_{n-1}=\operatorname{Vol}\left(S^{n-1}(1)\right)$. The basic content of the positive energy theorem, or this special case of it, is that if $R(g) \geq 0$ on all of $M$, then $E \geq 0$. Moreover, $E=0$ only if $(M, g)$ is isometric to Euclidean space.

For a compact manifold ( $M, g$ ) with $R(g)>0$ we can make the following construction. Given a point $P \in M$, there is a positive fundamental solution $G$ for the conformal Laplacian $L$ with pole at $P$. If we normalize $G$ so that

$$
\lim _{Q \rightarrow P} d(P, Q)^{n-2} G(Q)=1
$$

where $d(\cdot, \cdot)$ is the Riemannian distance function for $g$, then $G$ is unique. The manifold ( $M-$ $\{P\}, G^{4 /(n-2)} g$ ) is then asymptotically flat. If we let $y^{1}, \ldots, y^{n}$ denote a normal coordinate system for $g$ centered at $P$, then we have $g_{i j}=\delta_{i j}+O\left(|y|^{-2}\right)$. It is not difficult to show that $G(y)=|y|^{2-n}+O\left(|y|^{p+2-n}\right)$ where $p$ is any number less than two. If we let $x=|y|^{-2} y$, then we have the metric components of $g$ in the $x$-coordinates given by $|x|^{-4} g_{i j}\left(|x|^{-2} x\right)$. In particular, if we let

$$
G^{4 /(n-2)} g=\sum_{i, j} \bar{g}_{i j}(x) d x^{i} d x^{j}
$$

we have $\bar{g}_{i j}(x)=\delta_{i j}+O\left(|x|^{-p}\right)$ as $x \rightarrow \infty$. Also we have $R\left(G^{4 /(n-2)} g\right)=0$ since $L G=0$ on $M-\{P\}$. In particular, if $p>(n-2) / 2$, then the total energy can be defined. Since $(n-2) / 2<2$ for $n=3,4,5$ we see that in these dimensions we can assign to each point $P \in M$ a number $E(P)$ which is the total energy of $\left(M-\{P\}, G^{4 /(n-2)} g\right)$.

We now let $\left(M^{n}, g\right)$ denote a general asymptotically flat manifold. We are going to present the minimal hypersurface proof of the positive energy theorem which is joint with S.T. Yau and appears in [25], [26]. Our presentation will simplify the original proofs in a few technical respects. It is convenient to first simplify the asymptotic behavior of $g$ so that $g$ is conformally flat near infinity. We carried out this argument for $n=3$ in [27], and we present here the $n$-dimensional version.

Proposition 4.1. Let $(M, g)$ be asymptotically flat with $p>(n-2) / 2$ and $q>n$. Assume also that $R(g) \geq 0$. For any $\varepsilon>0$ there is a metric $\bar{g}$ such that $(M, \bar{g})$ is asymptotically flat and conformally flat near infinity with $R(\bar{g}) \equiv 0$ and such that $E(\bar{g}) \leq E(g)+\varepsilon$.

Proof: We first observe that we may take $R(g) \equiv 0$, since generally, we can solve $L u=0$, $u>0$ with $u \sim 1$ at infinity. In fact, we have $u(x)=1+A|x|^{2 \sim n}+O\left(|x|^{1+n}\right)$ where $A \leq 0$ since $0<u<1$ on $M$. (See [2] for the existence and expansion.) The metric $u^{4 /(n-2)} g$ is then scalar fiat and has total energy given by $E(g)+A \leq E(g)$. Thus $g$ may be replaced by $u^{4 /(n-2)} g$.

Now assume $R(g) \equiv 0$, and deform $g$ near infinity to the Euclidean metric. To accomplish this, choose a function $\Psi_{\sigma}(x)$ with the properties, $\Psi_{\sigma}(x)=1$ for $|x| \leq \sigma, \Psi_{\sigma}(x)=0$ for $|x| \geq 2 \sigma, \Psi_{\sigma}$ is a decreasing function of $|x|$, and $\sigma\left|\Psi_{\sigma}^{\prime}\right|+\sigma^{2}\left|\Psi_{\sigma}^{\prime \prime}\right| \leq c$. Now consider the metric ${ }^{(\sigma)} g$ given by ${ }^{(\sigma)} g=\psi_{\sigma} g+\left(1-\Psi_{\sigma}\right) \delta$ where $\delta=\sum_{i, j} \delta_{i j} d x^{i} d x^{j}$ denotes the Euclidean metric. Observe that ${ }^{(\sigma)} g=\delta+O\left(|x|^{-p}\right)$ uniformly in $\sigma$ for $\sigma$ large, and also $R\left(^{(\sigma)} g\right)=O\left(|x|^{-2-p}\right)$ for $\sigma \leq|x| \leq 2 \sigma$ uniformly in $\sigma$. In particular we have

