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# Varieties in positive characteristic with trivial tangent bundle

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Let $X$ be a smooth projective variety over an algebraically closed field $k$ whose tangent bundle is trivial (as an algebraic vector bundle). If char $k = 0$, then the Albanese mapping $X \to \text{Alb } X$ induces an isomorphism on the space of global 1-forms, and is hence an étale covering. But any étale covering of an abelian variety is another abelian variety; from the universal property of the albanese variety, we conclude that $X \cong \text{Alb } X$ i.e., $X$ is an abelian variety.

If char $k = p > 0$, then the above argument breaks down, and in fact the conclusion is false. Thus if $E$ is an ordinary elliptic curve in characteristic 2, and $t \in E$ the non-trivial point of order 2, then the hyper-elliptic surface $X = E \times E/(\mathbb{Z}/2\mathbb{Z})$, (where the involution is $(x, y) \to (x + t, -y)$) has trivial tangent bundle, but is not an abelian variety. This example is due to Igusa [6]; he also gives an example of a free $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on an abelian 3 fold such that the quotient variety has trivial tangent bundle and trivial Albanese variety (see the end of [6], and [7]).

The next hope might be that in positive characteristic, if $\Omega^1_X$ is trivial, then $X$ has a finite étale cover by an abelian variety. We can prove only the following weaker statement. For a smooth, projective variety $X$ with $\omega_X \cong \mathcal{O}_X$, we say that $X$ is *ordinary* if the Frobenius acts non-trivially on $H^n(X, \mathcal{O}_X)$; equivalently $X$ has a non-zero global $n$-form fixed by the Cartier operator (here $n = \dim X$). We prove:

THEOREM 1: *Let $X$ be a smooth, projective variety over an algebraically closed field $k$ of characteristic $p > 0$, such that the tangent bundle of $X$ is trivial. Suppose $X$ is ordinary. Then there exists a Galois étale covering $Y \to X$ of degree $p^m$* (for some $m \geq 0$) *such that $Y$ is an abelian variety.*

The above theorem allows us to give the following characterisation of varieties obtained by quotients of free actions of finite groups on ordinary abelian varieties in positive characteristics. Recall that if $X$ is a smooth variety in characteristic $p$, if $Z_X^1$, $B_X^1$ are respectively the sheaves of closed and exact 1-forms, we have an exact sequence

$$0 \longrightarrow B_X^1 \longrightarrow Z_X^1 \xrightarrow{C} \Omega_X^1 \longrightarrow 0$$

where $C$ is the Cartier operator, and $B_X^1$, $Z_X^1$ are locally free $\mathcal{O}_X$-submodules of $F_*\Omega_X^1$ (where $F$: $X \to X$ is the absolute Frobenius). The above extension corresponds to a class $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X^1, B_X^1) = H^1(X, T_X \otimes B_X^1)$ where $T_X$ is the tangent sheaf of $X$.

THEOREM 2: *For a smooth, projective variety $X$ over an algebraically closed field $k$ of characteristic $p$, the following are equivalent.*

(i) *$X \cong Y/C$, where $Y$ is an ordinary abelian variety, $G$ a finite group acting freely on $Y$*

(ii) *the extension class $\xi = 0$*, and *$\omega_X$ is a torsion line bundle of order prime to $p$*

(iii) *the extension class $\xi = 0$, and $\omega_X$ is numerically trivial.*

A crucial step in the proof of Theorem 1 is to show that an ordinary variety with trivial tangent bundle can be lifted to characteristic 0. In fact, such a variety has a "canonical lifting" characterised by being the unique lifting such that the Frobenius morphism also lifts to a morphism over the Frobenius of the Witt ring $W(k)$. This also gives another construction of the canonical lifting of Serre and Tate for ordinary abelian varieties, without any serious use of the theory of Barsotti–Tate groups ($p$-divisible groups). For other constructions of the Serre-Tate lifting, see [2, 4]; our construction is given in an appendix by M.V. Nori and the second author.

We also make use of the following result from complex differential geometry, which is a consequence of Yau's proof of the Calabi conjecture (see Theorem (1.5) of [10]): if $M$ is a compact Kähler manifold with $C_1 = 0$ and $C_2 = 0$ (in $H^*(M, \mathbb{R})$), then $M$ has a finite étale covering by a torus. This suggests that if $X$ is a smooth, projective variety in characteristic $P$ which is "ordinary" in some sufficiently strong sense, with $C_1$ and $C_2$ numerically trivial, then $X$ has a finite étale cover by an abelian variety. We are unable to deduce this from our results, but do not know any counter examples.

## §1. Proof of Theorem 1

Recall that a variety $X$ in characteristic $p$ is *Frobenius split* if the $\mathcal{O}_X$-linear map $\mathcal{O}_X \to F_*\mathcal{O}_X$ is $\mathcal{O}_X$-split, where $F$: $X \to X$ is the absolute Frobenius morphism (see [13]), for some applications of this notion). Illusie and Raynaud [9] have introduced the notion of an ordinary variety in a more general context; according to their definition, a smooth projective variety $X$ is ordinary if $H^i(X, B^j_X) = 0$ for all $i \geqslant 0, j > 0$. Here

$$B^j_X = \text{image } (d: \Omega^{j-1}_X \to \Omega^j_X).$$

On such a variety, all global $i$-forms are closed, for each $i > 0$.

LEMMA (1.1): *Let $X$ be a smooth, projective variety over $k = \bar{k}$ of characteristic $p > 0$ with trivial contangent bundle. Then $X$ is ordinary $\Leftrightarrow$ $X$ is ordinary in the sense of Illusie–Raynaud $\Leftrightarrow$ $X$ is Frobenius split $\Leftrightarrow$ the extension*

$$0 \to B^1_X \to Z^1_X \to \Omega^1_X \to 0$$

*is split*

*Proof:* We have exact sequences of locally free $\mathcal{O}_X$-modules ($n = \dim X$)

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow B^1_X \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow B^n_X \longrightarrow F_*\Omega^n_X \xrightarrow{C} \Omega^n_X \longrightarrow 0 \tag{2}$$

where $C$ is the Cartier operator. Since $C$ is also the trace map of Grothendieck duality for the finite flat map $F: X \to X$ we have a perfect $\mathcal{O}_X$-bilinear pairing $F_*\mathcal{O}_X \otimes F_*\Omega^n_X \to \Omega^n_X$ by $(f, \omega) \to C(f\omega)$. This induces perfect pairings $B^1_X \otimes B^n_X \to \Omega^n_X$ and $\mathcal{O}_X \otimes \Omega^n_X \to \Omega^n_X$ such that the exact sequences (1) and (2) above are interchanged by the functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \Omega^n_X)$. In particular, (1) is split $\Leftrightarrow$ (2) is split. Since $\Omega^n_X \cong \mathcal{O}_X$, a splitting of (2) is given by the choice of an $n$-form on $X$ fixed by the Cartier operator, equivalently by the choice of a (non-zero) class in $H^n(X, \mathcal{O}_X)$ fixed by Frobenius. Hence $X$ is ordinary $\Leftrightarrow$ $X$ is Frobenius split.

Next, we show that $X$ is Frobenius split $\Leftrightarrow$ the extension

$$0 \longrightarrow B^1_X \longrightarrow Z^1_X \xrightarrow{C} \Omega^1_X \longrightarrow 0$$

is split. Indeed, from the exact sequence (1), if $X$ is Frobenius split, then $H^i(X, B^1_X) = 0$ for all $i$. In particular $C$ induces an isomorphism of $H^0(X, Z^1_X)$ with $H^0(X, \Omega^1_X)$. Since $\Omega^1_X$ is trivial, this splits the above sequence. Conversely, if the above extension $\xi \in \mathrm{Ext}^1(\Omega^1_X, B^1_X)$ is trivial, then the

$p^{-1}$-linear map

$$C\colon H^0(X, Z_X^1) \to H^0(X, \Omega_X^1)$$

is surjective. Since $H^0(X, Z_X^1) \subset H^0(X, F_*\Omega_X^1) \cong H^0(X, \Omega_X^1)$, we see that $C$ must be bijective, and so $H^0(X, \Omega_X^1)$ has a basis consisting of forms fixed by $C$. Since $C(\omega_1 \wedge \omega_2) = C(\omega_1) \wedge C(\omega_2)$ for any two closed forms $\omega_1$, $\omega_2$ and $\wedge^n H^0(X, \Omega_X^1) \cong H^0(X, \Omega_X^n)$ we see that $H^0(X, \Omega_X^n)$ has a generator fixed by $C$ i.e., the sequence (2) above splits.

Finally, we check that if the extension $\xi$ above is trivial, then $X$ is ordinary in the sense of Illusie-Raynaud i.e., $H^i(X, B_X^j) = 0$ for all $i \geqslant 0$, $j > 0$. Since $\wedge^j H^0(X, \Omega_X^1) \cong H^0(X, \Omega_X^j)$ for each $j > 0$, we deduce that $H^0(X, \Omega_X^j)$ has a basis of forms fixed by the Cartier operator, so that the sequences

$$0 \longrightarrow B_X^j \longrightarrow Z_X^j \xrightarrow{C} \Omega_X^j \longrightarrow 0 \qquad (3)_j$$

are split for all $j > 0$. We have perfect $\mathcal{O}_X$-bilinear pairings $\forall\, j \geqslant 0$

$$F_*\Omega_X^j \otimes F_*\Omega_X^{n-j} \to \Omega_X^n$$

given by $(\omega_1, \omega_2) \to C(\omega_1 \wedge \omega_2)$. These induce perfect pairings

$$Z_X^j \otimes F_*\Omega_X^{n-j}/B_X^{n-j} \to \Omega_X^n, \quad B_X^j \otimes F_*\Omega_X^{n-j}/Z_X^{n-j} \to \Omega_X^n$$

where $F_*\Omega_X^{n-j}/Z_X^{n-j} \cong B_X^{n-j+1}$. Hence applying the functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \Omega_X^n)$ to the sequences $(3)_j$, the dual exact sequences

$$0 \longrightarrow \Omega_X^{n-j} \xrightarrow{C} F_*\Omega_X^{n-j}/B_X^{n-j} \xrightarrow{d} B_X^{n-j+1} \longrightarrow 0 \qquad (4)_{n-j}$$

are also all split. By ascending induction on $n - j$ i.e., by descending induction on $j$ (and starting at $n - j = 0$, with $B_X^0 = 0$), if we know that $H^i(X, B_X^{n-j}) = 0$ for all $i$, then $H^i(X, F_*\Omega_X^{n-j}/B_X^{n-j})$ has the same dimension as $H^i(X, \Omega_X^{n-j})$. Hence from the split sequence $(4)_{n-j}$, $H^i(X, B_X^{n-j+1}) = 0$ for all $i \geqslant 0$. Thus $X$ is ordinary in the sense of Illusie–Reynaud.

LEMMA (1.2): *Let $X$ satisfy the hypothesis of Theorem 1, $f\colon X' \to X$ a finite étale cover. Then $X'$ also satisfies the hypothesis of Theorem 1.*

*Proof:* Clearly $\Omega_{X'}^1$ is trivial since $\Omega_X^1$ is, and if $\omega$ is a non-zero $n$-form ($n = \dim X$) fixed by $C$, so is $f^*\omega$.

