

# Variations of gwistor space

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## Abstract

We study natural variations of the  $G_2$  structure  $\sigma_0 \in \Lambda_+^3$  existing on the unit tangent sphere bundle  $SM$  of any oriented Riemannian 4-manifold  $M$ . We find a circle of structures for which the induced metric is the usual one, the so-called Sasaki metric, and prove how the original structure has a preferred role in the theory. We deduce the equations of calibration and cocalibration, as well as those of  $W_3$  pure type and nearly-parallel type.

**Key Words:** calibration, Einstein manifold,  $G_2$ -structure, gwistor space.

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## 1 Introduction

In [6] it was shown how a natural  $G_2 = \text{Aut } \mathbb{O}$  structure is associated to the unit tangent sphere bundle  $\pi : SM \rightarrow M$  of any given oriented Riemannian 4-manifold  $M$ . The techniques are twistorial, such as those learned by the author from [11], so we have chosen to give the name of  $G_2$ -twistors or simply *gwistors* to the new spaces.

The theory starts by a construction of the octonions inside  $TTM$ , restricted to the 3-sphere fibre bundle  $SM$ , which we take a moment to explain. Recall the Levi-Civita connection of the base induces a canonical splitting of the tangent bundle of  $TM$ . Both vertical

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and horizontal subbundles  $V, H$  become isometric to  $\pi^*TM$  with the pull-back metric. The direct-sum metric over  $TM$  is called the Sasaki metric of this manifold. Independently of the metric,  $V$  has a tautological section, denoted  $U$  and defined by  $U_u = u$ ; hence also a vertical vector field on  $SM = \{u \in TM : \|u\| = 1\}$ . Now each point  $U_u$  is identified with the identity element, the generator of the real line in  $T_uTM \simeq \mathbb{O}$ . Then we use the volume form  $\pi^*\text{vol}_M$  coupled with  $U$ , to induce a cross-product on  $u^\perp \subset V$ . A conjugation map is equally trivial to define. Together these induce a quaternionic structure on  $V$ . Then, applying the well-known Cayley-Dickson process, we obtain the structure of  $\mathbb{O}$  in  $V \oplus H$ .

The pull-back of  $TM$  also inherits a metric connection  $\nabla^* = \pi^*\nabla$  and hence parallel identifications of horizontals with verticals, passing through  $\pi^*TM$ , cf. loc. cit. and [15]. The manifold  $SM$  is endowed with the induced metric from the canonical or Sasaki metric on  $TM$ . Clearly  $TSM$  coincides with  $V_1 \oplus H$  where  $V_1 = \{v \in V : \langle U_u, v \rangle = 0\}$  at each point  $u$ . Since  $u$  is pointing outwards, our space  $SM$  inherits a  $G_2$ -structure, for which it receives the name of gwistor space. Recall  $G_2 = \text{Aut } \mathbb{O}$ , but clearly the structure is the extension of an  $SO(3)$  structure. The connection induces a projection  $\nabla^*U : TSM \rightarrow V$  with kernel  $H$  and the identity on  $V$ .

By a Theorem of Y. Tashiro in [7] it is known that  $SM$  has a metric almost contact structure for a Riemannian base of arbitrary dimension. As these are rigid geometrical objects, the contact structure is bound to be K-contact if and only if  $M$  has constant sectional curvature 1. Then it turns out also to be Sasakian. Locally the space is the same as the Stiefel manifold  $V_{5,2} = SO(5)/SO(3)$ .

Now we leave aside the Cayley-Dickson process and concentrate on the five invariant 3-forms which are naturally defined on  $SM$ . Then we may try to find other interesting  $G_2$  structures. This article is devoted to them, the *variations* of gwistor space, which may also be called  $g$ -natural  $G_2$ -structures on the unit tangent sphere bundle, in analogy with the terms for the metrics used by [1],[2] and many references therein. On the other hand, the terms deformation or perturbation are also used in similar context by other authors, so we made a choice.

We readily announce the support of some computer algebra software for the proof of Theorem 1.6 below. It is a polynomial computation of the 7th order in four variables which we believe anyone can reproduce easily.

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The author dedicates this work to Marta Barata.

## 1.1 The basic 3-forms

We start by abbreviating the notation and write  $SM = \mathcal{G}$ . There is, as we have seen, an isometry connecting  $H$  with  $V$ , which we denote by  $B$ . We extend it by 0 to  $V$ , thus defining an endomorphism  $B$  of  $TTM$ . Then the transpose tangent vector field  $B^tU$  generates a real line bundle, contained in  $T\mathcal{G}$ , and a 1-form  $\theta = (B^tU)^b$ . We may write a splitting, with  $H_1 = B^tV_1$ :

$$T^*\mathcal{G} = \mathbb{R}\theta \oplus H_1^* \oplus V_1^*. \quad (1)$$

We pass to the language of differential forms. The 1-form  $\theta$  is the aforementioned almost contact structure, satisfying:

$$\theta_u(v) = \langle u, d\pi(v) \rangle, \quad \forall u \in \mathcal{G}, v \in T\mathcal{G}. \quad (2)$$

The usual pull-back (horizontal) of the volume form of  $M$  is also denoted by  $\text{vol}$ . The vertical pull-back of  $\text{vol} \in \Omega^4(M)$  contracted with  $U$  is denoted by  $\alpha$ ; then we define analogously a 3-form  $\alpha_3 = (B^tU) \lrcorner \text{vol}$ . Of course,

$$\theta \wedge \alpha_3 = \text{vol}, \quad \text{vol} \wedge \alpha = \text{Vol}_{\mathcal{G}}. \quad (3)$$

As shown in [4], it is possible to find locally an ‘adapted’ frame, i.e. an oriented orthonormal frame  $e_0, e_1, \dots, e_6$  respecting (1). In particular such that (with usual notation for the co-framing,  $e^{ab\dots c} = e^a \wedge e^b \wedge \dots \wedge e^c$ )

$$\theta = e^0, \quad \alpha_3 = e^{123}, \quad \alpha = e^{456}. \quad (4)$$

It is easy to compute that  $d\theta(v, w) = \langle (B^t - B)v, w \rangle$ ,  $\forall v, w \in T\mathcal{G}$ , which restricts to a symplectic 2-form on the vector bundle  $H_1 \oplus V_1$  and hence is written as  $d\theta = e^{41} + e^{52} + e^{63}$ .