$$
\int_{M}\left|R\left({ }^{(\sigma)} g\right)\right|^{n / 2} d \omega_{g}=O\left(\sigma^{-n p / 2}\right)
$$

and so for $\sigma$ large there is a unique solution $u_{\sigma}$ of $L_{\sigma} u_{\sigma}=0, u_{\sigma}>0, u_{\sigma} \rightarrow 1$ as $|x| \rightarrow \infty$. (See [2] for the existence.) The metric ${ }^{(\sigma)} \bar{g}=u_{\sigma}^{4 /(n-2)(\sigma)} g$ is then scalar flat and conformally Euclidean near infinity. We show $\lim _{\sigma \rightarrow \infty} E\left({ }^{(\sigma)} \bar{g}\right)=E(g)$ and then for $\sigma$ sufficiently large the
metric ${ }^{(\sigma)} \bar{g}$ will give the desired metric. From the uniform decay estimates on $u_{\sigma}$ and ${ }^{(\sigma)} g$, we see that given $\varepsilon>0$ there is a $\sigma_{0}$ independent of $\sigma$ such that

$$
\begin{array}{r}
\left|E\left({ }^{(\sigma)} \bar{g}\right)-\left(4(n-1) w_{n-1}\right)^{-1} \int_{S_{\sigma_{0}}} \sum_{i, j}\left({ }^{(\sigma)} \bar{g}_{i j, i} \nu_{j}-{ }^{(\sigma)} \bar{g}_{i i, j} \nu_{j}\right) d \xi\left(\sigma_{0}\right)\right| \leq \frac{\varepsilon}{3} \\
\left|E(g)-\left(4(n-1) w_{n-1}\right)^{-1} \int_{S_{\sigma_{0}}} \sum_{i, j}\left(g_{i j, i} \nu_{j}-g_{i i, j} \nu_{j}\right) d \xi\left(\sigma_{0}\right)\right| \leq \frac{\varepsilon}{3} .
\end{array}
$$

On the other hand, we have $\lim _{\sigma \rightarrow \infty} u_{\sigma}=1$ on compact subsets of $M$, and hence the two surface integrals above are within $\varepsilon / 3$ when $\sigma$ is sufficiently large. Thus we get $\left|E\left({ }^{(\sigma)} \bar{g}\right)-E(g)\right|<\varepsilon$ for $\sigma$ large. This completes the proof of Proposition 4.1.

Note that if $(M, g)$ is asymptotically flat and conformally flat near infinity we have $g_{i j}=$ $h^{4 /(n-2)}(x) \delta_{i j}$ for $|x|$ large where $h(x) \rightarrow 1$ as $x \rightarrow \infty$. If $R(g) \equiv 0$, then $h$ is a harmonic function for $|x|$ large and hence $h(x)=1+E|x|^{2-n}+O\left(|x|^{1-n}\right)$ where we have normalized the energy so that $E$ is the energy of the metric $h^{4 /(n-2)} \delta$. Thus by Proposition 4.1 we may assume $g$ to be of this form.

Theorem 4.2. Let $(M, g)$ be asymptotically flat with $p>(n-2) / 2, q>n$, and $R(g) \geq 0$ on $M$. Then $E(g) \geq 0$ and $E(g)=0$ only if $(M, g)$ is isometric to $\left(\mathrm{R}^{n}, \delta\right)$.

We will give the proof of this theorem for $n \leq 7$. This proof can be extended to arbitrary dimensions with an additional technical complication arising from singular sets of area minimizing hypersurfaces which appear for $n \geq 8$. We do not deal with this here, but leave it to a forthcoming work of the author and S.T. Yau. In any case, this is not required for the Yamabe problem as the remaining case of locally conformally flat manifolds of arbitrary dimension has been treated by a different argument in [29]. For the case in which $M$ is a spin manifold a different proof of Theorem 4.2 was given by E. Witten [34]. This proof was carried over to arbitrary dimensions in [15].