LEMMA (1.3) (Igusa [5]): *Let $Y/k = \bar{k}$ be a smooth, projective variety, char $k = p > 0$, and let $f$: $Y \to A$ be the Albanses map. Then $f^*$: $H^0(A, \Omega_A^1) \to H^0(Y, \Omega_Y^1)$ and $f^*$: $H^1(A, \mathcal{O}_A) \to H^1(Y, \mathcal{O}_Y)$ are injective.*

*Proof:* We give a proof rather different in spirit from Igusa's argument. Since all global 1-forms on an abelian variety are closed, we have a diagram

$$\begin{tikzcd} H^0(Y, Z_Y^1) \arrow[r, "C_Y"] & H^0(Y, \Omega_Y^1) \\ H^0(A, \Omega_Y^1) \arrow[u, "f^*"] \arrow[r, "C_A"] & H^0(A, \Omega_A^1) \arrow[u] \end{tikzcd}$$

where $C_Y$, $C_A$ are the respective Cartier operators. Hence $(\ker f^*)$ is a subspace invariant under $C_A$. In particular, if $(\ker f^*) \neq 0$, we can find $\omega \in H^0(A, \Omega_A^1)$ satisfying $f^*\omega = 0$, and either $C\omega = \omega$ or $C\omega = 0$. If $\varphi$: $A' \to A$ is the corresponding $\mu_p$ or $\alpha_p$ cover, then $\varphi$ has the following universal property: given any variety $Z$ together with a morphism $g$: $Z \to A$ such that $g^*\omega = 0$, there is a rational map $h$: $Z \to A'$ such that $g = \varphi h$. Now $A'$ is a smooth, projective variety since $\omega$ has no zeros; from the universal property of $\varphi$, the group law on $A$ induces one on $A'$ so that $\varphi$ is a homomorphism of abelian varieties. Since $f^*\omega = 0$, $f$ factors through $\varphi$, contradicting the universal property of the Albanese variety.

To show $f^*H^1(A, \mathcal{O}_A) \to H^1(X, \mathcal{O}_X)$ is injective, since $f^*$ commutes with the Frobenius $F^*$, if $\ker f^* \neq 0$, then we can find $\alpha \in \ker f^*$ such that either $F^*\alpha = \alpha$ or $F^*\alpha = 0$. Again, $f$: $X \to A$ would have to factor through an étale $\mathbb{Z}/p\mathbb{Z}$-cover or an $\alpha_P$-cover of $A$, which contradicts the universal property of $f$. This result has also been proved by J.-P. Serre [cf. 16].

LEMMA (1.4): *Let $X$ be a smooth, projective variety with $\Omega_X^1$ trivial. Then the Albanese map $f$: $X \to$ Alb $X$ is a surjective smooth morphism with $f_*\mathcal{O}_X \cong \mathcal{O}_{\mathrm{Alb}\,X}$. Also $\Omega^1_{X/\mathrm{Alb}\,X}$ is trivial.*

*Proof:* By the previous lemma $f^*$: $H^0(\mathrm{Alb}\,X, \Omega^1_{\mathrm{Alb}\,X}) \to H^0(X, \Omega_X^1)$ is injective; hence $f^*\Omega^1_{\mathrm{Alb}\,X} \to \Omega_X^1$ is an injection of trivial bundles whose cokernel $\Omega^1_{X/\mathrm{Alb}\,X}$ is hence trivial of rank dim $X$ − dim (Alb $X$). The injectivity of the above sheaf map implies that $f$ is dominant and separable; hence it is smooth, as $\Omega^1_{X/\mathrm{Alb}\,X}$ is trivial. If $X \to Y \to$ Alb $X$ is the Stein factorisation, then $Y \to$ Alb $X$ is smooth, hence an étale cover i.e., an abelian variety. The universal property of Alb $X$ forces $Y =$ Alb $X$ i.e., $f_*\mathcal{O}_X = \mathcal{O}_{\mathrm{Alb}\,\mathrm{X}}$.

LEMMA (1.5): *Suppose $X$ is an ordinary smooth projective variety in characteristic* p *with trivial cotangent bundle. Then $H^1(X, \mathcal{O}_X) \neq 0$; hence $H^1_{\text{et}}(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$.*

*Proof:* Since $X$ is ordinary, by lemma (1.1)

$$H^1(X, Z^1_X) \cong H^1(X, \Omega^1_X) \cong H^1(X, \mathcal{O}_X)^{\oplus n}, \quad n = \dim X.$$

We have an exact sequence of etale sheaves ([8], Ch. 0, 2.1.23)

$$0 \longrightarrow \nu(1) \longrightarrow Z^1_X \xrightarrow{1-C} \Omega^1_X \longrightarrow 0,$$

where $\nu(1)$ is the cokernel of multiplication by $p$ on $G_m$. In particular, we have an exact sequence

$$0 \to (\text{Pic } X) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to H^1_{et}(X, \nu(1)) \to {}_p\text{Br}'(X) \to 0$$

(where ${}_p\text{Br}'(X)$ is the $p$-torsion in $H^2(X, G_m)$). At any rate, $(\text{Pic } X) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow NS(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \neq 0$. Hence from the exact sequence

$$H^0(X, Z^1_X) \xrightarrow{1-C} H^0(X, \Omega^1_X) \longrightarrow H^1_{et}(X, \nu(1)) \longrightarrow H^1(X, Z^1_X)$$

(where $H^i(X, Z^j_X) \cong H^i_{et}(X, Z^j_X)$ since $Z^j_X \subset F_*\Omega^j_X$ is a coherent $\mathcal{O}_X$-module, and similarly for $\Omega^j_X$), to show $H^1(X, Z^1_X) \neq 0$, it suffices to show $1 - C$ is surjective, which is clear since $C$ is $p^{-1}$-linear, and $H^0(X, \Omega^1_X)$ has a basis fixed by $C$.

To see that $H^1_{et}(X, \mathbb{Z}/p\mathbb{Z}) = 0$, we see from the Artin-Schrier sequence that $H^1_{et}(X, \mathbb{Z}/p\mathbb{Z}) \subset H^1(X, \mathcal{O}_X)$ is the subgroup of elements fixed by the Frobenius. Since $X$ is Frobenius split, the Frobenius is bijective, and so $H^1(X, \mathcal{O}_X)$ has a basis of elements fixed by Frobenius.

LEMMA (1.6): *Let $X$ satisfy the hypothesis of Theorem 1. Then $X$ has a finite etale cover $Y \to X$ with* Alb $Y \neq 0$.

*Proof:* From the results proved in the appendix, there is a smooth, proper scheme f: $\mathcal{X} \to \text{Spec } W(k)$ with special fiber $X \to \text{Spec } k$. Let $\tilde{X}$ be the generic fiber, over Spec $K$, where $K = W(k)[1/p]$ is the quotient field. Since $R^if_*\mathbb{Q}_l(j)$ are all locally constant étale sheaves on Spec $W(k)$, if $C_1(\tilde{X}) \in H^2(\tilde{X}_{\bar{K}}, \mathbb{Q}_l(1))$, $C_2(\tilde{X}) \in H^4(\tilde{X}_{\bar{K}}, \mathbb{Q}_l(2))$ are the first two Chern classes of $T_{\tilde{X}}$, then $C_1(\tilde{X})$ and $C_2(\tilde{X})$ reduce to 0 in $H^*(X, Q_l(*))$; hence $C_1(\tilde{X}) = 0$, $C_2(\tilde{X}) = 0$. Now we use the result from complex differential geometry implied by the Calabi conjecture, which we mentioned in the introduction, to see that $\tilde{X}_{\bar{K}}$ has a finite étale covering by an abelian variety. Thus the algebraic fundamental group $\pi_1^{\mathrm{alg}}(\tilde{X})$ has the following property: let $\pi_1^{\mathrm{alg}}(\tilde{X})^0$ be the kernel of the natural map $\pi_1^{\mathrm{alg}}(\tilde{X}) \to \mathrm{Gal}\,(\bar{K}/K)$. Then $\pi_1^{\mathrm{alg}}(\tilde{X})^0$ has a closed normal subgroup of finite index of the form $\hat{\mathbb{Z}}^n$. Clearly $\pi_1^{\mathrm{alg}}(\tilde{X})^0 \twoheadrightarrow \pi_1^{\mathrm{alg}}(\mathscr{X})$, while $\pi_1^{\mathrm{alg}}(X) \to \pi_1^{\mathrm{alg}}(\mathscr{X})$ is an isomorphism since $W(k)$ is complete, by a result of Grothendieck ([4], X). Hence $\pi_1^{\mathrm{alg}}(X)$ has the property that it has a closed abelian normal subgroup (a quotient of $\hat{\mathbb{Z}}^n$) as a subgroup of finite index. In particular, replacing $X$ by a finite étale cover, we may assume $\pi_1^{\mathrm{alg}}(X)$ is abelian, and topologically finitely generated. Let $G$ be the maximal pro-p-quotient i.e., $G \cong \varprojlim_n (\pi_1^{\mathrm{alg}}(X) \otimes \mathbb{Z}/p^n\mathbb{Z})$. Then $G$ is a finitely generated $\mathbb{Z}_p$-module. Hence either $G$ is finite, or else there is a surjection $G \twoheadrightarrow \mathbb{Z}_p$ i.e., $H^1_{et}(X, \mathbb{Z}_p) \neq 0$. But $G$ cannot be finite, since by Lemmas (1.2), (1.5), every finite etale cover $X' \to X$ satisfies $H^1_{et}(X', \mathbb{Z}/p\mathbb{Z}) \neq 0$. Hence we must have $H^1_{et}(X, \mathbb{Z}_p) \neq 0$. Hence by [8], Chapter II, Theorem (5.2), $H^1_{\mathrm{Cris}}(X/W) \neq 0$, and so the first Betti number of $X$ is non-zero i.e. Alb $X \neq 0$.

LEMMA (1.7): *Let $X$ be as in Theorem 1, $Y \to X$ a Galois étale cover with* Alb $Y \neq 0$. *Then there is an intermediate Galois étale cover $Z \to X$ of degree $p^m$, for some* m, *such that $Y \to Z$ induces an isogeny on Albanese varieties; in particular* Alb $Z \neq 0$.