The endomorphism  $B$  allows one to construct two other 3-forms. We turn the reader to [4] for the invariant definition, ie. to see these forms depend only of the metric on  $M$  and not of the choice of adapted frame. They are:

$$\alpha_1 = e^{156} + e^{264} + e^{345} \quad (5)$$

and

$$\alpha_2 = e^{126} + e^{234} + e^{315}. \quad (6)$$

One can prove the five 3-forms  $\alpha, \alpha_1, \dots, \alpha_3, \theta \wedge d\theta$  correspond to a basis for the space of invariants in  $\Lambda^3(\mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$  under the action of  $\text{SO}(3)$ , the underlying structure group of  $\mathcal{G}$ , ie. there are five irreducible 1-dimensional submodules<sup>2</sup>.

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The five 3-forms satisfy the ‘first structure equations’,  $\forall i = 1, 2, 3$ :

$$\begin{aligned}
 *\alpha &= \theta \wedge \alpha_3 = \text{vol} = \pi^* \text{vol}_M, & *\alpha_1 &= -\theta \wedge \alpha_2, & *\alpha_2 &= \theta \wedge \alpha_1, \\
 *d\theta &= \frac{1}{2}\theta \wedge (d\theta)^2, & *(d\theta)^2 &= 2\theta \wedge d\theta, & *(d\theta)^3 &= 6\theta, \\
 \alpha_1 \wedge \alpha_2 &= 3\alpha_3 \wedge \alpha = 3*\theta = \frac{1}{2}(d\theta)^3, & d\theta \wedge \alpha_i &= d\theta \wedge *\alpha_i = \alpha_3 \wedge \alpha_i = 0, \\
 & & d\theta \wedge \alpha &= d\theta \wedge *\alpha = \alpha \wedge \alpha_1 = \alpha \wedge \alpha_2 = 0.
 \end{aligned} \tag{7}$$

The natural  $G_2$  structure on  $\mathcal{G}$  to which we have referred is given<sup>3</sup> by the 3-form

$$\sigma_0 = \alpha_2 - \alpha + \theta \wedge d\theta. \tag{8}$$

This form gives the *canonical* representation theory without changing the canonical orientation of  $\mathcal{G}$ ; namely it gives the usual  $G_2$ -modules  $\Lambda_7^2, \Lambda_{14}^2$  (which appeared from opposite highest weights in [4],[5],[6]).

The integrability of  $\sigma_0$  was studied in the case of the Levi-Civita connection on  $M$  in the seminal article [6], and in the case of metric connections with torsion, which clearly allow the same construction, in [4]. For the first case we know that the  $G_2$ -twistor structure is cocalibrated, ie.  $d*\sigma_0 = 0$ , if and only if the base  $M$  is an Einstein manifold.

## 1.2 Variations of $G_2$ structures

Let us recall the definition of stable forms from the theory of  $G_2$ -manifolds, [8],[9].

Let  $\sigma$  denote a linear  $G_2$  structure on a 7-dimensional oriented vector space  $V$ , ie. some identification of  $V$  with the canonical  $\mathbb{R}^7$  is assumed. A consequence of the study of the Lie group  $G_2 = \text{Aut } \sigma \subset \text{SO}(7)$  is that it is connected and 14 dimensional; henceforth, that the orbit of  $\sigma$  under  $\text{GL}(7, \mathbb{R})$  is an open set inside the module  $\Lambda^3 V^*$ . This orbit is denoted  $\Lambda_{\pm}^3$  and known as the space of stable  $G_2$ -structures on  $V$ . We detect the boundaries of such stability by the non-degeneracy of the induced Euclidean product. Indeed, the inner product  $\langle \cdot, \cdot \rangle_{\sigma}$  is given by the clearly symmetric map  $v \otimes w \mapsto v \lrcorner \sigma \wedge w \lrcorner \sigma \wedge \sigma$  — with this image 7-form required to be, on the diagonal of  $V$ , a positive multiple of the chosen orientation. The given  $\sigma$  satisfies this condition by assumption. Allowing also  $\sigma$  to vary, we do have a  $\text{GL}(7, \mathbb{R})$ -equivariant map

$$V \otimes V \otimes \Lambda^3 V^* \longrightarrow \Lambda^7 V^*.$$

Since  $\Lambda^7 V^* \setminus \{0\}$  has two connected components, we conclude  $\Lambda_{\pm}^3$  is the union of two open orbits under the action of the subgroup  $\text{GL}^+(7, \mathbb{R})$ , identified bijectively by a  $-$  sign because

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<sup>3</sup>Actually the structure was given first by the opposite,  $-\sigma_0$ , but we take the opportunity here to make the change.



Notice that  $\sigma_0$  corresponds to the identity 1<sub>7</sub>. Computing determinants, the metric is positive-definite if  $f_4 > 0$ ,  $x > 0$  and  $xy - z^2 > 0$ . This proves the following result.