Proof of Theorem 4.2: We first show that $E \geq 0$. Suppose on the contrary that $E<0$. Then by Proposition 4.1 we may assume $R(g) \equiv 0$ and $g_{i j}=h^{4 /(n-2)} \delta_{i j}$ where $h(x)$ has the expansion $h(x)=1+E|x|^{2-n}+\left(|x|^{1-n}\right)$ for $x$ large. It will be convenient to have $R(g)>0$ on $M$. This can be accomplished by replacing $g$ by $u^{4 /(n-2)} g$ where $u$ satisfies $L u=-g$ with $g>0$ on $M, g$ small and $g$ decaying rapidly. The solution $u$ will then satisfy $u(x)=$ $1+\delta|x|^{2-n}+O\left(|x|^{1-n}\right)$ with $\delta$ arbitrarily small. Thus the negativity of the energy is preserved. We compute the divergence of the unit vector field $\eta=h^{-2 /(n-2)} \frac{\partial}{\partial x^{n}}$ with respect to $g$. We find

$$
\begin{aligned}
\operatorname{div}_{g}(\eta) & =h^{-2 n /(n-2)} \frac{\partial}{\partial x^{n}}\left(h^{2 n /(n-2)} h^{-2 /(n-2)}\right) \\
& =\frac{2(n-1)}{n-2} E \frac{\partial}{\partial x^{n}}\left(|x|^{2-n}\right)+O\left(|x|^{-n}\right) \\
& =-2(n-1) E \frac{x^{n}}{|x|^{n}}+O\left(|x|^{-n}\right)
\end{aligned}
$$

In particular we see that $\operatorname{div}_{g}(\eta)>0$ for $x^{n} \geq a_{0}$ and div ${ }_{g}(-\eta)>0$ for $x^{n} \leq-a_{0}$ for some constant $a_{0}$. Now let $\sigma$ be a large radius, and tet $\Gamma_{\sigma, a}$ denote the ( $n-2$ )-dimensional sphere

$$
\Gamma_{\sigma, a}=\left\{x=\left(x^{\prime}, x^{n}\right):\left|x^{\prime}\right|=\sigma, x^{n}=a\right\}
$$

Let $C_{\sigma}$ denote the $(n-1)$-dimensional cylinder $C_{\sigma}=\left\{\left(x^{\prime}, x^{n}\right):\left|x^{\prime}\right|=\sigma\right\}$. We orient $\Gamma_{\sigma, a}$ as the boundary of the portion of $C_{\sigma}$ lying below $\Gamma_{\sigma, a}$. Let $\Sigma_{\sigma_{, a}}$ be an $(n-1)$-dimensional surface of least area with $\partial \Sigma_{\sigma, a}=\Gamma_{\sigma, a}$. The cylinder $C_{\sigma}$ bounds an interior region $\Omega_{\sigma}$ in $M$, and $\Sigma_{\sigma, a} \subset \Omega_{\sigma}$. Since $n \leq 7, \Sigma_{\sigma, a}$ will be free of singularities (see [11,30] for relevant results on the Plateau problem). For any $\sigma$, let

$$
V(\sigma)=\min \left\{V_{o} \mid\left(\Sigma_{\sigma, n}\right): a \in\left[-a_{0}, a_{0}\right]\right\}
$$

where we note that the function $a \mapsto \operatorname{Vol}\left(\Sigma_{\sigma_{a}}\right)$ is continuous. We now assert that there exists $a=a(\sigma) \in\left(-a_{0}, a_{0}\right)$ such that $\operatorname{Vol}\left(\mathcal{Y}_{\sigma, a}\right)=V(\sigma)$. To show that $a(\sigma)<a_{0}$, write $\Sigma_{\sigma, u}=\left(\partial \Omega_{\sigma, a}\right) \cap \Omega_{\sigma}$ where $\Omega_{\sigma, a}$ is the subregion of $\Omega_{\sigma}$ lying below $\Sigma_{\sigma, a}$. Let

$$
U_{\sigma, a}=\left\{\left(x^{\prime}, x^{n}\right) \in \Omega_{\sigma, a}: x^{n}>a_{0}-\delta\right\}
$$

where $\delta$ is chosen so small that $\operatorname{div}_{g}(\eta)>0$ for $x^{n}>a_{0}-\delta$. We show that $U_{\sigma, a}=\emptyset$ by applying the divergence theorem in $U_{r, n}$. Since $\eta$ is tangent to $C_{\sigma}$, we get

