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Variations on a Conjecture of C. C. Yang Concerning Periodicity

Xinling Liu[1,2] · Risto Korhonen[2] · Kai Liu[1]

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Abstract
The generalized Yang's Conjecture states that if, given an entire function $f(z)$ and positive integers n and k, $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function. In this paper, it is shown that the generalized Yang's conjecture is true for meromorphic functions in the case $k = 1$. When $k \geq 2$ the conjecture is shown to be true under certain conditions even if n is allowed to have negative integer values.

Keywords Meromorphic functions · Periodicity · Yang's Conjecture

Mathematical Subject Classification 30D35 · 39A05

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✉ Risto Korhonen
risto.korhonen@uef.fi

Xinling Liu
liuxinling@ncu.edu.cn; xinling.liu@uef.fi

Kai Liu
liukai418@126.com; liukai@ncu.edu.cn

1 Department of Mathematics, Nanchang University, Nanchang 330031, Jiangxi, People's Republic of China

2 Department of Physics and Mathematics, University of Eastern Finland, P. O. Box 111, 80101 Joensuu, Finland

1 Introduction and Main Results

Yang's Conjecture on the periodicity of transcendental entire functions, proposed in [8] and [15, Conj. 1.1], has recently prompted intensive research activity in the field of periodic entire functions, see, e.g., [10–12,15], which include results both on Yang's Conjecture and its difference version. For related problems on the periodicity of transcendental meromorphic functions, we refer the reader to [1,4–6,9,13,14,16,17].

Yang's Conjecture. Let $f(z)$ be a transcendental entire function and k be a positive integer. If $f(z)f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

The generalized Yang's Conjecture has been considered by Liu, Wei and Yu [10].

Generalized Yang's Conjecture. Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

Liu, Wei and Yu [10, Thm. 1.2] have shown that if f is a transcendental entire function and, in addition, if $k = 1$, or if $f(z) = e^{h(z)}$ where $h(z)$ is a non-constant polynomial, or if $f(z)$ has a non-zero Picard exceptional value and $f(z)$ is of finite order, then the Generalized Yang's Conjecture is true. The aforementioned conjectures and results are all concerned with transcendental entire functions. The following results are, to the best of our knowledge, the first to address the meromorphic case. In addition, we allow the exponent n to have negative values as an extension of the Generalized Yang's Conjecture. Actually, we consider the following question.

Question 1 Let $f(z)$ be a transcendental meromorphic function, let $n \in \mathbb{Z}$, and let $k \in \mathbb{N}$. If $f(z)^n f^{(k)}(z)$ is a periodic function, does it follow that $f(z)$ is also a periodic function?

Our first theorem gives a complete description of the case $k = 1$ of Question 1.

Theorem 1.1 *Let $f(z)$ be a transcendental meromorphic function and $n \in \mathbb{Z}$. Suppose that $f(z)^n f'(z)$ is a periodic function with period c.*

(1) If $n \geq 1$ or $n \leq -3$, then $f(z)$ is a periodic function with period $(n+1)c$.
(2) If $n = 0$, then $f(z) = \varphi(z) + Az/c$, where $\varphi(z)$ is a periodic function with period c and A is a constant.
(3) If $n = -1$, then $f(z) = e^{Az/c}\varphi(z)$, where $\varphi(z)$ is a periodic function with period c and A is a constant.
(4) If $n = -2$, then $f(z) = 1/(\varphi(z) + Az/c)$, where $\varphi(z)$ is a periodic function with period c and A is a constant. Furthermore, if $f(z)$ is transcendental entire, then $f(z)$ is a periodic function with period c.

Remark 1.2 (1) In Theorem 1.1 (1), f may have period qc for a minimal divisor q of $n+1$, where $\omega^q = 1$ and $\omega^{n+1} = 1$ for some root of unity ω.

(2) The function $f(z) = \varphi(z) + Az/c$, $A \neq 0$, in Theorem 1.1 (2) is called a pseudo-periodic function mod Az/c, see [2, Def. 3.4].

We proceed to consider Question 1. Here we will restrict our consideration to the entire case. The claim in case of $n = -1$ and $k \geq 1$ in Question 1 is not true, which can be seen by looking at $f(z) = e^{e^{iz}+z}$. Here $f(z)$ is not a periodic function, but $f^{(k)}/f$

is a periodic function with period $2m\pi$, where m is a non-zero integer. Observe that the transcendental entire function $f(z) = e^{e^{iz}+z}$ has a Picard exceptional value 0. We give the following result to show that this is, in a sense, essentially the only type of an example to be given here.

Theorem 1.3 *Let f be a transcendental entire function with a Picard exceptional value 0. If f''/f is a periodic function with period c, then $f(z) = e^{h_1(z)+Az/(2c)}$, where $h_1(z)$ is a periodic function with period $2c$ and A is a constant.*

Remark 1.4 (1) If f is of finite order in Theorem 1.3, then $f(z) = e^{Az+B}$, where A is a non-zero constant.