**Theorem 1.1.** *If a set of scalar functions  $f_0, \dots, f_4$  induces a  $G_2$  structure on  $\mathcal{G}$ , then it satisfies  $f_4 > 0$ ,  $f_2^2 - f_1 f_3 > 0$  and*

$$3f_0 f_1 f_2 f_3 - f_0 f_2^3 - f_0^2 f_3^2 - f_3 f_1^3 > 0. \quad (13)$$

**Remarks.** 1. The homogeneous fourth degree polynomial is irreducible and has no critical values in the domain. 2. The metrics obtained are all natural metrics in the sense of [1, 2] and other references therein.

Using the Gram-Schmidt process on the new metric, we obtain the oriented orthonormal frame, for  $i = 1, 2, 3$ ,

$$\tilde{e}_0 = \frac{1}{f_4 \sqrt{t}} e_0, \quad \tilde{e}_i = \frac{1}{\sqrt{tx}} e_i, \quad \tilde{e}_{i+3} = \sqrt{\frac{x}{th}} \left( e_{i+3} - \frac{z}{x} e_i \right), \quad (14)$$

where  $h$  is the polynomial in (13):

$$h = xy - z^2. \quad (15)$$

A dual co-frame is then

$$\tilde{e}^0 = f_4 \sqrt{t} e^0, \quad \tilde{e}^i = \sqrt{tx} e^i + z \sqrt{\frac{t}{x}} e^{i+3}, \quad \tilde{e}^{i+3} = \sqrt{\frac{th}{x}} e^{i+3}. \quad (16)$$

We obtain also the useful formulas

$$e^0 = \frac{1}{f_4 \sqrt{t}} \tilde{e}^0, \quad e^i = \frac{1}{\sqrt{txh}} (\sqrt{h} \tilde{e}^i - z \tilde{e}^{i+3}), \quad e^{i+3} = \sqrt{\frac{x}{th}} \tilde{e}^{i+3}. \quad (17)$$

Indeed the frame (14) is oriented, ie.  $\tilde{e}^{0123456} = m e^{0123456}$  is a positive multiple of the chosen orientation. Immediately through (12) and (16) we find that

$$m = f_4 h^{\frac{1}{3}}. \quad (18)$$

### 1.3 $G_2$ -structures $\sigma$ compatible with the Sasaki metric

Let  $\sigma$  be a variation of  $\sigma_0$ .

**Proposition 1.1.** *The metric induced by  $\sigma$  coincides with the Sasaki metric on  $\mathcal{G}$  if and only if*

$$f_0^2 + f_1^2 = 1, \quad f_2 = -f_0, \quad f_3 = -f_1, \quad f_4 = 1. \quad (19)$$

*Under the action of  $SO(7)$  the orbit of 3-forms which can be written in the form (9) is a circle  $S^1$ .*

*Proof.* By hypothesis, we have  $tf_4^2 = tx = ty = 1$  and  $z = 0$ . Hence  $f_4^3 = f_4x = f_4y = m$  and  $h = xy = f_4^4$ . Knowing  $m$  must equal 1 or equating through (18) we get all these equal to 1, except for  $z$ . Now solving the system (12) we deduce the equivalence in the first part of the result. The second follows from the first (as the metric is preserved) and the analysis of the orbit of  $\sigma_0 = \alpha_2 - \alpha + \theta \wedge d\theta$  through known methods. So, we note that already  $U(3) \subset SO(7)$  acts as a real group, fixing  $e_0$ , on the vector space  $E = H_1 \oplus V_1$ , which has a natural complex structure. Moreover,

$$(e^1 + \sqrt{-1}e^4) \wedge (e^2 + \sqrt{-1}e^5) \wedge (e^3 + \sqrt{-1}e^6) = \alpha_3 - \alpha_1 + \sqrt{-1}(\alpha_2 - \alpha) =: \eta \in \Lambda^3 E^{(1,0)*}.$$

Since  $SU(3) \subset G_2$ , we have only to consider maps  $g$  such that  $g|_E = e^{is}1_E$  for some  $s \in \mathbb{R}$ . One finds easily the role of  $g$  as a real map. Immediately we deduce  $g$  fixes the 3-form  $\theta \wedge d\theta = e^{041} + e^{052} + e^{063}$ . On the other hand  $g \cdot \eta = g^3\eta$ . Letting  $g$  be such that  $g^3 = f_0 + \sqrt{-1}f_1 \in S^1$  we find that this real map solves ( $\Im$  denotes imaginary part)

$$g \cdot \sigma_0 = g \cdot (\Im\eta + \theta \wedge d\theta) = \Im(g^3\eta) + \theta \wedge d\theta = -f_0\alpha - f_1\alpha_1 + f_0\alpha_2 + f_1\alpha_3 + \theta \wedge d\theta.$$

The result follows (notice the space  $SO(7)/G_2$  is 7 dimensional so we have to restrict our statement to the specific forms). ■

For the following computations we apply formulas which have been deduced in [4, 6]. We start by the particular case found above, when the Sasaki metric is preserved.