$$
\int_{\mathrm{\Sigma}_{\sigma . a} \cap\left\{r^{n} \geq a_{0}-\delta\right\}}\langle\eta . \nu\rangle_{g} d \mathcal{H}^{n-1}-\operatorname{Vol}\left(\Omega_{\sigma . a} \cap\left\{x^{n}=a_{0}-\delta\right\}\right)>0
$$

provided $U_{\sigma, a} \neq \emptyset$. Here $\nu$ denotes the unit normal of $\Sigma_{\theta, a}$. Thus we may apply the Schwarz inequality to assert

$$
\operatorname{Vol}\left(\Omega_{\sigma, a} \cap\left\{x^{n}=a_{0}-\delta\right\}\right)<\operatorname{Vol}\left(\cup_{\sigma, a} \cap\left\{x^{n} \geq a_{0}-\delta\right\}\right) .
$$

Therefore, if $U_{\sigma, a} \neq \emptyset$, then the hypersurface $\Xi$ given $b^{\text {g }}$

$$
\mathbb{Z}=\partial\left(\Omega_{\sigma, \delta} \cap\left\{r^{n}<a_{0}-\delta\right\}\right) \cap \Omega_{\sigma}
$$

has smaller volume than $\Xi_{\sigma, a}$ and $\partial \Sigma=1_{\tau, a,}$ where $a_{1}=\min \left\{a, a_{0}-\delta\right\}$. This contradiction shows that $U_{\sigma, a}=\emptyset$ and in particular $a(\sigma) \leq a_{0}-\delta$. An analogous argument shows that $a(\sigma) \geq-a_{0}+\delta$ for some $\delta>0$.

Let $\Sigma_{\sigma}=\Sigma_{\sigma, a(\sigma)}$ be one of the hypersurfaces which realizes the minimum volume $V(\sigma)$. Let $\mathrm{X}_{1}$ be a fixed vector field on $M$ which is cqual to $\frac{0}{\partial a^{10}}$ outside a compact set. Let $\mathrm{X}_{0}$ be a vector field of compact support, and let $\mathbf{X}=\mathbf{X}_{0}+\alpha \mathbf{X}_{1}$ where $\alpha \in \mathbf{R}$. Let $F_{t}$ be the one parameter group of difeomorphisms generated by $\mathbf{X}$ (or alternatively any curve of diffeomorphisms whose tangent vector at $t=0$ is $\mathbf{X}$ ). If $\sigma$ is sufficiently large that the support of $\mathrm{X}_{0}$ is compactly contained in $\Omega_{\sigma}$, then X gives a valid variation of $\Sigma_{\sigma}$; that is, we have

$$
\left.\frac{d}{d t} \operatorname{Vol}\left(F_{t}\left(\Sigma_{\sigma}\right)\right)\right|_{t=0}=0,\left.\quad \frac{d^{2}}{d l^{2}} \operatorname{Vol}\left(F_{t}\left(\Sigma_{\sigma}\right)\right)\right|_{t=0} \geq 0
$$

The second variation is the integral of the function $F_{\mathrm{x}, \sigma}$ given by

$$
F_{\mathrm{X}, \sigma}(P)=\left.\frac{d^{2}}{d t^{2}}\left\|\left(F_{t}\right)_{*}\left(T_{P} \Sigma_{\sigma}\right)\right\|_{s}\right|_{t=0}
$$

where $T_{P} \Sigma_{\sigma}$ denotes the oriented tangent plane of $\Sigma_{\sigma}$ at $P$, and $\left(F_{t}\right)_{*}$ denotes the differential of the map $F_{t}$. For $|x|$ large we have $F_{\mathrm{X}, \sigma}(x)=O\left(|x|^{-n}\right)$ uniformly in $\sigma$ because of the decay property which is assumed on $g$. The regularity theory implies that outside a fixed compact set $\Sigma_{\sigma}$ is the graph of a function $f_{\sigma}\left(x^{\prime}\right), x^{\prime}=\left(x^{\prime}, \ldots, x^{n-1}\right)$ having bounded gradient. We choose a sequence $\sigma_{i} \rightarrow \infty$ such that $\left\{\Sigma_{\sigma_{i}}\right\}$ converges to a limiting area minimizing hypersurface $\Sigma \subset M$. Because of the uniform decay condition on $F_{\mathbf{X}, \sigma_{1}}$, we get $\int_{\Sigma} F_{\mathbf{X}} d \mathcal{H}^{n-1} \geq 0$ where