(2) Assume that $f^{(k)}/f = A(z)$, that is $f^{(k)} - A(z)f = 0$, where $A(z)$ is a periodic function. Hence, we also can consider the periodicity of $f^{(k)}/f$ from the perspective of differential equations with periodic coefficients. Such results can be found in, e.g., [7, Lem. 5.19] and [3, Thm. 1].

We proceed to consider the case of $f(z)$ admitting a non-zero Picard exceptional value d, i.e. $f(z) = e^{h(z)} + d$, where $h(z)$ is an entire function.

Theorem 1.5 *Let f be a transcendental entire function with a Picard exceptional value $d \neq 0$. If $f^{(k)}/f$ is a periodic function with period c, then $f(z)$ is a periodic function with period c.*

The method of proof of Theorem 1.5 below cannot be used to extend the statement to hold for $f^{(k)}/f^n$ instead of $f^{(k)}/f$. However, we obtain the following result.

Theorem 1.6 *Let $f(z)$ be a transcendental meromorphic function, let $n \in \mathbb{Z}$, and let $k \in \mathbb{N}$. Suppose that $f^{(k)}(z)/f(z)^n$ and $f^{(k+1)}(z)/f(z)^n$ are periodic functions with period c.*

(1) If $n = 0$, then $f(z) = \varphi(z) + p(z)$, where $\varphi(z)$ is a periodic function with period c and $p(z)$ is a polynomial with degree at most k.
(2) If $n = 1$, then $f(z) = e^{Az/c}\varphi(z)$, where $\varphi(z)$ is a periodic function with period c and A is a constant.
(3) If $n \neq 0, 1$, then $f(z)$ is a periodic function with period $(n-1)c$.

Remark 1.7 We conjecture that there doesn't exist a transcendental meromorphic function f such that $f^{(k)}/f^n$ and $f^{(k+1)}/f^n$ are periodic functions, where d_1 is a period of $f^{(k)}/f^n$ but not of $f^{(k+1)}/f^n$ while d_2 is a period of $f^{(k+1)}/f^n$ but not of $f^{(k)}/f^n$, and d_1/d_2 is not a rational number.

2 Proofs of Theorems

Proof of Theorem 1.1 We first consider the case $n \geq 1$. Assume that $f(z)^n f'(z)$ is a periodic function with period c, then

$$f(z)^n f'(z) = f(z+c)^n f'(z+c).$$

Integrating the above equation, we have

$$f(z)^{n+1} - f(z+c)^{n+1} = B, \tag{2.1}$$

where B is a constant. Let $F(z) = f(z)^{n+1}$. Thus $F(z)$ is a non-constant meromorphic function with no simple zeros. From (2.1), we obtain

$$F(z+c) = F(z) - B.$$

Suppose that $B \neq 0$ and let $2 \leq m \in \mathbb{N}$. Then

$$\begin{aligned} F(z+mc) &= F(z+(m-1)c) - B \\ &= F(z+(m-2)c) - 2B = \ldots = F(z) - mB. \end{aligned}$$

This implies that all roots of $F(z) = mB$ are multiple for all $m \geq 2$, which is impossible, since a transcendental meromorphic function has at most four completely ramified values. Hence, $B = 0$ and $F(z)$ is a periodic function with period c. So $f(z+c)^{n+1} = f(z)^{n+1}$, and it follows that $f(z+c) = \omega f(z)$, where $\omega^{n+1} = 1$. Therefore, $f(z)$ is a periodic function with period $(n+1)c$.

The case of $n = 0$ is trivial. Namely, the assertion follows from $f'(z+c) = f'(z)$ by integrating and then solving the resulting first order non-homogeneous difference equation.

We proceed to consider the case $n = -1$. Now f'/f is a periodic function with period c, and so

$$\frac{f'(z)}{f(z)} = \frac{f'(z+c)}{f(z+c)}.$$

Integrating the above equation, we have $f(z+c) = e^{A} f(z)$, where A is a constant. Thus $f(z)$ can be written as $f(z) = e^{Az/c}\varphi(z)$, where $\varphi(z)$ is a periodic function with period c.

In the remaining case $n \leq -2$, we may write $f^n f' = -g^{-n-2}g'$ where $f = 1/g$. Hence $f^n f'$ and $g^{-n-2}g'$ have the same periodicity. Thus, if $n = -2$, we immediately obtain, by applying the case $n = 0$, that $f(z) = 1/(\varphi(z) + Az/c)$, where $\varphi(z)$ is a periodic function with period c and A is a constant. If $n \leq -3$, then, by the first part of the proof, $g(z)$ is a periodic function with period $(-1-n)c$, thus $(n+1)c$ is also a period of $g(z)$ and so of $f(z)$ as well.