**Theorem 1.2.** *Suppose the Riemannian manifold  $M$  is connected. Let  $\sigma$  be a variation of gwistor space satisfying the condition that the induced metric coincides with the Sasaki metric on  $\mathcal{G}$ , that is,  $\sigma = -f_0\alpha - f_1\alpha_1 + f_0\alpha_2 + f_1\alpha_3 + \theta \wedge d\theta$  with  $(f_0, f_1) : \mathcal{G} \rightarrow S^1$  a smooth function. Then we have:*

1. Always  $d\sigma \neq 0$ .
2. If  $(f_0, f_1) \neq (\pm 1, 0)$ , then  $d * \sigma = 0$  if and only if the functions  $f_0, f_1$  are constant and the Riemannian base  $M$  has constant sectional curvature.
3. If  $(f_0, f_1) = (\pm 1, 0)$ , then  $d * \sigma = 0$  if and only if  $M$  is Einstein.

The proof follows by recalling the list of derivatives of the fundamental 3-forms in (33), which were deduced in [4, Proposition 2.3]. Result (1) is the particular case of Theorem 1.5 (below). For (2) we may easily compute  $d * \sigma$ . If it is to vanish, then we deduce a curvature equation  $R_{0123} = 0$ , which implies constant sectional curvature on the base, and that  $f_0df_0 = -f_1df_1$  is a multiple of  $\theta$ , which implies  $(f_0, f_1)$  is constant. Finally, if the base metric has constant sectional curvature  $k$ , then another curvature term appearing satisfies  $\mathcal{R}^U\alpha = -k\theta \wedge \alpha_1$ , and we find this is the solution required in case  $f_1 \neq 0$ .

Theorem 1.2 shows that the original gwistor space structure we found, the standard  $\sigma_0$ , is indeed preferred; it has greater interest than the others on the circle (of course, besides the antipodal of  $\sigma_0$ , a duality which as explained in section 1.2 we shall not explore here).

We shall now see a result concerning the type of  $d\sigma$  with respect to the  $G_2$ -decomposition of  $\Lambda^4 T^* \mathcal{G}$ . We follow the description by [10] also found in several good references such as [3, 8, 9]. A structure is said to be of pure type  $W_3$  if  $d\sigma = *\tau_3$  with  $\tau_3$  the  $W_3$  part, that is satisfying  $\tau_3 \wedge \sigma = \tau_3 \wedge *\sigma = 0$ .

**Theorem 1.3.** *The gwistor space  $(\mathcal{G}, \sigma)$  of a constant sectional curvature  $k$  manifold with  $\sigma$  given as before and  $f_0, f_1$  constant, is of pure type  $W_3$  if and only if  $k = -2$ .*

*Proof.* Our invoked Riemann tensor satisfies  $R_{ijpq} = k(\delta_i^q \delta_j^p - \delta_i^p \delta_j^q)$  for a constant sectional curvature metric (this is not a sign convention; it is a compatibility condition between required tensors on  $\mathcal{G}$  and tensors on the base manifold). By definitions in (34,35), seen below but known from [4], we have  $\mathcal{R}^U \alpha = -k\theta \wedge \alpha_1$ ,  $\mathcal{R}^U \alpha_1 = -2k\theta \wedge \alpha_2$ .

Now, since the metric is Einstein we have  $d*\sigma = 0$  by Theorem 1.2 and thence  $d\sigma = \lambda*\sigma + *\tau_3$  (in other words, cf. [9], we have  $\tau_1 = \tau_2 = 0$ ). The condition of pure type  $W_3$ , equivalently  $\lambda = 0 \in \mathbb{R}$ , corresponds by a simple argument to  $(d\sigma) \wedge \sigma = 0$ .

With  $\sigma = -f_0\alpha - f_1\alpha_1 + f_0\alpha_2 + f_1\alpha_3 + \theta \wedge d\theta$ , we get the following formula:

$$d\sigma = \theta \wedge (-3f_1\alpha + f_0(k+2)\alpha_1 + f_1(2k+1)\alpha_2 - 3f_0k\alpha_3) + (d\theta)^2. \quad (20)$$

Using the ‘first structure equations’ from (7) or [4, Proposition 2.1] and  $f_0^2 + f_1^2 = 1$ , we have

$$\begin{aligned} d\sigma \wedge \sigma &= (3f_1^2 + 3f_0^2(k+2) + 3f_1^2(2k+1) + 3f_0^2k + 6)\text{Vol}_{\mathcal{G}} \\ &= (6f_1^2 + 6f_0^2 + 6(f_1^2 + f_0^2)k + 6)\text{Vol}_{\mathcal{G}} \\ &= 6(2+k)\text{Vol}_{\mathcal{G}}. \end{aligned}$$

Hence the result. ■

We recover, in particular, the result in [4, Corollary 3.1] for the preferred  $\sigma_0 = \alpha_2 - \alpha + \theta \wedge d\theta$  on hyperbolic space of sectional curvature  $-2$ . Notice however the independency from the pair  $(f_0, f_1) \in S^1$ . The same is true with the following quite noticeable formula.

**Proposition 1.2.** *Assuming the above conditions,  $\|d\sigma\|^2 = 12(k^2 + k + 2)$ . In particular,  $\|d\sigma\|^2 = 48$  if and only if  $k = -2$  or  $k = 1$ .*

*Proof.* Immediate from (20). ■



## 1.4 Properties of the general case

Let us consider some metric problems related with the variations of gwistor space.

Suppose  $(f_0, \dots, f_4) : \mathcal{G} \rightarrow \mathbb{R}^5$  is a function satisfying the conditions in Theorem 1.1. We study those 3-forms

$$\sigma = f_0\alpha + f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 + f_4\theta \wedge d\theta \quad (21)$$

which define  $G_2$ -structures on  $\mathcal{G} \rightarrow M$ .