$$
F_{\mathbf{X}}(P)=\left.\frac{d^{2}}{d t^{2}}\left\|\left(F_{t}\right)_{*}\left(T_{P} \Sigma\right)\right\|\right|_{t=0}
$$

and $\mathbf{X}=\mathbf{X}_{0}+\alpha \mathbf{X}_{1}$ for vector fields $\mathbf{X}_{0}$ of compact support and $\mathbf{X}_{1}$ fixed as above. Outside a compact subset of $M$ the surface $\Sigma$ is represented as the graph of a function $f\left(x^{2}\right)$ of bounded gradient. In fact, we easily get $|\partial f|(x)=\left(\left|x^{\prime}\right|^{-1}\right)$ from the regularity theory since we have a uniform bound on $f,\left|f\left(x^{\prime}\right)\right| \leq a_{0}$. On the other hand $f$ satisfies the minimal surface equation

$$
\sum_{i, j}\left(\delta_{i j}-\frac{f_{, i} f_{, j}}{1+|\partial f|^{2}}\right) f_{i j}+\sqrt{1+|\partial f|^{2}} \frac{\partial}{\partial \nu_{0}} \log h=0
$$

where

$$
h(x)=1+E|x|^{2-n}+O\left(|x|^{1-n}\right)
$$

and

$$
\nu_{0}=\left(1+|\partial f|^{2}\right)^{-1 / 2}(-\partial f, 1)
$$

is the Euclidean unit normal vector. Applying linear theory (see [10]) we get $\int\left(x^{\prime}\right)=a+$ $O\left(\left|x^{\prime}\right|^{3-n}\right)$ for $n \geq 4$, and $f\left(x^{\prime}\right)=a+O\left(\left|x^{\prime}\right|^{-1}\right)$ for $n=3$ for some constant $a$. The function $F_{X}$ can be calculated in terms of the geometry of $\Sigma$ (see [30])

$$
\begin{aligned}
F_{\mathbf{X}}=- & \sum_{i=1}^{n-1}
\end{aligned} \quad\left\langle R\left(\mathbf{X}, e_{i}\right) \mathbf{X}, e_{i}\right\rangle+\operatorname{div}_{M} Z+\left(\operatorname{div}_{M} \mathbf{X}\right)^{2} .
$$

where $Z=D_{\frac{\theta}{\partial t}} \frac{\partial F}{\partial t}$ is the acceleration vector field of the deformation, and $D$ is used to denote covariant differentiation in $M$ with respect to $g$. We use the notation

$$
\operatorname{div}_{M} \mathbf{X}=\sum_{i=1}^{n-1}\left\langle D_{e_{\mathbf{i}}} \mathbf{X}, e_{i}\right\rangle
$$

where $\mathbf{X}$ is a (not necessarily tangent) vector field along $\Sigma$ and $e_{1}, \ldots, e_{n-1}$ denotes an orthonormal basis for the tangent space to $\Sigma$. We write $\mathbf{X}=\hat{\mathbf{X}}+\varphi \nu$ where $\hat{\mathbf{X}}$ is tangent to $\Sigma$
and $\nu$ is the unit normal. Similarly $Z=\hat{Z}+\psi \nu$. Since $\Sigma$. is minimal we have $\operatorname{div}_{M} \chi^{\nu}=0$ for any function $\chi$. We then have

$$
F_{\mathrm{X}}=-\varphi^{2} \operatorname{Ric}(\nu, \nu)-\varphi^{2}\|B\|^{2}+\|\nabla \varphi\|^{2}+G
$$

where $G$ is given by

$$
\begin{aligned}
G=- & \sum_{i=1}^{n-1} \varphi\left(R\left(\hat{\mathbf{X}}, e_{i}\right) \nu, e_{i}\right\rangle-\sum_{i=1}^{n-1} \varphi\left(R\left(\hat{\mathbf{X}}, e_{i}\right) \hat{\mathbf{X}}, e_{i}\right\rangle \\
& +\operatorname{div}_{M} \hat{Z}+\left(\operatorname{div}_{M} \hat{\mathbf{X}}\right)^{2}-2 B(\nabla \varphi, \hat{\mathbf{X}})+\sum_{i=1}^{n-1} B\left(e_{i}, \dot{\mathbf{X}}\right)^{2} \\
& -2 \varphi \sum_{i, j=1}^{n-1} b_{i j} \hat{\mathbf{X}}_{i ; j}-\sum_{i, j=1}^{n-1} \hat{\mathbf{X}}_{i ; j} \hat{\mathbf{X}}_{j ; i} .
\end{aligned}
$$

