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VARIATIONS ON A THEME OF KAPLANSKY

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Abstract. We explore the relationship between Kaplansky's Lemma about locally algebraic operators, a dual to Kaplansky's lemma, and non singularity for pairs of operators in the sense of Joseph L. Taylor.

Kaplansky's Lemma ([7] Lemma 14;[10] Theorem 4.8;[11] (3.5)) says that, for bounded linear operators on Banach spaces,

0.1 $$\textit{locally algebraic} \Longrightarrow \textit{algebraic} .$$

Here an *algebraic* operator satisfies a non trivial polynomial identity, while a *locally algebraic* operator satisfies such a thing separately at each point. Thus for a locally algebraic operator $T : X \to X$ there is a family $(p_x)_{x\in X}$ of non trivial polynomials for which each $p_x(T)$ kills its own x: $p_x(T)x = 0 \in X$. When all the p_x are the same then the operator T is algebraic; if more generally all the p_x can be taken with the same degree then we shall call $T : X \to X$ *boundedly locally algebraic.* Formally

1. Definition *The polynomial kernel and the polynomial range of a linear operator $T : X \to X$ are the subspaces*

1.1 $$E_X(T) = \bigcup_{n=1}^{\infty} E_X^n(T) \textit{ and } F_X(T) = \bigcap_{n=1}^{\infty} F_X^n(T) \ ,$$

where for each $n \in \mathbf{N}$

1.2 $$\begin{aligned} &E_X^n(T) = \bigcup\{p(T)^{-1}(0) : 1 \le \mathrm{degree}(p) \le n\} \textit{ and} \\ &F_X^n(T) = \bigcap\{p(T)(X) : 1 \le \mathrm{degree}(p) \le n\} \ . \end{aligned}$$

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T is said to be locally algebraic iff

1.3 $$E_X(T) = X\ ,$$

and boundedly locally algebraic iff

1.4 $$\exists n \in \mathbf{N}\ ,\ E_X^n(T) = X\ .$$

T is said to be algebraic iff there is a non trivial polynomial $0 \neq p \in \mathrm{Poly}$ *for which*

1.5 $$p(T) = 0\ .$$

The subspaces $E_X(T)$ and $F_X(T)$ are linear and "hyperinvariant" under T, but for bounded operators on Banach spaces not necessarily closed. It is evident that

1.6 *algebraic* $\Longrightarrow$ *boundedly locally algebraic* $\Longrightarrow$ *locally algebraic* ;

Kaplansky's Lemma reverses these in two orthogonal bites. For arbitrary linear operators on linear spaces

1.7 *boundedly locally algebraic* $\Longrightarrow$ *algebraic* ;

for bounded linear operators on Banach spaces

1.8 *locally algebraic* $\Longrightarrow$ *boundedly locally algebraic* .

The first of these implications comes from the *Euclidean algorithm* (partial fractions), while the second calls on *Baire's theorem.*

In finite dimensions it is familiar that every linear operator is algebraic: if $\dim(X) = n$ then $(I, T, T^2, \ldots, T^{n^2})$ is linearly dependent. Of course the Cayley-Hamilton theorem says that T satisfies a polynomial identity $p(T) = 0$ with a polynomial p of degree n - but that is a long story: at any rate Kaplansky's Lemma is not of great interest in finite dimensions.

The implication (1.8) uses *Baire's theorem* ([2] Theorem 4.6.1), which says that in a Banach space countable unions of nowhere dense sets have empty interior:

2. Theorem *If X is a Banach space and if subsets (K_n) in X satisfy*

2.1 $$\bigcup_{n=1}^{\infty} \text{int cl } K_n = \emptyset,$$

then also

2.2 $$\text{int} \bigcup_{n=1}^{\infty} K_n = \emptyset.$$

Proof. The argument is a version of Cantor's diagonal technique: we claim that if (2.1) holds and not (2.2) the space X cannot be complete. Recursively we construct a Cauchy sequence whose limit has nowhere to go: if (2.1) holds and not (2.2) there is $x_1 \in X$ and $\varepsilon_1 > 0$ for which

$$\text{Disc}(x_1, \varepsilon_1) \subseteq \Big(\bigcup_{n=1}^{\infty} K_n\Big) \setminus \text{cl } K_1 \textit{ and } \varepsilon_1 \leq 2^{-1} \ ,$$

else int cl $K_1 \supseteq \bigcup_{n=1}^{\infty} K_n \neq \emptyset$, and recursively there are (x_n) in X and $\varepsilon_n > 0$ for which

$$\text{Disc}(x_{n+1}, \varepsilon_{n+1}) \subseteq \text{int Disc}(x_n, \varepsilon_n) \setminus \text{cl } K_{n+1} \textit{ and } \varepsilon_n \leq 2^{-n} \ .$$