*Proof:* Let $G = \mathrm{Gal}(Y/X)$. Since $\Gamma(X, \Omega^1_X) \to \Gamma(Y, \Omega^1_Y)$ is an isomorphism, $G$ acts trivially on $\Gamma(Y, \Omega^1_Y)$. Fix a base point $y \in Y$; then there is a map $f$: $Y \to \mathrm{Alb}\ Y$ with $f(y) = 0$, universal for maps from $Y$ to abelian varieties which map $y$ to 0. If $\sigma \in G$, we have another map $f^\sigma$: $Y \to \mathrm{Alb}\ Y$ given by $f^\sigma(z) = f(\sigma(z))$ for $z \in Y$. If $a \in \mathrm{Alb}\ Y$, let $T_a$: Alb $Y \to$ Alb $Y$ denote translation by a. Then $T_{-f^\sigma(y)} \circ f^\sigma$: $Y \to \mathrm{Alb}\ Y$ maps $y$ to 0. Hence by the universal property of $f$, there is an automorphism $\sigma'$ of Alb $Y$ such that $T_{-f^\sigma(y)} \circ f^\sigma = \sigma' \circ f$. Hence $f^\sigma = (T_{f^\sigma(y)} \circ \sigma') \circ f$ and $\sigma \to T_{f^\sigma(y)} \circ \sigma'$ gives an action of $G$ on Alb $Y$ making $f$ an equivariant map. Let $H \subset G$ be the normal subgroup of elements acting by translations, and let $Z = Y/H$. Then Alb $Y \to$ Alb $Z$ is an isogeny, and $Z \to X$ is an étale cover such that no non-trivial element of Gal $(Z/X)$ acts by translation on Alb $Z$. Replacing $Y$ by $Z$, we assume $Y$ itself has this property. In the notation above, this means the representation $\sigma \to \sigma'$ of $G$, as group automorphisms of Alb $Y$, is faithful. We claim this forces $G$ to be a $p$-group. Indeed, let $\sigma \in G$ satisfy $\sigma^l = 1$ for some prime $l \neq p$. Let $\mathcal{O}$ be the local ring at the origin in Alb $Y$, $m \subset \mathcal{O}$ the maximal ideal. Then

$$\Gamma(\mathrm{Alb}\ Y, \Omega^1_{\mathrm{Alb}\,Y}) \to \Omega^1_{\mathcal{O}/k} \to \Omega^1_{\mathcal{O}/k} \otimes \mathcal{O}/m$$

gives a $G$-equivariant isomorphism

$$\Gamma(\text{Alb } Y, \Omega^1_{\text{Alb } Y}) \to m/m^2.$$

Hence $\sigma$ acts trivially on $m/m^2$. But $m^i/m^{i+1} \cong S^i(m/m^2)$ as $G$-modules; hence $\sigma$ acts trivially on $m^i/m^{i+1}$ for all $i$. Thus $\sigma$ acts unipotently on $\mathcal{O}/m^i$, which is a finite dimensional k-vector space, while $\sigma^l = 1$. Hence $\sigma$ acts trivially on $\mathcal{O}/m^i$, for each $i$, and hence also on $\mathcal{O}$, and on Alb $Y$.

LEMMA (1.8): *Let $X$ satisfy the hypothesis of Theorem 1, and let $\pi: X \to A$ be the Albanese mapping. Then (i) all geometric fibers of $\pi$ are ordinary (ii) for each $i \geqslant 0$, $R^i_{\pi_*\mathcal{O}_X}$ is a locally free $\mathcal{O}_A$-module which becomes free on a finite étale cover of $A$ (iii) $A$ is an ordinary abelian variety.*

*Proof:* We have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\pi"'] & X' \arrow[r, "g"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ & A \arrow[r, "F"'] & A \end{tikzcd}$$

where $F: A \to A$ and $gf: X \to X$ are the absolute Frobenius morphisms, and $f: X \to X'$ is the relative Frobenius of $\pi$. Let $B^i_{X/A} = \text{image}\,(d: \Omega^{i-1}_{X/A} \to \Omega^i_{X/A})$, and let $d$ be the relative dimension of $X/A$. Let $\omega \in H^0(X, \Omega^d_X)$ map to a generator $\bar{\omega} \in H^0(X, \omega_{X/A}) \cong H^0(X, \mathcal{O}_X)$. If $\eta$ is a generator of $H^0(A, \omega_A)$, then $\pi^*\eta \wedge \omega \in H^0(X, \omega_X)$ is a generator.

Consider the relative Cartier operator

$$C_{X/A}: H^0(X', f_*\omega_{X/A}) \to H^0(X', \omega_{X'/A}).$$

If $C_{X/A}(\bar{\omega}) = 0$, there exists a meromorphic section $\bar{\omega}_1$ of $\Omega^{d-1}_{X/A}$ such that $d\bar{\omega}_1 = \bar{\omega}$. Let $\omega_1$ be a meromorphic section of $\Omega^{d-1}_X$ restricting to $\bar{\omega}_1$; then $d\omega_1 = \omega + \omega_2$ where $\omega_2 \to 0$ in $\omega_{X/A}$. But then $\pi^*\eta \wedge \omega_2 = 0$, so that $\pi^*\eta \wedge \omega = \pi^*\eta \wedge d\omega_1 = d(\pi^*\eta \wedge \omega_1)$. Hence $C(\pi^*\eta \wedge \omega) = 0$, contradicting that $X$ is ordinary. Hence $C_{X/A}(\bar{\omega})$ is a non-zero scalar multiple of $\bar{\omega}$, and by replacing $\bar{\omega}$ by a scalar multiple, we may assume $C_{X/A}(\bar{\omega}) \neq 0$. Thus the exact sequence of $\mathcal{O}_{X'}$-modules

$$0 \longrightarrow f_*B^d_{X/A} \longrightarrow f_*\omega_{X/A} \xrightarrow{C_{X/A}} \omega_{X'/A} \longrightarrow 0$$

is split. Thus the dual exact sequence (see the proof of Lemma (1.1)), obtained by applying $\mathrm{Hom}_{\mathcal{O}_{X'}}(-, \omega_{X'/A})$,

$$0 \to \mathcal{O}_{X'} \to f_*\mathcal{O}_X \to f_*B^1_{X/A} \to 0$$

is also split. Thus all geometric fibers of $\pi$ are Frobenius split, hence ordinary. Also, $C(\pi^*\eta \wedge \omega) = \pi^*C\eta \wedge C\omega \neq 0$ shows $C\eta \neq 0$, so that $A$ is an ordinary abelian variety (Lemma (1.1)). Note that in the above calculation, $C\omega$ makes sense, since by Lemma (1.1) all global d-forms on $X$ are closed.

Finally, since $F$: $A \to A$ is flat, there are natural isomorphisms

$$F^*R^i\pi_*\mathcal{O}_X \cong R^i\pi'_*\mathcal{O}_{X'}.$$

Also, the natural maps

$$R^i\pi'_*\mathcal{O}_{X'} \to R^i\pi'_* f_*\mathcal{O}_X \cong R^i\pi_*\mathcal{O}_X$$

are isomorphisms, since $R^i\pi'_* f_*B^1_{X/A} = 0$ for all $i \geqslant 0$ (as all geometric fibers of $\pi$ are ordinary, and hence ordinary in the sense of Illusie–Raynaud; now use the formal function theorem).

Hence $F^*R^i\pi_*\mathcal{O}_X \cong R^i\pi_*\mathcal{O}_X$ for all $i \geqslant 0$. Thus by Lemma (1.4) of [12], $R^i\pi_*\mathcal{O}_X$ is locally free for all $i \geqslant 0$, while by Lemma (1.4) of [11], $R^i\pi_*\mathcal{O}_X$ becomes free on a finite étale cover of $A$.

LEMMA (1.9): *Let $X$ satisfy the hypothesis of Theorem 1, and let $\pi$: $X \to A$ be the Albanese mapping. Let $s \in A$ be a closed point, $X_s$ the fiber over $s$. Then given any étale $\mathbb{Z}/p\mathbb{Z}$-cover of $X_s$, we can find an étale cover of $X$ (not necessarily a $\mathbb{Z}/p\mathbb{Z}$-cover) inducing the given cover on $X_s$.*

*Proof:* The étale cover of $X_s$ corresponds to a class in $H^1_{et}(X, \mathbb{Z}/p\mathbb{Z}) = \ker(1 - F^*: H^1(X_s, \mathcal{O}_{X_s}) \to H^1(X_s, \mathcal{O}_{X_s}))$, where $F$ is the Frobenius on $X_s$. So we first study the map on cohomology $H^1(X, \mathcal{O}_X) \to H^1(X_s, \mathcal{O}_{X_s})$.

The Leray spectral sequence for $\pi$: $X \to A$, together with the formula $\pi_*\mathcal{O}_X \cong \mathcal{O}_A$ (Lemma (1.4)), yields an exact sequence

$$0 \to H^1(A, \mathcal{O}_A) \to H^1(X, \mathcal{O}_X) \to \Gamma(A, R^1\pi_*\mathcal{O}_X) \to H^2(A, \mathcal{O}_A).$$

Now $R^1\pi_*\mathcal{O}_X$ is locally free on $A$, and becomes free on a finite étale cover $A' \to A$. Let $X' = X \times_A A'$, $\pi'$: $X' \to A'$; then $R^1\pi_*\mathcal{O}_{X'}$ is free. Thus if $A'' \to A'$ is another etale cover, and $X'' = X' \times_{A'} A''$, then $R^1\pi''_*\mathcal{O}_{X''}$ is free, and the natural map

$$\Gamma(A', R^1\pi'_*\mathcal{O}_{X'}) \to \Gamma(A'', R^1\pi'_*\mathcal{O}_{X''})$$

is an isomorphism. However, if $A'' = A' \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,k$ where $\mathrm{Spec}\,k \to \mathrm{Spec}\,k$ is the Frobenius, and if $A' \to A''$ is the $k$-linear Frobenius (i.e., relative Frobenius over $k$), then there is an étale cover $A'' \to A'$ such that the composite $A'' \to A''$ is multiplication by $p$. This claim follows because $A$, and hence $A'$, $A''$, are ordinary abelian varieties. In fact, if $\dim A' = m$, then $H^1_{et}(A', \mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^m$; then $A'' \to A'$ is the corresponding étale Galois cover with group $\mathbb{Z}/p\mathbb{Z})^m$. In particular

$$H^1_{et}(A', \mathbb{Z}/p\mathbb{Z}) \to H^1_{et}(A'', \mathbb{Z}/p\mathbb{Z})$$

is 0. Hence $H^1(A', \mathcal{O}_{A'}) \to H^1(A'', \mathcal{O}_{A''})$ is 0, and by the known cup-product structure of $\oplus\, H^i(A', \mathcal{O}_{A'})$ we see that

$$H^i(A', \mathcal{O}_{A'}) \to H^i(A'', \mathcal{O}_{A''})$$

is 0, for all $i > 0$.

We thus have a diagram

$$\begin{array}{ccccc} H^1(X'', \mathcal{O}_{X''}) & \longrightarrow & \Gamma(A'', R^1\pi''_*\mathcal{O}_{X''}) & \longrightarrow & H^2(A'', \mathcal{O}_{A''}) \\ \uparrow & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle 0} \\ \mathrm{H}^1(X', \mathcal{O}_{X'}) & \longrightarrow & \Gamma(A', R^1\pi'_*\mathcal{O}_{X'}) & \longrightarrow & H^2(A', \mathcal{O}_{A'}) \end{array}$$

Hence we obtain an exact sequence

$$0 \to H^1(A'', \mathcal{O}_{A''}) \to H^1(X'', \mathcal{O}_{X''}) \to \Gamma(A'', R^1\pi''_*\mathcal{O}_{X''}) \to 0.$$

Since $R^i\pi''_*\mathcal{O}_{X''}$ is locally free for each $i$, the theorem on cohomology and base change implies (by descending induction on $i$) that for any $a \in A''$,

$$(R^i\pi''_*\mathcal{O}_{X''})_a \otimes_{\mathcal{O}_{a,A''}} k(a) \cong H^i(X''_a, \mathcal{O}_{X''_a}),$$

where $X''_a$ is the fiber over $a$. In particular, for any $a \in A''$,

$$\Gamma(A'', R^1\pi''_*\mathcal{O}_{X''}) \to H^1(X''_a, \mathcal{O}_{X''_a})$$

is an isomorphism.