**Remarks.** 1. Recall a metric almost contact structure is said to be K-contact if the characteristic vector field is Killing. In the case of the Sasaki metric,  $(\mathcal{G}, \theta, B^tU)$  is K-contact if and only if  $M$  is locally isometric to  $S^4$  of radius 1, a result due to Y. Tashiro. In general, our metrics  $\langle \cdot, \cdot \rangle_\sigma$  induced from  $\sigma$  turned out to be ‘ $g$ -natural’ contact metrics in the sense of e.g. [1] (in particular the immediate question of  $\langle \cdot, \cdot \rangle_\sigma$  being K-contact is solved in the same reference). 2. Another feature of gwistor theory is that  $\sigma$  seems to be never preserved by the vector field  $B^tU$ . This is known both as the geodesic spray or the geodesic flow vector field, cf. [14, 15]. Indeed, computations for constant  $f_i$  have shown that the equation  $\mathcal{L}_{B^tU}\sigma = 0$  has no solution  $\sigma \in \Lambda^3_+$ . For any  $f_i$  defined on  $\mathcal{G}$ , or even just the pull-back of functions on  $M$ , one may write interesting differential equations.

Now we shall compute the exterior derivatives of the  $G_2$ -structures. From the formulas in (17) we deduce

$$\theta = \frac{1}{f_4 t^{\frac{1}{2}}} \tilde{\theta}, \quad d\theta = \frac{1}{th^{\frac{1}{2}}} \widetilde{d\theta}, \quad \alpha = \frac{x^{\frac{3}{2}}}{(th)^{\frac{3}{2}}} \tilde{\alpha}, \quad (22)$$

$$\alpha_1 = \frac{x^{\frac{1}{2}}}{t^{\frac{3}{2}}h} \left( \tilde{\alpha}_1 - \frac{z}{h^{\frac{1}{2}}} \tilde{\alpha} \right), \quad \alpha_2 = \frac{1}{x^{\frac{1}{2}}(th)^{\frac{3}{2}}} (h\tilde{\alpha}_2 - 2h^{\frac{1}{2}}z\tilde{\alpha}_1 + 3z^2\tilde{\alpha}), \quad (23)$$

$$\alpha_3 = \frac{1}{(txh)^{\frac{3}{2}}} (h^{\frac{3}{2}}\tilde{\alpha}_3 - hz\tilde{\alpha}_2 + h^{\frac{1}{2}}z^2\tilde{\alpha}_1 - z^3\tilde{\alpha}). \quad (24)$$

The forms with a tilde are defined algebraically using the orthonormal basis for  $\sigma$ , formally introduced as the respective  $\theta, d\theta, \alpha, \dots, \alpha_3$ . For instance  $\tilde{\theta} = \tilde{e}^0$ ,  $\widetilde{d\theta} = \tilde{e}^{41} + \tilde{e}^{52} + \tilde{e}^{63}$ , cf. (4). In particular, we note that we may use the already mentioned ‘first structure equations’ from (7) but with a tilde!

We also need the inverse formulas of the above:

$$\widetilde{\theta \wedge d\theta} = f_4 t^{\frac{3}{2}} h^{\frac{1}{2}} \theta \wedge d\theta, \quad \tilde{\alpha} = \frac{(th)^{\frac{3}{2}}}{x^{\frac{3}{2}}} \alpha, \quad (25)$$

$$\tilde{\alpha}_1 = \frac{ht^{\frac{3}{2}}}{x^{\frac{3}{2}}}(x\alpha_1 + 3z\alpha), \quad \tilde{\alpha}_2 = \frac{h^{\frac{1}{2}}t^{\frac{3}{2}}}{x^{\frac{3}{2}}}(x^2\alpha_2 + 2xz\alpha_1 + 3z^2\alpha), \quad (26)$$

$$\tilde{\alpha}_3 = \frac{t^{\frac{3}{2}}}{x^{\frac{3}{2}}}(x^3\alpha_3 + x^2z\alpha_2 + xz^2\alpha_1 + z^3\alpha). \quad (27)$$

Using the ‘first structure equations’ for the Hodge operator of the metric and orientation induced by  $\sigma$ , and writing back in terms of the usual frame, we obtain the following result.

**Theorem 1.4.**

$$*_\sigma(\theta \wedge d\theta) = \frac{t^{\frac{1}{2}}h^{\frac{1}{2}}}{2f_4}(d\theta)^2, \quad (28)$$

$$*_\sigma\alpha = \frac{f_4t^{\frac{1}{2}}}{h^{\frac{3}{2}}}\theta \wedge (x^3\alpha_3 + x^2z\alpha_2 + xz^2\alpha_1 + z^3\alpha), \quad (29)$$

$$*_\sigma\alpha_1 = -\frac{f_4t^{\frac{1}{2}}}{xh^{\frac{3}{2}}}\theta \wedge (3x^3z\alpha_3 + x^2(h + 3z^2)\alpha_2 + x(2hz + 3z^3)\alpha_1 + (3hz^2 + 3z^4)\alpha), \quad (30)$$

$$*_\sigma\alpha_2 = \frac{f_4t^{\frac{1}{2}}}{x^2h^{\frac{3}{2}}}\theta \wedge (3x^3z^2\alpha_3 + x^2(2hz + 3z^3)\alpha_2 + x(h^2 + 4hz^2 + 3z^4)\alpha_1 + (3h^2z + 6hz^3 + 3z^5)\alpha), \quad (31)$$

$$*_\sigma\alpha_3 = -\frac{f_4t^{\frac{1}{2}}}{x^3h^{\frac{3}{2}}}\theta \wedge (x^3z^3\alpha_3 + x^2(hz^2 + z^4)\alpha_2 + x(h^2z + 2hz^3 + z^5)\alpha_1 + (h^3 + 3h^2z^2 + 3hz^4 + z^6)\alpha). \quad (32)$$