Evidently (x_n) is Cauchy and hence by completeness has a limit x_∞: but now

$$x_\infty \in \bigcap_{n=1}^{\infty} \text{Disc}(x_n, \varepsilon_n) \subseteq \big(\text{int } H\big) \setminus H \textit{ with } H = \bigcup_{n=1}^{\infty} K_n \ \bullet$$

To use this to deduce (1.4) from (1.3) we must cunningly choose (K_n):

3. Theorem *If $T \in BL(X, X)$ is bounded and linear on a Banach space, and locally algebraic, then it is boundedly locally algebraic.*

Proof. For each $m \in \mathbf{N}$ recall the spaces $E^n_X(T)$ of (1.2): by the locally algebraic assumption

3.1 $$X = \bigcup_{m=1}^{\infty} E_X^m(T) \ .$$

Baire's theorem will now tell us that the interior of the closure of one of the $E_X^m(T)$ must be non empty: but first we notice that each of the $E_X^m(T)$ is already closed. Indeed if $x_\infty = \lim_n x_n$ with $x_n \in E_X^m(T)$ then for each n there is a polynomial p_n, of degree $\leq m$, for which $p_n(T)x_n = 0$: by scalar multiplication we arrange that norm of its sequence of coefficients is 1. Thus the polynomials p_n give rise to sequences from the unit ball of $\mathbf{C}^{m+1}$ and by compactness have a convergent subsequence $(p'_n) = (p_{\phi(n)})$, whose limit gives rise to a polynomial p'_∞: evidently

3.2 $$p'_\infty(T)x_\infty = \lim_n p'_n(T)(x'_n) = 0 \ .$$

Baire's theorem now tells us that there is $m \in \mathbf{N}$ for which $E_X^m(T)$ has non empty interior, and hence $w \in X$ and $\varepsilon > 0$ for which

3.3 $$\|x\| < \varepsilon \Longrightarrow w + x \in E_X^m(T) \ .$$

Thus there are q_w and q_{w+x} for which $q_{w+x}(T)(w+x) = 0 = q_w(T)w$ and hence, for arbitrary $\lambda \in \mathbf{C}$,

$$p_x = q_w q_{w+x} \textit{ gives } p_x(T)(\lambda x) = 0 \textit{ with } \mathrm{degree}(p_x) \leq 2m \ \bullet$$

The completeness of X is necessary not just for the proof of Theorem 3 but also for the result: for example on the normed space c_{00} of terminating sequences the standard weight operator

3.4 $$W : (x_n) \mapsto (\tfrac{1}{n} x_n)$$

is locally algebraic but not boundedly locally algebraic.

The implication (1.7) uses the Euclidean algorithm for polynomials. To explore the background to this, look at *Taylor nonsingularity* for pairs of commuting operators:

4. Definition *If $R : X \to X$ and $S : X \to X$ are linear we shall call the pair (R, S) left non singular provided*

4.1 $$R^{-1}(0)\cap S^{-1}(0) = \{0\} \ ,$$

right non singular provided

4.2 $$R(X) + S(X) = X \ ,$$

and middle non singular provided

4.3 $$(\,-R \quad S\,)^{-1}\,(0) \subseteq \begin{pmatrix} S \\ R \end{pmatrix} (X) \ .$$

For bounded operators on Banach spaces, we shall call (R, S) *left invertible if there are bounded* R' *and* S' *for which*

4.4 $$R'R + S'S = I \ ,$$

right invertible if there are bounded R'' *and* S'' *for which*

4.5 $$RR'' + SS'' + I \ ,$$

and middle invertible if there are bounded R', S', R'' *and* S'' *for which*

4.6 $$\begin{pmatrix} -R'' \\ S'' \end{pmatrix} (\,-R \quad S\,) + \begin{pmatrix} S \\ R \end{pmatrix} (\,S' \quad R'\,) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} .$$

These definitions work better when R and S commute, so that the inclusion (4.3) becomes equality. It is obvious that the invertibility conditions each imply their nonsingular counterparts; the converse holds provided the row and column matrices derived from (R, S) have (bounded) "generalized inverses". Sufficient for all three splitting conditions is that there are bounded U and V on X for which

4.7 $$\{U, V\} \subseteq \mathrm{comm}(R, S) \text{ and } VR - SU = I \ :$$

derivation is left to the reader. We collect the following list of conditions which ([3] Theorem 4) are equivalent to middle non singularity :