In these formulas we work in an orthonormal frame, $B(\cdot, \cdot)$ denotes the second fundamental form given by $B(V, W)=\left\langle D_{V} W, \nu\right\rangle$ for tangent vector fields $V, W$. We let $b_{i j}=B\left(e_{i}, e_{j}\right)$ in our orthonormal basis, and for a tangent vector field $V=\sum V_{i} e_{i}, V_{i ; j}$ denotes the covariant derivative in the induced metric on $\Sigma$. Any term which involves $Z$ or $\hat{\mathbf{X}}$ must reduce to a boundary term. If $D \subset \Sigma$ is a bounded domain, we see

$$
\begin{aligned}
\int_{D} G d \mathcal{H}^{n-1}=\int_{\partial D}\{ & (\operatorname{div} \hat{\mathbf{X}})\langle\hat{\mathbf{X}}, \eta\rangle-\sum_{i, j} \hat{\mathbf{X}}_{i ; j} \hat{\mathbf{X}}_{j} \eta_{i} \\
& \left.-2 \varphi \sum_{i, j} b_{i j} \hat{\mathbf{X}}_{i} \eta_{j}+\langle\hat{Z}, \eta\rangle\right\} d \mathcal{H}^{n-2}
\end{aligned}
$$

where $\eta$ is the outward normal to $\partial D$ in $\Sigma$. To see that the interior terms drop out one must use the Gauß and Codazzi equations as well as the Ricci formula. For $\sigma>0$, let $D_{\sigma}=\Omega_{\sigma} \cap \Sigma$ where $\Omega_{\sigma}$ is the interior region bounded by $C_{\sigma}$ as above. From the decay conditions on $f$ and $h$ one checks that each of the boundary terms above decays faster than $\sigma^{2-n}$, and hence the boundary term tends to zero as $\sigma \rightarrow \infty$. Therefore we conclude

$$
\begin{equation*}
\int_{\Sigma}\left(\operatorname{Ric}(\nu, \nu)+\|B\|^{2}\right) \varphi^{2} d \mathcal{H}^{n-1} \leq \int_{\Sigma}|\nabla \varphi|^{2} d \mathcal{H}^{n-1} \tag{4.1}
\end{equation*}
$$

where $\varphi=\langle\mathbf{X}, \nu\rangle$ and $\mathbf{X}=\mathbf{X}_{0}+\alpha \mathbf{X}_{1}$ as above. Since $\mathbf{X}$ can be chosen to be arbitrary except that $\mathbf{X}=\alpha \frac{\partial}{\partial x^{n}}$ outside a compact set, we see that $\varphi$ is arbitrary except that

$$
\varphi=\alpha\left\langle\frac{\partial}{\partial x^{n}}, \nu\right\rangle=\alpha h^{2 /(n-2)}\left(1+|\partial f|^{2}\right)^{-1 / 2}
$$

outside a compact set for a constant $\alpha$. Since $\varphi-\alpha=O\left(\left|x^{\prime}\right|^{2-n}\right)$ we see that $\varphi-\alpha$ has finite energy and therefore we can take $\varphi$ to be any function for which $\varphi-\alpha$ has compact support (or finite energy) for some constant $\alpha$. As in [28] we can use the Gauß equation to write

$$
\begin{equation*}
\operatorname{Ric}(\nu, \nu)+\|B\|^{2}=\frac{1}{2} R_{M}-\frac{1}{2} R_{\Sigma}+\frac{1}{2}\|B\|^{2} \tag{4.2}
\end{equation*}
$$

where $R_{\Sigma}$ is the (intrinsic) scalar curvature of $\Sigma$ in the induced metric.
To complete the proof, we first suppose $n=3$ and choose $\varphi \equiv 1$ in (4.1) to obtain $\frac{1}{2} \int_{\Sigma} R_{\Sigma} d \mathcal{H}^{2}>0$. Now $\frac{1}{2} R_{\Sigma}$ is simply the Gaussian curvature of $\Sigma$. The decay estimates for $f, h$ easily imply that the total geodesic curvature of $\partial D_{\sigma}$ converges to $2 \pi$ where $D_{\sigma}=\Sigma \cap \Omega_{\sigma}$. Therefore we may apply the Gauß-Bonnet theorem on $D_{\sigma}$ and let $\sigma$ tend to infinity to get

$$
\frac{1}{2} \int_{\Sigma} R_{\Sigma} d \mathcal{H}^{2}=2 \pi \chi(\Sigma)-2 \pi
$$

Since $\chi(\Sigma) \leq 1$ for an open surface $\Sigma$, the right hand side is nonpositive. This contradicts the previous inequality and completes the proof for $n=3$.