**Corollary 1.1.** *The Hodge  $*$  operator is homogeneous of degree  $\frac{1}{3}$  on 3-forms viewed as a map  $\sigma \rightsquigarrow *_\sigma$ .*

*Proof.* From definitions, we see  $x, y, z$  have degree 2 and thence  $h$  has degree 4; then  $m = f_4h^{\frac{1}{3}}$  and  $\text{Vol}_\sigma$  have degree  $\frac{7}{3}$  and finally  $t = f_4/m$  has degree  $-\frac{4}{3}$ . Finally, observing (28) the result follows (though quite easily seen as a corollary from the above, this result also follows from the definition of  $*_\sigma$ ).  $\blacksquare$

Now we recall the formulas from [4, Proposition 2.3]:

$$\begin{aligned} d\alpha &= \mathcal{R}^U\alpha, & d\alpha_1 &= 3\theta \wedge \alpha + \mathcal{R}^U\alpha_1, \\ d\alpha_2 &= 2\theta \wedge \alpha_1 - \underline{r} \text{vol}, & d\alpha_3 &= \theta \wedge \alpha_2. \end{aligned} \quad (33)$$

$\mathcal{R}^U\alpha, \mathcal{R}^U\alpha_1$  are linearly independent forms depending on the curvature  $R$  of  $M$ , and  $\underline{r}$  is a scalar function on  $\mathcal{G}$  defined by  $\underline{r}(u) = r(u, u)$ , with  $R$  and  $r$  the usual Riemann and Ricci curvature tensors. Concretely, cf. [4, formulas 25 and 26],

$$\mathcal{R}^U\alpha = \sum_{0 \leq i < j < 3} R_{ij01}e^{ij56} + R_{ij02}e^{ij64} + R_{ij03}e^{ij45}, \quad (34)$$

$$\mathcal{R}^U \alpha_1 = \sum_{0 \leq i < j \leq 3} R_{ij01}(e^{ij26} + e^{ij53}) + R_{ij02}(e^{ij61} + e^{ij34}) + R_{ij03}(e^{ij15} + e^{ij42}). \quad (35)$$

In particular  $\theta \wedge \mathcal{R}^U \alpha_1 = -\rho \wedge \text{vol}$  where  $\rho = \sum_{i=1}^3 r(e_i, e_0)e^{i+3}$ .

**Theorem 1.5.** *For any functions  $f_0, \dots, f_4$ , we have  $d\sigma \neq 0$ .*

*Proof.* Indeed, since  $d\theta \wedge \alpha_i = 0, \forall i = 0, 1, 2, 3$ ,  $\alpha_0 = \alpha$ , we have by the Bianchi identity

$$\begin{aligned} \theta \wedge d\theta \wedge d\sigma &= \theta \wedge d\theta \wedge (f_4(d\theta)^2 + \sum df_i \wedge \alpha_i + f_i d\alpha_i) \\ &= (6f_4 + f_0(R_{2301} + R_{3102} + R_{1203}))\text{Vol}_G = 6f_4 \text{Vol}_G. \end{aligned}$$

However, we saw  $f_4$  must be positive. ■

From now on we assume the functions  $f_0, \dots, f_4$  are constant.

Returning to the Hodge duals of Theorem 1.4, then we have by simple reasons

$$d(*_\sigma(\theta \wedge d\theta)) = 0, \quad (36)$$

$$d(*_\sigma \alpha) = -\frac{f_4 t^{\frac{1}{2}}}{h^{\frac{3}{2}}} \theta \wedge (xz^2 \mathcal{R}^U \alpha_1 + z^3 \mathcal{R}^U \alpha), \quad (37)$$

$$d(*_\sigma \alpha_1) = \frac{f_4 t^{\frac{1}{2}}}{x h^{\frac{3}{2}}} \theta \wedge (x(2hz + 3z^3) \mathcal{R}^U \alpha_1 + (3hz^2 + 3z^4) \mathcal{R}^U \alpha), \quad (38)$$

$$d(*_\sigma \alpha_2) = -\frac{f_4 t^{\frac{1}{2}}}{x^2 h^{\frac{3}{2}}} \theta \wedge (x(h^2 + 4hz^2 + 3z^4) \mathcal{R}^U \alpha_1 + (3h^2 z + 6hz^3 + 3z^5) \mathcal{R}^U \alpha), \quad (39)$$

$$d(*_\sigma \alpha_3) = \frac{f_4 t^{\frac{1}{2}}}{x^3 h^{\frac{3}{2}}} \theta \wedge (x(h^2 z + 2hz^3 + z^5) \mathcal{R}^U \alpha_1 + (h^3 + 3h^2 z^2 + 3hz^4 + z^6) \mathcal{R}^U \alpha). \quad (40)$$

Adding up the above with the respective coefficients from (9), we find the vanishing of the two polynomials

$$\mathfrak{p}_1 = -f_0 x^3 z^2 + f_1 x^2 (2hz + 3z^3) - f_2 x (h^2 + 4hz^2 + 3z^4) + f_3 (h^2 z + 2hz^3 + z^5), \quad (41)$$

$$\mathfrak{p}_2 = f_0 x^3 z^3 - f_1 x^2 (3hz^2 + 3z^4) + f_2 x (3h^2 z + 6hz^3 + 3z^5) - f_3 (h^3 + 3h^2 z^2 + 3hz^4 + z^6) \quad (42)$$

is a sufficient condition for the vanishing of  $d(*_\sigma \sigma)$ :

$$d(*_\sigma \sigma) = \frac{f_4 t^{\frac{1}{2}}}{x^3 h^{\frac{3}{2}}} \theta \wedge (x \mathfrak{p}_1 \mathcal{R}^U \alpha_1 - \mathfrak{p}_2 \mathcal{R}^U \alpha). \quad (43)$$