Choose a point $t \in A''$ mapping to $s \in A$; then $X''_t \cong X_s$, and it suffices to prove

$$H^1_{et}(X'', \mathbb{Z}/p\mathbb{Z}) \twoheadrightarrow H^1_{et}(X''_t, \mathbb{Z}/p\mathbb{Z}).$$

we have an exact sequence

$$0 \to H^1(A'', \mathcal{O}_{A''}) \to H^1(X'', \mathcal{O}_{X''}) \to H^1(X''_t, \mathcal{O}_{X''_t}) \to 0$$

compatible with the respective Frobenius actions. On each space, $1 - F^*$ is a surjective endomorphism; hence by the snake lemma we have an exact sequence of kernels of $1 - F^*$

$$0 \to H^1_{et}(A'', \mathbb{Z}/p\mathbb{Z}) \to H^1_{et}(X'', \mathbb{Z}/p\mathbb{Z}) \to H^1_{et}(X'', \mathbb{Z}/p\mathbb{Z}) \to 0.$$

This proves the lemma.

LEMMA (1.10): *Let $X$ satisfy the hypothesis of Theorem 1, $\pi\colon X \to A$ the Albanese mapping, $l$ a prime $\neq p$. Then $R^1\pi_*\mathbb{Q}_l$ becomes trivial on a finite étale cover of $A$.*

*Proof:* Once more we use the existence of a lifting to characteristic 0, and then appeal to Kähler differential geometry. (However, a proof using only characteristic p methods can be given, using among other things the ampleness of the "Hodge bundles", given by tensor powers of the direct image, on the moduli of abelian varieties, of the relative canonical bundle of the "universal abelian variety" – see the remarks after the proof of the lemma.) Let $f\colon \mathscr{X} \to \operatorname{Spec} W(k)$, $g\colon \mathscr{A} \to \operatorname{Spec} W(k)$ be the respective canonical liftings of $X$ and $A$. Then the morphism $\pi\colon X \to A$ lifts uniquely to a morphism $h\colon \mathscr{X} \to \mathscr{A}$ over $\operatorname{Spec} W(k)$ (which is compatible with the liftings of the Frobenius morphisms of $X$ and $A$). Since $\pi\colon X \to A$ is smooth and surjective and $f$, $g$ are flat, $h$ is flat (by the local criterion for flatness) and hence smooth and surjective. Now $R^1h_*\mathbb{Q}_l$ is a locally constant etale sheaf on $\mathscr{A}$, coming from a representation of $\pi_1^{\mathrm{alg}}(\mathscr{A}) \cong \pi_1^{\mathrm{alg}}(A)$, since $W(k)$ is Hensel; it suffices to show this representation has finite image. If $K = W(k)[p^{-1}]$ is the quotient field of $W(k)$, and $\bar{K}$ is the algebraic closure of $K$, let $\bar{f}\colon \bar{X} \to \operatorname{Spec} \bar{K}$, $\bar{g}\colon \bar{A} \to \operatorname{Spec} \bar{K}$ be the geometric generic fibers, and let $\bar{h}\colon \bar{X} \to \bar{A}$ be the corresponding map. Since $\pi_1^{\mathrm{alg}}(\bar{A}) \to \pi_1^{\mathrm{alg}}(\mathscr{A})$ is surjective, it suffices to show that the locally constant étale sheaf $R^1\bar{h}_*\mathbb{Q}_l$ becomes trivial on a finite étale cover of $\bar{A}$. Now as in the proof of Lemma (1.6), we see that $\bar{X} \cong B/G$, where $B$ is an abelian variety over $\bar{K}$, and $G$ is a finite group acting freely on $B$. If $q\colon B \to \bar{X}$ is the quotient map, $r\colon B \to \bar{A}$ the composite with $\bar{h}$, then by Poincaré reducibility, clearly $R^1 r_* \mathbb{Q}_l$ becomes trivial on a finite étale cover of $\bar{A}$. From the spectral sequence for the higher direct image of a composite, $R^1 \bar{h}_*(q_* \mathbb{Q}_l) \hookrightarrow R^1 r_* \mathbb{Q}_l$, and since $\mathbb{Q}_l$ is a direct summand of $q_* \mathbb{Q}_l$, $R^1 \bar{h}_* \mathbb{Q}_l \hookrightarrow R^1 r_* \mathbb{Q}_l$ too. Thus $R^1 \bar{h}_* \mathbb{Q}_l$ becomes trivial on a finite étale cover.

REMARK: We sketch another proof of this lemma which does not appeal to complex differential geometry. Replace the given family $\pi\colon X \to A$ by a family $\varphi\colon Y \to C$, where $Y = X \times_A C$, and $C \to A$ is a morphism from a smooth curve such that $\varphi$ has a section, and image $(\pi_1^{\mathrm{alg}}(C) \to \pi_1^{\mathrm{alg}}(A))$ has finite index. Then all fibers of $\varphi$ are ordinary, and $\Omega^1_{Y/C}$ is trivial. The Albanese variety of the generic fiber of $\varphi$ spreads out to an abelian scheme $B \to C$, by the Serre-Tate criterion [15, 17] for good reduction, and the Albanese map on the generic fiber extends to a $C$-morphism $Y \to B$ (if $Y' \to Y$ is birational, all fibers of $Y' \to Y$ are linearly connected). Thus all fibers of $B \to C$ are ordinary, being isogenous to the Albanese varieties of the corresponding fibers of $Y \to C$.

So it suffices to prove that any family of ordinary abelian varieties over a complete smooth curve is trivial on a finite étale cover of the base. Since any abelian variety over an algebraically closed field is isogenous to a principally polarized abelian variety, one reduces easily to the case when the given family is principally polarized.

Let $\mathfrak{A}_{g,n}$ be the fine moduli space of principally polarized abelian varieties/ $k$ of dimension $g$ with level $n$ structure (for some suitable $n$) and let $\psi\colon \mathscr{A}_{g,n} \to \mathfrak{A}_{g,n}$ be the universal family of abelian varieties. We have a diagram

$$\begin{tikzcd}
\mathscr{A}_{g,n} \arrow[rr, "f"] \arrow[dr, "\psi"'] & & \mathscr{A}'_{g,n} \arrow[rr, "g"] \arrow[dl, "\psi'"] & & \mathscr{A}_{g,n} \arrow[dl, "\psi"] \\
& \mathfrak{A}_{g,n} \arrow[rr, "F"'] & & \mathfrak{A}_{g,n} &
\end{tikzcd}$$

where $F$ is the absolute Frobenius, and $f$ is the relative Frobenius. The relative Cartier operator gives a map

$$\psi'_* f_* \omega_{\mathscr{A}_{g,n}/\mathfrak{A}_{g,n}} \to \psi'_* \omega_{\mathscr{A}'_{g,n}/\mathfrak{A}_{g,n}}$$

i.e., a map

$$\psi_* \omega_{\mathscr{A}_{g,n}/\mathfrak{A}_{g,n}} \to F^* \psi_* \omega_{\mathscr{A}_{g,n}/\mathfrak{A}_{g,n}} \cong (\psi_* \omega_{\mathscr{A}_{g,n}/\mathfrak{A}_{g,n}})^{\otimes p}.$$

The zeros of this map correspond precisely to the non-ordinary abelian varieties. But the map vanishes along the divisor of zeros of a section of $(\psi_*\omega_{\mathscr{A}_{g,n}/\mathfrak{A}_{g,n}})^{\otimes p-1}$, which is known to be an *ample* line bundle on $\mathfrak{A}_{g,n}$, (see [1] pg. 209, [18] pg. 210). In particular, the open set of ordinary abelian varieties contains no complete curves, a result which is due to Raynaud ([18], Chapter XI, Theorem 5.1).

*Proof of Theorem 1.* Let $X$ be a smooth, projective variety satisfying the hypothesis of Theorem 1 i.e., $X$ is ordinary, and $\Omega^1_X$ is trivial. It suffices to show that $X$ has some étale cover by an abelian variety; then it clearly has a Galois étale cover by one, and then Lemma (1.7) gives a Galois étale p-cover. We work by induction on $N(X) = \dim X - \dim \operatorname{Alb} X$. By Lemma (1.4), $X \to \operatorname{Alb} X$ is a smooth surjective morphism with connected fibers. In particular $N(X) \geqslant 0$, and $N(X) = 0$ if and only if $X$ is an abelian variety.

By Lemma (1.6), $X$ has a finite étale cover with non-zero Albanese variety. If $\operatorname{Alb} X = 0$, then such an étale cover $X' \to X$ has $N(X') < N(X)$, so we are done by induction. So we may as well assume $\operatorname{Alb} X \neq 0$.

By Lemmas (1.4) and (1.8), the fibers of $X \to \operatorname{Alb} X$ satisfy the hypothesis of the theorem. Thus if $s \in \operatorname{Alb} X$ is a closed point, and $X_s$ is the corresponding fiber, then $X_s$ has a finite étale cover $X'_s \to X_s$ with $\operatorname{Alb} X'_s \neq 0$, by Lemma (1.6). Thus $N(X'_s) < \dim X'_s = \dim X_s = N(X)$; hence by induction, $X'_s$ has a finite étale cover by an abelian variety. Hence $X_s$ has the same property; as remarked above, this implies that $X_s$ has an étale Galois p-cover by an abelian variety. Since any $p$-group is solvable, this cover is a tower of étale $\mathbb{Z}/p\mathbb{Z}$-covers. By successive applications of Lemma (1.9), we see that there is an etale cover $X' \to X$, such that if $\pi$: $X' \to A'$ is the Stein factorisation of the composite $X' \to \operatorname{Alb} X$, then $R^1\pi_*\mathbb{Q}_l$ is a locally constant étale sheaf of rank $2 \cdot N(X)$ on $A'$.

If $N(X') < N(X)$, we are done, so we may assume $\pi$: $X' \to A'$ is the Albanese map. Then by Lemma (1.10), $R^1\pi_*\mathbb{Q}_l$ becomes trivial on a finite étale cover of $A'$. Let $B \to A'$ be such an étale cover, $Y = B \times_{A'} X'$ and $\alpha$: $Y \to B$ the induced map. The Leray spectral sequence for $\alpha$ yields

$$0 \to H^1_{et}(B, \mathbb{Q}_l) \to H^1_{et}(Y, \mathbb{Q}_l) \to \Gamma(B, R^1\alpha_*\mathbb{Q}_l) \to H^2_{et}(B, \mathbb{Q}_l) \to H^2_{et}(Y, \mathbb{Q}_l).$$

The map $H^2_{et}(B, \mathbb{Q}_l) \to H^2_{et}(Y, \mathbb{Q}_l)$ is injective, since $Y \to B$ has multisections which are smooth over $k$. Also, by construction, if $r = 2N(X)$, then $\Gamma(B, R^1\alpha_*\mathbb{Q}_l) \cong \mathbb{Q}_l^r$. Hence $H^1_{et}(Y, \mathbb{Q}_l) \cong \mathbb{Q}_l^{2n}$, $n = \dim Y = \dim X$ i.e., $N(Y) = 0$, and $Y$ is an abelian variety.

## §2. Proof of Theorem 2

We recall the statement of Theorem 2 below. Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic $p$. We denote by $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, B^1_X)$ the class of the extension

$$0 \longrightarrow B^1_X \longrightarrow Z^1_X \xrightarrow{C} \Omega^1_X \longrightarrow 0$$

where $C$ is the Cartier operator.