Now suppose $n \geq 4$, and observe that the induced metric $\bar{g}$ on $\Sigma$ satisfies (in terms of coordinates $x^{1}, \ldots, x^{n-1}$ )

$$
\bar{g}_{i j}=h\left(x^{\prime}, f\left(x^{\prime}\right)\right)^{4 /(n-2)}\left(\delta_{i j}+f_{, i} f_{, j}\right)=\delta_{i j}+O\left(|x|^{2-n}\right)
$$

Therefore ( $\Sigma, \bar{g}$ ) is asymptotically flat and has energy zero. Inequality (4.1) together with (4.2) and the inequality $R_{M} \geq 0$ imply that the lowest Dirichlet eigenvalue for $L_{\bar{g}}$ on any compact domain in $\Sigma$ is positive because $c(n)=\frac{n-2}{4(n-1)}<\frac{1}{2}$ for $n \geq 3$. Linear theory then enables us to solve $L_{\vec{g}} u=0$ on $\Sigma, u>0$ on $\Sigma$, and $u \rightarrow 1$ at infinity. Moreover, $u$ has the expansion

$$
u\left(x^{\prime}\right)=1+E_{0}\left|x^{\prime}\right|^{3-n}+O\left(\left|x^{\prime}\right|^{2-n}\right)
$$

In particular, $u-1$ has finite energy on $\Sigma$, and we may take $\varphi=u$ in (4.1). Using (4.2) and the fact that $R_{M}>0$ we get

$$
-\int_{\Sigma} R_{\Sigma} u^{2} d \mathcal{H}^{n-1}<2 \int_{\Sigma}|\nabla u|^{2} d \mathcal{H}^{n-1} \leq c(n)^{-1} \int_{\Sigma}|\nabla u|^{2} d \mathcal{H}^{n-1}
$$

We may then write

$$
\begin{aligned}
\int_{\Sigma}|\nabla u|^{2} d \mathcal{H}^{n} & =\lim _{\sigma \rightarrow \infty} \int_{D_{\sigma}}|\nabla u|^{2} d \mathcal{H}^{n} \\
& =-c(n) \lim _{\sigma \rightarrow \infty} \int_{D_{\sigma}} R_{\Sigma} u^{2} d \mathcal{H}^{n-1}+\lim _{\sigma \rightarrow \infty} \int_{\partial D_{\sigma}} u \frac{\partial u}{\partial \eta} d \mathcal{H}^{n-1}
\end{aligned}
$$

where $\eta$ denotes the outward unit normal to $\partial D_{\sigma}$. From the expansion for $u$ we then find $E_{0}<0$. Thus ( $\left.\Sigma, u^{4 /(n-3)} \bar{g}\right)$ is asymptotically flat, has zero scalar curvature, and negative total energy. The contradiction now follows inductively from $n=3$. This completes the proof that $E \geq 0$. The statement that $E=0$ only if $(M, g)$ is isometric to $\mathbf{R}^{n}$ is proven in [23, Lemma 3 and Proposition 2]. We omit the details. This completes the proof of Theorem 4.2.

## 5 Noncompact manifolds and weak solutions

One of the results of [29] is that a simply connected, complete, locally conformally flat manifold $(M, g)$ with $R(g) \geq 0$ is conformally diffeomorphic to a domain $\Omega \subset S^{n}$ with the Hausdorff dimension of $S^{n}-\Omega$ being at most $(n-2) / 2$. In particular, any compact locally conformally flat manifold $\left(M, g_{0}\right)$ with $R\left(g_{0}\right) \geq 0$ is conformally covered by a simply connected domain $\Omega \subset S^{n}$ with $\operatorname{dim}\left(S^{n}-\Omega\right) \leq(n-2) / 2$. Thus by lifting solutions of (1.9) from $M$ to $\Omega$ we get solutions $u>0$ on $\Omega$ of the equation