Also the reader understands now why we chose constant coefficients. If  $z \neq 0$ , we may multiply the first polynomial by  $z$ , add to the second and factor out a  $h(> 0)$  from the result, to obtain:

$$-f_1 x^2 z^2 + 2f_2 x h z + 2f_2 z^3 x - f_3 h^2 - 2f_3 h z^2 - f_3 z^4. \quad (44)$$

Finally, introducing equations (12,15) and resorting to some computer algebra software, we are able to find two independent expressions in the original parameters  $f_0, \dots, f_3$ :

$$\mathbf{p}_1 = -f_0 (f_1^2 - f_0 f_2) (-f_2^2 + f_1 f_3)^2 \quad (45)$$

$$\begin{aligned} \mathbf{p}_2 = (f_2^2 - f_1 f_3)^3 & (-2f_0 f_1^3 f_2^3 + 3f_0^2 f_1 f_2^4 - f_1^6 f_3 + 6f_0 f_1^4 f_2 f_3 - 6f_0^2 f_1^2 f_2^2 f_3 \\ & - 2f_0^3 f_2^3 f_3 - 3f_0^2 f_1^3 f_3^2 + 6f_0^3 f_1 f_2 f_3^2 - f_0^4 f_3^3) \end{aligned} \quad (46)$$

Notice they are homogeneous, as expected, and notice the factor  $y = f_1^2 - f_0 f_2$  in the second polynomial and the common factor  $x = f_2^2 - f_1 f_3$ , which must both be positive by hypothesis. From equivalence we get the simple expression

$$(f_1^3 - 2f_0 f_1 f_2 + f_0^2 f_3)(f_2^2 - f_1 f_3)^3 \quad (= (44)). \quad (47)$$

**Theorem 1.6.** *A 3-form  $\sigma$  as above defining a  $G_2$ -structure, with  $f_0, \dots, f_4$  constant, satisfies  $d *_{\sigma} \sigma = 0$  if and only if any one of the following occurs:*

- (i) *the polynomial  $\mathbf{p}_2$  from (46) vanishes and  $M$  is Einstein.*
- (ii)  *$M$  has constant sectional curvature.*

*Proof.* Notice first that  $d *_{\sigma} \sigma = 0$  if and only if both  $\theta \wedge \mathbf{p}_1 \mathcal{R}^U \alpha_1$  and  $\theta \wedge \mathbf{p}_2 \mathcal{R}^U \alpha$  vanish. Also we note that, if  $f_0 = 0$ , then neither  $f_1$  or  $f_3$  can vanish (otherwise we would get  $y = 0$  or  $h = 0$  from definition). So the two main polynomials cannot vanish simultaneously, as we see directly, or from the implied equation (47).

Now, if  $\mathbf{p}_2$  vanishes, then we may conclude that  $f_0 \neq 0$ , ie. the first polynomial  $\mathbf{p}_1$  does not vanish. So the coccalibration equation becomes equivalent to the vanishing of  $\theta \wedge \mathcal{R}^U \alpha_1 = -\rho \wedge \text{vol}$ , which happens if and only if  $M$  is Einstein. Conversely, if the polynomial  $\mathbf{p}_2$  does not vanish, then the equation relies on a metric such that  $\theta \wedge \mathcal{R}^U \alpha = 0$ ; equivalently,  $R_{1201} = R_{2301} = 0$ , etc. This is the same as  $M$  having constant sectional curvature. In particular,  $M$  being Einstein. ■

For example, if  $f_0 = 0$ , then we are certainly bound to the second case.

Noteworthy is the case when  $f_1 f_2 = f_0 f_3$  (or  $z = 0$ ), which generalizes Proposition 1.2. By formulas (36...40) we see

$$d *_{\sigma} \sigma = f_3 \frac{f_4 t^{\frac{1}{2}}}{x^3 h^{\frac{3}{2}}} h^3 \theta \wedge \mathcal{R}^U \alpha = \frac{f_3 f_4 t^{\frac{1}{2}} h^{\frac{3}{2}}}{x^3} \theta \wedge \mathcal{R}^U \alpha. \quad (48)$$

A question put to the author by colleagues was: if we could always find, invariant of the metric on  $M$ , a natural  $G_2$  structure which would be co-closed. The answer is no, because the two polynomials do not vanish simultaneously. By the contrary we stress the relevance of  $G_2$  coccalibration goes much beyond the known cases and examples.

## 1.5 Nearly-parallel $G_2$ -structures

Nearly-parallel  $G_2$ -structures on 7-dimensional manifolds are defined by  $d *_{\sigma} \sigma = 0$  and  $d\sigma = c *_{\sigma} \sigma$  for some constant  $c$ . Clearly, if  $c \neq 0$ , the condition is simply the latter equation.

We consider a variation of the  $G_2$  structure on  $\mathcal{G}$ , as in (21). In order to find a nearly-parallel structure  $\sigma$ , we may assume already that it is cocalibrated ( $c \neq 0$ ). Recall the Hodge  $*$  operator is homogeneous of degree  $1/3$  on 3-forms viewed as a map  $\sigma \rightsquigarrow *_{\sigma} \sigma$ . Hence if we find a solution to the above in our subspace of  $\sigma \in \Lambda_+^3$ , we find a line of solutions:

$$d(s\sigma) = cs *_{\sigma} \sigma = cs^{-\frac{1}{3}} *_{s\sigma} s\sigma, \quad s \in \mathbb{R}^+. \quad (49)$$

We restrict here to the case  $z = f_1 f_2 - f_0 f_3 = 0$ , the less ‘prohibitive’ condition. And continue to assume the coefficients are constants.