THEOREM 2: *The following conditions on a smooth, projective variety $X$ in characteristic $p$ are equivalent:*

(i) $X = Y/G$, *where $Y$ is an ordinary abelian variety, $G$ a finite group acting freely on $Y$.*

(ii) *the extension class* $\xi = 0$, *and $\omega_X$ is a torsion line bundle or order prime to $p$.*

(iii) *the extension class* $\xi = 0$, *and $\omega_X$ is numerically trivial.*

*Proof:* Clearly (ii) $\Rightarrow$ (iii). Given (i), $\omega_X$ is the line bundle associated to a character of $G$, which must have order prime to $p$ in characteristic $p$. Also. The Cartier operator $C$: $H^0(Y, Z^1_Y) \to H^0(Y, \Omega^1_Y)$ is an isomorphism, since $Y$ is an ordinary abelian variety; hence (since $\Omega^1_Y$ is trivial)

$$0 \longrightarrow B^1_Y \longrightarrow Z^1_Y \xrightarrow{C} \Omega^1_Y \longrightarrow 0$$

has a *unique* splitting, which is equivariant for any automorphism of $Y$, and in particular for the $G$-action. Hence $\xi = 0$. Thus (i) $\Rightarrow$ (ii).

To prove (iii) $\Rightarrow$ (i), if $\xi = 0$, choose a splitting $\varphi$: $\Omega^1_X \to Z^1_X$. Then $\Lambda^i\varphi$: $\Omega^i_X \to Z^i_X$ splits the Cartier operator $C$: $Z^i_X \to \Omega^i_X$ for each $i$, and in particular for $i = n = \dim X$. The map $\Lambda^n\varphi$: $\omega_X \to F_*\omega_X$ is adjoint to a non-zero map $F^*\omega_X \to \omega_X$ i.e., $\omega_X^{\otimes(1-p)}$ has a non-zero section. Since $\omega_X$ is numerically trivial, this forces $\omega_X^{\otimes 1-p} \cong \mathcal{O}_X$. Thus, there is an étale cover $X' \to X$ on which $\omega_X$ becomes trivial.

Replacing $X$ by this cover, we may assume $\omega_X \cong \mathcal{O}_X$. The map $\varphi$: $\Omega^1_X \to Z^1_X$, together with the inclusion $Z^1_X \subset F_*\Omega^1_X$, give an adjoint map $F^*\Omega^1_X \to \Omega^1_X$. The determinant of this map is just the adjoint map $F^*\omega_X \to \omega_X$ given by the map $\Lambda^n\varphi$: $\omega_X \to F_*\omega_X$. But $F^*\omega_X \cong \omega_X \cong \mathcal{O}_X$; since $\Lambda^n\varphi \neq 0$, the adjoint $F^*\omega_X \to \omega_X$ must be non-zero, hence an isomorphism. Hence $F^*\Omega^1_X \to \Omega^1_X$ is an isomorphism. Now by Lemma (1.4) of [11], $\Omega^1_X$ becomes trivial on a finite étale cover of $X$. If $X' \to X$ is this étale cover, then $\Omega^1_X$ becomes trivial on a finite étale cover of $X$. If $X' \to X$ is this étale cover, then $\Omega^1_{X'}$ is trivial, and $X'$ is ordinary, since $X$ (and hence $X'$) has an $n$-form fixed by the Cartier operator. Hence by Theorem 1, $X'$ has a finite étale cover by an abelian variety. Hence $X$ has such a cover; since any étale cover of an abelian variety is an abelian variety, $X$ has a Galois étale cover by an abelian variety.

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## Appendix: Canonical liftings

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In this appendix, we construct the "canonical liftings" used in the foregoing paper, and prove certain nice properties of these liftings. The proofs depend on three propositions from deformation theory. The trick of always looking at liftings together with liftings of Frobenius, or of liftings compatible with Frobenius, was suggested by Deligne-Illusie's recent characteristic $p$ proof of the degeneration of the Hodge to De Rham spectral sequence.

Given a scheme $X$ over a perfect field $k$, a *lifting* of $X$ over $W_n(k)$ (or over $W(k)$) is a scheme $X_n$ over Spec $W_n(k)$) (or a scheme $\mathscr{X}$ over Spec $W(k)$) which is flat, together with an isomorphism of the closed fiber with $X$. Given such a lifting $X_n \to$ Spec $W_n(k)$ (or $\mathscr{X} \to$ Spec $W(k)$), a *lifting of Frobenius* morphism is a morphism $X_n \to X_n$ which is compatible with the Frobenius on Spec $W_n(k)$, and induces the absolute Frobenius morphism on $X/k$ (a similar definition holds for $\mathscr{X} \to$ Spec $W(k)$). Recall $B^1_X = F_*\mathcal{O}_X/\mathcal{O}_X \cong F_*$ (image $d$: $\mathcal{O}_X \to \Omega^1_X$). Let $T_X$ be the tangent sheaf of $X$.

PROPOSITION 1: *Let $X/k$ be a smooth variety, $(Y, F_Y)/W_{n-1}(k)$ a lifting of $X$ to $W_{n-1}(k)$ together with a lifting of Frobenius. Then the obstruction to the existence of a pair $(Z, F_Z)/W_n(k)$, consisting of a lifting $Z$ of $X$ to $W_n(k)$, and a lifting of Frobenius $F_Z$, such that $Z|_{W_{n-1}(k)} \cong Y$, and $F_{Z|Y} = F_Y$, is given by a class in $H^1(X, T_X \otimes B^1_X)$. The various liftings are a principal homogeneous space under $H^0(X, T_X \otimes B^1_X)$.*

PROPOSITION 2: *Let $X/k$ be a variety, $(Z, F_Z)$ a lifting of $X$ to $W_n(k)$ together with a lifting of Frobenius, and let $Y = Z|_{W_{n-1}(k)}$, $F_Y = F_{Z|Y}$. Suppose the Frobenius action is bijective on $H^i(X, \mathcal{O}_X)$ for $i = 1, 2$. Then given $\mathscr{L} \in$* Pic *$Y$ such that $F_Y^*\mathscr{L} \cong \mathscr{L}^{\otimes p}$, there exists a unique $\mathscr{M} \in$* Pic *$Z$ such that $\mathscr{M}|_Y \cong \mathscr{L}$, and $F_Z^*\mathscr{M} \cong \mathscr{M}^{\otimes p}$.*

PROPOSITION 3: *Let $X_1$, $X_2$ be smooth k-varieties, $(Z_1, F_{Z_1})$ and $(Z_2, F_{Z_2})$ lifts of $X_1$, $X_2$ respectively to $W_n(k)$, together with liftings of Frobenius. Suppose $Y_1 = Z_1|_{W_{n-1}(k)}$, $Y_2 = Z_2|_{W_{n-1}(k)}$, $F_{Y_i} = F_{Z_i}|_{Y_i}$. Let $\varphi$: $X_1 \to X_2$ be a k-morphism, and let $\psi$: $Y_1 \to Y_2$ be a $W_{n-1}(k)$ morphism with $\psi|_{X_1} = \varphi$. Suppose $\psi$ is compativle with the given liftings of Frobenius i.e., $F_{Y_2} \circ \psi = \psi \circ F_{Y_1}$. Then the obstruction to lifting $\psi$ to a morphism $\chi$: $Z_1 \to Z_2$ over $W_n(k)$, satisfying $\chi|_{Y_1} = \psi$, and $F_{Z_2} \circ \chi = \chi \circ F_{Z_1}$, is a class in $H^0(X_1, \varphi^*T_{X_2} \otimes B^1_{X_1})$. If $\chi$ exists, it is unique.*

Given the three propositions above, it is easy to prove the following theorem.

THEOREM 1: *Let $X$ be a smooth, projective variety over a perfect field $k$, such that $\Omega^1_X$ is trivial, and $X$ is ordinary. Then there is a scheme $\mathscr{X}$, smooth and projective over Spec $W(k)$, with special fiber $X$ together with a morphism $F_{\mathscr{X}}$: $\mathscr{X} \to \mathscr{X}$ over the Frobenius* Spec $W(k) \to$ Spec $W(k)$, *such that $F_{\mathscr{X}|X} = F_X$, the absolute Frobenius. Such a pair $(\mathscr{X}, F_{\mathscr{X}})$ is unique upto unique isomoriso-mormorphism (inducing the identity on $X$). $(\mathscr{X}, F_{\mathscr{X}})$ is called "canonical lifting" of $X$.*

2) *Let $X_1$, $X_2$ be smooth, projective $k$-varieties, with trivial contangent bundle, which are ordinary, and let $\varphi$: $X_1 \to X_2$ be a $k$-morphism. Then there is a unique morphism of canonical liftings $\psi$: $\mathscr{X}_1 \to \mathscr{X}_2$ compatible with the liftings of Frobenius i.e. satisfying $F_{\mathscr{X}_2} \circ \psi = \psi \circ F_{\mathscr{X}_1}$, with $\psi|_{X_1} = \varphi$.*

3) *Let $X$ be a smooth projective $k$-variety with $\Omega^1_X$ trivial, such that $X$ is ordinary, and let $(\mathscr{X}, F_{\mathscr{X}})$ be its canonical lifting. Let* Pic $(\mathscr{X})_{F_{\mathscr{X}}} = \{\mathscr{L} \in$ Pic $\mathscr{X} | F_{\mathscr{X}}^* \mathscr{L} \cong \mathscr{L}^{\otimes p}\}$. *Then* Pic $(\mathscr{X})_{F_{\mathscr{X}}} \to$ Pic $X$ *is an isomorphism.*

*Proof of the Theorem*: 1) Existence of the lifting: suppose we have already constructed a lifting $(Y, F_Y)$ over $W_{n-1}(k)$, and proved uniqueness. By Proposition 1, the obstruction to construct-ing a lifting $(Z, F_Z)$ over $W_n(k)$ with $Z|_{W_{n-1}(k)} \cong Y$, and $F_Z|_Y \cong F_Y$, is a class in $H^1(X, T_X \otimes B^1_X)$, where $T_X$ is the tangent sheat of $X$, and $B^1_X = F_* \mathcal{O}_X / \mathcal{O}_X$. Since $T_X$ is trivial,

$$H^1(X, T_X \otimes B^1_X) \cong H^0(X, T_X) \otimes H^1(X, B^1_X).$$

Consider the exact sequence

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to B^1_X \to 0.$$

Since the Frobenius action on $H^i(X, \mathcal{O}_X)$ is bijective, since $X$ is ordinary (see Lemma (1.1) in the accompanying paper), $H^i(X, B^1_X) = 0$ for all $i$. Hence the obstruction to lifting $(Y, F_Y)$ to $(Z, F_Z)$ vanishes. The lifting is unique, because $H^0(X, T_X \otimes B^1_X) = H^0(X, T_X) \otimes H^0(X, B^1_X) = 0$ (again using triviality of $T_X$).