5. Theorem *If $R : X \to X$ and $S : X \to X$ are linear then the following two conditions are equivalent to middle non singularity for (R, S):*

5.1 $$R^{-1}(0) \subseteq S\ R^{-1}(0) \text{ and } S^{-1}(0) \subseteq R\ S^{-1}(0)\ ;$$

5.2 $$R(X)_\cap S(X) \subseteq RS\big((SR - RS)^{-1}(0)\big) \subseteq (RS)(X)_\cap (SR)(X)\ .$$

If (5.1) holds then also

5.3 $$(RS)^{-1}(0) + (SR)^{-1}(0) \subseteq R^{-1}(0) + S^{-1}(0)\ .$$

Proof. If middle non singularity (4.3) holds then

$$Ry = 0 \Longrightarrow (\,-R \quad S\,)\begin{pmatrix} y \\ 0 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} S \\ R \end{pmatrix} x$$

giving $Ry = Sx$ with $x \in R^{-1}(0)$: this is the first part of (5.1) and the second is similar. Also

$$w = Sx = Ry \Longrightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} S \\ R \end{pmatrix} z \Longrightarrow w = RSz = SRz$$

giving (5.2). Conversely if these conditions hold then, using first (5.2),

$$\begin{pmatrix} y \\ x \end{pmatrix} \in (\,-R \quad S\,)^{-1}\,(0) \Longrightarrow Ry = Sx = RSz = SRz$$

giving $y - Sx \in R^{-1}(0) \subseteq SR^{-1}(0)$ and $x - Rz \in S^{-1}(0) \subseteq RS^{-1}(0)$, so that there are u and v for which

$$y - Sz = Su \text{ with } Ru = 0 \text{ and } x - Rz = Rv \text{ with } Sv = 0\, :$$

but now $\begin{pmatrix} S \\ R \end{pmatrix}(z + u + v) = \begin{pmatrix} y \\ x \end{pmatrix}$ as required by (4.3).

Towards the last part we assume only (5.1) and claim that $(RS)^{-1}(0)$ is a subset of the sum of the null spaces: for if $RSx = 0$ then $Sx \in SR^{-1}(0)$ giving $Sx = Sz$ with $Rz = 0$ so that $x = (x - z) + z$ •

It is a trivial consequence of (5.1) that both sequences (R, S) and (S, R) are themselves "exact":

5.4 $$R^{-1}(0) \subseteq S(X) \text{ and } S^{-1}(0) \subseteq R(X) \ ;$$

similarly the middle invertibility (4.6) makes them "split exact":

5.5 $$R''R + SS' = I = S''S + RR' \ .$$

Theorem 5 interacts ([3] Lemma 11) with the "Euclidean algorithm" for polynomials:

6. Theorem *If $T : X \to X$ is complex linear then there is equality*

6.1 $$E_X(T) = \sum_{\lambda \in \mathbf{C}} (T - \lambda I)^{-\infty}(0) \text{ and } F_X(T) = \bigcap_{\lambda \in \mathbf{C}} (T - \lambda I)^{\infty}(X) \ ,$$

where we write

6.2 $$T^{-\infty}(X) = \bigcup_{n=1}^{\infty} T^{-n}(0) \text{ and } T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X)$$

for the "hyperkernel" and "hyperrange" of T. If $\lambda \neq \mu$ then

6.3 $$\begin{aligned} &(T - \lambda I)^{-\infty}(0) \cap (T - \mu I)^{-\infty}(0) = \{0\}, \\ &(T - \lambda I)^{\infty}(X) + (T - \mu I)^{\infty}(X) = X, \quad \text{and} \\ &(T - \lambda I)^{-\infty}(0) \subseteq (T - \mu I)^{\infty}(X). \end{aligned}$$

Proof. The right hand side of the first part of (6.1) is obviously included in the left; conversely if $p = qr \in \text{Poly}$ with $\text{hcf}(q, r) = 1$ then by the Euclidean algorithm there are polynomials q' and r' for which $q'q - r'r = 1$ and hence also

6.4 $$q'(T)q(T) - r(T)r'(T) = I \ .$$

Thus the pair $(q(T), r(T))$ satisfies the condition (4.7) and hence in particular is middle exact (4.3). Thus the conditions (5.1), (5.2) and (5.3) all hold.

Inductively if $p = q_1 q_2 \dots q_k$ with $\mathrm{hcf}(q_i, q_j) = 1$ whenever $i \neq j$ then the inductive extension of (5.3) says that

6.5 $$p(T)^{-1}(0) = \sum_{j=1}^{k} q_j(T)^{-1}(0) \ .$$

By the fundamental theorem of algebra this happens with

6.6 $$q_j = (z - \lambda_j)^{\nu_j} \ \ :$$

thus the left hand side of the first part of (6.1) is included in the right.

