

Variations on an Ordering Theme with Constraints

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INSTANCE: Finite set A , collection C of ordered triples from A^3 .

QUESTION: Is there a total order $<$ of A , such that for all $(a, b, c) \in C$,

- $a < b < c$ or $c < b < a$
(Betweenness)
- neither $a < b < c$ nor $c < b < a$
(Nonbetweenness)
- $a < b < c$ or $b < c < a$ or $c < a < b$
(Cyclic Ordering)
- $b < a$ or $c < a$

Extended topological sorting

input finite set A , partial order \leq , collection of disjoint triples C

output ordering of A that extends \leq such that the first element of each triple in C comes after the second or the third

method $(S, P, T) \leftarrow (A, \leq, C)$
Ordering \leftarrow empty sequence
while $S \neq \emptyset$ **do**
 find $e \in S$ such that $\forall y, z \in S : (y, e) \notin P \wedge (e, y, z) \notin T$
 if such an e exists **then**
 $T \leftarrow \{(x, y, z) \in T \mid y \neq e \wedge z \neq e\}$
 $P \leftarrow \{(x, y) \in P \mid x \neq e\}$
 $S \leftarrow S \setminus \{e\}$
 append e to Ordering
 else
 output "there is no ordering"; **halt**
end
output Ordering

An output is correct, because

- (x, y) stays in P until x is appended.
- if $(x, y) \in P$, then x is appended to the ordering before y .
- (x, y, z) stays in T until y or z is appended.
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If a solution exists, it is found.

Assume there is no output. Then

find $e \in S$ such that $\forall y, z \in S : (y, e) \notin P \wedge (e, y, z) \notin T$

fails to find an e .

\Rightarrow A subset $S \subseteq A$ exists where no minimum can be identified.

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QUESTION: Is there a one-to-one function $f : A \rightarrow \{1, \dots, |A|\}$, such that for all $(t_1, t_2, t_3) \in C$, there exists a $p \in P$ with $f(t_{p(1)}) < f(t_{p(2)}) < f(t_{p(3)})$.

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Example 1

$P = \{123, 321\}$ specifies the problem Betweenness.

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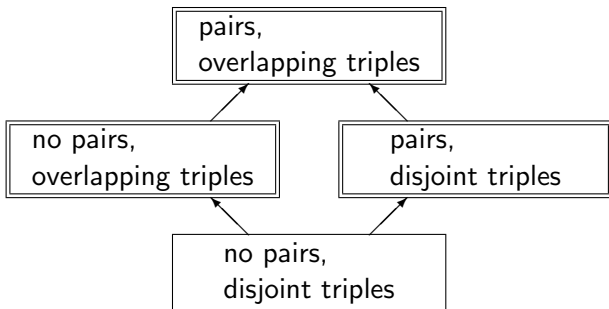
$P = \{213, 231, 132, 312\}$ specifies the problem Nonbetweenness.

- allow/disallow pairs

Variations

- allow/disallow pairs
- only use disjoint triples.

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From triples to k -tuples

For $k \in \mathbb{N}^+$ and $P \subseteq \mathcal{S}_k$ the problem (k, P) -CO is:

- INSTANCE: Finite set A , collection C of k -tuples (t_1, \dots, t_k) of distinct elements from A .
- QUESTION: Is there a one-to-one function $f : A \rightarrow \{1, 2, \dots, |A|\}$ such that for each $(t_1, \dots, t_k) \in C$, there is a $p \in P$ with $f(t_{p(i)}) < f(t_{p(j)})$ for all $1 \leq i < j \leq k$?

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$P = \{1234, 4321\}$ specifies the generalisation of the problem Betweenness to quadruples.

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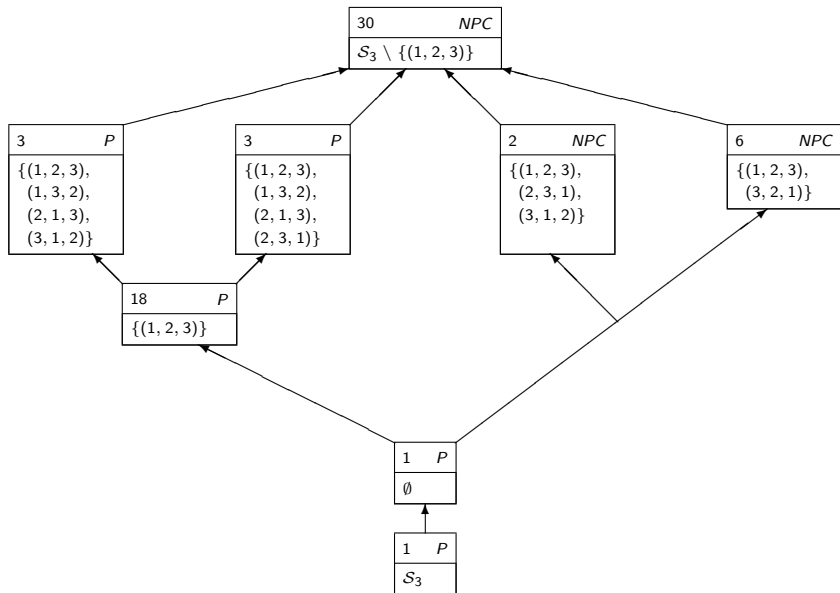
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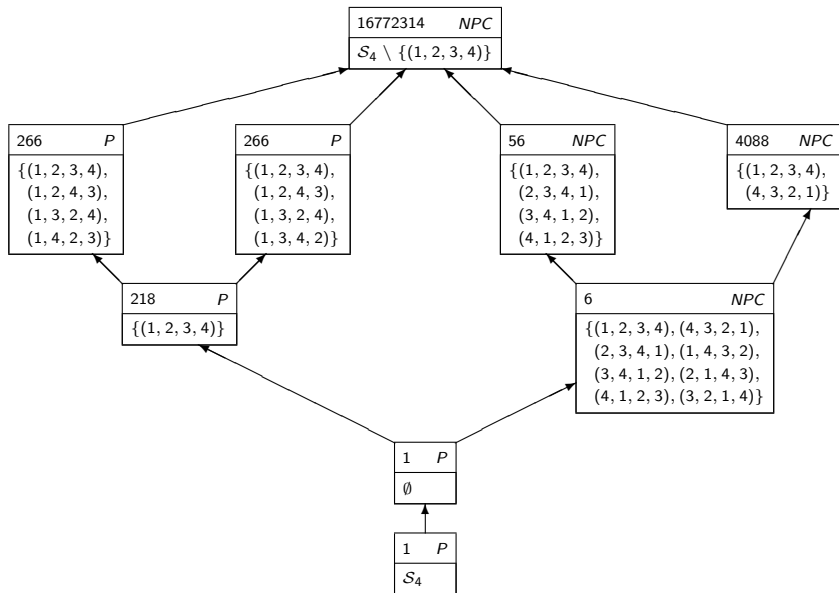
Example 4

$P = \{1234, 2341, 3412, 4123\}$ specifies the problem Cyclic Ordering generalised to quadruples.

Class structure among $(3, \cdot)$ -CO



Class structure among $(4, \cdot)$ -CO



- for $k \leq 4$, clear “NP-complete” vs. “linear time solvable” dichotomy
- similar class structures for $k = 3$ and $k = 4$
- one new class for $k = 4$

Open problems:

- Can we decide if a given problem A can be expressed in terms of another problem B ?
- class structure for (k, \cdot) -CO with $k \geq 5$
- weak order instead of strict order