Hence there exists a formal scheme $\hat{\mathscr{X}} \to$ Sp $f$ $W(k)$ lifting $X/k$, together with a lifting $F_{\hat{\mathscr{X}}}$ of the Frobenius, which is unique upto unique isomorphism. From Proposition 2, if $\mathscr{L} \in$ Pic $X$ is ample, since $F_X^* \mathscr{L} \cong \mathscr{L}^{\otimes p}$, we can lift $\mathscr{L}$ to a formal line bundle $\hat{\mathscr{M}} \in$ Pic $(\hat{\mathscr{X}})$, satisfying $F_{\mathscr{X}}^* \hat{\mathscr{M}} \cong \hat{\mathscr{M}}^{\otimes p}$, and such an $\hat{\mathscr{M}}$ is unique. By Grothendieck's algebraization theorem ([3] EGA III) $\hat{\mathscr{X}}$ is algebraizable i.e., there exists a smooth projective scheme $\mathscr{X}$ over Spec $W(k)$ whose formal completion along the closed fiber is $\hat{\mathscr{X}}$. Further the morphism $F_{\hat{\mathscr{X}}}$ is obtained by completion from a morphism $F_{\mathscr{X}}$: $\mathscr{X} \to \mathscr{X}$ of schemes. This proves 1).

2) Lifting of morphisms: Let $\varphi$: $X_1 \to X_2$ be a morphism, and let $\psi$: $Y_1 \to Y_2$ be a morphism lifting $\varphi$ to the canonical lifts $Y_i$ of $X_i$ over $W_{n-1}(k)$, such that $\psi$ is compatible with lifts of Frobenius. According to Proposition 3, $\psi$ is unique, and if $Z_i$ are the canonical lifts over $W_n(k)$, the obstruction to lifting $\psi$ to a morphism $\mathscr{X}$: $Z_1 \to Z_2$, compatible with the liftings of Frobenius, is a class in $H^0(X_1, \varphi^* T_{X_2} \otimes B^1_{X_1})$. Since $T_{X_2}$ is trivial, so is $\varphi^* T_{X_2}$, and so

$$H^0(X_1, \varphi^* T_{X_2} \otimes B^1_X) = H^0(X_1, \varphi^* T_{X_2}) \otimes_k H^0(X_1, B^1_{X_1}) = 0, \text{ since } X_1 \text{ is ordinary.}$$

Hence a lifting $\mathscr{X}$: $Z_1 \to Z_2$ exists which is compatible with Frobenius, and is unique, by Proposition 3.

3) Lifting of line bundles: This is immediate from Proposition 2, and the Grothendieck Existence Theorem.

## The Canonical lifting for ordinary abelian varieties

If $X$ is an ordinary abelian variety over a perfect field of characteristic $p$, then from the Theorem, there is a projective smooth $W(k)$-scheme $\Pi: \mathscr{X} \to \operatorname{Spec} W(k)$ with special fiber $X$. There is a unique lifting of the origin of $X$ to a section of $\Pi$ invariant under the lifting $F_{\mathscr{X}}$ of Frobenius to $\mathscr{X}$. Similarly, the multiplication $\mu: X \times X \to X$ and inverse $\gamma: X \to X$ lift uniquely to morphisms $\tilde{\mu}: \mathscr{X} \times_{W(k)} \mathscr{X} \to \mathscr{X}$ and $\tilde{\gamma}: \mathscr{X} \to \mathscr{X}$ over $W(k)$, compatible with the liftings of Frobenius; the uniqueness of liftings then forces $\tilde{\mu}$ to be a group law on $\mathscr{X}/W(k)$ with the given section as 0-section and $\tilde{\gamma}$ as the inverse. In particular $\mathscr{X}$ is an abelian scheme over $W(k)$ in a canonical way.

Let $f_m: X \to X_1^{(m)}$ be the $m$th iterated relative Frobenius morphism over $k$ i.e., $f_m$ is the quotient modulo the connected subgroup scheme of $X(p^m)$, the kernel of multiplication by $p^m$ on $X$. Let $K_m = \ker f_m$. Then $X(p^m) = K_m \times K_m'$ is étale, and isomorphic (over $\bar{k}$) to $(\mathbb{Z}/p^m\mathbb{Z})^n$, since $X$ is ordinary of dimension $n$. Let $V_m: X \to X_2^{(m)}$ be the quotient modulo $k_m'$. Then $X_1^{(m)}, X_2^{(m)}$ are ordinary abelian varieties over $k$. Let $\mathscr{X}_1^{(m)}, \mathscr{X}_2^{(m)}$ be their canonical liftings, with the canonical abelian scheme structure given above. There are unique liftings of $f_m$, $V_m$ to isogenies $\tilde{f}_m: \mathscr{X} \to \mathscr{X}_1^{(m)}$, $\tilde{V}_m: \mathscr{X} \to \mathscr{X}_2^{(m)}$ such that $\tilde{f}_m$, $\tilde{V}_m$ are compatible with the respective liftings of Frobenius. Since multiplication by $p^m$ on $X$ factors through both $f_m$ and $V_m$, multiplication by $p^m$ on $\mathscr{X}$ factors through $\tilde{f}_m$ and $\tilde{V}_m$ – indeed, if $f_m': X_1^{(m)} \to X$ is such that $f_m' \circ f_m$ is multiplication by $p^m$, $f_m'$ lifts to $\tilde{f}_m': \mathscr{X}_1^{(m)} \to \mathscr{X}$, and $\tilde{f}_m' \circ \tilde{f}_m$ lifts multiplication by $p_m$ on $X$ and is compatible with $F_{\mathscr{X}}$, as-does multiplication by $p^m$ on $\mathscr{X}$ (the argument for $\tilde{V}_m$ is similar). Thus if $\mathscr{X}(p^m)$ is the kernel of multiplication by $p^m$, and $K_1^{(m)} = \ker \tilde{f}_m$, $K_2^{(m)} = \ker \tilde{V}_m$, then $K_1^{(m)}, K_2^{(m)}$ are flat $W(k)$-group schemes of order $p^{mn}$, such that $K_2^{(m)}$ is étale, while the special fiber of $K_1^{(m)}$ is connected, so that $K_1^{(m)}$ is connected. Thus $K_1^{(m)} \cap K_2^{(m)} = (0)$, and since $K_i^{(m)} \subset \mathscr{X}(p^m)$, we must have $K_1^{(m)} \oplus K_2^{(m)} \cong \mathscr{X}(p^m)$. Multiplication by $p$ on $\mathscr{X}$ carries $K_1^{(m)}$ into $K_1^{(m-1)}$ and $K_2^{(m)}$ into $K_2^{(m-1)}$, since a quotient of a finite etale group scheme must be etale, and a quotient of a connected group scheme must be connected. Since $\mathscr{X}(p^m)$ maps onto $\mathscr{X}(p^{m-1})$, $K_i^{(m)} \to K_i^{(m-1)}$ must be onto. Thus the $p$-divisible group of $\mathscr{X}$ is the product of an etale $p$-divisible group and a connected $p$-divisible group. This is the standard property characterising the canonical lifting of an ordinary abelian variety.

*Proof of Proposition 1*: Let $A$ be a commutative $k$-algebra. A *lifting* of $A$ over $W_n(k)$is a flat $W_n(k)$-algebra $B$, together with an isomorphism $B/pB \cong A$. Then for $0 < i\, n$, $p^iB/p^{i+1}B \cong B/pB \cong A$ as $B$-modules. We have the following elementary (and standard) observations:

i) if $\varphi: B \to B$ is a $W_n(k)$-algebra automorphism satisfying $\varphi(b) \equiv b \pmod{p^{n-1}B}$, for all $b \in B$, then there is a unique $k$-derivation $\delta: A \to A$ such that $\varphi(b) = b + p^{n-1}\delta(\bar{b})$, where $p^{n-1}B \cong B/p\,B \cong A$, and $b \to \bar{b} \in B/p\,B \cong A$.

ii) If $F: B \to B$ is a ring homomorphism restricting to the Frobenius on $W_n(k)$, and inducing Frobenius on $B/p\,B \cong A$, then there is a unique function $\psi: B \to B/p^{n-1}B$ such that $F(b) = b^p + p\psi(b)$, where

$$\psi(b_1 + b_2) = \psi(b_1) + \psi(b_2) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \bar{b}_1^i \bar{b}_2^{p-i}$$

$$\psi(b_1 b_2) = \bar{b}_1^p \psi(b_2) + \bar{b}_2^p \psi(b_1) + p\psi(b_1)\psi(b_2)$$

where $b_i \to \bar{b}_i \in B/p^{n-1}B \cong pB$. Such an $F$ is called a lifting of Frobenius to the lifting $B$.

iii) If $F$: $B \to B$ is a lifting of Frobenius to the lifting $B$, then any other lifting of Frobenius $F'$ to $B$ such that $F' \equiv F(\text{mod } p^{n-1}B)$ is of the form $F'(b) = F(b) + p^{n-1}\eta\,(\bar{b})$ for a unique function $\eta$: $A \to A$ satisfying

$$\eta(\bar{b}_1 + \bar{b}_2) = \eta(\bar{b}_1) + \eta(\bar{b}_2)$$

$$\eta(\bar{b}_1\bar{b}_2) = \bar{b}_1^p\eta(\bar{b}_2) + \bar{b}_2^p\eta(\bar{b}_1).$$

Since $F$, $F'$ agree on $W_n(k)$, $\eta$ vanishes on $k \subset A$. Thus if $F$: $A \to A$ is the Frobenius, and we denote by $F_*(A)$ the resulting $A$-module structure on $A$, then $\eta$ corresponds uniquely to an element of $\text{Hom}_A(\Omega^1_{A/k}, F_*(A))$, which we again denote by $\eta$.

iv) Let $\varphi$: $B \to B$ be an automorphism such that $\varphi$ induces the identity on $B/p^{n-1}B$, so that $\varphi(b) = b + p^{n-1}\,\delta(\bar{b})$ for a $k$-derivation $\delta$: $A \to A$, as in (i). Let $F$: $B \to B$ be a lifting of Frobenius, $F' = \varphi^{-1} \circ F \circ \varphi$ another lifting such that $F' \equiv F\,(\text{mod } p^{n-1}B)$, as in (iii). If $F' = F + p^{n-1}\eta$ then $\eta(\bar{b}) = \delta(\bar{b})^p$ for all $\bar{b} \in A$; hence $\eta$ is the image of $\delta$ under the natural map $\text{Hom}_A(\Omega^1_{A/k}, A) \to \text{Hom}_A(\Omega^1_{A/k}, F_*(A))$ induced by $A \to F_*(A)$. This follows from the calculation

$$\begin{aligned}
\varphi^{-1}F\varphi(b) &= \varphi^{-1}F(b + p^{n-1}\,\delta(\bar{b}))\\
&= \varphi^{-1}(F(b) + p^{n-1}\,\delta(\bar{b})^p)\\
&= F(b) - p^{n-1}\,\delta(\overline{F(b)}) + p^{n-1}\,\delta(\bar{b})^p\\
&= F(b) + p^{n-1}\,\delta(\bar{b})^p,
\end{aligned}$$

since $\overline{F(b)}$ is a $p$th power in $A$, and $\delta$: $A \to A$ a derivation.