$$
\begin{equation*}
L u+\frac{n(n-2)}{4} u^{(n+2) /(n-2)}=0 \tag{5.1}
\end{equation*}
$$

where $L u=\Delta_{S^{n}} u-\frac{n(n-2)}{4} u$. These solutions satisfy the "boundary condition" that $\left(\Omega, u^{4 /(n-2)} g_{0}\right)$ is a complete Riemannian manifold. Here we take $g_{0}$ to be the metric on the unit sphere. The theorem of Obata discussed in $\S 1$ classifies the global regular solutions of (5.1). The first example of a domain $\Omega$ arising from the above construction is $S^{n}-\{P, Q\}$ for two points $P, Q \in S^{n}$. After a conformal transformation, we can take $Q=-P$ and think of $S^{n}=\mathbf{R}^{n} \cup\{\infty\}$ with $P=0 . Q=\infty$. We explicitly analyzed the solutions of (5.1) for this domain $\Omega$ in $\S 2$. In general, any domain $\Omega$ arising as the universal cover (or any covering) of a compact manifold is invariant under a discrete subgroup $\Gamma$ of the conformal group of $S^{n}$ and is the domain of discontinuity of this group. From Kleinian group theory we know that if the limit set $\Lambda=S^{n}-\Omega$ contains more than two points, then it must contain a Cantor set. It is a theorem in [29] that for a domain $\Omega$ which covers a compact manifold, the quotient manifold $\Omega / \Gamma$ has a conformal metric of positive scalar curvature if and only if the Hausdorff dimension of $S^{n}-\Omega$ is less than $(n-2) / 2$.

Generally, if $u$ is a solution of (5.1) on a domain $\Omega \subset S^{n}$ such that $\left(\Omega, u^{4 /(n-2)} g_{0}\right)$ is a complete manifold, then it is shown in [29] that $u$ is integrable on $5^{n}$ to the power $(n+2) /(n-2)$ and that $u$ defines a global weak solution of (5.1) on $S^{n}$. Thus the problem of constructing complete solutions of (5.1) on $\Omega$ is closely related to the problem of constructing weak solutions of (5.1) on $S^{n}$ with prescribed singular set $\Lambda=S^{n}-\Omega$. We have seen that many solutions of (5.1) exist which are singular at two specified points; in fact, such solutions can be classified. The question of specifying more than two singular points has been posed in various contexts over the years. (Solutions do not exist with one singular point.) An obvious approach to this problem would be to fix the asymptotic behavior near $k$ specified points of $S^{n}$ and to construct a solution which is essentially a compact perturbation of a given function with the correct asymptotics. The difficulties in this approach are apparent from analysis of the solutions singular at $0, \infty$. Let $x \in \mathbf{R}^{n}, t=\log |x|$ as in $\S 2$. The simplest solution of (2.2) is the constant solution $u(t) \equiv u_{0}$. This gives rise to the solution $v(x)=u_{0}|x|^{-(n-2) / 2}$ in $\mathbf{R}^{n}-\{0\}$ of the equation $\Delta v+\frac{n(n-2)}{4} v^{(n+2) /(n-2)}=0$ which is equivalent to equation (5.1). If we consider solutions which are near $u_{0}$ on a large piece of $\mathbf{R} \times S^{n-1}$, then we would expect the linearized
equation at $u_{0}$ to dictate their behavior. The linearized operator is $\mathcal{L} \eta=\Delta \eta+(n-2) \eta$ where $\Delta$ is with respect to the metric $d t^{2}+d \xi^{2}$ on the cylinder. In particular, we see that zero is embedded in the continuous spectrum for $\mathcal{L}$ on $\mathrm{R} \times S^{n-1}$. Thus controlling $\mathcal{L}^{-1}$ on large regions of $\mathbf{R} \times S^{n-1}$ will be a difficult problem. It is not known whether solutions exist with asymptotic behavior given by the constant solution $u_{0}$. In [24] we proved a general existence theorem for weak solutions which implies that one can specify any $k$ points of $S^{n}$ and construct solutions singular at these points and asymptotic to solutions described in $\S 2$ with $\alpha$ near one. Roughly speaking, the spectrum of the linearized operator for such solutions ( $\alpha \approx 1$ ) contains a small interval near 0 , and the spectral subspace corresponding to this interval imposes an infinite number of geometric "balancing" conditions on the way in which spherical pieces of solutions are attached. We refer the reader to [24] for details,

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