Finally, we consider the case $n = -2$ and $f(z)$ is transcendental entire. By

$$\frac{f'(z)}{f(z)^2} = \frac{f'(z+c)}{f(z+c)^2},$$

we obtain

$$\frac{1}{f(z)} - \frac{1}{f(z+c)} = A. \tag{2.2}$$

We affirm that $A = 0$, and thus $f(z)$ is a periodic function with period c. If f is entire, then $1/f$ has no zeros. Let $G = 1/f$. Then,

$$G(z) - G(z + c) = A,$$

and G has no zeros. Let $2 \le m \in \mathbb{N}$ again. So the fact that $G(z) - mA$ has no zeros follows by the same argument we used following (2.1). But this is impossible, since a meromorphic function has at most two finite Picard exceptional values. □

Proof of Theorem 1.3. By assuming that $f(z) = e^{h(z)}$ and f''/f is a periodic function with period c, it follows that

$$[h'(z)]^2 + h''(z) = [h'(z + c)]^2 + h''(z + c).$$

By a recent result given by Liu, Wei and Yu [10, Thm. 1.6], then $h'(z)$ must be a periodic function with period $2c$. Thus, $h(z) = h_1(z) + Az/(2c)$, where $h_1(z)$ is a periodic function with period $2c$ and A is a constant. □

Proof of Theorem 1.5. Since $f(z) = e^{h(z)} + d$ and $f^{(k)}/f$ is a periodic function with period c, it follows by an elementary computation, that

$$\frac{H_k(z)e^{h(z)}}{e^{h(z)} + d} = \frac{H_k(z + c)e^{h(z+c)}}{e^{h(z+c)} + d}, \tag{2.3}$$

where $H_k(z)$ is a differential polynomial of $h(z)$ and

$$T(r, H_k(z)) = S(r, e^{h(z)}). \tag{2.4}$$

From (2.3), we also have

$$T(r, e^{h(z+c)}) = T(r, e^{h(z)}) + S(r, e^{h(z)}). \tag{2.5}$$

The Eq. (2.3) can be written as

$$\frac{[H_k(z) - H_k(z + c)]}{dH_k(z + c)}e^{h(z)} + \frac{H_k(z)}{H_k(z + c)}e^{h(z)-h(z+c)} = 1. \tag{2.6}$$

Let

$$f_1 = \frac{[H_k(z) - H_k(z + c)]}{dH_k(z + c)}e^{h(z)} \quad \text{and} \quad f_2 = \frac{H_k(z)}{H_k(z + c)}e^{h(z)-h(z+c)}.$$

Using the second main theorem, (2.4), (2.5) and (2.6), we obtain

$$T(r, f_1) \le \overline{N}(r, f_1) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_1 - 1}\right) + S(r, f_1)$$

$$
\begin{aligned}
&= \overline{N}(r, f_1) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + S(r, f_1) \\
&\leq S(r, e^{h(z)}).
\end{aligned}
$$

In order to avoid a contradiction, we see that f_1 must be a constant. Since $f(z)$ is a transcendental entire function, we have

$$H_k(z) - H_k(z+c) \equiv 0$$

and

$$\frac{H_k(z)}{H_k(z+c)} e^{h(z)-h(z+c)} \equiv 1.$$

Hence, the fact that $f(z)$ is a periodic function with period c follows from (2.3). □

Proof of Theorem 1.6 We first consider the case $n = 0$, that is $f^{(k)}(z)$ and $f^{(k+1)}(z)$ are periodic functions with period c. Then $f^{(k)}(z) \equiv f^{(k)}(z+c)$, thus $f(z+c) - f(z) = Q(z)$, where $Q(z)$ is a polynomial of degree less than k. By choosing a polynomial $p(z)$ with the degree $\deg(Q) + 1$ such that $p(z+c) - p(z) = Q(z)$ it follows that $f(z) = \varphi(z) + p(z)$, where $\varphi(z)$ is a periodic function with period c and $p(z)$ is a polynomial with degree at most k.

We proceed to consider the case $n \neq 0$. Then we know that

$$\left(\frac{f^{(k)}}{f^n}\right)' = \frac{f^{(k+1)}}{f^n} - n\frac{f^{(k)}}{f^n}\frac{f'}{f}. \tag{2.7}$$

Since $f^{(k)}(z)/f(z)^n$ and $f^{(k+1)}(z)/f(z)^n$ are periodic functions with period c, we obtain f'/f is also a periodic function with period c. From Theorem 1.1 (3) we have that $f(z) = e^{Az/c}\varphi(z)$, where φ is a periodic function with period c and A is a constant. Furthermore, if $n \neq 1$, rewrite (2.7) as

$$\left(\frac{f^{(k)}}{f^n}\right)' = \frac{f^{(k+1)}}{f^n} - n\frac{1}{f^{n-1}}\frac{f^{(k)}}{f}\frac{f'}{f}. \tag{2.8}$$

By the classical formula, see, e.g., [7, Lem. 2.3.7], we have

$$\frac{f^{(k)}}{f} = \left(\frac{f'}{f}\right)^k + P_{k-1}\left(\frac{f'}{f}\right), \tag{2.9}$$

where $P_{k-1}\left(f'/f\right)$ is a differential polynomial in f'/f and its derivatives with constant coefficients and of total degree $\leq k-1$. Recall that f'/f is periodic with period c, thus $f^{(k)}/f$ is also a periodic function with period c. From (2.8), we see that $1/f^{n-1}$ is also a periodic function with period c, which implies that $f(z)$ has period $(n-1)c$, and case (3) is thus proved. □

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