Similarly it is obvious that the left hand side of the second part of (6.1) is included in the right, and for equality go to the inductive extension of (5.2). For (6.3) note that if $\lambda \neq \mu$ then the polynomials $(z-\lambda)^n$ and $(z-\mu)^m$ are relatively prime •

We can now see the implication (1.7):

7. Theorem *If $T : X \to X$ is linear and boundedly locally algebraic then it is algebraic.*

Proof. We make two observations: necessary and sufficient for T to be algebraic is that it be boundedly locally algebraic with finite point spectrum

7.1 $$\begin{aligned} \pi^{left}(T) &= \{\lambda \in \mathbf{C} : (T - \lambda I)^{-1}(0) \neq \{0\}\} \\ &= \{\lambda \in \mathbf{C} : (T - \lambda I)^{-\infty}(0) \neq \{0\}\} \ ; \end{aligned}$$

also a boundedly locally algebraic linear operator necessarily has finite point spectrum. For the first observation simply notice that if the point spectrum (6.4) is finite then so is the sum in the first part of (6.1); for the second claim that if we can find m distinct eigenvalues for T then we can find a vector which can only be killed by a polynomial of degree $\geq m$. Indeed suppose that $(\lambda_1, \lambda_2, \dots, \lambda_m)$ are pairwise distinct eigenvalues of T, with corresponding eigenvectors $(x_1, x_2, \dots, x_m)$: then for a non trivial polynomial $0 \neq p \in \mathrm{Poly}$ we claim implication

7.2 $$\begin{aligned} p(T)\sum_{j=1}^{m} x_m = 0 &\Longrightarrow \{x_1, x_2, \dots, x_m\} \subseteq p(T)^{-1}(0) \\ &\Longrightarrow \{\lambda_1, \lambda_2, \dots, \lambda_m\} \subseteq p^{-1}(0) \ , \end{aligned}$$

thus forcing degree$(p) \geq m$. To see why the first implication in (7.2) holds argue

$$7.3 \qquad \mathrm{hcf}(q, r) = 1 \ , \ q(T)y = r(T)z = 0 \ , \ y + z = 0 \implies y = z = 0 \ :$$

this is because condition (4.1) gives $y = -z \in q(T)^{-1}(0) \cap r(T)^{-1}(0) = \{0\}$. To apply (7.3) to the first part of (7.2) take, correcting a misprint in the proof of Theorem 12 of [3],

$$y = p(T)x_j \ , \ z = p(T)\sum_{i \neq j} x_i \ , \ q = z - \lambda_j \ , \ r = \prod_{i \neq j} z - \lambda_i \ .$$

To see why the second implication in (7.2) holds argue

$$p(\lambda_j) \neq 0 \Longrightarrow \mathrm{hcf}(p, z - \lambda_j) = 1 \Longrightarrow p(T)^{-1}(0) \cap (T - \lambda_j I)^{-1}(0) = \{0\} \ \bullet$$

For real spaces we replace (6.1) with equality

$$7.4 \qquad E_X(T) = \sum_{\lambda \in \mathbf{R}} (T - \lambda I)^{-\infty}(0) + \sum_{\mu, \nu \in \mathbf{R}} (T^2 - 2\mu T + \nu I)^{-\infty}(0)$$

and

$$7.5 \qquad F_X(T) = \bigcap_{\lambda \in \mathbf{R}} (T - \lambda I)^{\infty}(X) \cap \bigcap_{\mu, \nu \in \mathbf{R}} (T^2 - 2\mu T + \nu I)^{\infty}(X) \ ,$$

The Euclidean algorithm also gives equality

$$7.6 \qquad E_X(T) = \{x \in X : \dim \mathrm{Poly}(T)x < \infty\} \ .$$

The condition

$$7.7 \qquad F_X(T) = \{0\}$$

is not sufficient for T to be algebraic: if ([4] Example 13) $T = U : X \to X$ is the forward shift $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$ on $X = \ell_p$ or $X = c_0$ then $F_X(T) \subseteq T^{\infty}(X) = \{0\}$ while T is not algebraic. The "bounded" analogue of (7.7) works however:

8. Theorem *If there is $m \in \mathbf{N}$ for which*

8.1 $$F_X^m(T) = \{0\}$$

then $T : X \to X$ is algebraic.