v) Assume now that $A$ is reduced, and let $B_1$, $B_2$ be two isomorphic lifts of $A$ over $W_n(k)$, such that there exist liftings $F_1$, $F_2$ of the Frobenius over $W_n(k)$. Suppose $\varphi$: $B_1 \to B_2$ is an isomorphism inducing the identity on $A \cong B_i/pB_i$, and assume $\varphi F_1 - F_2\varphi \equiv 0\ (\text{mod } p^{n-1}B_2)$. Then by (iii) above, $\varphi F_1 - F_2\varphi = p^{n-1}\eta$ where $\eta$: $A \to F_*(A)$ is a derivation i.e., $\eta \in \text{Hom}_A(\Omega^1_{A/k}, F_*(X))$. By (iv), if we change $\varphi$ to an isomorphism $\varphi' \equiv \varphi(\text{mod } p^{n-1}B_2)$, then $\varphi' = \varphi + p^{n-1}\delta$ for some derivation $\delta$: $A \to A$ i.e.,

$$\delta \in \text{Hom}_A(\Omega^1_{A/k}, A); \quad \text{and if} \quad \varphi'F_1 - F_2\varphi' = p^{n-1}\eta', \quad \text{then} \quad \eta' = \eta + (\delta)^p,$$

where $(\delta)^p$ is the image of $\delta$ under

$$\text{Hom}_A(\Omega^1_{A/k}, A) \to \text{Hom}_A(\Omega^1_{A/k}, \ F_*(A)).$$

Since $A$ is reduced, $A \to F_*(A)$ is injective. Hence there exists at most 1 choice of $\varphi$ satisfying $\varphi F_1 = F_2\varphi$. Further, the image of $\eta$ in $\text{Hom}_A(\Omega^1_{A/k}, F_*(A)/A)$ is independent of the choice of $\varphi$.

vi) If $A \cong k[x_1, \ldots, x_m, f_0^{-1}]/(f_1, \ldots, f_r)$ where $f_1, \ldots, f_r$ form a regular sequence (so that $(f_1, \ldots, f_r)$ has height $r$) in the localised polynomial ring, then $A$ has a lifting over $W_n(k)$ for any $n > 0$ to a complete intersection quotient of $W_n(k)[x_1, \ldots, x_m, g^{-1}]$, where $g$ is any lift of $f_0$ (the localised polynomial $W_n(k)$-algebra is independent of the lift $g$, upto isomorphism). If $A$ is smooth over $k$, then any such lifting is smooth over $W_n(k)$, and hence unique upto isomorphism by formal smoothness: if $B_1$, $B_2$ are smooth lifts, the identity map $A \to A$ lifts to $W_n(k)$-algebra maps $\varphi: B_1 \to B_2$, $\psi: B_2 \to B_1$. Since $\varphi\psi$ and $\psi\varphi$ are congruent to the identity modulo a nilpotent ideal, they must be isomorphisms. If $\bar{\varphi}: B_1/p^{n-1} B_1 \to B/p^{n-2}B_2$ is a given isomorphism, then we can lift it to $\varphi: B_1 \to B_2$ by formal smoothness, and again $\varphi$ must be an isomorphism since we can similarly lift $\bar{\varphi}^{-1}$. Finally, if $B$ is a lifting of a smooth $k$-algebra $A$ over $W_n(k)$, the absolute Frobenius $F: A \to A$ lifts to a homomorphism $F_B: B \to B$ restricting to the Frobenius on $W_n(k)$ (this lifting is of course not unique); further if $E: B/p^{n-1}B \to B/p^{n-1}B$ is a given lift, then we can choose $F_B$ lifting $E$.

vii) Now let $X$ be a smooth $k$-variety, admitting a lifting $(Y, F_Y)$ over $W_{n-1}(k)$. Choose an open cover of $X$ by affine open sets $U_i$ such that each $U_i$ is a complete intersection in an open subset of some affine space which is the complement of a hypersurface (thus $U_i \cong \operatorname{Spec} k[x_1, \ldots, x_m, f_0^{-1}]/(f_1, \ldots, f_r)$ for a regular sequence $f_1, \ldots, f_r$ in the localised polynomial ring as in (vi) above). Let $V_i \subset Y$ be the corresponding open sets. By (vi), we can choose liftings $W_i$ of $V_i$ over $W_n(k)$, together with liftings of Frobenius $F_i: W_i \to W_i$ over the Frobenius of $W_n(k)$, such that $F_i|V_i = F_Y|_{V_i}$. Also, if $W_{ij} \subset W_i$ is the open subscheme corresponding to $U_i \cap U_j$, there are isomorphisms $\varphi_{ij}: W_{ij} \to W_{ji}$, such that $\varphi_{ij}|_{V_{ij}}$ is the identity, where $V_{ij} = V_i \cap V_j = V_{ji}$; we may assume them chosen so that $\varphi_{ji} = \varphi_{ij}^{-1}$.

Now $\varphi_{ij} \circ F_i|_{W_{ij}} \circ \varphi_{ji}$ and $F_j|_{W_{ji}}$ are two liftings of Frobenius on $W_{ji}$, agreeing on $V_i \cap V_j$, and hence differ by class $\eta_{ij} \in \operatorname{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega^1_{\mathcal{O}_{U_{ij}}/k}, F_*\mathcal{O}_{U_{ij}})$. The image $\bar{\eta}_{ij}$ of $\eta_{ij}$ in $\operatorname{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega^1_{\mathcal{O}_{U_{ij}}/k}, F_*\mathcal{O}_{U_{ij}}) = H^0(U_{ij}, T_X \otimes B^1_X)$ is independent of the choice of the isomorphisms $\varphi_{ij}$. Hence $\bar{\eta}_{ij} + \bar{\eta}_{jk} = \bar{\eta}_{ik}$ in $H^0(U_{ijk}, T_X \otimes B^1_X)$, where $U_{ijk} = U_i \cap U_j \cap U_k$, and $\{\bar{\eta}_{ij}\}$ give a Cech 1-cocycle representing a class $\xi \in H^1(X, T_X \otimes B^1_X)$. We claim $\xi$ is precisely the obstruction to lifting $(Y, F_Y)$ to $(Z, F_Z)$ over $W_n(k)$. Indeed, if a lifting $(Z, F_Z)$ exists, let $W_i \subset Z$ be the open subscheme corresponding to $U_i$, and $\varphi_{ij}: W_i \cap W_j \to W_i \cap W_j$ the identity map; then $\eta_{ij} = 0$. Conversely, if $\xi = 0$, then there exist $\bar{\eta}_i \in H^0(U_i, T_X \otimes B^1_X)$ with $\bar{\eta}_{ij} = \bar{\eta}_i - \bar{\eta}_j$. Lift $\bar{\eta}_i$ to $\eta_i \in H^0(U_i, T_X \otimes F_*\mathcal{O}_X) = \operatorname{Hom}_{\mathcal{O}_{U_i}}(\Omega^1_{\mathcal{O}_{U_i}/k}, F_*\mathcal{O}_{U_i})$. Then $\eta_{ij} = \eta_i - \eta_j + \mu^p_{ij}$, for some $\mu_{ij} \in \operatorname{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega^1_{\mathcal{O}_{U_{ij}}/k}, \mathcal{O}_{U_{ij}})$. Changing the lifting of Frobenius $F_i$ on $W_i$ to $F'_i$, where $(F'_i)^* - (F_i)^*$ corresponds to $-\eta_i$, and changing $\varphi_{ij}$ to $\varphi'_{ij}$ so that $(\varphi'_{ij})^* - \varphi^*_{ij}$ corresponds to $-\mu_{ij}$, we see that $\varphi'_{ij} F'_i \mathcal{O}'_{ji} = F'_j$ on $W_{ji}$. Hence $\varphi'_{jk}\varphi'_{ij} F'_i \varphi'_{ji}\varphi'_{kj} = F'_k = \varphi'_{ik}F'_i\varphi'_{ki}$ on $W_{ki} \cap W_{kj}$. This forces $\varphi'_{jk}\varphi'_{ij} = \varphi'_{ik}$ on $W_{ij} \cap W_{ik}$, by the uniqueness of the map intertwining the respective Frobenius maps. Thus the $W_i$ patch together to give a scheme $Z/W_n(k)$, such that the local lifts $F'_i$ of Frobenius patch to a morphism $F_Z$ $Z$.

We leave it to the reader to verify that the collection of all such $(Z, F_Z)$ is a principal homogeneous spacunder $H^0(X, T_X \otimes B^1_X)$.

*Proof of Proposition 2*: The ideal sheaf of $Y$ on $Z$ is isomorphic to $\mathcal{O}_X$ as an $\mathcal{O}_Z$-module. We have an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}^*_Z \to \mathcal{O}^*_Y \to 0$$

giving a cohomology sequence (compatible with the Frobenius actions)

$$\begin{array}{ccccccc} H^1(X, \mathcal{O}_X) & \xrightarrow{\alpha} & H^1(X, \mathcal{O}^*_Z) & \longrightarrow & H^1(Y, \mathcal{O}^*_Y) & \xrightarrow{\delta} & H^2(X, \mathcal{O}_X) \\ & & \wr\Vert & & \wr\Vert & & \\ & & \operatorname{Pic} Z & & \operatorname{Pic} Y & & \end{array}$$

Now $F_Y^*\mathscr{L} \cong \mathscr{L}^{\otimes p}$, and $\delta$ is a homomorphism compatible with Frobenius, so that $F^*\delta[\mathscr{L}] = \delta[F_Y^*\mathscr{L}] = \delta([\mathscr{L}^{\otimes p}] = p.\ \delta[\mathscr{L}] = 0$. Since $F^*\colon H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)$ is injective, $\delta[\mathscr{L}] = 0$. Hence there exists $\mathscr{M} \in \operatorname{Pic} Z$ with $\mathscr{M}|_Y = \mathscr{L}$. Now $F_Z^*\mathscr{M} \otimes (\mathscr{M}^{-1})^{\otimes p} = \alpha(t)$, for some $t \in H^1(X, \mathcal{O}_X)$. Since $F^*\colon H^1(X, \mathcal{O}_X)$ is injective, hence bijective, $t = F^*s$ for some $s \in H^1(X, \mathcal{O}_X)$. Hence $F_Z^*\alpha(s) = F_Z^*\mathscr{M} \otimes (\mathscr{M}^{-1})^{\otimes p}$. Since $\alpha(s)^{\otimes p} = 0$ (as $\mathcal{O}_X \subset \mathcal{O}_Z^*$ is of exponent $p$), $F_Z^*(\mathscr{M} \otimes \alpha(s)^{-1}) \cong (\mathscr{M} \otimes \alpha(s)^{-1})^{\otimes p}$. Thus $\mathscr{M} \otimes \alpha(s)^{-1}$ is the desired lifting. This is unique, because for any $s' \in H^1(X, \mathcal{O}_X)$ with $\alpha(s') \neq 0$, we claim $F_Z^*\alpha(s') \neq 0$ (indeed, $F^*$ is bijective on $H^1(X, \mathcal{O}_X)$, and maps ker $\alpha$ into itself, and is hence bijective on ker $\alpha$). This implies $F_Z^*\alpha(s') \otimes (\alpha(s')^{-1})^{\otimes p} \neq 0$.

*Proof of Proposition 3*: As in the proof of Proposition 1, we first make an appropriate local calculation with rings. Let $A_1$, $A_2$ be $k$-algebras, $B_1$, $B_2$ liftings to $W_n(k)$, $\varphi\colon A_1 \to A_2$ a $k$-algebra map, and $\psi, \psi'\colon B_1 \to B_2$ $W_n(k)$-algebra maps lifting $\varphi$ such that $\psi \equiv \psi' \pmod{p^{n-1}}$. Then $\psi' = \psi + p^{n-1}\delta$ where $\delta\colon A_1 \to A_2$ is a $k$-derivation.