**Theorem 1.7.** *Under the previous condition, the only metric on an oriented Riemannian 4-manifold  $M$  for which a  $(\mathcal{G}, \sigma)$  is nearly-parallel is the constant sectional curvature 1 metric. Then there are two classes of solutions, represented by the following two  $G_2$ -structures:*

$$\sigma_{\pm} = \pm \frac{\sqrt{2}}{2} (\alpha_2 - \alpha + \alpha_3 - \alpha_1) + \sqrt{\frac{3}{2}} \theta \wedge d\theta, \quad (50)$$

both satisfying  $d\sigma = \sqrt{6} *_{\sigma} \sigma$ .

*Proof.* Since we assume  $z = 0$  and this is maintained on the line  $\mathbb{R}^+ \sigma$ , there exists a positive multiple of  $\sigma$  such that  $(f_0, f_1)$  is in the unit circle. Then we easily deduce  $x = y = 1$  and  $f_2 = -f_0$ ,  $f_3 = -f_1$ . Hence  $h = 1 = t$  and  $m = f_4$ , cf. (18).

From formulas (28...32) and the hypothesis of  $\sigma$  being nearly-parallel, we see the 4-form  $d\sigma$  is again  $SO(3)$ -invariant. Then we easily deduce the curvature restriction: it must be of the constant kind. The equation  $d\sigma = c *_{\sigma} \sigma$  is solved using those same formulas. Looking at components, we find a system ( $k$  is the sectional curvature)

$$\begin{cases} c = 2f_4 \\ f_0 f_1 - k f_0^2 = 0 \\ 2f_0 f_1 k + f_0 f_1 - 3f_1^2 = 0 \\ 3f_1 - 2f_0 f_4^2 = 0 \\ 2f_0 + k f_0 - 2f_0 f_4^2 = 0 \end{cases} .$$

This yields  $f_0 = f_1$ , which occurs twice in the circle; and  $k = 1$ ,  $f_4 = \sqrt{3/2}$ ,  $c = \sqrt{6}$ . The given 3-forms satisfy the equation and are genuine  $G_2$ -structures.  $\blacksquare$

Notice the metric on  $\mathcal{G}$  is the same on both solutions. Now we recall the classification of nearly-parallel  $G_2$  structures in [12]. The ones we got correspond to the Stiefel manifold  $V_{5,2} = SO(5)/SO(3)$  in their Table 2, which is of course the unit tangent sphere bundle of  $S^4$ . The  $G_2$  structure is constructed as a  $U(1)$ -bundle over the complex quadric  $G_{5,2}$ , the Grassmannian of 2-planes, with a Kähler-Einstein metric. The resulting nearly-parallel  $G_2$  structure is said to be Einstein-Sasakian for *some* homogeneous  $SO(5)$ -invariant metric. We have thus found more detail of this case. It is also most interesting to see that our result gives a metric coinciding precisely with the Einstein metric on  $V_{5,2}$  deduced in [2, Theorem 4]. It has Riemannian scalar curvature  $\frac{63}{4}$ , by a formula there.

## References

- [1] M. T. K. Abbassi and G. Calvaruso, *g-Natural Contact Metrics on Unit Tangent Sphere Bundles*, Monatsh. Math. 151 (2006), 89–109.
- [2] M.T.K. Abbassi and O.Kowalski, *On Einstein Riemannian g-natural metrics on unit tangent sphere bundles*, Ann. Global Anal. Geom. 38(1), (2010), 11–20.
- [3] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Mathematicum (Brno), Tomus 42 (2006), Suppl., 5–84.
- [4] R. Albuquerque, *On the  $G_2$  bundle of a Riemannian 4-manifold*, J. Geom. Phys. 60 (2010), 924–939.
- [5] R. Albuquerque, *On the characteristic connection of gwistor space*, to appear in Central European Journal of Mathematics.
- [6] R. Albuquerque and I. Salavessa, *The  $G_2$  sphere of a 4-manifold*, Monatsh. Math. 158, Issue 4 (2009), 335–348.
- [7] D. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, vol. 203, Birkhäuser Boston Inc., Boston, MA, 2002.
- [8] R. L. Bryant, *Metrics with exceptional holonomy*, Annals of Mathematics (2), vol. 126 no. 3 (1987), 525–576.
- [9] R. L. Bryant, *Some remarks on  $G_2$  structures*, Proceedings of the 2004 Gokova Conference on Geometry and Topology (May, 2003).

- [10] M. Fernández and A. Gray, *Riemannian manifolds with structure group  $G_2$* , Ann. Mat. Pura Appl. 4 132 (1982), 19–45.
- [11] N. O’Brian and J. Rawnsley, *Twistor spaces*, Ann. Global Anal. Geom. 3(1) (1985), 29–58.
- [12] Th. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On nearly parallel  $G_2$ -structures*, J. Geom. Phys. 23 (1997), 259–286.
- [13] Th. Friedrich and S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable  $G_2$ -manifolds*, J. Geom. Phys. 48 (2003), 1–11.
- [14] H. Geiges, *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics 109, Cambridge University Press, 2008.
- [15] T. Sakai, *Riemannian Geometry*, Transl. Math. Mono. Vol 149 AMS, 1996.