Proof. The condition (8.1) together with the finiteness of the "defect spectrum"

8.2 $$\tau^{right}(T) = \{\lambda \in \mathbf{C} : (T - \lambda I)X \neq X\} = \{\lambda \in \mathbf{C} : (T - \lambda I)^\infty X \neq X\}$$

are equivalent to T being algebraic; also we claim that if (8.2) is not finite then (8.1) fails. Indeed if there are distinct $(\lambda_1, \lambda_2, \dots, \lambda_m)$ in $\tau^{right}(T)$ then there are linear functionals $f_j : X \to \mathbf{C}$ - no boundedness involved here - for which

8.3 $$\sum_{j=1}^m f_j \circ p(T) = 0 : X \to \mathbf{C},$$

so that, as in (7.2), $f_j \circ (T - \lambda_j I) = 0$ and then $p(\lambda_j) = 0$ for each j. Thus if the degree of p is less than m then (8.3) cannot happen. There must therefore be $x \in X$ for which $\sum_j f_j(p(T)x) \neq 0$ •

We recall that $T : X \to X$ is *of finite ascent* if there is $k \in \mathbf{N}$ for which

8.4 $$T^k(X) = T^{k+1}(X)\ ,\ \textit{equivalently}\ T^k(X) \cap T^{-1}(0) = \{0\}\ ,$$

and *of finite descent* if there is $k \in \mathbf{N}$ for which

8.5 $$T^{-k}(0) = T^{-k-1}(0)\ ,\ \textit{equivalently}\ T^{-k}(0) + T(X) = X\ .$$

9. Theorem *If $T \in BL(X, X)$ is bounded and linear on a Banach space, then necessary and sufficient for T to have finite descent is that*

9.1 $$E_X(T) + T(X) = X\ .$$

Proof. We claim ([6] Lemma 2.4) that

9.2 $$E_X(T) + T(X) = T^{-\infty}(0) + T(X)$$

and that ([6] Lemma 2.5), provided the first space is complete, there is $k \in \mathbf{N}$ for which

9.3 $$T^{-\infty}(0) + T(X) = T^{-k}(0) + T(X)\ .$$

For (9.2) note (6.3) that if $\lambda \neq 0$ then each pair $((T - \lambda I)^n, T)$ is exact in the sense (5.4). For (9.3) we follow part of the proof of the open mapping theorem: we claim that if $R_k : Y_k \to X$ are bounded and linear on Banach spaces Y_k and satisfy

9.4 $$\bigcup_{k=1}^{\infty} R_k(Y_k) = X$$

then for at least one $k \in \mathbf{N}$ the mapping R_k is "almost open" and hence onto. Indeed from (9.4) it follows

$$X = \bigcup_{k,h} K_{kh} \ \textit{with} \ K_{kh} = \{R_k y : \|y\| \leq h\}$$

and hence (cf [2] Theorem 4.6.2)

$$0 \in \text{int cl } R_k \text{Disc}(0,1) = \frac{1}{h} \text{int cl } K_{kh} \ ,$$

making R_k almost open. To apply this to (9.3) take

9.5 $$R_k : (y, x) \mapsto y + Tx \ \left(T^{-k}(0) \times X \to X\right) \bullet$$

Dually, using again (6.3),

9.6 $$F_X(T)_\cap T^{-1}(0) = T^\infty(X)_\cap T^{-1}(0) \ ;$$

we cannot however expect implication

9.7 $$T^\infty(X)_\cap T^{-1}(0) = \{0\} \Longrightarrow \exists k \in \mathbf{N} \ , \ T^k(X)_\cap T^{-1}(0) = \{0\} \ :$$

10. Example *If $U : Y \to Y$ and $V : Y \to Y$ are the forward and the backward shift and*

10.1 $$T = V \otimes U : Y \otimes Y = X \to X$$

then for arbitrary $k \in \mathbf{N}$

10.2 $$T^\infty(X)_\cap T^{-1}(0) = \{0\} \neq T^k(X)_\cap T^{-1}(0) \ .$$

Proof. We can represent T as an infinite operator matrix

10.3
$$T \sim \begin{pmatrix} 0 & U & 0 & \cdots \\ 0 & 0 & U & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} : \begin{pmatrix} Y \\ Y \\ Y \\ \cdots \end{pmatrix} \to \begin{pmatrix} Y \\ Y \\ Y \\ \cdots \end{pmatrix},$$

and notice

10.4
$$T^k(X)_\cap T^{-1}(0) \sim \begin{pmatrix} U^kY \\ U^kY \\ U^kY \\ \cdots \end{pmatrix} \cap \begin{pmatrix} Y \\ \{0\} \\ \{0\} \\ \cdots \end{pmatrix} \neq \begin{pmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \cdots \end{pmatrix} \sim T^\infty(X)_\cap T^{-1}(0) \bullet$$

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