Now assume given lifts $F_i\colon B_i \to B_i$ of the absolute Frobenius on the $A_i$. Then $F_1(b) = b^p + p\,\theta_1(b)$, $F_2(b') = (b')^p + p\,\theta_2(b')$, for uniquely defined functions

$$\begin{array}{l} \theta_i\colon B_i \to B_i/p^{n-1}B_i \\ \qquad\qquad\ \ \wr\Vert \\ \qquad\qquad\ \ p\,B_i \end{array}$$

Here

$$\theta_1(b_1 + b_2) = \theta_1(b_1) + \theta_1(b_2) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \bar{b}_1^i \bar{b}_2^{p-i},$$

$$\theta_1(b_1 b_2) = \bar{b}_1^p\theta_1(b_2) + \bar{b}_1^p\theta_1(b_1) + p\,\theta_1(b_1)\,\theta_1(b_2),$$

where $b_i \to \bar{b}_i$ in $B_1/p^{n-1}B_1$; the function $\theta_2$ satisfies similar identities. Hence if $\psi\colon B_1 \to B_2$ is a homomorphism of $W_n(k)$-algebras satisfying $\psi \otimes \mathbb{Z}/p\mathbb{Z} = \varphi$, then

$$\psi F_1 - F_2\psi = p(\psi\theta_1 - \theta_2\psi),$$

where $\psi\theta_1 - \theta_2\psi\colon B_1 \to B_2/p^{n-1}B_2$ is an additive homomorphism, and satisfies

$$\begin{aligned}(\psi\theta_1 - \theta_2\psi)(b_1 b_2) &= \overline{\psi(b_1)}^p[(\psi\theta_1 - \theta_2\psi)(b_2) + \theta\,\overline{\psi(b_2)}^p(\psi\theta_1 - \theta_2\psi)(b_1)] \\ &\quad + p[\psi\theta_1(b_1)(\psi\theta_1 - \theta_2\psi)(b_2 + \theta_2\psi(b_2)(\psi\theta_1 - \theta_2\psi)(b_1)].\end{aligned}$$

In particular, if we know that $\psi\,F_1 \equiv F_2\psi \pmod{p^{n-1}}$, then $\psi\,F_1 - F_2\psi = p^{n-1}\eta$, where $\eta\colon A_1 \to F_*A_2$ is a $k$-derivation. Here $F_*A_2$ is the $A_1$-module where $a \cdot b = (a)^p b$ for $a \in A_1$, $b \in A_2$.

If we change $\psi$ to $\psi'\colon B_1 \to B_2$, with $\psi \equiv \psi' \pmod{p^{n-1}}$, and $\psi' = \psi + p^{n-1}\delta$, then we compute

$$\begin{aligned}(\psi'\,F_1 - F_2\psi') &= (\psi + p^{n-1}\delta)F_1 - F_2(\psi + p^{n-1}\delta) \\ &= \psi\,F_1 + p^{n-1}\delta F_1 - F_2\psi - F_2(p^{n-1}\delta) \\ &= (\psi\,F_1 - F_2\psi) - p^{n-1}\delta^p.\end{aligned}$$

Hence $\eta' - \eta$ lies in the image of

$$\mathrm{Hom}_{A_1}(\Omega^1_{A_1/k}, A_2) \to \mathrm{Hom}_{A_1}(\Omega^1_{A_1/k}, F_* A_2).$$

If $A_2$ is reduced, this map is injective. Hence if $\bar{\psi}: \bar{B}_1 \to \bar{B}_2$ is given, where $\bar{B}_i = B_i \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$, compatible with the liftings of Frobenius to the $\bar{B}_i$, there is atmost 1 choice of $\psi: B_1 \to B_2$ lifting $\bar{\psi}$ which will be compatible with the liftings of Frobenius to $B_i$. Also, the image $\bar{\eta}$ of $\eta$ in $\mathrm{Hom}_{A_1}(\Omega^1_{A_1/k}, F_* A_2/A_2)$ is independent of $\psi$.

As usual, if $A_1$, $A_2$ are smooth over $k$, $B_1$, $B_2$ given liftings, $\varphi: A_1 \to A_2$ a given $k$-algebra map, $\bar{\psi}: \bar{B}_1 \to \bar{B}_2$ a given lifting of $\varphi$, where $\bar{B}_i = B_i \otimes \mathbb{Z}/p^{n-1}\mathbb{Z}$, then $\bar{\psi}$ admits lifts $\psi$: $B_1 \to B_2$, by formal smoothness of $B_1/W_n(k)$.

Now let $X_1$, $X_2$ be smooth $k$-varieties, $\varphi: X_1 \to X_2$ a morphism, $Z_i$ a lifting of $X_i$ over $W_n(k)$, $F_i: Z_i \to Z_i$ a lifting of Frobenius to $Z_i$. Let $Y_i = Z_i|_{W_{n-1}(k)}$, $F_{Y_i} = F_i|_{Y_i}$, and let $\bar{\psi}: Y_1 \to Y_2$ be a $W_{n-1}(k)$-morphism lifting $\varphi$ such that $\bar{\psi} F_{Y_1} {}_{Y_2}\bar{\psi}$. Let $\{U_i\}_{i\in I}$ be an affine open cover of $X_2$, $\{V_j\}_{j\in J}$ an affine open cover of $X_1$ subordinate to the open cover $\{\varphi^{-1}(U_i)\}_{i\in I}$. Choose a mapping $\pi: J \to I$ of index sets so that $V_j \subset \varphi^{-1}(U_{\pi(j)})$. Let $W_j \subset Z_1$ be the open subscheme determined by $V_j$, and let $S_i \subset Z_2$ be the open subschemes determined by $U_i \subset Z_2$. Let $\bar{W}_j \subset Y_1$, $\bar{S}_i \subset Y_2$ be the corresponding open subschemes over $W_{n-1}(k)$. The given map $\bar{\psi}$: $Y_1 \to Y_2$ compatible with $F_{Y_1}$ and $F_{Y_2}$ gives a map $\bar{\psi}_j: \bar{W}_j \to \bar{S}_{\pi(j)}$ compatible with $F_{Y_1}|_{\bar{W}_j}$ and $F_{Y_2}|_{\bar{S}_{\pi(j)}}$. The associated map on rings $F_1|_{W_j}\psi_j^* - \psi_j^* F_2|^*_{S_{\pi(j)}}$ gives a class $\eta_j \in \mathrm{Hom}_{\mathcal{O}_{U_{\pi(j)}}}(\Omega^1_{\mathcal{O}_{U_{\pi(j)}}/k}$, $F_*\mathcal{O}_{V_j})$, whose image $\bar{\eta}_j$ in $\mathrm{Hom}_{\mathcal{O}_{U_{\pi(j)}}}(\Omega^1_{\mathcal{O}_{U_{\pi(j)}}/k}, F_*\mathcal{O}_{V_j}/\mathcal{O}_{V_j}) \cong H^0(V_j, \varphi^* T_{X_2} \otimes B^1_X)$, is independent of the lifting $\psi_j$ of $\bar{\psi}_j$. In particular, $\bar{\eta}_j = \bar{\eta}_k$ on $V_j \cap V_k$ i.e., $\{\bar{\eta}_j\}$ give a global section $\xi$ of $\varphi^* T_{X_2} \otimes B^1_{X_1}$.

Clearly $\xi = 0$ if there exists $\psi: Z_1 \to Z_2$ compatible with $F_{Z_1}$ and $F_{Z_2}$. Conversely if $\xi = 0$, then $\bar{\eta}_j = 0$ on $V_j$ for all $j \in J$. Hence there exists unique local lifts $\psi_j'$ of $\bar{\psi}_j$ such that $\psi_j'$: $W_j \to S_{\pi(j)}$ are compatible with $F_1|_{W_j}$ and $F_2|_{S_{\pi(j)}}$. In particular, since $\psi_j'$, $\psi_k'$ both give lifts of $\bar{\psi}$ over $W_j \cap W_k$ compatible with Frobenius, we must have $\psi_j' = \psi_k'$ ib $Q_j \cap W_k$ i.e., they patch to give $\psi: Z_1 \to Z_2$ compatible with $F_1$, $F_2$ and lifting $\bar{\psi}$.

## References


1. Ching-Li Chai: Compactification of Siegel moduli schemes, *London Math. Soc. Lect. Note Ser.* No. 107, Cambridge (1985).
2. V.G. Drinfeld, Coverings of p-adic symmetric regions, *Funct. Anal. Appl.* 10 (1976) 107–115.
3. A. Grothendieck, Elements de Geometrie Algebrique, III, *Publ. Math. IHES* 11 (1961), 17 (1963).
4. A. Grothendieck, Reventements etale et group fondamental (SGA 1), *Lect. Notes in Math.* No. 224, Springer-Verlag, Berlin (1971).
5. J.-I. Igusa, A fundamental inequality in the theory of Picard varieties, *Proc. Nat. Acad. Sci. USA* 41 (1955) 318–320.
6. J.-I. Igusa, On some problems in Abstract Algebraic Geometry, *Proc. Nat. Acad. Sci. USA* 41 (1955) 964–967.
7. J.-I. Igusa, On the structure of a certain class of Kähler varieties, *Am. J. Math.* 76 (1954) 669–678.
8. L. Illusia, Complex de De Rham-Witt et cohomologie cristalline, *Ann. Sci. Ecole Norm. Sup.* 12 (1979) 501–661.
9. L. Illusie and M. Raynaud: Les suites spectrales associees au complexe de De Rham-Publ. Math. IHES *57 (1983) 73–212.*

10. S. Kobayashi, Recent results in Complex Differential Geometry, *Jber. d. Dt. Math.-Verein* 83 (1981) 147–158.
11. H. Lange and U. Stuhler: Vektorbündel auf Kurven and Darstellungen der algebraischen Fundamentalgruppe, *Math. Zeit.* 156 (1977) 73–83.
12. V.B. Mehta and M. Nori: Semistable sheaves on homogeneous spaces and Abelian varieties, *Proc. Indian Acad. Sci. (Math. Sci.)* 93 (1984) 1–12.
13. V.B. Mehta and A. Ramanathan: Frobenius splitting and cohomology vanishing for Schubert varieties, *Ann. Math.* 122 (1985) 27–40.
14. W. Messing, The Crystals associated to Barsotti-Tate groups: with applications to Abelian schemes, *Lect. Notes in Math.* No. 264, Springer-Verlag, Berlin (1972).
15. J.-P. Serre and J. Tate: Good reduction of Abelian varieties, *Ann. Math.* 88 (1968) 492–517.
16. J.-P. Serre, Quelques propriétés des variétés abéliennes en caractéristique p. *Am. J. Math.* 80 (1958) 715–739.
17. A. Grothendieck, Groupes de Monodromie en Géometrie Algébrique, SGA 7, *Lect. Notes in Math.* No. 288, Springer-Verlag, Berlin (1972).
18. L. Moret-Bailly, Pinceaux de variétés abéliennes, *Astérisque* 129 (1985).
19. P. Deligne, L. Illusie: Relèvements modulo $p^2$ et décomposotion du complexe de de Rham, *Inventiones Mathematicae* 89 (1987) 247–270.